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# NONLINEAR STABILITY OF WAVEFRONTS FOR A DELAYED STAGE-STRUCTURED POPULATION MODEL ON A 2-D LATTICE

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**Abstract**

In this paper, we study the nonlinear stability of wavefronts in a delayed stage-structured population model on a 2-D spatial lattice. For all wavefronts with the speed

$$c > \max\{c(\eta_0), c_*(\theta)\},$$

where $\eta_0$ is some positive constant, $c_*(\theta) > 0$ is the critical wave speed and $\theta$ is the direction of propagation, we prove that these wavefronts are asymptotically stable, when the initial perturbation around the wavefronts decays exponentially as $i\cos\theta + j\sin\theta \to -\infty$, but it can be arbitrary large in other locations. This essentially improves the previous work with more strongly restricted wave speed and the small initial perturbation. Our approach adopted in this paper is the weighted energy method and the squeezing technique.

## 1. Introduction

In this paper, we consider a time-delayed population model with stage structure on a 2-D spatial lattice

$$\frac{dw_{i,j}(t)}{dt} = D[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)] - dw_{i,j}(t) + \varepsilon b(w_{i,j}(t-r)) \tag{1.1}$$

with the initial condition

$$w_{i,j}(s) = w^0_{i,j}(s), \quad s \in [-r, 0], \quad i, j \in \mathbb{Z}, \tag{1.2}$$

which describes the population density of a single species with stage structure in a 2-D patchy environment, where $D > 0$ and $d > 0$ are the diffusion and the death rate

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of the population, respectively, and $\varepsilon > 0$ represents the impact of the death rate for the immature. More details can be found in [1, 2, 3, 4]. The nonlinear function $b(w)$ denotes the birth rates of the mature population and satisfies the following hypotheses:
(H1) There exist 0 and $w^+ > 0$ such that $b(0) = 0$, $\varepsilon b(w^+) = dw^+$, $b \in C^2[0, w^+]$ and for all $w \in (0, w^+)$, $\varepsilon b(w) > dw$;
(H2) $\varepsilon b'(0) > d$ and for all $w \in [0, w^+]$, $0 \leq b'(w) \leq b'(0)$ and $b''(w) \leq 0$;
(H3) $d > \varepsilon b'(w^+)$.
As we know, the reaction term in a lattice equation, which is some what like the birth function in our equation (1.1), plays an important role to depict the character of the equation. Several interesting type of such functions are the logistic type $b(w) = w(1 - w)$ and Ricker type $b(w) = pwe^{-aw^q}$, which satisfy assumptions (H1)–(H3) under the suitable parameters (see, [5, 7, 8, 9] and the references therein). Recently, 2-D lattice dynamical systems have been paid close attention. Authors [10] studied the existence of traveling waves for the following bistable systems

$$\begin{aligned}\frac{dw_{i,j}(t)}{dt} &= D[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)] \\ &\quad + f(w_{i,j}(t)).\end{aligned} \tag{1.3}$$

For (1.3) with the monostable nonlinearity, authors [11] showed that the existence, uniqueness and asymptotic behavior of traveling waves. In [1], authors used the methods developed in [3, 12] for 1-D lattice differential equations to investigate the spreading speeds and traveling waves of the following 2-D lattice equation

$$\begin{aligned}\frac{dw_{i,j}(t)}{dt} &= D[w_{i+1,j}(t) + w_{i-1,j}(t) + w_{i,j+1}(t) + w_{i,j-1}(t) - 4w_{i,j}(t)] - dw_{i,j}(t) \\ &\quad + \sum_{l=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \beta(l)\gamma(q)b(w_{i+l,j+q}(t-r)).\end{aligned} \tag{1.4}$$

Authors [4] studied the uniqueness and asymptotic behavior of (critical) traveling waves for (1.4). Especially, when the immature population is non-mobile, (1.4) could reduce to (1.1). In [2], the stability of traveling waves for (1.1) with the wave speed $c > 4D(e-1) + 2(\varepsilon b'(0) - d)$ (larger than the minimal wave speed $c_*(\theta)$) and the small initial perturbation was investigated by using the weighted energy method. It is well known that for delayed reaction diffusion equations (see, for example [13, 14, 15, 16] and some references therein), the small initial perturbation can be improved to be arbitrarily large in a weighted energy Sobolev space by the comparison principle and the squeeze technique. Motivated by the above references [13, 14, 15, 16], we will discuss the nonlinear stability of wavefronts with the speed

$$c > \max\{c(\eta_0), c_*(\theta)\}, \quad \text{where} \quad c(\eta_0) = \min_{\eta \in (0, +\infty)} \frac{2D(e^\eta - 1) + 2(\varepsilon b'(0) - d)}{\eta},$$

