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CONGRUENCE SOLUTIONS TO THE LOCALIZED INDUCTION HIERARCHY IN THREE-DIMENSIONAL SPACE FORMS

Dedicated to Professor Norihito Koiso on the occasion of his 60th birthday

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Abstract

The localized induction hierarchy in three-dimensional space forms is studied. In particular, we determine all the generating curves of congruence solutions to each evolution equation belonging to the localized induction hierarchy. Here, a congruence solution means a solution moving without changing shape. Also, we give the characterization of some low-order soliton curves.

1. Introduction

We consider the motion of thin vortex filaments in ideal three-dimensional fluids. In the so-called localized induction approximation, the motion of vortex filaments is governed by the following evolution equation:

$$\tilde{\gamma}_t = \tilde{\gamma}_s \times \tilde{\gamma}_{ss}, \tag{1.1}$$

where $\tilde{\gamma} = \tilde{\gamma}(s, t) \in \mathbf{R}^3$ represents the position of the centerline of a vortex filament in the three-dimensional Euclidean space $\mathbf{R}^3$, and t is time and s is the arclength parameter along the centerline of the vortex filament ([1]). The equation (1.1) is called the *localized induction equation* (LIE). (It is also called the *Betchov–Da Rios equation*, the *vortex filament equation*, the *filament model*, etc.)

In [9], Hasimoto discovered a transformation which associates the LIE with the cubic nonlinear Schrödinger equation, a well-known example of a soliton equation. Also, Marsden–Weinstein ([27]) showed that the LIE is described as a Hamiltonian flow on an appropriate space of curves in $\mathbf{R}^3$, the Hamiltonian being just the arclength functional. By using a similar framework to [27] and the Hasimoto transformation, Langer-Perline ([23]) constructed an infinite sequence of commuting Hamiltonian vector fields $X_1, X_2, \ldots$ and the associated infinite sequence of flows starting with (1.1). Therefore, the LIE (1.1) is viewed as an infinite-dimensional completely integrable Hamiltonian system.

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In [23], the sequence X_n had been constructed by successively applying an integro-differential operator $\mathcal{R}$, called the *recursion operator*, to the vector field $X_0 = -\gamma_s$. In [22], however, Langer obtained an inductive formula of X_n without integral operators. According to this formula, the first few vector fields are calculated as follows:

$$\begin{aligned}
X_0 &= -\gamma_s,\\
X_1 &= -\frac{C_1}{2}\gamma_s + \gamma_s \times \gamma_{ss},\\
X_2 &= \left(\frac{3}{2}|\gamma_{ss}|^2 + \frac{C_1^2}{8} - \frac{C_2}{2}\right)\gamma_s + \gamma_{sss} + \frac{C_1}{2}\gamma_s \times \gamma_{ss},
\end{aligned}$$

where C_1, C_2 are real constants. Each $X_n = X_n[\gamma]$ can be viewed as an $(n+1)$-st order ordinary differential operator with respect to s depending on n real parameters $C_1, \dots, C_n$. Then the evolution equation $\tilde{\gamma}_t = X_n[\tilde{\gamma}]$, with $C_1, \dots, C_n$ fixed, is called the *n-th localized induction equation*, and the infinite sequence of these equations is called the *localized induction hierarchy* (LIH). The first localized induction equation $\tilde{\gamma}_t = X_1[\tilde{\gamma}]$ with $C_1 = 0$ is equal to the original localized induction equation (1.1). Furthermore, it is known that the second and third localized induction equations also arise in contexts of fluid mechanics (cf. [4], [5], [6], [30], [35], etc.).

An arclength-parametrized curve $\gamma = \gamma(s)$ is said to be an *n-th soliton curve* if γ is a solution to the n-th stationary equation $X_n[\gamma] = 0$ for some constants $C_1, \dots, C_n$. It is known that low-order soliton curves are applied to surface theory in a wide variety of contexts. In these cases, it is often useful to consider the natural generalization of soliton curves in *non-Euclidean space forms* S^2, H^2, S^3, etc. ([3], [7], [25], [31], etc.). For example, by using elasticae (a type of third soliton curves) in S^2, Pinkall ([31]) constructed the first examples of Willmore surfaces in $\mathbf{R}^3$ not coming from stereographic projections of minimal surfaces in S^3. Also, some fourth soliton curves in H^2 were applied to construct explicit examples of the Konopelchenko-Taimanov motions of immersed Riemann surfaces in $\mathbf{R}^3$ ([7]). These examples imply the significance of investigating the localized induction hierarchy and soliton curves in space forms, or in general Riemannian manifolds.

In this paper, we consider the localized induction hierarchy in oriented three-dimensional Riemannian manifolds. In particular, we investigate *congruence solutions* to the n-th localized induction equation. Here, a congruence solution is defined to be a solution which evolves without changing shape. This permanent shape is called the *generating curve* of the congruence solution.

Our main results are as follows: we prove that in the case of the three-dimensional space forms $\mathbf{R}^3$, S^3, H^3, the set of all the generating curves of congruence solutions to the n-th ($n \geqslant 1$) localized induction equation coincides with the set of all $(n+2)$-nd soliton curves (Theorem 5.1). Also, we investigate some low-order soliton curves. In particular, we prove that the set of all first (resp. second) soliton curves in an oriented three-dimensional Riemannian manifold coincides with the set of all geodesics (resp. helices). Further, in the case of $\mathbf{R}^3$, S^3, H^3, we prove that the set of all third soliton curves coincides with the set of all *Kirchhoff rod centerlines* (Theorem 3.3).

Unless otherwise specified, all manifolds, curves, vector fields, etc., are assumed to be C^∞ throughout this paper. Let $\mathscr{M}$ be an oriented three-dimensional Riemannian manifold. We denote by $\langle\ ,\ \rangle$ the Riemannian metric, by ∇ the Levi-Civita connection and by $\times$ the vector product.

In Section 2, we define the localized induction hierarchy in $\mathscr{M}$ (Definition 2.6). Let $\gamma = \gamma(s)$ be an arclength-parametrized curve in $\mathscr{M}$. We denote by $T(s) = \partial\gamma(s)/\partial s$ the unit tangent vector to γ. Let $\{C_n\}_{n=1}^\infty$ be an arbitrary real sequence. We inductively define a sequence $\{X_n\}_{n=0}^\infty$ of vector fields along γ by

$$X_0 = -T, \quad X_n = \left(-\frac{C_n}{2} + \frac{1}{2}\sum_{k=1}^{n-1}\langle X_k, X_{n-k}\rangle\right)T - T \times \nabla_T X_{n-1}.$$

Here, when $n = 1$, we treat the term $(1/2)\sum_{k=1}^{n-1}\langle X_k, X_{n-k}\rangle$ as zero. For example, X_2 is calculated as follows:

$$X_2 = \left(\frac{3}{2}|\nabla_T T|^2 + \frac{C_1^2}{8} - \frac{C_2}{2}\right)T + (\nabla_T)^2 T + \frac{C_1}{2} T \times \nabla_T T.$$

The evolution equation

$$\frac{\partial\tilde{\gamma}(s,t)}{\partial t} = X_n[\tilde{\gamma}(s,t)] \tag{1.2}$$

of arclength-parametrized curves in $\mathscr{M}$ is called the *n-th localized induction equation* (*n*-th LIE), and the infinite sequence of the equations (1.2), $n = 0, 1, 2, \ldots$, is called the *localized induction hierarchy* (LIH).

We introduce the definition of a congruence solution to the n-th LIE as follows:

DEFINITION 2.9. Let $\mathscr{M}$ be an oriented three-dimensional Riemannian manifold. A solution $\tilde{\gamma}\colon \mathbf{R}\times\mathbf{R} \to \mathscr{M}$ to the n-th localized induction equation $\tilde{\gamma}_t = X_n[\tilde{\gamma}]$ is called a *congruence solution* if $\tilde{\gamma}$ is expressed as follows: there exist an arclength-parametrized curve $\gamma\colon \mathbf{R} \to \mathscr{M}$, a constant $c \in \mathbf{R}$ and a one-parameter group $\{\varphi^t\}_{t\in\mathbf{R}}$ of isometries of $\mathscr{M}$ such that $\tilde{\gamma}(s,t) = \varphi^t(\gamma(s-ct))$. This γ is uniquely determined by $\tilde{\gamma}$ and is called the *generating curve* of the congruence solution $\tilde{\gamma}$.

In Section 3, in the same way as [22], we define an n-th soliton curve as a solution to the n-th stationary equation (Definition 3.1). That is, an arclength-parametrized curve $\gamma\colon \mathbf{R} \to \mathscr{M}$ is called an *n-th soliton curve* if $X_n[\gamma] = 0$ holds for some $C_1, \ldots, C_n \in \mathbf{R}$. The set of all n-th soliton curves is called the *n-th soliton class*, and is denoted by Γ_n. Then we obtain the following theorem, which is a generalization of the Euclidean case obtained by Langer ([22]).

Theorem 3.3. *Let $\mathscr{M}$ be an oriented three-dimensional Riemannian manifold. Then $\Gamma_n \subset \Gamma_{n+1}$ holds for any positive integer n. Also, let $\gamma\colon \mathbf{R} \to \mathscr{M}$ be an arclength-parametrized curve. Then the following holds*:
(i) $\gamma \in \Gamma_1$ *if and only if* γ *is a geodesic.*
(ii) $\gamma \in \Gamma_2$ *if and only if* γ *is a helix.*
(iii) *Suppose that* $\mathscr{M} = \mathbf{R}^3, S^3, H^3$. *Then* $\gamma \in \Gamma_3$ *if and only if* γ *is a Kirchhoff rod centerline.*

Here, a *Kirchhoff rod centerline* is one of the mathematical models of thin elastic rods. We defer the precise definition of a Kirchhoff rod centerline and the proof of (iii) of Theorem 3.3 to Section 6.

In Section 4, as a preparation for the proof of the main theorem (Theorem 5.1), we summarize some fundamental properties of the *natural frame*, *natural curvatures* and *complex curvature* of a curve in three-dimensional space forms.

In Section 5, we state and prove the main theorem. We consider the problem of determining the generating curves of congruence solutions to the n-th LIE. First, by the definition of an n-th soliton curve, it is easily shown that any $\gamma \in \Gamma_n$ is the generating curve of a congruence solution to the n-th LIE with some $C_1, \dots, C_n \in \mathbf{R}$. This is valid for the case where $\mathscr{M}$ is a general oriented three-dimensional Riemannian manifold.

In the case where $\mathscr{M} = \mathbf{R}^3, S^3, H^3$, however, we can prove that any curve γ in Γ_{n+2} ($\supset \Gamma_n$) is the generating curve of a congruence solution to the n-th LIE. More precisely, the following main theorem holds.

Theorem 5.1. *Let* $\mathscr{M} = \mathbf{R}^3, S^3, H^3$, *and* $n \geqslant 1$. *Let* $\gamma\colon \mathbf{R} \to \mathscr{M}$ *be an arclength-parametrized curve. Then the following* (i) *and* (ii) *are equivalent.*
(i) $\gamma \in \Gamma_{n+2}$.
(ii) γ *is the generating curve of a congruence solution to the n-th localized induction equation with some* $C_1, \dots, C_n \in \mathbf{R}$.

In the case where $\mathscr{M} = \mathbf{R}^3$, the part of (i) $\Rightarrow$ (ii) of Theorem 5.1 is proven in (b) of Proposition 12 of [22]. One of the different points of the proof of the case of $\mathscr{M} = S^3, H^3$ from that of the Euclidean case is that finding the constants $C_1, \dots, C_n$ in (ii) is not trivial.

