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A COMBINATORIAL DECOMPOSITION OF HIGHER LEVEL FOCK SPACES

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Abstract

We give a simple characterization of the highest weight vertices in the crystal graph of the level l Fock spaces. This characterization is based on the notion of totally periodic symbols viewed as affine analogues of reverse lattice words classically used in the decomposition of tensor products of fundamental $\mathfrak{sl}_n$-modules. This yields a combinatorial decomposition of the Fock spaces in their irreducible components and the branching law for the restriction of the irreducible highest weight $\mathfrak{sl}_\infty$-modules to $\widehat{\mathfrak{sl}}_e$.

1. Introduction

To any l-tuple $\mathbf{s} \in \mathbb{Z}^l$ is associated a Fock space $\mathcal{F}_\mathbf{s}$ which is a $\mathbb{C}(q)$-vector space with basis the set of l-partitions (i.e. the set of l-tuples of partitions). This level l Fock space was introduced in [7] in order to construct the irreducible highest weight representations of the quantum groups $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_q(\mathfrak{sl}_\infty)$. It provides a natural frame for the simultaneous study of the representation theories of $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_q(\mathfrak{sl}_\infty)$. It moreover permits to categorify the representation theory of the Ariki–Koike algebras (some generalizations of the Hecke algebras of the symmetric groups) in the nonsemisimple case (see [1]).

The Fock space $\mathcal{F}_\mathbf{s}$ has two structures of $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_q(\mathfrak{sl}_\infty)$-modules. For these two structures, the empty l-partition $\boldsymbol{\emptyset}$ is a highest weight vector with dominant weights $\Lambda_{\mathbf{s},e}$ and $\Lambda_{\mathbf{s},\infty}$. We denote by $V_e(\mathbf{s})$ and $V_\infty(\mathbf{s})$ the corresponding highest weight $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_q(\mathfrak{sl}_\infty)$-modules. In fact, any highest weight irreducible $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ or $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module can be realized in this way as the irreducible component of a Fock space with highest weight vector $\boldsymbol{\emptyset}$. It is also known [2, 7] that the two modules structures are compatible. This means that the action of any Chevalley generator for $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ can be obtained from the actions of the Chevalley generators for $\mathcal{U}_q(\mathfrak{sl}_\infty)$. In particular, $V_\infty(\mathbf{s})$ admits the structure of a $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-module.

The first purpose of this paper is to give a simple combinatorial description of the decomposition of $\mathcal{F}_\mathbf{s}$ in its irreducible $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_q(\mathfrak{sl}_\infty)$-components. For the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module structure, this problem is very similar to the decomposition of a tensor product

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of fundamental $\mathcal{U}_q(\mathfrak{sl}_n)$-modules into irreducible ones. It is well-known that this decomposition can be obtained by using the notion of reverse lattice (or Yamanouchi) words. Here our description of the decomposition into irreducible is based on the notion of totally periodic symbols which can be regarded as affine analogues of reverse lattice words.

The Kashiwara crystal associated to the Fock space $\mathcal{F}_{\mathbf{s}}$ admits as set of vertices, the set $\mathcal{G}_{\mathbf{s}}$ of all l-partitions. According to Kashiwara crystal basis theory, it suffices to characterize the highest weight vertices in $\mathcal{G}_{\mathbf{s}}$ to obtain the decomposition of $\mathcal{F}_{\mathbf{s}}$ into its irreducible components. We prove in fact that the totally periodic symbols label the highest weight vertices of $\mathcal{G}_{\mathbf{s}}$. It is also worth mentioning that, according to recent papers by Gordon–Losev and Shan–Vasserot [5, 12], there should exist a natural labelling of the finite dimensional irreducible representations of the rational Cherednik algebras by highest weight vertices of $\mathcal{G}_{\mathbf{s}}$, thus by a subset of the set of totally periodic symbols (see also [11, Remark 6.4]). Nevertheless the combinatorial characterization of this subset seems not immediate.