in a delayed stage-structured population model on a 2-D spatial lattice when the initial perturbation around the wavefronts decays exponentially as $i\cos\theta + j\sin\theta \to -\infty$, but it can be arbitrary large in other locations. On the other hand, since $2D(e-1) + 2(\varepsilon b'(0) - d) = c(1) \geq c(\eta_0)$, then $4D(e-1) + 2(\varepsilon b'(0) - d) > \max\{c(\eta_0), c_*(\theta)\}$. Thus, the stability obtained in [2] is based on a stronger restriction on the wave speed. Our results essentially improve the stability results obtained in [2] for the wave speed $c > 4D(e-1) + 2(\varepsilon b'(0) - d)$ with the small initial perturbation.

Throughout this paper, $l_v^2$ denotes the weighted $l^2$-space with weight $0 < v(\xi) \in C(\mathbb{R})$ and a fixed $\theta \in [0, \pi/2]$, that is,

$$l_v^2 := \left\{ \zeta = \{\zeta_{i,j}\}_{i,j\in\mathbb{Z}},\ \zeta_{i,j} \in \mathbb{R} \ \middle|\ \sum_{i,j} v(i\cos\theta + j\sin\theta)\zeta_{i,j}^2 < \infty \right\}$$

and its norm is defined by

$$\|\zeta\|_{l_v^2} = \left( \sum_{i,j} v(i\cos\theta + j\sin\theta)\zeta_{i,j}^2 \right)^{1/2}, \quad \text{for} \quad \zeta \in l_v^2.$$

In particular, when $v \equiv 1$, we denote $l_v^2$ by $l^2$.

The rest of this paper is organized as follows. In Section 2, we introduce some basic results and then state our stability result. Section 3 is devoted to proving our main result.

## 2. Preliminaries and main theorem

In this section, we first recall some known results, then define a weighted function and state our stability result.

A wavefront of (1.1) is a solution of the form

$$w_{i,j}(t) = \phi(\xi) \tag{2.1}$$

satisfying the boundary conditions

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = w^+, \tag{2.2}$$

where $\phi$ is monotone and

$$\xi = i\cos\theta + j\sin\theta + ct, \quad \theta \in \left[0, \frac{\pi}{2}\right]. \tag{2.3}$$

Hence, the wave profile equation of (1.1) is given by

$$
\begin{aligned}
c\frac{d\phi(\xi)}{d\xi} &= D[\phi(\xi+\cos\theta)+\phi(\xi-\cos\theta)+\phi(\xi+\sin\theta)+\phi(\xi-\sin\theta)-4\phi(\xi)] \\
&\quad -d\phi(\xi)+\varepsilon b(\phi(\xi-cr)).
\end{aligned} \tag{2.4}
$$

Define

$$
\begin{aligned}
\Delta(c,\lambda) &= -c\lambda + D(e^{\lambda\cos\theta}+e^{-\lambda\cos\theta}+e^{\lambda\sin\theta}+e^{-\lambda\sin\theta}-4) \\
&\quad +\varepsilon b'(0)e^{-\lambda cr}-d.
\end{aligned} \tag{2.5}
$$

The following lemma and proposition come from Lemma 4.2 and Theorem 5.4 of [1], respectively.

**Lemma 2.1.** *Assume that $\varepsilon b'(0) > d$ holds. Then, for any fixed $\theta \in [0,\pi/2]$ there exist a unique pair of $c_*(\theta) > 0$ and $\lambda_*(\theta) > 0$ such that the following assertions hold.*
(i) $\Delta(c_*(\theta),\lambda_*(\theta)) = 0$, $\partial\Delta(c,\lambda)/\partial\lambda|_{c=c_*(\theta),\lambda=\lambda_*(\theta)} = 0$;
(ii) *For any $c \in (0, c_*(\theta))$ and $\lambda > 0$, $\Delta(c,\lambda) > 0$;*
(iii) *For any $c > c_*(\theta)$, $\Delta(c,\lambda) = 0$ has two positive roots $0 < \lambda_1 < \lambda_2$. Moreover, $\Delta(c,\lambda) < 0$ for any $\lambda \in (\lambda_1,\lambda_2)$.*

**Proposition 2.1.** *Assume that* (H1)–(H2) *hold. Then, for every $\theta \in [0,\pi/2]$, there exists $c_*(\theta) > 0$ such that for any $c \geq c_*(\theta)$,* (1.1) *admits a monotone wavefront $\phi(i\cos\theta + j\sin\theta + ct)$ satisfying the boundary conditions*

$$
\lim_{\xi\to-\infty}\phi(\xi) = 0 \quad \text{and} \quad \lim_{\xi\to+\infty}\phi(\xi) = w^+,
$$

*and for any $c \in (0, c_*(\theta))$, there are no non-trivial wavefront $\phi(i\cos\theta + j\sin\theta + ct)$ satisfying $\phi(\xi) \in [0, w^+]$. Moreover, for $c > c_*(\theta)$, $\phi$ satisfies*

$$
\lim_{\xi\to-\infty}\phi(\xi)e^{-\lambda_1\xi} = 1 \quad \textit{and} \quad \lim_{\xi\to-\infty}\phi'(\xi)e^{-\lambda_1\xi} = \lambda_1,
$$

*where $c_*(\theta)$ and $\lambda_1$ are defined as in* Lemma 2.1.