In the last part of this section, we consider special congruence solutions. Suppose that $\mathscr{M} = \mathbf{R}^3$. Then we can consider congruence solutions evolving by *translation*. A solution $\tilde{\gamma}(s, t)$ to the n-th LIE is called a *translation solution* if $\tilde{\gamma}(s, t)$ satisfies the condition obtained by replacing "isometries of $\mathscr{M}$" in Definition 2.9 by "translations of $\mathbf{R}^3$" (Definition 5.6). We give the translation solution version of Theorem 5.1 (Proposition 5.7).

In Section 6, we review the concept of a *Kirchhoff elastic rod* and define a *Kirchhoff rod centerline*, and give the proof of (iii) of Theorem 3.3.

This paper is the detailed version of the former part of the announcement [16]. In the present paper, the author uses one different terminology from [16]. The "traveling wave solution" in the abstract of [16] is identical to our "congruence solution".

2. The localized induction hierarchy

In this section, we give the definition of the localized induction hierarchy $\tilde{\gamma}_t = X_n[\tilde{\gamma}]$, $n = 0, 1, 2, \ldots$, in an oriented three-dimensional Riemannian manifold (Definition 2.6), which is obtained by replacing the ordinary differentiation of the Euclidean case in Section 2 of [22] by the covariant differentiation. Also, we define a congruence solution to the n-th localized induction equation (Definition 2.9). In the last part of this section, we introduce the normalization of X_n, which will be used in the following sections. Although the contents of this section are basically parallel to the Euclidean case in Section 2 of [22], we describe the details for self-containedness.

Unless otherwise specified, all manifolds, curves, vector fields, etc., are assumed to be C^∞. Let $\mathscr{M}$ be an oriented three-dimensional Riemannian manifold. We denote by $\langle\ ,\ \rangle$ the Riemannian metric, by ∇ the Levi-Civita connection and by $\times$ the vector product.

Let I $(\subset \mathbf{R})$ be an open interval, and let $\gamma\colon I \to \mathscr{M}$ be a curve parametrized by arc length s. We denote by $T(s) = \partial\gamma(s)/\partial s$ the unit tangent vector to γ. Also, we denote by ∂_s the differentiation with respect to s, and by ∂_s^{-1} the antidifferentiation with respect to s, that is, $\partial_s^{-1} f$ is a function whose differentiation with respect to s is equal to f.

We define a sequence $\{X_n\}_{n=0}^{\infty}$ of vector fields along γ. This $X_n = X_n[\gamma]$ is the direction in which a solution of the n-th LIE evolves. Since the n-th LIE is an evolution equation of arclength-parametrized curves, the vector field X_n must satisfy one condition mentioned below (Definition 2.2). First, the following lemma holds.

Lemma 2.1. *Let $\gamma(u,t) = \gamma_t(u)$ $(|t| \ll 1)$ be a variation of curves in $\mathscr{M}$ with variation parameter t, and let $W = \partial\gamma/\partial t$ be the variation vector field. We assume that the curve $u \mapsto \gamma_0(u)$ is unit-speed. Then the curve $u \mapsto \gamma_t(u)$ is unit-speed for any fixed t if and only if W satisfies the condition $\langle \nabla_{\partial/\partial u} W, \partial\gamma/\partial u\rangle = 0$.*

Proof. Suppose that for any fixed t, the curve $u \mapsto \gamma_t(u)$ is unit-speed. That is, $|\partial\gamma/\partial u|^2 = 1$ for all (u,t). Then $(\partial/\partial t)\langle \partial\gamma/\partial u, \partial\gamma/\partial u\rangle = 0$. Since

$$\frac{\partial}{\partial t}\left\langle \frac{\partial\gamma}{\partial u}, \frac{\partial\gamma}{\partial u}\right\rangle = 2\left\langle \nabla_{\partial/\partial t}\frac{\partial\gamma}{\partial u}, \frac{\partial\gamma}{\partial u}\right\rangle = 2\left\langle \nabla_{\partial/\partial u}\frac{\partial\gamma}{\partial t}, \frac{\partial\gamma}{\partial u}\right\rangle = 2\left\langle \nabla_{\partial/\partial u} W, \frac{\partial\gamma}{\partial u}\right\rangle,$$

we see $\langle \nabla_{\partial/\partial u} W, \partial\gamma/\partial u\rangle = 0$. The converse also holds immediately. $\square$

And so we define as follows:

DEFINITION 2.2. Let W be a vector field along an arclength-parametrized curve γ. Then W is called *locally arclength-preserving* (LAP) if W satisfies $\langle \nabla_T W, T\rangle = 0$.

A trivial example of a LAP vector field is given by the vector field cT along γ, where c is a constant.

Before giving the definition of $\{X_n\}_{n=0}^{\infty}$, we introduce the *recursion operator* $\mathcal{R}$, which gives a method making another LAP vector field from one LAP vector field.

First, let W be a not necessarily LAP vector field along γ. Then we can make W LAP by modifying its tangential component. More precisely, the following proposition holds.

Proposition 2.3. *Let W be a vector field along γ. Let $W = fT + W^{\perp}$ be the decomposition of W into the tangential and normal components, where f is a function. Then the following holds*:
(i) *W is a LAP vector field along γ if and only if $\partial_s f = \langle W^{\perp}, \nabla_T T\rangle$ holds.*
(ii) *We set $\mathcal{P}W = (\partial_s^{-1}\langle W^{\perp}, \nabla_T T\rangle)T + W^{\perp}$. Then $\mathcal{P}W$ is a LAP vector field along γ.*

Proof. By using the Leibniz rule, we see $\langle \nabla_T W, T\rangle = \partial_s f + \langle \nabla_T W^{\perp}, T\rangle = \partial_s f - \langle W^{\perp}, \nabla_T T\rangle$. Therefore (1) holds. Also, (2) immediately follows from (1). □

We note that $\mathcal{P}W$ is only defined up to addition of a constant multiple of T. The operator $\mathcal{P}$ is called the *reparametrization operator*.

Let J denote the operator which takes vector product with the unit tangent T. That is, $JX = T \times X$ for $X \in T_{\gamma(s)}\mathscr{M}$, where $s \in I$. Note that $\langle JX, Y\rangle = -\langle X, JY\rangle$ and $J^2 X = J(JX) = -X^{\perp}$ holds, where $X, Y \in T_{\gamma(s)}\mathscr{M}$. Then the following proposition holds.

Proposition 2.4. *Let W be a LAP vector field along γ. Then there exists a LAP vector field X along γ satisfying $JX = \nabla_T W$. Such X is uniquely determined up to addition of a constant multiple of T, and X is expressed as follows*: $X = \mathcal{P}(-J(\nabla_T W))$.

Proof. First, we seek for all vector fields X along γ satisfying $JX = \nabla_T W$. Suppose that a not necessarily LAP vector field X satisfies $JX = \nabla_T W$. Then $X^{\perp} = -J^2 X = -J(\nabla_T W)$, and so X is expressed as

$$X = fT - J(\nabla_T W), \tag{2.1}$$

where f is a function of s. Conversely, for any function f, the vector field X defined by (2.1) satisfies $JX = -J^2(\nabla_T W) = (\nabla_T W)^{\perp} = \nabla_T W$. Consequently, X satisfies $JX = \nabla_T W$ if and only if X is expressed as (2.1), where f is any function.

Therefore, by Proposition 2.3, we obtain that the X defined by (2.1) is LAP if and only if X is expressed as $X = \mathcal{P}(-J(\nabla_T W))$. This completes the proof. □

For a LAP vector field W along γ, we set

$$\mathcal{R}W = \mathcal{P}(-J(\nabla_T W)) = (\partial_s^{-1}\langle -J(\nabla_T W), \nabla_T T\rangle)T - J(\nabla_T W).$$

The operator $\mathcal{R}$, which sends a LAP vector field to another LAP vector field, is called the *recursion operator*. Note that in the same way as $\mathcal{P}W$, $\mathcal{R}W$ is only defined up to addition of a constant multiple of T.

In later sections, we also use the following notation $\mathcal{P}_c$, $\mathcal{R}_c$ to eliminate the ambiguity of the definitions of $\mathcal{P}$, $\mathcal{R}$. We first fix a point $s_0 \in I$. Let $c \in \mathbf{R}$, and we set

$$\begin{aligned}\mathcal{P}_c W &= \left(\int_{s_0}^{s} \langle W^{\perp}, \nabla_T T\rangle\, ds + c\right)T + W^{\perp},\\ \mathcal{R}_c W &= \mathcal{P}_c(-J(\nabla_T W)) = \left(\int_{s_0}^{s} \langle -J(\nabla_T W), \nabla_T T\rangle\, ds + c\right)T - J(\nabla_T W).\end{aligned}$$

If two LAP vector fields X, W satisfy $JX = \nabla_T W$, then there exists a unique $c \in \mathbf{R}$ such that $X = \mathcal{R}_c W$. Also, for a fixed c, $\mathcal{R}_c$ is viewed as a map of the vector space of all LAP vector fields along γ into itself. In particular, $\mathcal{R}_0$ is a linear transformation on this vector space. Noting that $\mathcal{R}_c W = \mathcal{R}_0 W + cT$ holds, we can verify that

$$\mathcal{R}_{a_1+a_2}(W_1 + W_2) = \mathcal{R}_{a_1} W_1 + \mathcal{R}_{a_2} W_2, \quad \mathcal{R}_{ca}(cW) = c\mathcal{R}_a W \tag{2.2}$$

for any LAP vector fields W_1, W_2, W along γ and any $a_1, a_2, c, a \in \mathbf{R}$. We note that $\mathcal{R}_c$ is determined only after a reference point s_0 in the domain of γ is fixed. In what follows, however, we use the notation $\mathcal{R}_c$ without mentioning each time that we take a reference point s_0.

Now let us construct a sequence $\{X_n\}_{n=0}^{\infty}$ of LAP vector fields satisfying

$$\begin{cases} X_0 = -T,\\ JX_n = \nabla_T X_{n-1} \quad (n \geqslant 1).\end{cases} \tag{2.3}$$

By Proposition 2.4, such $\{X_n\}_{n=0}^{\infty}$ is constructed as follows: $X_n = \mathcal{R}_{d_n}(X_{n-1})$, where $\{d_n\}_{n=1}^{\infty}$ is any real sequence. Note that this formula is written in terms of the integro-differential operator $\mathcal{R}_{d_n}$. In Section 2 of [22], however, Langer expressed X_n by $X_0, \dots, X_{n-1}$ without integral, in the case of $\mathscr{M} = \mathbf{R}^3$. This result is naturally extended to the case of an oriented three-dimensional Riemannian manifold (Proposition 2.5).

To prove Proposition 2.5, it is convenient to use the concept of a formal power series with coefficients of vector fields along a curve. Let $C^{\infty}(I)$ denote the algebra of all C^{∞} functions on I, and let $\mathscr{X}(\gamma)$ denote the vector space of all C^{∞} vector fields along γ. Then $\mathscr{X}(\gamma)$ is a $C^{\infty}(I)$-module. Let $C^{\infty}(I)[[\lambda]]$ denote the algebra of all formal power series in the indeterminate λ with coefficients in $C^{\infty}(I)$ and let $\mathscr{X}(\gamma)[[\lambda]]$ denote the vector space of all formal power series in the indeterminate λ with coefficients in $\mathscr{X}(\gamma)$. For $(g, X) = \left(\sum_{n=0}^{\infty} \lambda^n g_n, \sum_{n=0}^{\infty} \lambda^n X_n\right) \in C^{\infty}(I)[[\lambda]] \times \mathscr{X}(\gamma)[[\lambda]]$, where

$g_n \in C^\infty(I)$, $X_n \in \mathscr{X}(\gamma)$, we define $gX \in \mathscr{X}(\gamma)[[\lambda]]$ by $gX = \sum_{n=0}^{\infty} \lambda^n \bigl(\sum_{k=0}^{n} g_k X_{n-k}\bigr)$. Under this action, $\mathscr{X}(\gamma)[[\lambda]]$ becomes a $C^\infty(I)[[\lambda]]$-module.