The set $\mathcal{G}_{\mathbf{s}}$ admits two $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal structures. In [6] we established that the $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-structure of graph on $\mathcal{G}_{\mathbf{s}}$ is in fact a subgraph of the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-structure. This implies that each $\mathcal{U}_q(\mathfrak{sl}_\infty)$-connected component decomposes into $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-connected components. Each l-partition then admits a $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$ and a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weight. In particular, We can consider the decomposition of the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-connected component $\mathcal{G}_{\mathbf{s},\infty}(\boldsymbol{\emptyset})$ with highest weight vertex $\boldsymbol{\emptyset}$ in its $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-connected components, that is the decomposition of the crystal graph of $V_\infty(\mathbf{s})$ in $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-crystals. We prove that this decomposition gives the branching law for the restriction of the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module $V_\infty(\mathbf{s})$ to $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$. Observe this does not follow immediately from crystal basis theory since the root system of affine type $A_{e-1}^{(1)}$ is not parabolic in the root system of type A_∞. We also establish that the number of highest weight $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-vertices in $\mathcal{G}_{\mathbf{s},\infty}(\boldsymbol{\emptyset})$ with fixed $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weight is counted by some particular (skew semistandard) tableaux we call totally periodic. These tableaux can be regarded as affine analogues of the usual semistandard skew tableaux relevant for computing the branching coefficients associated to the restriction of the irreducible $\mathfrak{gl}_n$-modules to $\mathfrak{gl}_m \oplus \mathfrak{gl}_{n-m}$ with $m < n$ some positive integers.

It also follows that the number of $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-highest weight vertices in $\mathcal{G}_{\mathbf{s}}$ with fixed $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weight is finite and can be expressed in terms of the Kostka numbers and the number of totally periodic tableaux of fixed shape and weight.

The paper is organized as follows. In Section 2, we introduce the notion of symbol of an l-partition. Section 3 is devoted to some background on $\mathcal{F}_{\mathbf{s}}$, its two module structures and the corresponding crystal bases theory. In Section 4, we show that the two crystal bases on $\mathcal{F}_{\mathbf{s}}$ for $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$ and for $\mathcal{U}_q(\mathfrak{sl}_\infty)$ are compatible. This implies that the decomposition of $\mathcal{G}_{\mathbf{s},e}(\boldsymbol{\emptyset})$ into its $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-connected components yields the desired branching law. Section 5 characterizes the highest weight vertices in $\mathcal{G}_{\mathbf{s}}$ by totally periodic symbols. Finally in Section 6, we first express the multiplicities of the irreducible $\mathcal{U}_q(\mathfrak{sl}_\infty)$-modules appearing in the decomposition of $\mathcal{F}_{\mathbf{s}}$ in terms of the

Kostka numbers. Next, we establish that the branching coefficients for the restriction of $V_\infty(\mathbf{s})$ to $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ can be graded by the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weights and then counted by totally periodic semistandard tableaux. This gives the decomposition of $\mathcal{F}_{\mathbf{s}}$ in its irreducible $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-components.

2. Preliminaries on multipartitions and their symbols

2.1. Nodes in multipartitions. Let $n \in \mathbb{N}$, $l \in \mathbb{Z}$ and $\mathbf{s} = (s_0, s_1, \dots, s_{l-1}) \in \mathbb{Z}^l$. A *partition* λ is a sequence $(\lambda_1, \dots, \lambda_r)$ of decreasing non negative integers. An *l-partition* (or multipartition) $\boldsymbol{\lambda}$ is an l-tuple of partitions $(\lambda^0, \dots, \lambda^{l-1})$. We write $\boldsymbol{\lambda} \vdash_l n$ when $\boldsymbol{\lambda}$ is an l-partition of total rank n. The empty l-partition (which is the l-tuple of empty partitions) is denoted by $\boldsymbol{\emptyset}$.

If $\boldsymbol{\lambda}$ is not the empty multipartition, the *height* of $\boldsymbol{\lambda}$ is by definition the minimal non negative integer i such that there exists $c \in \{0, \dots, l-1\}$ satisfying $\lambda_i^c \neq 0$. By convention, the height of $\boldsymbol{\emptyset}$ is 0.