Let $\delta = (d - \varepsilon b'(w^+))/(2\varepsilon) > 0$. According to $b'(\phi(\xi)) \to b'(w^+)$ as $\xi \to +\infty$, there exists a large enough number $\xi_*$ such that for $\xi \geq \xi_*$,

$$
b'(\phi(\xi)) < b'(w^+) + \delta. \tag{2.6}
$$

Define the function

$$
c(\eta) = \frac{2D(e^\eta - 1) + 2(\varepsilon b'(0) - d)}{\eta}.
$$

Since $c(0+) = +\infty$ and $c(+\infty) = +\infty$, then there exists $\eta_0 \in (0, +\infty)$ such that

$$c(\eta_0) = \min_{\eta \in (0,+\infty)} \frac{2D(e^{\eta} - 1) + 2(\varepsilon b'(0) - d)}{\eta} > 0. \tag{2.7}$$

Now, we define a weight function $v(\xi)$ as

$$v(\xi) = \begin{cases} e^{-\eta_0(\xi - \xi_*)}, & \xi \le \xi_*; \\ 1, & \xi \ge \xi_*. \end{cases} \tag{2.8}$$

We are in the position to state our main result.

**Theorem 2.1.** *Assume that* (H1)–(H3) *hold. For any given wavefront* $\phi(i \cos\theta + j \sin\theta + ct)$ *with the wave speed*

$$c > \max\{c(\eta_0), c_*(\theta)\},$$

*if the initial data satisfies*

$$0 \le w_{i,j}^0(s) \le w^+ \quad \text{for} \quad s \in [-r, 0],\ i,\ j \in \mathbb{Z}, \tag{2.9}$$

*and the initial perturbation* $w_{i,j}^0(s) - \phi(i \cos\theta + j \sin\theta + cs)$ *is in* $C([-r, 0], l_v^2)$, *where* $v = v(i \cos\theta + j \sin\theta + ct)$ *is the weight function given in* (2.8), *then the solution* $\{w_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *of* (1.1) *and* (1.2) *satisfies*

$$0 \le w_{i,j}(t) \le w^+, \quad \text{for} \quad t \in [0, +\infty),\ i,\ j \in \mathbb{Z}$$

*and*

$$\{w_{i,j}(t) - \phi(i \cos\theta + j \sin\theta + ct)\}_{i,j\in\mathbb{Z}} \in C([0, \infty), l_v^2).$$

*In particular, the solution* $\{w_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *converges to the wavefront* $\phi(i \cos\theta + j \sin\theta + ct)$ *exponentially in time* $t$, *that is,*

$$\sup_{i,j\in\mathbb{Z}} |w_{i,j}(t) - \phi(i \cos\theta + j \sin\theta + ct)| \le Ce^{-\mu t}, \quad t \ge 0,$$

*for some positive constants* $C$ *and* $\mu$.

REMARK 2.1. (i) Noting the weight function $v(\xi)$ given in (2.8), we recognize from Theorem 2.1 that, as the sufficient condition, the initial perturbation must converge to 0 in the form

$$|w_{i,j}^0(s) - \phi(i \cos\theta + j \sin\theta + cs)| = O(1)e^{-(\eta_0/2)|i \cos\theta + j \sin\theta|}$$

as $i \cos\theta + j \sin\theta \to -\infty$, $s \in [-r, 0]$.

(ii) In [2], the stability of traveling waves for (1.1) with the small initial perturbation in a weighted energy Sobolev space was investigated, while in the present paper, the small initial perturbation can be improved to be arbitrarily large.
(iii) By (2.7), we have $2D(e-1)+2(\varepsilon b'(0)-d)=c(1)\geq c(\eta_0)$. On the other hand, it follows from the argument in p. 564 of [2] that $4D(e-1)+2(\varepsilon b'(0)-d)>c_*(\theta)$. Then $4D(e-1)+2(\varepsilon b'(0)-d)>\max\{c(\eta_0), c_*(\theta)\}$. Thus, the stability was obtained in [2] with a stronger restriction on the wave speed $c>4D(e-1)+2(\varepsilon b'(0)-d)$. For the wave speed $c\in[c_*(\theta), \max\{c(\eta_0), c_*(\theta)\}]$, the stability problem is investigated in the further work by adopting the $L^1$-weighted energy method (see [6, 16, 17]).
(iv) The condition $d>\varepsilon b'(w^+)+2D(e-1)$ (i.e. (H3) in [2]) can be weaken to the present condition (H3): $d>\varepsilon b'(w^+)$.