The linear operator ∂_s on $C^\infty(I)$ naturally extends to that on $C^\infty(I)[[\lambda]]$, which is also denoted by the same notation. The linear operators J and ∇_T on $\mathscr{X}(\gamma)$ naturally extend to those on $\mathscr{X}(\gamma)[[\lambda]]$, which are also denoted by the same notation. That is, for $X = \sum_{n=0}^{\infty} \lambda^n X_n \in \mathscr{X}(\gamma)[[\lambda]]$, where $X_n \in \mathscr{X}(\gamma)$, we define JX, $\nabla_T X$ as follows: $JX = \sum_{n=0}^{\infty} \lambda^n JX_n$, $\nabla_T X = \sum_{n=0}^{\infty} \lambda^n \nabla_T X_n$. Similarly, the symmetric bilinear form $\langle\ ,\ \rangle$ on the $C^\infty(I)$-module $\mathscr{X}(\gamma)$ naturally extends to that on the $C^\infty(I)[[\lambda]]$-module $\mathscr{X}(\gamma)[[\lambda]]$. That is, for $X = \sum_{n=0}^{\infty} \lambda^n X_n$, $Y = \sum_{n=0}^{\infty} \lambda^n Y_n \in \mathscr{X}(\gamma)[[\lambda]]$, where $X_n, Y_n \in \mathscr{X}(\gamma)$, we define $\langle X, Y\rangle$ $(\in C^\infty(I)[[\lambda]])$ by $\langle X, Y\rangle = \sum_{n=0}^{\infty} \lambda^n \bigl(\sum_{k=0}^{n} \langle X_k, Y_{n-k}\rangle\bigr)$.

Proposition 2.5. *Let $\{C_n\}_{n=1}^{\infty}$ be an arbitrary real sequence. We inductively define a sequence $\{X_n\}_{n=0}^{\infty}$ of vector fields along γ as follows*:

$$X_0 = -T, \tag{2.4}$$

$$X_n = f_n T - J(\nabla_T X_{n-1}) \quad (n \geqslant 1), \tag{2.5}$$

where

$$f_n = \begin{cases} -\dfrac{C_1}{2} & (n = 1), \\ -\dfrac{C_n}{2} + \dfrac{1}{2} \displaystyle\sum_{k=1}^{n-1} \langle X_k, X_{n-k}\rangle & (n \geqslant 2). \end{cases} \tag{2.6}$$

Then $\{X_n\}_{n=0}^{\infty}$ is a sequence of LAP vector fields satisfying (2.3).

Conversely, if $\{X_n\}_{n=0}^{\infty}$ is a sequence of LAP vector fields satisfying (2.3), *then there exists a unique real sequence $\{C_n\}_{n=1}^{\infty}$ such that* (2.5) *and* (2.6) *hold. The $\{C_n\}_{n=1}^{\infty}$ is given by $C_n = \sum_{k=0}^{n} \langle X_k, X_{n-k}\rangle$.*

Under the above relation, the set of all real sequences $\{C_n\}_{n=1}^{\infty}$ is in one-to-one correspondence with the set of all sequences $\{X_n\}_{n=0}^{\infty}$ of LAP vector fields satisfying (2.3).

Proof. We first prove the latter part. Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of LAP vector fields satisfying (2.3). Then $X_n^\perp = -J(JX_n) = -J\nabla_T X_{n-1}$. We express the tangential component $\langle X_n, T\rangle$ of X_n by $X_0, \dots, X_{n-1}$. We set $X = \sum_{n=0}^{\infty} \lambda^n X_n$ $(\in \mathscr{X}(\gamma)[[\lambda]])$. Then (2.3) implies

$$JX = JX_0 + \sum_{n=1}^{\infty} \lambda^n JX_n = \sum_{n=1}^{\infty} \lambda^n \nabla_T X_{n-1} = \lambda \nabla_T X. \tag{2.7}$$

Now, we show $\partial_s \langle X, X\rangle = 0$. Note that J is skew-adjoint, that is, $\langle JX, Y\rangle = -\langle X, JY\rangle$ holds for any $X, Y \in \mathscr{X}(\gamma)[[\lambda]]$. The skew-adjointness of J and (2.7) yield $\lambda\partial_s\langle X, X\rangle = 2\langle \lambda\nabla_T X, X\rangle = 2\langle JX, X\rangle = 0$. Thus we have $\partial_s\langle X, X\rangle = 0$. Therefore, the coefficient of each λ^n term of $\langle X, X\rangle$ is independent of s, that is, there exists a real sequence $C_0, C_1, C_2, \ldots$ such that

$$\langle X, X\rangle = \sum_{n=0}^{\infty} \lambda^n C_n. \tag{2.8}$$

Hence the definition of $\langle X, X\rangle$ yields $C_n = \sum_{k=0}^{n}\langle X_k, X_{n-k}\rangle$. Since $X_0 = -T$, we see $C_0 = 1$ and $C_1 = -2\langle X_1, T\rangle$. Thus $\langle X_1, T\rangle = -C_1/2$. When $n \geqslant 2$, we see that

$$C_n = \sum_{k=0}^{n}\langle X_k, X_{n-k}\rangle = -2\langle X_n, T\rangle + \sum_{k=1}^{n-1}\langle X_k, X_{n-k}\rangle,$$

from which it follows that $\langle X_n, T\rangle$ is expressed as the right hand side of (2.6). Thus X_n ($n \geqslant 1$) is expressed as (2.5). Also, we can easily check the uniqueness of $\{C_n\}_{n=1}^{\infty}$, and hence the proof of the latter part is completed.

Next we show the former part. Let $\{C_n\}_{n=1}^{\infty}$ be an arbitrary real sequence, and $\{X_n\}_{n=0}^{\infty}$ the sequence of vector fields along γ defined by (2.4), (2.5) and (2.6). First, $X_0 = -T$ is LAP, as stated earlier.

We show, by induction, that for all $n \geqslant 1$, $JX_n = \nabla_T X_{n-1}$ holds and X_n is LAP. First, $\mathcal{R}X_0 = \mathcal{P}(-J(\nabla_T X_0)) = (\partial_s^{-1}\langle J\nabla_T T, \nabla_T T\rangle)T + J\nabla_T T = CT + J\nabla_T T$, where $C \in \mathbf{R}$. On the other hand, (2.5) yields $X_1 = -(C_1/2)T + J\nabla_T T$. Therefore, it follows from Proposition 2.4 that X_1 is LAP and $JX_1 = \nabla_T X_0$. Next, we assume that $JX_l = \nabla_T X_{l-1}$ holds for $l = 1, 2, \ldots, n$ and X_n is LAP. We show that $JX_{n+1} = \nabla_T X_n$ and X_{n+1} is LAP. First, by (2.5) together with the assumption that X_n is LAP, we have $JX_{n+1} = -J^2(\nabla_T X_n) = (\nabla_T X_n)^{\perp} = \nabla_T X_n$. It remains only to show that X_{n+1} is LAP. By the assumption $JX_l = \nabla_T X_{l-1}$ ($l = 1, 2, \ldots, n$) and $JX_{n+1} = \nabla_T X_n$, we see

$$\begin{aligned} 2\partial_s f_{n+1} &= \sum_{k=1}^{n}(\langle JX_{k+1}, X_{n+1-k}\rangle + \langle X_k, JX_{n+2-k}\rangle) \\ &= \sum_{k=1}^{n}(\langle JX_{k+1}, X_{n+1-k}\rangle - \langle JX_k, X_{n+2-k}\rangle) \\ &= \langle JX_{n+1}, X_1\rangle - \langle JX_1, X_{n+1}\rangle = -2\langle X_{n+1}, JX_1\rangle. \end{aligned}$$

Therefore, $\partial_s f_{n+1} = \langle X_{n+1}, \nabla_T T\rangle$, which implies that X_{n+1} is LAP. Hence the proof of the former part is completed.

It is easily checked that under the above relation, the set of all real sequences $\{C_n\}_{n=1}^{\infty}$ is in one-to-one correspondence with the set of all sequences $\{X_n\}_{n=0}^{\infty}$ of LAP vector fields satisfying (2.3). This completes the proof of the proposition. $\square$

In what follows, we set $C_0 = 1$, and instead of a sequence $C_1, C_2, \dots$, we consider the sequence $C_0 = 1$, C_1, C_2, ... whose index starts from zero. We set

$$\mathcal{S} = \{\{C_n\}_{n=0}^{\infty} \mid C_n \in \mathbf{R}\ (\forall n \geqslant 0),\ C_0 = 1\}. \tag{2.9}$$

Since X_n does not depend on C_{n+1}, C_{n+2}, ..., we also write X_n as $X_n^{C_0, C_1, \dots, C_n}$. According to (2.5), we can calculate X_0, X_1, X_2 as follows:

$$\begin{aligned}
X_0 &= -T, \\
X_1 &= -\frac{C_1}{2} T + T \times \nabla_T T, \\
X_2 &= \left(\frac{3}{2} |\nabla_T T|^2 + \frac{C_1^2}{8} - \frac{C_2}{2} \right) T + (\nabla_T)^2 T + \frac{C_1}{2} T \times \nabla_T T.
\end{aligned}$$

Each $X_n^{C_0, \dots, C_n} = X_n^{C_0, \dots, C_n}[\gamma]$ can be viewed as an $(n+1)$-st order ordinary differential operator with respect to s.

We define the n-th localized induction equation and the localized induction hierarchy in $\mathscr{M}$ as follows:

DEFINITION 2.6. Let $\{C_n\}_{n=0}^{\infty} \in \mathcal{S}$ and let $\{X_n\}_{n=0}^{\infty}$ be the sequence of the differential operators defined by (2.4), (2.5) and (2.6). The evolution equation

$$\frac{\partial \tilde{\gamma}(s, t)}{\partial t} = X_n^{C_0, \dots, C_n}[\tilde{\gamma}(s, t)] \tag{2.10}$$

of arclength-parametrized curves in $\mathscr{M}$ is called the *n-th localized induction equation* (n-th LIE). Here, s is the arclength parameter and t is time. Also, the infinite sequence of the equations (2.10), $n = 0, 1, 2, \dots$, is called the *localized induction hierarchy* (LIH).

REMARK 2.7. As mentioned in the introduction, in $\mathbf{R}^3$, the LIE is described as a Hamiltonian flow on an appropriate space of curves, and the LIH is interpreted as a sequence of commuting Hamiltonian flows of this Hamiltonian system. As for the Hamiltonian formulation in the case of a general oriented three-dimensional Riemannian manifold, we refer the reader to [36]. In this general case, the existence of a sequence of commuting Hamiltonian flows is not expected. However, in the case of three-dimensional space forms, it is shown in [36] that there exists a sequence of commuting Hamiltonian flows, which is essentially equivalent to the LIH in Definition 2.6.