For all $\boldsymbol{\lambda} \vdash_l n$, we consider its *Young diagram*:

$$[\boldsymbol{\lambda}] = \{(a, b, c) \;\; a \geq 1, \; c \in \{0, \dots, l-1\}, \; 1 \leq b \leq \lambda_a^c\}.$$

The *nodes* of $\boldsymbol{\lambda}$ are usually defined as the elements of $[\boldsymbol{\lambda}]$. However, by slightly abuse the notation, they will be regarded in the sequel as the elements of the (infinite) set:

$$\{(a, b, c) \;\; a \geq 1, \;\; c \in \{0, \dots, l-1\}, 0 \leq b \leq \lambda_a^c\}.$$

We define the *content* of a node $\gamma = (a, b, c) \in [\boldsymbol{\lambda}]$ as follows:

$$\mathrm{cont}(\gamma) = b - a + s_c,$$

and the *residue* $\mathrm{res}(\gamma)$ is by definition the content of γ taken modulo e. An *i-node* is then a node with residue $i \in \mathbb{Z}/e\mathbb{Z}$. The nodes of *the right rim* of $\boldsymbol{\lambda}$ are the nodes (a, λ_a^c, c) with $\lambda_a^c \neq 0$. We will say that γ is an i-node of $\boldsymbol{\lambda}$ when $\mathrm{res}(\gamma) \equiv i \pmod{e}$. Finally, We say that γ is *removable* when $\gamma = (a, b, c) \in \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \setminus \{\gamma\}$ is an l-partition. Similarly γ is *addable* when $\gamma = (a, b, c) \notin \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \cup \{\gamma\}$ is an l-partition.

2.2. Symbol of a multipartition. Let $\boldsymbol{\lambda} \vdash_l n$. Then one can associate to $\boldsymbol{\lambda}$ its *shifted* **s**-*symbol* denoted by $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$. Our notation slightly differs from the one used in [4, §5.5.5] because the symbols we use here are semi-infinite with possible negative values. Thus, the symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ is the l-tuple

$$(\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^0, \mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^1, \dots, \mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^{l-1})$$

where for each $c \in \{0, 1, \dots, l-1\}$, and $i = 1, 2, \dots,$ we have

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_i^c = \lambda_i^c - i + s_c + 1.$$

This symbol is usually represented as an l-row tableau whose c-th row (counted from bottom) is $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^c$.

EXAMPLE 2.1. With $\boldsymbol{\lambda} = (3, 2.2.2, 2.1)$ and $\mathbf{s} = (1, 0, 2)$, we obtain

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) = \begin{pmatrix} \cdots & -3 & -2 & -1 & 0 & 2 & 4 \\ \cdots & -3 & 0 & 1 & 2 & & \\ \cdots & -3 & -2 & -1 & 0 & 4 & \end{pmatrix}.$$

We make the following observations.

- It is easy to recover the multipartition $\boldsymbol{\lambda}$ and the multicharge $\mathbf{s}$ from the datum of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$.
- For all $c \in \{0, \ldots, l-1\}$, let j_c be the maximal integer such that $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^c_{j_c} \neq -j_c + s_c + 1$, if it exists, we set $j_c := 0$ otherwise. Then the entries $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^c_j$ of the symbol such that $0 \leq c \leq l-1$ and $j \leq j_c$ are bijectively associated with the nodes (j, λ^c_j, c) of the right rim of $\boldsymbol{\lambda}$.

2.3. Period in a symbol. We now introduce the notion of period in a symbol which is crucial for the sequel.

DEFINITION 2.2. Consider a pair $(\boldsymbol{\lambda}, \mathbf{s})$ and its symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$. We say that $(\boldsymbol{\lambda}, \mathbf{s})$ is *e-periodic* if there exists a sequence $(i_1, c_1), (i_2, c_2), \ldots, (i_e, c_e)$ in $\mathbb{N} \times \{0, 1, \ldots, l-1\}$ and $k \in \mathbb{Z}$ satisfying

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^{c_1}_{i_1} = k,\ \mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^{c_2}_{i_2} = k-1, \ldots, \mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^{c_e}_{i_e} = k-e+1$$

and such that

1. $c_1 \geq c_2 \geq \cdots \geq c_e$,
2. for all $0 \leq c \leq l-1$ and $i \in \mathbb{N}$, we have $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^c_i \leq k$ (i.e. k is the largest entry of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$),
3. given $t \in \{1, \ldots, e\}$ and (j, d) such that $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^d_j = k - t + 1$, we have $c_t \leq d$. (i.e. there is no entry $k-t+1$ in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ strictly below than the one corresponding to (i_t, c_t)).