## 3. Proof of main theorem

We first state the boundedness and the comparison principle for the Cauchy problem (1.1) and (1.2) which comes from the results in [1] and then prove the main theorem by using the weighted energy method and the squeezing technique. The method we used here is similar to that in [14, 15], which is applied in delayed reaction diffusion equations.

**Lemma 3.1** (Boundedness). *Let*

$$0\leq w_{i,j}^0(s)\leq w^+,\quad \textit{for}\quad s\in[-r,0],\ i,j\in\mathbb{Z},$$

*then the solution* $\{w_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *of* (1.1) *and* (1.2) *satisfies*

$$0\leq w_{i,j}(t)\leq w^+,\quad \textit{for}\quad t\in[0,+\infty),\ i,j\in\mathbb{Z}.$$

**Lemma 3.2** (Comparison principle). *Let* $\{\overline{w}_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *and* $\{\underline{w}_{i,j}(t)\}_{i,j\in\mathbb{Z}}$ *be the solutions of* (1.1) *and* (1.2) *with the initial data* $\{\overline{w}_{i,j}^0(t)\}_{i,j\in\mathbb{Z}}$ *and* $\{\underline{w}_{i,j}^0(t)\}_{i,j\in\mathbb{Z}}$, *respectively. If*

$$0\leq \underline{w}_{i,j}^0(s)\leq \overline{w}_{i,j}^0(s)\leq w^+\quad \textit{for}\quad s\in[-r,0],\ i,j\in\mathbb{Z}$$

*then*

$$0\leq \underline{w}_{i,j}(t)\leq \overline{w}_{i,j}(t)\leq w^+\quad \textit{for}\quad t\in[0,+\infty),\ i,j\in\mathbb{Z}.$$

Let the initial data $w_{i,j}^0(s)$ satisfy

$$0\leq w_{i,j}^0(s)\leq w^+\quad \text{for}\quad s\in[-r,0],\ i,j\in\mathbb{Z},$$

and define

$$\begin{cases}\overline{W}_{i,j}^0(s)=\max\{w_{i,j}^0(s),\phi(i\cos\theta+j\sin\theta+cs)\},\\ \underline{W}_{i,j}^0(s)=\min\{w_{i,j}^0(s),\phi(i\cos\theta+j\sin\theta+cs)\},\end{cases}\quad \text{for}\quad s\in[-r,0],\ i,j\in\mathbb{Z}. \tag{3.1}$$

It is obvious that

(3.2)

$$\begin{cases} 0 \le \underline{W}^0_{i,j}(s) \le w^0_{i,j}(s) \le \overline{W}^0_{i,j}(s) \le w^+, \\ 0 \le \underline{W}^0_{i,j}(s) \le \phi(i\cos\theta + j\sin\theta + cs) \le \overline{W}^0_{i,j}(s) \le w^+, \end{cases} \quad \text{for} \quad s \in [-r, 0],\ i,\ j \in \mathbb{Z}.$$

Let $\overline{W}^0_{i,j}(t)$ and $\underline{W}^0_{i,j}(t)$ be the corresponding solutions of (1.1) and (1.2) with the initial data $\overline{W}^0_{i,j}(s)$ and $\underline{W}^0_{i,j}(s)$, respectively. According to Lemmas 3.1 and 3.2, we easily obtain

(3.3)

$$\begin{cases} 0 \le \underline{W}_{i,j}(t) \le w_{i,j}(t) \le \overline{W}_{i,j}(t) \le w^+, \\ 0 \le \underline{W}_{i,j}(t) \le \phi(i\cos\theta + j\sin\theta + ct) \le \overline{W}_{i,j}(t) \le w^+, \end{cases} \quad \text{for} \quad t \in [0, +\infty),\ i,\ j \in \mathbb{Z}.$$

Proof of Theorem 2.1. We divide the proof into three steps.