REMARK 2.8. The first LIE with $C_1 = 0$, that is,

$$\frac{\partial \tilde{\gamma}}{\partial t} = \frac{\partial \tilde{\gamma}}{\partial s} \times \nabla_{\partial/\partial s} \frac{\partial \tilde{\gamma}}{\partial s}, \tag{2.11}$$

is the natural generalization of the original LIE (1.1) from the point of view of Riemannian geometry. For the initial value problem of this evolution equation, we refer the reader to [19], [20], [21]. Also, the second LIE with $C_2 = C_1^2/4$ is the natural generalization of the Fukumoto–Miyazaki equation

$$\tilde{\gamma}_t = \frac{C_1}{2}\tilde{\gamma}_s \times \tilde{\gamma}_{ss} + \left(\tilde{\gamma}_{sss} + \frac{3}{2}|\tilde{\gamma}_{ss}|^2\tilde{\gamma}_s\right) \tag{2.12}$$

from the point of view of Riemannian geometry. The Fukumoto–Miyazaki equation (2.12) is a model of the motion of vortex filaments with the effect of axial flow. For details, we refer the reader to [5], [6], [28], [33], etc. Furthermore, it is known that the third LIE also arises in fluid mechanical contexts ([4], [30], [35], etc.).

Now we define a congruence solution, that is, a solution evolving without changing shape. We will investigate congruence solutions to the n-th LIE in Section 5.

DEFINITION 2.9. Let $\mathscr{M}$ be an oriented three-dimensional Riemannian manifold. A solution $\tilde{\gamma}\colon \mathbf{R}\times\mathbf{R} \to \mathscr{M}$ to the n-th localized induction equation $\tilde{\gamma}_t = X_n[\tilde{\gamma}]$ is called a *congruence solution* if $\tilde{\gamma}$ is expressed as follows: there exist an arclength-parametrized curve $\gamma\colon \mathbf{R} \to \mathscr{M}$, a constant $c \in \mathbf{R}$ and a one-parameter group $\{\varphi^t\}_{t\in\mathbf{R}}$ of isometries of $\mathscr{M}$ such that $\tilde{\gamma}(s, t) = \varphi^t(\gamma(s - ct))$. This γ is uniquely determined by $\tilde{\gamma}$ and is called the *generating curve* of the congruence solution $\tilde{\gamma}$.

For a convenience in Section 5, we consider the *normalization* Y of $X = \sum_{n=0}^{\infty}\lambda^n X_n$ in the same way as in the case of $\mathscr{M} = \mathbf{R}^3$ (p. 29 of [22]). When $C_n = 0$ for all $n \geqslant 1$, we denote the $X_n^{C_0,\dots,C_n}$ by Y_n. For example,

$$\begin{aligned} Y_0 &= -T, \\ Y_1 &= T \times \nabla_T T, \\ Y_2 &= \frac{3}{2}|\nabla_T T|^2 T + (\nabla_T)^2 T. \end{aligned}$$

We set $Y = \sum_{n=0}^{\infty}\lambda^n Y_n$. Then (2.8) implies

$$\langle Y, Y\rangle = 1.$$

We investigate the relation between X and its normalization Y. The following proposition holds.

Proposition 2.10. *If $\{A_n\}_{n=0}^{\infty} \in \mathcal{S}$, then there exists a unique $\{C_n\}_{n=0}^{\infty} \in \mathcal{S}$ such that $X = AY$, where $X = \sum_{n=0}^{\infty}\lambda^n X_n = \sum_{n=0}^{\infty}\lambda^n X_n^{C_0,\dots,C_n}$ and $A = \sum_{n=0}^{\infty}\lambda^n A_n$. Moreover, the map sending $\{A_n\}_{n=0}^{\infty} \in \mathcal{S}$ to $\{C_n\}_{n=0}^{\infty} \in \mathcal{S}$ is bijective. Also, the $\{C_n\}_{n=0}^{\infty}$ is given by the relation $C = A^2$, where $C = \sum_{n=0}^{\infty}\lambda^n C_n$.*

Proof. Let $\{A_n\}_{n=0}^{\infty}\in\mathcal{S}$. Then $AY=\sum_{n=0}^{\infty}\lambda^n Z_n$, where we set $Z_n=\sum_{k=0}^{n}A_{n-k}Y_k$. It follows from $Z_0=Y_0=-T$ and $JY_k=\nabla_T Y_{k-1}$ that $JZ_n=\sum_{k=0}^{n}A_{n-k}JY_k=\sum_{k=1}^{n}A_{n-k}\nabla_T Y_{k-1}=\nabla_T Z_{n-1}$, where $n\geqslant 1$. Also, each Z_n is LAP, because $\langle\nabla_T Z_n, T\rangle=\langle JZ_{n+1}, T\rangle=0$. Therefore, by the latter part of Proposition 2.5, there exists a unique $\{C_n\}_{n=0}^{\infty}\in\mathcal{S}$ such that $Z_n=X_n^{C_0,\dots,C_n}$ for all $n\geqslant 0$. Thus $X=AY$ holds.

Next we show $C=A^2$. Note that (2.8), that is, $\langle X, X\rangle=C$ holds. On the other hand, it follows from $X=AY$ and $\langle Y, Y\rangle=1$ that $\langle X, X\rangle=A^2\langle Y, Y\rangle=A^2$. Therefore, $C=A^2$ holds, that is, $C_n=\sum_{k=0}^{n}A_k A_{n-k}$ holds for all $n\geqslant 0$.

We show the bijectivity of the map sending $\{A_n\}_{n=0}^{\infty}\in\mathcal{S}$ to $\{C_n\}_{n=0}^{\infty}\in\mathcal{S}$. Let $\{C_n\}_{n=0}^{\infty}\in\mathcal{S}$. Then we can check that there exists a unique $\{A_n\}_{n=0}^{\infty}\in\mathcal{S}$ satisfying $C=A^2$. Thus, by the above assertion, this $\{A_n\}_{n=0}^{\infty}$ satisfies $X=AY$. Also, the uniqueness of $\{A_n\}_{n=0}^{\infty}$ satisfying $C=A^2$ yields the uniqueness of $\{A_n\}_{n=0}^{\infty}$ satisfying $X=AY$. This completes the proof of the bijectivity. □

3. Soliton curves

In this section, we define an n-th soliton curve as a solution to the stationary equation corresponding to the n-th LIE, and investigate some low-order soliton curves. In particular, we give the characterizations of first and second soliton curves in an oriented three-dimensional Riemannian manifold $\mathscr{M}$ and that of third soliton curves in $\mathbf{R}^3$, S^3, H^3 (Theorem 3.3).

DEFINITION 3.1. Let $n\geqslant 1$. An arclength-parametrized curve $\gamma\colon\mathbf{R}\to\mathscr{M}$ is called an *n-th soliton curve* if $X_n^{C_0,\dots,C_n}[\gamma]=0$ holds for some $C_1,\dots,C_n\in\mathbf{R}$. Also, the set of all n-th soliton curves is called the *n-th soliton class*, and is denoted by Γ_n.

An n-th soliton curve is also characterized in the following way.

Proposition 3.2. *Let $\gamma\colon\mathbf{R}\to\mathscr{M}$ be an arclength-parametrized curve. Then $\gamma\in\Gamma_n$ if and only if there exists $(C_0, C_1,\dots,C_{n-1})\in\mathbf{R}^n$ with $C_0=1$ such that $\nabla_T X_{n-1}^{C_0,\dots,C_{n-1}}=0$.*

Proof. Let $\gamma\in\Gamma_n$. Then there exists $(C_0,\dots,C_n)\in\mathbf{R}^{n+1}$ with $C_0=1$ such that $X_n^{C_0,\dots,C_n}[\gamma]=0$. Thus the normal component $X_n^{\perp}=-J\nabla_T X_{n-1}$ of X_n is equal to zero. Therefore, $(\nabla_T X_{n-1})^{\perp}=-J(J\nabla_T X_{n-1})=0$. On the other hand, since X_{n-1} is a LAP vector field, the tangential component of $\nabla_T X_{n-1}$ is also equal to zero. Hence $\nabla_T X_{n-1}=0$.

We show the converse. Suppose that there exists $(C_0,C_1,\dots,C_{n-1})\in\mathbf{R}^n$ with $C_0=1$ such that $\nabla_T X_{n-1}^{C_0,\dots,C_{n-1}}=0$. Then it follows from the definition of $\mathcal{R}_0$ that $\mathcal{R}_0(X_{n-1})=0$. On the other hand, there exists $C_n\in\mathbf{R}$ such that $X_n^{C_0,\dots,C_n}=\mathcal{R}_0(X_{n-1}^{C_0,\dots,C_{n-1}})$. Thus $X_n^{C_0,\dots,C_n}=0$, and hence $\gamma\in\Gamma_n$. □

We investigate some low-order soliton classes. An arclength-parametrized curve γ in $\mathscr{M}$ is said to be a *helix* if γ is a geodesic or has a constant Frenet curvature (> 0) and a constant Frenet torsion.

Theorem 3.3. *Let $\mathscr{M}$ be an oriented three-dimensional Riemannian manifold. Then $\Gamma_n \subset \Gamma_{n+1}$ holds for any positive integer n. Also, let $\gamma\colon \mathbf{R} \to \mathscr{M}$ be an arclength-parametrized curve. Then the following holds*:
(i) *$\gamma \in \Gamma_1$ if and only if γ is a geodesic.*
(ii) *$\gamma \in \Gamma_2$ if and only if γ is a helix.*
(iii) *Suppose that $\mathscr{M} = \mathbf{R}^3, S^3, H^3$. Then $\gamma \in \Gamma_3$ if and only if γ is a Kirchhoff rod centerline.*

Here, we do not give the proof of (iii). We will describe the definition of a Kirchhoff rod centerline and give the proof of (iii) in Section 6.

Proof of Theorem 3.3. First we show $\Gamma_n \subset \Gamma_{n+1}$. Let $\gamma \in \Gamma_n$. Then $X_n^{C_0,\ldots,C_n}[\gamma] = 0$ for some $(C_0, C_1, \ldots, C_n) \in \mathbf{R}^{n+1}$ with $C_0 = 1$, and hence $\nabla_T X_n^{C_0,\ldots,C_n} = 0$. Therefore, Proposition 3.2 yields $\gamma \in \Gamma_{n+1}$. Hence $\Gamma_n \subset \Gamma_{n+1}$ follows.

Next we show (i). By Proposition 3.2, $\gamma \in \Gamma_1$ holds if and only if $\nabla_T X_0^{C_0} = -\nabla_T T = 0$. Hence the proof of (i) is completed.

We show (ii). By Proposition 3.2, $\gamma \in \Gamma_2$ holds if and only if $\nabla_T X_1^{C_0,C_1} = \nabla_T(-(C_1/2)T + T \times \nabla_T T) = 0$ for some $C_1 \in \mathbf{R}$. Let $\gamma \in \Gamma_2$. We denote by $\kappa = |\nabla_T T|$ the Frenet curvature of γ. Suppose that γ is not a geodesic. Let $s_0 \in \mathbf{R}$ be a point such that $\kappa(s_0) > 0$. We denote by (T, N, B) the Frenet frame along γ and by τ the Frenet torsion of γ around $s = s_0$. A straightforward calculation yields $(-(C_1/2) - \tau)\kappa N + \kappa' B = 0$. Therefore, $\kappa = \kappa_0$ and $\tau = -C_1/2$ around s_0, where κ_0 is a positive constant. By using the continuity of κ, we see that $\kappa(s) = \kappa_0$ on the whole $\mathbf{R}$, and hence the Frenet frame is defined on the whole $\mathbf{R}$. Thus $\tau(s) = -C_1/2$ on $\mathbf{R}$, and hence γ is a helix. Conversely, we can check that if γ is a helix, then $\gamma \in \Gamma_2$. Hence the proof of (ii) is completed. □

4. Natural curvatures and complex curvature

In this section, we define the natural frame, natural curvatures and complex curvature of a curve in three-dimensional space forms, and describe some fundamental properties. Although these notions are originally defined for a curve in the three-dimensional Euclidean space, they are naturally extended to a curve in three-dimensional space forms. Since the proofs of the facts in this section are similar to those of the Euclidean case, we omit them. For more details about these notions, we refer the reader to [2], [22], [24], [26], etc.