STEP 1. We first prove that $\overline{W}_{i,j}(t)$ converges to $\phi(i\cos\theta + j\sin\theta + ct)$ for $t \in [0, +\infty)$, $i, j \in \mathbb{Z}$. For the sake of convenience, we always take $\xi = \xi(t, i, j) := i\cos\theta + j\sin\theta + ct$. Let

$$u_{i,j}(t) = \overline{W}_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct), \quad t \in [0, +\infty),\ i,\ j \in \mathbb{Z}$$

and

$$u^0_{i,j}(s) = \overline{W}^0_{i,j}(s) - \phi(i\cos\theta + j\sin\theta + cs), \quad s \in [-r, 0],\ i,\ j \in \mathbb{Z}.$$

Therefore, it follows from (3.2) and (3.3) that

$$u_{i,j}(t) \ge 0 \quad \text{and} \quad u_{i,j}(s) \ge 0. \tag{3.4}$$

From (1.1) and (1.2), $u_{i,j}(t)$ satisfies

(3.5)

$$\begin{cases} \dfrac{du_{i,j}(t)}{dt} = D[u_{i+1,j}(t) + u_{i-1,j}(t) + u_{i,j+1}(t) + u_{i,j-1}(t) - 4u_{i,j}(t)] \\ \qquad\qquad - du_{i,j}(t) + \varepsilon b'(\phi(\xi(t,i,j) - cr))u_{i,j}(t-r) + Q_{i,j}(t-r), \quad t > 0, \\ u^0_{i,j}(s) = \overline{W}^0_{i,j}(s) - \phi(\xi(s,i,j)),\ i,\ j \in \mathbb{Z}, \qquad s \in [-r, 0], \end{cases}$$

where

$$\begin{aligned} Q_{i,j}(t-r) = \varepsilon[b(u_{i,j}(t-r) + \phi(\xi(t,i,j) - cr)) - b(\phi(\xi(t,i,j) - cr))] \\ - \varepsilon b'(\phi(\xi(t,i,j) - cr))u_{i,j}(t-r), \end{aligned} \tag{3.6}$$

$$u_{i\pm1,j}(t) = \overline{W}_{i\pm1,j}(t) - \phi(\xi(t,i,j) \pm \cos\theta)$$

and

$$u_{i,j\pm1}(t) = \overline{W}_{i,j\pm1}(t) - \phi(\xi(t,i,j) \pm \sin\theta).$$

Let $v(\xi) > 0$ be the weight function defined in (2.8). Multiplying (3.5) by $e^{2\mu t}u_{i,j}(t)v(\xi(t, i, j))$, where $\mu > 0$ will be given later in Lemma 3.4, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}[e^{2\mu t}u_{i,j}^2(t)v(\xi(t, i, j))] \\
&-De^{2\mu t}u_{i,j}(t)v(\xi(t, i, j))[u_{i+1,j}(t)+u_{i-1,j}(t)+u_{i,j+1}(t)+u_{i,j-1}(t)] \\
&+e^{2\mu t}u_{i,j}^2(t)v(\xi(t, i, j))\left[-c\frac{v'_\xi(\xi(t, i, j))}{2v(\xi(t, i, j))}+4D+d-\mu\right] \\
&-\varepsilon b'(\phi(\xi(t, i, j)-cr))e^{2\mu t}v(\xi(t, i, j))u_{i,j}(t)u_{i,j}(t-r) \\
&= Q_{i,j}(t-r)e^{2\mu t}u_{i,j}(t)v(\xi(t, i, j)).
\end{aligned} \tag{3.7}$$

By the Cauchy–Schwartz inequality $|xy| \le (\kappa/2)x^2 + (1/(2\kappa))y^2$ for any $\kappa > 0$ and then taking $\kappa = 1$, we obtain

$$u_{i,j}(t)u_{i\pm1,j\pm1}(t) \le \frac{u_{i,j}^2(t)}{2} + \frac{u_{i\pm1,j\pm1}^2(t)}{2}. \tag{3.8}$$

Summing about all $i, j \in \mathbb{Z}$ and integrating over $[0, t]$ yield

$$\begin{aligned}
&\int_0^t \sum_{i,j} e^{2\mu s}u_{i,j}^2(s)v(\xi(s, i, j))\left\{-c\frac{v'_\xi(\xi(s, i, j))}{v(\xi(s, i, j))}-D[\mathcal{L}v(\xi(s, i, j))-4]+2d-2\mu\right\}ds \\
&-2\int_0^t \sum_{i,j} \varepsilon b'(\phi(\xi(s, i, j)-cr))e^{2\mu s}v(\xi(s, i, j))u_{i,j}(s)u_{i,j}(s-r)\,ds+e^{2\mu t}\|u(t)\|_{l_v^2}^2 \\
&\le 2\int_0^t \sum_{i,j} Q_{i,j}(s-r)e^{2\mu s}u_{i,j}(s)v(\xi(s, i, j))\,ds+\|u^0(0)\|_{l_v^2}^2,
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
\mathcal{L}v(\xi(t, i, j)) = &\frac{v(\xi(t, i, j)}{v(\xi(t, i+1, j)} + \frac{v(\xi(t, i, j)}{v(\xi(t, i-1, j)} \\
&+ \frac{v(\xi(t, i, j)}{v(\xi(t, i, j-1)} + \frac{v(\xi(t, i, j)}{v(\xi(t, i, j+1)}.
\end{aligned} \tag{3.10}$$

Note that there exists some positive number $C_1 > 0$ such that

$$\varepsilon b'(0)e^{2\mu r}\frac{v(\xi(s, i, j)+cr)}{v(\xi(s, i, j))} \le C_1 \quad \text{for all} \quad i, j \in \mathbb{Z}.$$

Since $0 \le b'(w) \le b'(0)$ for any $w \in [0, w^+]$ and $u_{i,j}(t)u_{i,j}(t-r) \le u_{i,j}^2(t)/2 + u_{i,j}^2(t-r)/2$, we obtain

(3.11)

$$\begin{aligned}
2\int_0^t \sum_{i,j} \varepsilon b'(\phi(\xi(s,i,j)-cr))e^{2\mu s}v(\xi(s,i,j))u_{i,j}(s)u_{i,j}(s-r)\,ds \\
\le \int_0^t \sum_{i,j} \varepsilon b'(\phi(\xi(s,i,j)-cr))e^{2\mu s}v(\xi(s,i,j))u_{i,j}^2(s)\,ds \\
+ \int_0^t \sum_{i,j} \varepsilon b'(\phi(\xi(s,i,j)-cr))e^{2\mu s}v(\xi(s,i,j))u_{i,j}^2(s-r)\,ds \\
= \int_0^t \sum_{i,j} \varepsilon b'(\phi(\xi(s,i,j)-cr))e^{2\mu s}v(\xi(s,i,j))u_{i,j}^2(s)\,ds \\
+ \int_{-r}^{t-r} \sum_{i,j} \varepsilon b'(\phi(\xi(s,i,j)))e^{2\mu s}v(\xi(s,i,j)+cr)u_{i,j}^2(s)e^{2\mu r}\,ds \\
= \int_0^t \sum_{i,j} \varepsilon b'(\phi(\xi(s,i,j)-cr)e^{2\mu s}v(\xi(s,i,j))u_{i,j}^2(s)\,ds \\
+ \int_0^{t-r} \sum_{i,j} \varepsilon b'(\phi(\xi(s,i,j))e^{2\mu s}v(\xi(s,i,j)+cr)u_{i,j}^2(s)e^{2\mu r}\,ds \\
+ \int_{-r}^{0} \sum_{i,j} \varepsilon b'(\phi(\xi(s,i,j))e^{2\mu s}\frac{v(\xi(s,i,j)+cr)}{v(\xi(s,i,j))}v(\xi(s,i,j))u_{i,j}^2(s)e^{2\mu r}\,ds \\
\le \int_0^t \sum_{i,j} \varepsilon b'(\phi(\xi(s,i,j)-cr)e^{2\mu s}v(\xi(s,i,j))u_{i,j}^2(s)\,ds \\
+ \int_0^t \sum_{i,j} \varepsilon b'(\phi(\xi(s,i,j))e^{2\mu s}v(\xi(s,i,j)+cr)u_{i,j}^2(s)e^{2\mu r}\,ds + C_1\int_{-r}^0 \|u^0(s)\|_{l_v^2}^2\,ds.
\end{aligned}$$

On the other hand, by Taylor's formula and assumption (H2), we have

$$\begin{aligned}
Q_{i,j}(t-r)) = \varepsilon[b(u_{i,j}(t-r) + \phi(\xi(t,i,j)-cr)) - b(\phi(\xi(t,i,j)-cr))] \\
- \varepsilon b'(\phi(\xi(t,i,j)-cr))u_{i,j}(t-r) \le 0.
\end{aligned} \tag{3.12}$$

Thus, it follows from (3.9) and (3.11)–(3.12) that

$$\begin{aligned}
e^{2\mu t}\|u(t)\|_{l_v^2}^2 + \int_0^t \sum_{i,j} e^{2\mu s}u_{i,j}^2(s)v(\xi(s,i,j))B_{\mu,v}(\xi(s,i,j))\,ds \\
\le \|u^0(0)\|_{l_v^2}^2 + C_1\int_{-r}^0 \|u^0(s)\|_{l_v^2}^2\,ds,
\end{aligned} \tag{3.13}$$

where

$$B_{\mu,v}(\xi(t,i,j)) := A_{\mu,v}(\xi(s,i,j)) - 2\mu - \varepsilon b'(\phi(\xi(s,i,j)))(e^{2\mu r}-1)\frac{v(\xi(t,i,j)+cr)}{v(\xi(t,i,j))}$$

and

$$A_{\mu,v}(\xi(t,i,j)) =: -c\frac{v'_\xi(\xi(t,i,j))}{v(\xi(t,i,j))} - D[\mathcal{L}v(\xi(t,i,j))-4] + 2d$$
$$- \varepsilon b'(\phi(\xi(s,i,j)-cr)) - \varepsilon b'(\phi(\xi(s,i,j)))\frac{v(\xi(t,i,j)+cr)}{v(\xi(t,i,j))}.$$

The most important step now is to prove $B_{i,j}(t) \geq C > 0$ for some constant $C$. In order to obtain this, the following lemma plays a key role in this paper.