Let γ be an arclength-parametrized curve in $\mathscr{M}=\mathbf{R}^3, S^3, H^3$. Then there exists a positive orthonormal frame (T, U_1, U_2) along γ such that

$$\nabla_T T = u_1 U_1 + u_2 U_2, \quad \nabla_T U_1 = -u_1 T, \quad \nabla_T U_2 = -u_2 T, \tag{4.1}$$

where u_1, u_2 are functions of s. Such (T, U_1, U_2) is called a *natural frame* (or *Bishop frame*) along γ. Also, the functions u_1, u_2 are called the *natural curvatures* of γ, and the complex-valued function $\psi=\psi(s)$ defined by $\psi = u_1 + iu_2$ is called the *complex curvature* of γ.

We denote by $T^{\perp}\mathscr{M}$ the normal bundle along γ, and by $\nabla^{\perp}$ the normal connection in $T^{\perp}\mathscr{M}$. That is, $\nabla^{\perp}_T U = \nabla_T U - \langle \nabla_T U, T\rangle T$ for a normal vector field U along γ. Then a positive orthonormal frame field (T, V_1, V_2) along γ is a natural frame if and only if $\nabla^{\perp}_T V_1 = \nabla^{\perp}_T V_2 = 0$.

Compared to the Frenet frame (T, N, B), the natural frame (T, U_1, U_2) has the advantage that it can be defined even on a point where $\nabla_T T = 0$. On the other hand, unlike the Frenet frame, the natural frame is not uniquely determined by γ. For a given γ, the natural frame is uniquely determined only up to rotation around T by a constant angle. Also, the complex curvature is determined only up to multiplication by a complex unit. To be precise, if (T, U_1, U_2), $(T, \hat{U}_1, \hat{U}_2)$ are two natural frames along γ, and ψ, $\hat{\psi}$ the corresponding complex curvatures, respectively, then there exists a unique $\theta \in \mathbf{R}/(2\pi\mathbf{Z})$ such that $\hat{U}_1 = (\cos\theta)U_1 + (\sin\theta)U_2$, $\hat{U}_2 = -(\sin\theta)U_1 + (\cos\theta)U_2$. Also, $\hat{\psi} = e^{-i\theta}\psi$ holds.

In the same way as curves in $\mathbf{R}^3$ (Theorem 3 of [2]), the following proposition (an analog of the classical fundamental theorem of curve theory) holds, whose proof is omitted. Two curves γ_1, γ_2 in $\mathscr{M}$ are called *properly congruent* if there exists an orientation-preserving isometry φ of $\mathscr{M}$ such that $\gamma_2 = \varphi \circ \gamma_1$.

Proposition 4.1. *Let $\mathscr{M}=\mathbf{R}^3, S^3, H^3$. Two arclength-parametrized curves in $\mathscr{M}$ are properly congruent if and only if their complex curvatures are identical up to multiplication of a complex unit. For any complex-valued function $\psi\colon I \to \mathbf{C}$, there exists an arclength-parametrized curve $\gamma\colon I \to \mathscr{M}$ whose complex curvature corresponds to ψ.*

For the following section, we introduce another notation. Let $\gamma=\gamma(s)$ be an arclength-parametrized curve in $\mathscr{M}=\mathbf{R}^3, S^3, H^3$, and (T, U_1, U_2) a natural frame along γ. For a vector field X along γ, we define a complex-valued function $\mathcal{Z}(X)$ of s by

$$\mathcal{Z}(X) = \langle X, U_1\rangle + i\langle X, U_2\rangle.$$

We call $\mathcal{Z}$ the *normal coordinate map* with respect to the natural frame (T, U_1, U_2). Then the complex curvature ψ of γ is expressed as $\psi = \mathcal{Z}(\nabla_T T)$.

5. Main theorem

In this section, we state and prove the main theorem (Theorem 5.1). Also, in the last part of this section, we show the translation solution version of the main theorem (Proposition 5.7).

We consider the problem of determining the generating curves of congruence solutions to the n-th LIE. First, let $\gamma \in \Gamma_n$. Then $X_n[\gamma] = 0$ holds for some $C_1, \ldots, C_n \in \mathbf{R}$. Therefore, by setting $\tilde{\gamma}(s,t) := \varphi^t(\gamma(s)) = \gamma(s)$, where $\varphi^t \colon \mathscr{M} \to \mathscr{M}$ is the identity map for all $t \in \mathbf{R}$, we see that the both sides of (2.10) are equal to zero. Thus, this $\tilde{\gamma}(s, t)$ is a congruence solution. Consequently, any $\gamma \in \Gamma_n$ is the generating curve of a congruence solution to the n-th LIE with some $C_1, \ldots, C_n \in \mathbf{R}$. This is valid for the case where $\mathscr{M}$ is a general oriented three-dimensional Riemannian manifold.

In the case where $\mathscr{M} = \mathbf{R}^3$, S^3, H^3, however, we can prove that any curve γ in Γ_{n+2} ($\supset \Gamma_n$) is the generating curve of a congruence solution to the n-th LIE. More precisely, the following main theorem holds.

Theorem 5.1. *Let $\mathscr{M} = \mathbf{R}^3, S^3, H^3$, and $n \geqslant 1$. Let $\gamma \colon \mathbf{R} \to \mathscr{M}$ be an arclength-parametrized curve. Then the following* (i) *and* (ii) *are equivalent.*
(i) $\gamma \in \Gamma_{n+2}$.
(ii) *γ is the generating curve of a congruence solution to the n-th localized induction equation with some $C_1, \ldots, C_n \in \mathbf{R}$.*

Let us sketch the basic idea of the proof of Theorem 5.1. We need three lemmas, that is, Lemma 5.3, Lemma 5.4 and Lemma 5.5. Before Lemma 5.3, we first seek for the variation formula for the complex curvature of a curve (Proposition 5.2). Using this formula, we derive the condition of a vector field along a curve in $\mathscr{M}$ to extend to a Killing vector field on $\mathscr{M}$ (Lemma 5.3).

Now, suppose that (i) holds. Then $X_{n+2}[\gamma] = 0$ for some $C_1, \ldots, C_{n+2} \in \mathbf{R}$. In the case where $\mathscr{M} = \mathbf{R}^3$, as is shown in Proposition 12 of [22], (ii) of Theorem 5.1 holds for the constants $C_1, \ldots, C_n$. However, in the case where $\mathscr{M} = S^3, H^3$, (ii) does not necessarily holds for the same constants $C_1, \ldots, C_n$. In order to replace them by appropriate different constants, we prove Lemma 5.5. Here, Lemma 5.4 is a preparation for proving Lemma 5.5. By using Lemma 5.3 and Lemma 5.5, we show that the X_n for these new constants extends to a Killing vector field on $\mathscr{M}$. This implies that the shape of γ does not change infinitesimally in the direction X_n. By letting $\{\varphi^t\}_{t \in \mathbf{R}}$ denote the one-parameter group of isometries of $\mathscr{M}$ generated by the Killing vector field, we see that $\tilde{\gamma}(s, t) := \varphi^t(\gamma(s))$ is a congruence solution to the n-th LIE, from which (ii) follows. The proof of (ii) $\Rightarrow$ (i) is carried out by reversing this process.

We first seek for the variation formula for the complex curvature of a curve in $\mathscr{M}$ (Proposition 5.2). Proposition 5.2 is the space form version of Theorem 14 of [22] (see also [24]).

Proposition 5.2. *Let $\gamma(s,t)$ be a variation of arclength-parametrized curves in $\mathscr{M} = \mathbf{R}^3, S^3, H^3$, where t is the variation parameter. Let $T(s,t), U_1(s,t), U_2(s,t)$ be vector fields along γ such that for each fixed t, (T, U_1, U_2) is a natural frame along the curve $s \mapsto \gamma(s,t)$. We denote by u_1, u_2 the corresponding natural curvatures and by $\psi = u_1 + iu_2$ the complex curvature. Let $W = \partial\gamma/\partial t$ be the variation vector field of the variation γ. Then the following variation formula of the complex curvature holds: there exists $b \in \mathbf{R}$ such that*

$$\left.\frac{\partial\psi}{\partial t}\right|_{t=0} = \mathcal{Z}(-\mathcal{R}_b^2 W + GW), \tag{5.1}$$

where $\mathcal{R}$ denotes the recursion operator with respect to the curve $s \mapsto \gamma(s,0)$ and $\mathcal{Z}$ denotes the normal coordinate map with respect to the natural frame $(T(s,0), U_1(s,0), U_2(s,0))$.

Proof. We denote the induced connection in the induced bundle $\gamma^{-1}T\mathscr{M}$ by $\nabla^{\gamma^{-1}T\mathscr{M}}$, and write $\nabla^{\gamma^{-1}T\mathscr{M}}_{\partial/\partial s}$ and $\nabla^{\gamma^{-1}T\mathscr{M}}_{\partial/\partial t}$ as ∇_T and ∇_W, respectively. We can verify that $\nabla_W T = \nabla_T W$ and

$$\nabla_T\nabla_W Y - \nabla_W\nabla_T Y = G(\langle W, Y\rangle T - \langle T, Y\rangle W), \tag{5.2}$$

where Y is an arbitrary vector field along γ.

Since (T, U_1, U_2) is an orthonormal frame field along γ, there exist three functions $A(s,t)$, $B(s,t)$, $C(s,t)$ such that

$$\nabla_W T = CU_1 - BU_2, \quad \nabla_W U_1 = -CT + AU_2, \quad \nabla_W U_2 = BT - AU_1.$$

Let $\omega = \omega(s,t)$ be the t-angular velocity of the orthonormal frame field (T, U_1, U_2), that is, let $\omega = AT + BU_1 + CU_2$. Then the following equations hold:

$$\nabla_W T = \omega \times T, \quad \nabla_W U_1 = \omega \times U_1, \quad \nabla_W U_2 = \omega \times U_2. \tag{5.3}$$

We show that ω is expressed by W as follows: for each fixed t, there exists a constant $b(t) \in \mathbf{R}$ such that

$$\omega = -\mathcal{R}_{b(t)} W, \tag{5.4}$$

where $\mathcal{R}$ denotes the recursion operator with respect to the arclength-parametrized curve $s \mapsto \gamma(s,t)$. First, the first equation of (5.3) and $\nabla_W T = \nabla_T W$ imply $T \times (-\omega) = \nabla_T W$. Also, we see that $-\omega$ is a LAP vector field along the curve $s \mapsto \gamma(s,t)$. (We give the proof, below.) Therefore, we obtain that there exists $b(t) \in \mathbf{R}$ such that $-\omega = \mathcal{R}_{b(t)} W$.