**Lemma 3.3.** *$A_{\mu,v}(\xi(t,i,j)) \geq C_2 > 0$ for some positive constant $C_2$.*

Proof. Without loss of generality, suppose that $0 \leq \cos\theta \leq \sin\theta$. It is obvious that

$$\xi(t,i,j-1) \leq \xi(t,i-1,j) \leq \xi(t,i,j) \leq \xi(t,i+1,j) \leq \xi(t,i,j+1)$$

and

$$v(\xi(t,i,j)) \leq v(\xi(t,i-1,j)),\ v(\xi(t,i,j))$$
$$\leq v(\xi(t,i,j-1)),\ v(\xi(t,i,j)+cr) \leq v(\xi(t,i,j)).$$

Thus, we have

(3.14)
$$A_{\mu,v}(\xi(t,i,j)) \geq -c\frac{v'_\xi(\xi(t,i,j))}{v(\xi(t,i,j))} - D\left[\frac{v(\xi(t,i,j))}{v(\xi(t,i+1,j))} + \frac{v(\xi(t,i,j))}{v(\xi(t,i,j+1))} - 2\right] + 2d$$
$$-\varepsilon b'(\phi(\xi(s,i,j)-cr)) - \varepsilon b'(\phi(\xi(s,i,j)))\frac{v(\xi(t,i,j)+cr)}{v(\xi(t,i,j))}.$$

(i) If $\xi_* \leq \xi(t,i,j)$, then $v(\xi(t,i,j)) = v(\xi(t,i+1,j)) = v(\xi(t,i,j+1)) = 1$. Hence, we have

$$A_{\mu,v}(\xi(t,i,j)) \geq 2d - 2[\varepsilon(b'(w^+) + \delta)] =: C_3 > 0$$

according to (2.6).

(ii) If $\xi(t,i,j) < \xi_* \leq \xi(t,i+1,j)$, then $v(\xi(t,i,j) = e^{-\eta_0(i\cos\theta + j\sin\theta + ct - \xi_*)}$ and $v(\xi(t,i+1,j)) = v(\xi(t,i,j+1)) = 1$. Thus, we obtain

$$A_{\mu,v}(\xi(t,i,j)) \geq c\eta_0 + 2d + 2(D - De^{\eta_0\cos\theta}) - 2\varepsilon b'(0)$$
$$> c\eta_0 + 2(D - De^{\eta_0}) + 2(d - \varepsilon b'(0)) =: C_4 > 0$$

according to

$$c > \frac{2D(e^{\eta_0}-1)+2(\varepsilon b'(0)-d)}{\eta_0}.$$

(iii) If $\xi(t,i+1,j) < \xi_* \le \xi(t,i,j+1)$, then $v(\xi(t,i,j)) = e^{-\eta_0(i\cos\theta+j\sin\theta+ct-\xi_*)}$, $v(\xi(t,i+1,j)) = e^{-\eta_0[(i+1)\cos\theta+j\sin\theta+ct-\xi_*]}$ and $v(\xi(t,i,j+1)) = 1$. In this case, we have

$$\begin{aligned}A_{\mu,v}(\xi(t,i,j)) &\ge c\eta_0 + 2d - D(e^{\eta_0\cos\theta}+e^{\eta_0\sin\theta}-2) - 2\varepsilon b'(0)\\ &\ge c\eta_0 + 2d - 2D(e^{\eta_0}-1) - 2\varepsilon b'(0) = C_4 > 0.\end{aligned}$$

(iv) If $\xi(t,i,j+1) < \xi_*$, then $v(\xi(t,i,j)) = e^{-\eta_0(i\cos\theta+j\sin\theta+ct-\xi_*)}$, $v(\xi(t,i+1,j)) = e^{-\eta_0[(i+1)\cos\theta+j\sin\theta+ct-\xi_*]}$ and $v(\xi(t,i,j+1)) = e^{-\eta_0[i\cos\theta+(j+1)\sin\theta+ct-\xi_*]}$. Similarly, it follows that

$$\begin{aligned}A_{\mu,v}(\xi(t,i,j)) &\ge c\eta_0 + 2d - D(e^{\eta_0\cos\theta}+e^{\eta_0\sin\theta}-2) - 2\varepsilon b'(0)\\ &\ge C_4.\end{aligned}$$