We prove that $-\omega$ is a LAP vector field along the curve $s \mapsto \gamma(s,t)$. By Proposition 2.3, it is sufficient to prove $\partial A/\partial s = \langle \omega, \nabla_T T\rangle$. By using (5.2), (4.1), (5.3), we have

$$\begin{aligned}\frac{\partial A}{\partial s} = \frac{\partial \langle \nabla_W U_1, U_2\rangle}{\partial s} &= \langle \nabla_T \nabla_W U_1, U_2\rangle + \langle \nabla_W U_1, \nabla_T U_2\rangle \\ &= \langle \nabla_W \nabla_T U_1, U_2\rangle + \langle \nabla_W U_1, \nabla_T U_2\rangle \\ &= \langle -u_1 \nabla_W T, U_2\rangle + \langle \nabla_W U_1, -u_2 T\rangle \\ &= \langle \omega, u_1 U_1 + u_2 U_2\rangle = \langle \omega, \nabla_T T\rangle.\end{aligned}$$

Now, we calculate $\partial u_j/\partial t$, where $j = 1, 2$. It follows from (5.2) and (5.3) that

$$\begin{aligned}\frac{\partial u_j}{\partial t} = \frac{\partial \langle \nabla_T T, U_j\rangle}{\partial t} &= \langle \nabla_T(\omega \times T) + GW, U_j\rangle + \langle \nabla_T T, \omega \times U_j\rangle \\ &= \langle (\nabla_T \omega) \times T + GW, U_j\rangle = \langle \mathcal{R}\omega + GW, U_j\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial \psi}{\partial t}\bigg|_{t=0} = \left[\frac{\partial u_1}{\partial t} + i\frac{\partial u_2}{\partial t}\right]_{t=0} &= \mathcal{Z}(\mathcal{R}\omega + GW) \\ &= \mathcal{Z}(\mathcal{R}(-\mathcal{R}_{b(0)}W) + GW) = \mathcal{Z}(-\mathcal{R}_{b(0)}^2 W + GW),\end{aligned}$$

which completes the proof of the proposition. $\square$

By using Proposition 5.2, we obtain the following

Lemma 5.3. *Let $\gamma : I \to \mathscr{M} = \mathbf{R}^3, S^3, H^3$ be an arclength-parametrized curve, and let W be a vector field along γ. If W extends to a Killing vector field on $\mathscr{M}$, then W is LAP and*

$$-\mathcal{R}_a \mathcal{R}_b W + GW = 0 \tag{5.5}$$

holds for some $a, b \in \mathbf{R}$. Also, if γ is not a geodesic, then the converse is again true. That is, if W is LAP and (5.5) *holds for some $a, b \in \mathbf{R}$, then W extends to a Killing vector field on $\mathscr{M}$. Moreover, the Killing vector field is uniquely determined.*

Proof. Suppose that W extends to a Killing vector field $\tilde{W}$ on $\mathscr{M}$. We prove that W is LAP and (5.5) holds for some $a, b \in \mathbf{R}$. Let $\{\varphi^t\}_{t\in\mathbf{R}}$ denote the one-parameter group of isometries of $\mathscr{M}$ generated by $\tilde{W}$, and φ^t_* the differential map of φ^t for each t. We set $\tilde{\gamma}(s,t) = \varphi^t(\gamma(s))$. Since $\tilde{\gamma}(s,t)$ is a curve with arclength parameter s for each fixed t, it follows from Lemma 2.1 that W is LAP. Next we show (5.5) holds for some $a, b \in \mathbf{R}$. Let $(T(s), U_1(s), U_2(s))$ be a natural frame along γ, $u_1(s), u_2(s)$ the corresponding natural curvatures and $\psi(s) = u_1(s) + iu_2(s)$ the complex curvature. Let $\tilde{T}(s,t) = \varphi^t_*(T(s)) = \partial\tilde{\gamma}/\partial s$, $\tilde{U}_1(s,t) = \varphi^t_*(U_1(s))$ and $\tilde{U}_2(s,t) = \varphi^t_*(U_2(s))$. Since φ^t is an isometry of $\mathscr{M}$, we see that for each fixed t, the frame $\{\tilde{T}(s,t), \tilde{U}_1(s,t), \tilde{U}_2(s,t)\}$ is a natural frame along the curve $s \mapsto \tilde{\gamma}(s,t)$, and the corresponding complex curvature $\tilde{\psi}(s,t)$ of the curve $s \mapsto \tilde{\gamma}(s,t)$ is equal to $\psi(s)$. Therefore, from Proposition 5.2, we see that $\mathcal{Z}(-\mathcal{R}_b^2 W + GW) = 0$ for some $b \in \mathbf{R}$, which implies that the normal component of the LAP vector field $-\mathcal{R}_b^2 W + GW$ is zero. On the other hand, (i) of Lemma 2.3 yields that a LAP vector field whose normal component is zero is a constant multiple of T. Therefore, $-\mathcal{R}_b^2 W + GW = qT$ holds for some $q \in \mathbf{R}$. Thus, $-\mathcal{R}_{b+q}\mathcal{R}_b W + GW = 0$, and hence we obtain (5.5) by setting $a = b + q$.

Before proving the rest part, we introduce some notation. Let $\mathscr{X}(\mathscr{M})$ denote the vector space of all vector fields on $\mathscr{M}$, and let $\gamma^*\colon \mathscr{X}(\mathscr{M}) \to \mathscr{X}(\gamma)$ denote the pull-back by γ. That is, for $X \in \mathscr{X}(\mathscr{M})$, $\gamma^* X \in \mathscr{X}(\gamma)$ is defined by $(\gamma^* X)(x) = X(\gamma(x))$, where $x \in \mathscr{M}$. Let $\mathscr{K}(\mathscr{M})$ denote the vector space of all Killing vector fields on $\mathscr{M}$. It is well known that $\mathscr{K}(\mathscr{M})$ is a 6-dimensional linear subspace of $\mathscr{X}(\mathscr{M})$. Also, let

$$\mathscr{K}(\gamma) = \{W \in \mathscr{X}(\gamma) \mid W \text{ is LAP and satisfies (5.5) for some } a, b \in \mathbf{R}\}. \tag{5.6}$$

Since (2.2) holds for any LAP vector fields W_1, W_2, W along γ and any $a_1, a_2, c, a \in \mathbf{R}$, it follows that $\mathscr{K}(\gamma)$ is a linear subspace of $\mathscr{X}(\gamma)$. Also, the former part of the proof implies that if $X \in \mathscr{K}(\mathscr{M})$, then $\gamma^* X \in \mathscr{K}(\gamma)$. Hence γ^* can be viewed as a linear map of $\mathscr{K}(\mathscr{M})$ into $\mathscr{K}(\gamma)$.

Suppose that γ is not a geodesic. We show that if W is LAP and (5.5) holds for some $a, b \in \mathbf{R}$, then W uniquely extends to a Killing vector field on $\mathscr{M}$. To prove this, it is sufficient to prove that the linear map $\gamma^*\colon \mathscr{K}(\mathscr{M}) \to \mathscr{K}(\gamma)$ is bijective. Thus it is sufficient to prove that γ^* is injective and $\dim \mathscr{K}(\gamma) \leqslant 6$. We note the fact that the set of all zeros of a Killing vector field on $\mathscr{M}$ is either the empty set, the whole $\mathscr{M}$, or one geodesic in $\mathscr{M}$. From this fact and the assumption that γ is not a geodesic, the injectivity of $\gamma^*\colon \mathscr{K}(\mathscr{M}) \to \mathscr{K}(\gamma)$ follows.

It remains only to show $\dim \mathscr{K}(\gamma) \leqslant 6$. Let (T, U_1, U_2) be a natural frame along γ, and u_1, u_2 the corresponding natural curvatures of γ. Then the condition of $W = fT + gU_1 + hU_2 \in \mathscr{X}(\gamma)$ to be LAP is $f_s = u_1 g + u_2 h$, where f, g, h are functions of s on I, and the subscript s denotes the derivative with respect to s.

Let $W \in \mathscr{K}(\gamma)$. First, we show that the a and b in (5.5) are uniquely determined by W. Suppose that $-\mathcal{R}_{\hat{a}}\mathcal{R}_{\hat{b}} W + GW = 0$ for some $\hat{a}$, $\hat{b} \in \mathbf{R}$. Since $\mathcal{R}_a\mathcal{R}_b W = \mathcal{R}_0^2 W - bT \times \nabla_T T + aT$, it follows that $-(b - \hat{b})T \times \nabla_T T + (a - \hat{a})T = 0$. Therefore, by the assumption that γ is not a geodesic, we see $a = \hat{a}$ and $b = \hat{b}$. Hence the uniqueness of a and b holds. In what follows, we also write the a and b as $a(W)$ and $b(W)$, respectively.

Let f, g, h denote the T, U_1, U_2 components of W, respectively, and set $A = \langle -\mathcal{R}_{b(W)}W, T\rangle$. Then f, g, h, A satisfy the following system of linear ordinary differential equations:

$$f_s = u_1 g + u_2 h, \tag{5.7}$$

$$A_s = u_2 g_s - u_1 h_s, \tag{5.8}$$

$$g_{ss} = -(u_1)_s f - (u_1^2 + G)g - u_1 u_2 h - u_2 A, \tag{5.9}$$

$$h_{ss} = -(u_2)_s f - u_1 u_2 g - (u_2^2 + G)h + u_1 A. \tag{5.10}$$

We show these four equations. First, (5.7) follows from the condition of W to be LAP. Also, by a straightforward calculation, we see

$$\mathcal{R}_b W = -AT + (h_s + u_2 f)U_1 - (g_s + u_1 f)U_2, \tag{5.11}$$

$$\langle -\mathcal{R}\mathcal{R}_b W + GW, U_1\rangle = g_{ss} + (u_1)_s f + (u_1^2 + G)g + u_1 u_2 h + u_2 A, \tag{5.12}$$

$$\langle -\mathcal{R}\mathcal{R}_b W + GW, U_2\rangle = h_{ss} + (u_2)_s f + u_1 u_2 g + (u_2^2 + G)h - u_1 A. \tag{5.13}$$

The condition of $\mathcal{R}_b W$ to be LAP yields (5.8). Also, by $-\mathcal{R}_a\mathcal{R}_b W + GW = 0$, we have (5.9) and (5.10).

Let $\mathscr{F}$ denote the set of all pairs $(f, g, h, A) \in (C^\infty(I))^4$ satisfying (5.7), (5.8), (5.9) and (5.10). Then $\mathscr{F}$ is a 6-dimensional subspace of $(C^\infty(I))^4$. Therefore, to prove $\dim \mathscr{K}(\gamma) \leqslant 6$, it is sufficient to prove that the map $W\ (\in \mathscr{K}(\gamma)) \mapsto (f, g, h, A) \in \mathscr{F}$ is linear and injective. First, by using (2.2), we see that the map $W\ (\in \mathscr{K}(\gamma)) \mapsto b(W) \in \mathbf{R}$ is linear, and that the map $W\ (\in \mathscr{K}(\gamma)) \mapsto (f, g, h, A) \in \mathscr{F}$ is also linear. Also, the injectivity of the linear map $W\ (\in \mathscr{K}(\gamma)) \mapsto (f, g, h, A) \in \mathscr{F}$ immediately follows. Hence we obtain $\dim \mathscr{K}(\gamma) \leqslant 6$. The proof of Lemma 5.3 is completed. □

We state the second lemma.

Lemma 5.4. *Let $\{B_n\}_{n=0}^\infty$, $\{C_n\}_{n=0}^\infty \in \mathcal{S}$, and set $B = \sum_{n=0}^\infty \lambda^n B_n$, $C = \sum_{n=0}^\infty \lambda^n C_n$ and $X = \sum_{n=0}^\infty \lambda^n X_n = \sum_{n=0}^\infty \lambda^n X_n^{C_0,\dots,C_n}$. Then there exists $\{\hat{C}_n\}_{n=0}^\infty \in \mathcal{S}$ such that $B\hat{X} = X$, where $\hat{X} = \sum_{n=0}^\infty \lambda^n \hat{X}_n$, $\hat{X}_n = X_n^{\hat{C}_0,\dots,\hat{C}_n}$. Also, there exists $\{\tilde{C}_n\}_{n=0}^\infty \in \mathcal{S}$ such that $BX = \tilde{X}$, where $\tilde{X} = \sum_{n=0}^\infty \lambda^n \tilde{X}_n$, $\tilde{X}_n = X_n^{\tilde{C}_0,\dots,\tilde{C}_n}$.*

Proof. We prove the former part. By Proposition 2.10, there exists a unique $\{A_n\}_{n=0}^\infty \in \mathcal{S}$ such that $X = AY$, where $A = \sum_{n=0}^\infty \lambda^n A_n$. We deform this A. We can check that there exists a unique $\{\hat{A}_n\}_{n=0}^\infty \in \mathcal{S}$ such that $B\hat{A} = A$, where $\hat{A} = \sum_{n=0}^\infty \lambda^n \hat{A}_n$. We set $\hat{C} = \hat{A}^2$, and define $\{\hat{C}_n\}_{n=0}^\infty \in \mathcal{S}$ by $\hat{C} = \sum_{n=0}^\infty \lambda^n \hat{C}_n$. Then Proposition 2.10 yields $\hat{X} = \hat{A}Y$. Therefore, $B\hat{X} = (B\hat{A})Y = AY = X$, which completes the proof of the former part.

We prove the latter part. Let $\{A_n\}_{n=0}^\infty$ and A be as above. We set $\tilde{A} = BA$ and define $\{\tilde{A}_n\}_{n=0}^\infty \in \mathcal{S}$ by $\tilde{A} = \sum_{n=0}^\infty \lambda^n \tilde{A}_n$. And we set $\tilde{C} = \tilde{A}^2$ and define $\{\tilde{C}_n\}_{n=0}^\infty \in \mathcal{S}$ by

$\tilde{C} = \sum_{n=0}^{\infty} \lambda^n \tilde{C}_n$. Then Proposition 2.10 yields $\tilde{X} = \tilde{A}Y$, and hence $\tilde{X} = B(AY) = BX$. $\square$

We state the third lemma.

Lemma 5.5. *Let $n \geqslant 0$ and let $\gamma\colon \mathbf{R} \to \mathscr{M}$ be an arclength-parametrized curve in an oriented three-dimensional Riemannian manifold $\mathscr{M}$. Let p be an arbitrary real constant. Then $\gamma \in \Gamma_{n+2}$ if and only if there exists $(\hat{C}_0, \hat{C}_1, \dots, \hat{C}_{n+2}) \in \mathbf{R}^{n+3}$ with $\hat{C}_0 = 1$ such that $X_{n+2}^{\hat{C}_0,\dots,\hat{C}_{n+2}}[\gamma] - pX_n^{\hat{C}_0,\dots,\hat{C}_n}[\gamma] = 0$.*

Proof. Let $\gamma \in \Gamma_{n+2}$. Then there exists $(C_0, \dots, C_{n+2}) \in \mathbf{R}^{n+3}$ with $C_0 = 1$ such that $X_{n+2}^{C_0,\dots,C_{n+2}}[\gamma] = 0$. Let $C_j = 0$ for $j \geqslant n+3$, and $X = \sum_{n=0}^{\infty} \lambda^n X_n^{C_0,\dots,C_n}$. We define $\{B_n\}_{n=0}^{\infty} \in \mathcal{S}$ by $B_0 = 1$, $B_1 = 0$, $B_2 = -p$ and $B_n = 0$ for $n \geqslant 3$, and set $B = \sum_{n=0}^{\infty} \lambda^n B_n$. Then it follows from the former part of Lemma 5.4 that there exists $\{\hat{C}_n\}_{n=0}^{\infty} \in \mathcal{S}$ such that $B\hat{X} = X$, where $\hat{X} = \sum_{n=0}^{\infty} \lambda^n X_n^{\hat{C}_0,\dots,\hat{C}_n}$. By taking the coefficients of the λ^{n+2} terms of the both sides of $B\hat{X} = X$, we have $X_{n+2}^{\hat{C}_0,\dots,\hat{C}_{n+2}} - pX_n^{\hat{C}_0,\dots,\hat{C}_n} = X_{n+2}^{C_0,\dots,C_{n+2}} = 0$.

We show the converse. Suppose that there exists $(\hat{C}_0, \hat{C}_1, \dots, \hat{C}_{n+2}) \in \mathbf{R}^{n+3}$ with $\hat{C}_0 = 1$ such that $X_{n+2}^{\hat{C}_0,\dots,\hat{C}_{n+2}}[\gamma] - pX_n^{\hat{C}_0,\dots,\hat{C}_n}[\gamma] = 0$. Let $\hat{C}_j = 0$ for $j \geqslant n+3$ and let $\hat{X} = \sum_{n=0}^{\infty} \lambda^n X_n^{\hat{C}_0,\dots,\hat{C}_n}$. We define $\{B_n\}_{n=0}^{\infty}$ and B in the same way as above. Then it follows from the latter part of Lemma 5.4 that there exists $\{C_n\}_{n=0}^{\infty} \in \mathcal{S}$ such that $B\hat{X} = X$, where $X = \sum_{n=0}^{\infty} \lambda^n X_n^{C_0,\dots,C_n}$. By taking the coefficients of the λ^{n+2} terms of the both sides of $B\hat{X} = X$, we have $0 = X_{n+2}^{\hat{C}_0,\dots,\hat{C}_{n+2}} - pX_n^{\hat{C}_0,\dots,\hat{C}_n} = X_{n+2}^{C_0,\dots,C_{n+2}}$. Hence $\gamma \in \Gamma_{n+2}$, which completes the proof. $\square$

Now we give the proof of the main theorem.

Proof of Theorem 5.1. We show (i) $\Rightarrow$ (ii). Let $\gamma \in \Gamma_{n+2}$. First we consider the case where γ is not a geodesic. By Lemma 5.5, there exists $(C_0, C_1, \dots, C_{n+2}) \in \mathbf{R}^{n+3}$ with $C_0 = 1$ such that $X_{n+2}^{C_0,\dots,C_{n+2}} - GX_n^{C_0,\dots,C_n} = 0$. We simply write $X_n^{C_0,\dots,C_n}$ etc. as X_n etc. Since there exist $a, b \in \mathbf{R}$ such that $X_{n+1} = \mathcal{R}_b X_n$ and $X_{n+2} = \mathcal{R}_a X_{n+1}$, we have $\mathcal{R}_a \mathcal{R}_b X_n - GX_n = 0$. Therefore, it follows from Lemma 5.3 that X_n uniquely extends to a Killing vector field $\tilde{X}_n$ on $\mathscr{M}$. Let $\{\varphi^t\}_{t\in\mathbf{R}}$ denote the one-parameter group of isometries of $\mathscr{M}$ generated by $\tilde{X}_n$, and set $\tilde{\gamma}(s,t) = \varphi^t(\gamma(s))$. Since φ^t is an orientation-preserving isometry, we see $X_n[\varphi^t \circ \gamma] = \varphi^t_*(X_n[\gamma])$. Thus, $\tilde{\gamma}(s,t)$ satisfies

$$\frac{\partial \tilde{\gamma}}{\partial t} = \tilde{X}_n(\varphi^t(\gamma(s))) = \varphi^t_*(X_n[\gamma](s)) = X_n[\varphi^t \circ \gamma](s) = X_n[\tilde{\gamma}(s,t)].$$

Therefore, $\tilde{\gamma}(s,t)$ is a congruence solution with generating curve γ, and hence (ii) holds.

Next we consider the case where γ is a geodesic. Then, by Theorem 3.3, $\gamma \in \Gamma_1 \subset \Gamma_n$. Thus, there exists $(C_0, C_1, \dots, C_n) \in \mathbf{R}^{n+1}$ with $C_0 = 1$ such that $X_n^{C_0,\dots,C_n} = 0$. By setting $\tilde{\gamma}(s, t) = \gamma(s)$, we have $\partial\tilde{\gamma}/\partial t = 0 = X_n^{C_0,\dots,C_n}[\tilde{\gamma}]$, and hence (ii) holds. The proof of (i) $\Rightarrow$ (ii) is completed.

We show (ii) $\Rightarrow$ (i). Suppose that γ is the generating curve of a congruence solution $\tilde{\gamma}(s, t)$ to (2.10) with some $C_1, \dots, C_n \in \mathbf{R}$. Then $\tilde{\gamma}(s, t)$ is expressed as $\tilde{\gamma}(s,t) = \varphi^t(\gamma(s-ct))$, where $c \in \mathbf{R}$ and $\{\varphi^t\}_{t\in\mathbf{R}}$ is a one-parameter group of isometries of $\mathscr{M}$. Let Z denote the Killing vector field on $\mathscr{M}$ corresponding to $\{\varphi^t\}_{t\in\mathbf{R}}$. Since $[\partial\varphi^t(\gamma(s-ct))/\partial t]_{t=0} = (\gamma^* Z - cT)(s)$, we have $\gamma^* Z = X_n + cT = X_n - cX_0$.

Now, we show that there exists $(\tilde{C}_0, \tilde{C}_1, \dots, \tilde{C}_n) \in \mathbf{R}^{n+1}$ with $\tilde{C}_0 = 1$ such that $X_n - cX_0 = X_n^{\tilde{C}_0,\dots,\tilde{C}_n}$. Let $C_j = 0$ for $j \geqslant n+1$ and set $X = \sum_{j=0}^{\infty} \lambda^j X_j^{C_0,\dots,C_j}$. We define $\{B_j\}_{j=0}^{\infty} \in \mathcal{S}$ by $B_0 = 1$, $B_n = -c$ and $B_j = 0$ for $j \neq 0, n$, and set $B = \sum_{j=0}^{\infty} \lambda^j B_j$. Then it follows from Lemma 5.4 that there exists $\{\tilde{C}_j\}_{j=0}^{\infty} \in \mathcal{S}$ such that $BX = \tilde{X}$, where $\tilde{X} = \sum_{j=0}^{\infty} \lambda^j X_j^{\tilde{C}_0,\dots,\tilde{C}_j}$. By taking the coefficients of the λ^n terms of the both sides of $BX = \tilde{X}$, we have $X_n - cX_0 = X_n^{\tilde{C}_0,\dots,\tilde{C}_n}$.