Finally, letting $C_2 = \min\{C_3, C_4\} > 0$, this implies that lemma holds. □

**Lemma 3.4.** *$B_{\mu,v}(\xi(t,i,j)) > 0$ for $0 < \mu < \mu_1$, where $\mu_1$ is the unique root of the following equation*

$$C_2 - 2\mu - \varepsilon b'(0)(e^{2\mu r}-1) = 0. \tag{3.15}$$

Proof. As shown in Lemma 3.3, we have $v(\xi(t,i,j)+cr) \le v(\xi(t,i,j))$. Thus, it follows immediately that

$$\begin{aligned}B_{\mu,v}(\xi(t,i,j)) &= A_{\mu,v}(\xi(t,i,j)) - 2\mu - \varepsilon b'(\phi(\xi(s,i,j))(e^{2\mu r}-1)\frac{v(\xi(t,i,j)+cr)}{v(\xi(t,i,j))}\\ &\ge C_2 - 2\mu - \varepsilon b'(0)(e^{2\mu r}-1) > 0\end{aligned}$$

for $0 < \mu < \mu_1$. □

According to Lemma 3.4 and dropping the positive term

$$\int_0^t \sum_{i,j} e^{2\mu s} u_{i,j}^2(s) v(\xi(s,i,j)) B_{i,j}(\xi(s,i,j))\, ds$$

in (3.13), we obtain the following basic energy estimate.

**Lemma 3.5.** *It holds that*

$$e^{2\mu t}\|u(t)\|_{l_v^2}^2 \le \|u^0(0)\|_{l_v^2}^2 + C_1\int_{-r}^0 \|u^0(s)\|_{l_v^2}^2\, ds, \quad t \ge 0. \tag{3.16}$$

Using the standard Sobolev's embedding inequality $l^2 \hookrightarrow l^\infty$ and $l^2_v \hookrightarrow l^2$ for $v(i\cos\theta + j\sin\theta + ct) \geq 1$ defined by (2.8), we finally have the following stability result.

**Lemma 3.6.** *It holds that*

$$\sup_{i,j} |\overline{W}_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)| = \sup_{i,j} |u_{i,j}(t)| \leq C_5 e^{-\mu t} \tag{3.17}$$

*for all* $t \geq 0$ *and some positive constant* $C_5$.

Proof. It follows from (3.16) and

$$|u_{i,j}(t)| \leq \sup_{i,j} |u_{i,j}(t)| \leq \|u(t)\|_{l^2} \leq \|u(t)\|_{l^2_v}$$

that the conclusion holds. □

STEP 2. Prove that $\underline{W}_{i,j}(t)$ converges to $\phi(i\cos\theta + j\sin\theta + ct)$. Let

$$u_{i,j}(t) = \underline{W}_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)$$

and

$$u^0_{i,j}(s) = \underline{W}^0_{i,j}(s) - \phi(i\cos\theta + j\sin\theta + cs).$$

Similar to the process in Step 1, we have the following lemma.

**Lemma 3.7.** *It holds that*

$$\sup_{i,j} |\underline{W}_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)| = \sup_{i,j} |u_{i,j}(t)| \leq C_6 e^{-\mu t} \tag{3.18}$$

*for all* $t \geq 0$ *and some positive constant* $C_6$.

STEP 3. Prove that $W_{i,j}(t)$ converges to $\phi(i\cos\theta + j\sin\theta + ct)$, i.e., the following lemma holds.

**Lemma 3.8.** *It holds that*

$$\sup_{i,j} |W_{i,j}(t) - \phi(i\cos\theta + j\sin\theta + ct)| \leq Ce^{-\mu t} \tag{3.19}$$

*for all* $t \geq 0$ *and some positive constant* $C$.

Proof. Since the initial data satisfy $\underline{W}_{i,j}(s) \leq W_{i,j}(s) \leq \overline{W}_{i,j}(s)$, $s \in [-r, 0]$, it follows from Lemma 3.2 that the corresponding solutions of (1.1) and (1.2) satisfy

$$\underline{W}_{i,j}(t) \leq W_{i,j}(t) \leq \overline{W}_{i,j}(t), \quad \text{for all} \quad t \geq 0,\ i,\ j \in \mathbb{Z}.$$

According to Lemmas 3.5–3.6, the squeeze method yields

$$\sup_{i,j} |W_{i,j}(t) - \phi(i \cos\theta + j \sin\theta + ct)| \le C e^{-\mu t} \tag{3.20}$$

for all $t \ge 0$ and some positive constant $C$. This completes the proof. □

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