In what follows, we rewrite $\tilde{C}_0, \tilde{C}_1, \dots, \tilde{C}_n$ as $C_0, C_1, \dots, C_n$, and write $X_n^{C_0,\dots,C_n}$ etc. as simply X_n etc. Thus $\gamma^* Z = X_n$. Therefore, by Lemma 5.3, there exist $a, b \in \mathbf{R}$ such that $\mathcal{R}_a\mathcal{R}_b X_n - GX_n = 0$. Since there exists a unique $C_{n+1} \in \mathbf{R}$ satisfying $X_{n+1}^{C_0,\dots,C_{n+1}} = \mathcal{R}_b X_n^{C_0,\dots,C_n}$ and there exists a unique $C_{n+2} \in \mathbf{R}$ satisfying $X_{n+2}^{C_0,\dots,C_{n+2}} = \mathcal{R}_a X_n^{C_0,\dots,C_{n+1}}$, we have $X_{n+2}^{C_0,\dots,C_{n+2}} - GX_n^{C_0,\dots,C_n} = 0$. Hence Lemma 5.5 implies $\gamma \in \Gamma_{n+2}$, which completes the proof. $\square$

We consider the case of $n = 1$. It is verified that if $\tilde{\gamma}(s,t)$ is a congruence solution to the first LIE $\tilde{\gamma}_t = X_1^{C_0,C_1}[\tilde{\gamma}]$, then $\tilde{\gamma}(s + (C_1/2)t, t)$ is a congruence solution to the first LIE with $C_1 = 0$, that is, the equation (2.11). This implies that the set of all the generating curves of congruence solutions to (2.11) coincides with the set of all Kirchhoff rod centerlines. In the case where $\mathscr{M} = \mathbf{R}^3$, the essentially equivalent result is obtained in [10], [26]. For details about congruence solutions to the original LIE (1.1), see also [8], [18], etc.

In the rest of this section, we investigate special congruence solutions to the n-th LIE in the case where $\mathscr{M} = \mathbf{R}^3$. An isometry $\varphi\colon \mathbf{R}^3 \to \mathbf{R}^3$ is called a *translation* if there exists $\boldsymbol{a} \in \mathbf{R}^3$ such that $\varphi(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{a}$ for all $\boldsymbol{x} \in \mathbf{R}^3$. We define a translation solution as follows:

DEFINITION 5.6. Let $\mathscr{M} = \mathbf{R}^3$. A solution $\tilde{\gamma}\colon \mathbf{R} \times \mathbf{R} \to \mathscr{M}$ to the n-th LIE $\tilde{\gamma}_t = X_n[\tilde{\gamma}]$ is called a *translation solution* if $\tilde{\gamma}$ is expressed as follows: there exist an arclength-parametrized curve $\gamma\colon \mathbf{R} \to \mathscr{M}$, a constant $c \in \mathbf{R}$ and a one-parameter group $\{\varphi^t\}_{t\in\mathbf{R}}$ of translations of $\mathbf{R}^3$ such that $\tilde{\gamma}(s, t) = \varphi^t(\gamma(s-ct))$. This γ is uniquely determined by $\tilde{\gamma}$ and is called the *generating curve* of the translation solution $\tilde{\gamma}$.

Then we have the translation solution version of Theorem 5.1 as follows. The part of (i) $\Rightarrow$ (ii) of Proposition 5.7 is obtained in (b) of Proposition 12 of [22]. Also, in [5], it is shown that the permanent form of a steadily translating vortex-jet filament is identical to a Kirchhoff rod centerline. The case of $n = 2$ of Proposition 5.7 is analogous to this result in [5].

Proposition 5.7. *Let $\mathscr{M} = \mathbf{R}^3$, and $n \geqslant 1$. Let $\gamma\colon \mathbf{R} \to \mathscr{M}$ be an arclength-parametrized curve. Then the following* (i) *and* (ii) *are equivalent.*
(i) $\gamma \in \Gamma_{n+1}$.
(ii) *γ is the generating curve of a translation solution to the n-th localized induction equation with some $C_1, \dots, C_n \in \mathbf{R}$.*

Proof. We show (i) $\Rightarrow$ (ii). Let $\gamma \in \Gamma_{n+1}$. By Proposition 3.2, there exists $(C_0, C_1, \dots, C_n) \in \mathbf{R}^{n+1}$ with $C_0 = 1$ such that $\nabla_T X_n^{C_0,\dots,C_n} = 0$. Since $\mathscr{M} = \mathbf{R}^3$, the vector field $X_n^{C_0,\dots,C_n}$ along γ extends to a constant vector field $\tilde{X}_n$ on $\mathscr{M}$. Let $\{\varphi^t\}_{t\in\mathbf{R}}$ denote the one-parameter group of translations of $\mathscr{M}$ generated by $\tilde{X}_n$, and let $\tilde{\gamma}(s, t) = \varphi^t(\gamma(s))$. By a similar argument to the proof of Theorem 5.1, we have $\partial\tilde{\gamma}/\partial t = X_n^{C_0,\dots,C_n}[\tilde{\gamma}]$, and hence (ii) holds.

Next we show (ii) $\Rightarrow$ (i). Suppose that γ is the generating curve of a translation solution $\tilde{\gamma}(s, t)$ to (2.10) with some $C_1, \dots, C_n \in \mathbf{R}$. Then $\tilde{\gamma}(s, t)$ is expressed as $\tilde{\gamma}(s,t) = \varphi^t(\gamma(s-ct))$, where $c \in \mathbf{R}$ and $\{\varphi^t\}_{t\in\mathbf{R}}$ is a one-parameter group of translations of $\mathscr{M}$. Let Z denote the constant vector field on $\mathscr{M}$ corresponding to $\{\varphi^t\}_{t\in\mathbf{R}}$. By a similar argument to the proof of Theorem 5.1, there exists $(\tilde{C}_0, \tilde{C}_1, \dots, \tilde{C}_n) \in \mathbf{R}^{n+1}$ with $\tilde{C}_0 = 1$ such that $\gamma^* Z = X_n^{\tilde{C}_0,\dots,\tilde{C}_n}$. Since Z is a constant vector field on $\mathscr{M}$, we see $\nabla_T X_n^{\tilde{C}_0,\dots,\tilde{C}_n} = 0$. Hence Proposition 3.2 implies $\gamma \in \Gamma_{n+1}$, which completes the proof. $\square$

6. Third soliton curves in space forms

In this section, we describe the definition of a Kirchhoff rod centerline, and give the proof of (iii) of Theorem 3.3. That is, we prove that in the case where $\mathscr{M} = \mathbf{R}^3, S^3, H^3$, the third soliton class Γ_3 coincides with the set of all Kirchhoff rod centerlines.

First, we recall the notion of a *Kirchhoff elastic rod* (or simply *Kirchhoff rod*), which is a mathematical model of an elastic rod with the effects of both bending and twisting. Let $\gamma = \gamma(s)\colon [s_1, s_2] \to \mathscr{M}$ be a unit-speed curve in $\mathscr{M} = \mathbf{R}^3$, S^3, H^3 and let $M = (M_1, M_2)$ be an orthonormal frame field in the normal bundle $T^\perp\mathscr{M}$ along γ. We call such a pair $\{\gamma, M\}$ a *unit-speed curve with adapted orthonormal frame*.

Let ν be a fixed positive constant, which is determined by the material of the elastic rod. We define the energy $\mathfrak{T}$, which includes the effects of both bending and twisting, as follows:

$$\mathfrak{T}(\{\gamma, M\}) = \int_{s_1}^{s_2} |\nabla_T T|^2\, ds + \nu \sum_{i=1}^{2} \int_{s_1}^{s_2} |\nabla_T^\perp M_i|^2\, ds.$$

Here, the first term of $\mathfrak{T}(\{\gamma, M\})$ expresses the energy of bending, and the second term that of twisting. We call $\{\gamma, M\}$ a Kirchhoff rod if $\{\gamma, M\}$ is a critical point of $\mathfrak{T}$ with respect to the variations of unit-speed curves with adapted orthonormal frames which preserve the end points $\gamma(s_1)$, $\gamma(s_2)$ and the orthonormal frames $(T(s_1), M(s_1))$, $(T(s_2), M(s_2))$ at the end points. More precisely, a Kirchhoff rod is defined to be a solution of the associated Euler–Lagrange equations.

DEFINITION 6.1 (Definition 2.1 of [15]). Let $\mathscr{M} = \mathbf{R}^3, S^3, H^3$. A unit-speed curve with adapted orthonormal frame $\{\gamma, M\}$ is called a *Kirchhoff rod* if the following two equations hold for some real constants a and μ.

$$\nabla_T\left[(\nabla_T)^2 T + \left(\frac{3}{2}|\nabla_T T|^2 - \frac{\mu}{2} + G + \nu a^2\right)T - 2\nu a T \times \nabla_T T\right] = 0, \tag{6.1}$$

$$\langle \nabla_T^{\perp} M_1, T \times M_1 \rangle = a. \tag{6.2}$$

The constant a is uniquely determined, and is called the *twist rate* of $\{\gamma, M\}$.

We define a Kirchhoff rod centerline as follows:

DEFINITION 6.2 (the case of $n = 3$ in Definition 2.4 of [17]). A unit-speed curve γ in $\mathscr{M} = \mathbf{R}^3$, S^3, H^3 is called a *Kirchhoff rod centerline* if there exists an orthonormal frame field $M = (M_1, M_2)$ in the normal bundle along γ such that $\{\gamma, M\}$ is a Kirchhoff rod.

Many authors have been studying explicit expressions of Kirchhoff rod centerlines (see, e.g., [11], [12], [13], [14], [15], [26], [29], [32], [34]). In [26], Langer–Singer obtained explicit formulas of Kirchhoff rod centerlines in $\mathbf{R}^3$ by Jacobi sn function and the elliptic integrals in terms of cylindrical coordinates. Also, in the case where $\mathscr{M} = S^3$, H^3, analogous explicit formulas of Kirchhoff rod centerlines are obtained in [15].

Before the proof of (iii) of Theorem 3.3, we give the following characterization of Kirchhoff rod centerlines.

Proposition 6.3. *A unit-speed curve γ in $\mathscr{M} = \mathbf{R}^3$, S^3, H^3 is a Kirchhoff rod centerline if and only if* (6.1) *holds for some $\mu, a \in \mathbf{R}$.*

Proof. Suppose that γ is a Kirchhoff rod centerline. Then it follows from Definitions 6.1 and 6.2 that (6.1) holds for some μ, $a \in \mathbf{R}$. Conversely, suppose that a unit-speed curve γ satisfies (6.1) for some $\mu, a \in \mathbf{R}$. We take a unit normal vector U^0 at a point $\gamma(s_0)$, and let $U = U(s)$ be the parallel translation of U^0 with respect to the normal connection $\nabla^{\perp}$. We define an orthonormal frame field $M = (M_1, M_2)$ in $T^{\perp}\mathscr{M}$ by setting $M_1 = (\cos sa)U + (\sin sa)T \times U$, $M_2 = -(\sin sa)U + (\cos sa)T \times U$. Then (6.2) holds, and hence $\{\gamma, M\}$ is a Kirchhoff rod. Therefore, γ is a Kirchhoff rod centerline, which completes the proof. □

Now, we give the proof of (iii) of Theorem 3.3.

Proof of (iii) of Theorem 3.3. By Proposition 3.2, $\gamma \in \Gamma_3$ holds if and only if $\nabla_T X_2^{C_0,C_1,C_2} = 0$, that is,

$$\nabla_T \left[(\nabla_T)^2 T + \left(\frac{3}{2} |\nabla_T T|^2 + \frac{C_1^2}{8} - \frac{C_2}{2} \right) T + \frac{C_1}{2} T \times \nabla_T T \right] = 0, \tag{6.3}$$

for some $C_1, C_2 \in \mathbf{R}$. On the other hand, by Proposition 6.3, γ is a Kirchhoff rod centerline if and only if (6.1) holds for some $\mu, a \in \mathbf{R}$. By comparing (6.1) with (6.3), we see that $\gamma \in \Gamma_3$ if and only if γ is a Kirchhoff rod centerline. □

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