The *e-period* of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ is the sequence $(i_1, \lambda^{c_1}_{i_1}, c_1), (i_2, \lambda^{c_2}_{i_2}, c_2), \ldots, (i_e, \lambda^{c_e}_{i_e}, c_e)$ and the *form* of the e-period is the associated sequence $(k, k-1, \ldots, k-e+1)$ which can be read in the symbol.

2.4. Reading of a symbol. An e-period can be easily read on the symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ associated with $(\boldsymbol{\lambda}, \mathbf{s})$ as follows. First, consider the truncated symbol $\mathfrak{B}^t(\boldsymbol{\lambda}, \mathbf{s})$. It is obtained by keeping only in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ the entries of the symbol of the form $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^c_j$ for $j = 1, \ldots, h_c + e$ (where h_c denotes the height of λ^c) and $c = 0, 1, \ldots, l-1$.

Denote by w the word with letters in $\mathbb{Z}$ obtained by reading the entries in the rows of $\mathfrak{B}^t(\boldsymbol{\lambda}, \mathbf{s})$ from right to left, next from top to bottom. We say that w is the reading of $\mathfrak{B}^t(\boldsymbol{\lambda}, \mathbf{s})$. Each letter of w encodes a node in $(\boldsymbol{\lambda}, \mathbf{s})$ (possibly associated with a part 0).

When it exists, the e-period of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ is the sequence of nodes corresponding to the subword u of w of the form $u = k(k-1)\cdots(k-e+1)$ where k is the largest integer appearing in w (and thus also in the symbol) and each letter $k-a$, $a = 0, \dots, e-1$ in t is the rightmost letter $k-a$ in w.

EXAMPLE 2.3. For $\mathbf{s} = (0, -1, 1)$ and $\boldsymbol{\lambda} = (3, 2.2.2, 2.1)$, the symbol

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) = \begin{pmatrix} \cdots & -4 & -3 & -2 & -1 & 1 & 3 \\ \cdots & -4 & -1 & 0 & 1 & & \\ \cdots & -4 & -3 & -2 & -1 & 3 & \end{pmatrix}$$

admits no 4-period. So $(\boldsymbol{\lambda}, \mathbf{s})$ is not 4-periodic.

For $\mathbf{s}' = (-1, -1, 1)$ and $\boldsymbol{\nu} = (3.3, 4.3, 4.4.2)$, we have:

$$\mathfrak{B}(\boldsymbol{\nu}, \mathbf{s}') = \begin{pmatrix} \cdots & -5 & -4 & -3 & -2 & 1 & \mathbf{4} & \mathbf{5} \\ \cdots & -5 & -4 & -3 & 1 & \mathbf{3} & & \\ \cdots & -5 & -4 & -3 & \mathbf{1} & \mathbf{2} & & \end{pmatrix}.$$

Thus $\boldsymbol{\lambda}$ admits a 5-period with form $(5, 4, 3, 2, 1)$. The word associated w described in §2.4 is:

$$w = \mathbf{54}1\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\,\mathbf{3}1\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\,\mathbf{21}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}\bar{8}$$

where we write $\bar{x}$ for $-x$ for any $x \in \mathbb{Z}_{>0}$. So $(\boldsymbol{\nu}, \mathbf{s}')$ is 5-periodic.

REMARK 2.4. A pair $(\boldsymbol{\emptyset}, \mathbf{s})$ is always e-periodic with form of the e-period M, $M-1, \dots, M-e+1$ where $M = \max(\mathbf{s})$.

2.5. Removing periods in $\mathfrak{B}(\boldsymbol{\emptyset}, \mathbf{s})$. For $l \in \mathbb{N}$ and $e \in \mathbb{N}$, we denote

$$\mathcal{T}_{l,e} = \{\mathbf{t} = (t_0, \dots, t_{l-1}) \in \mathbb{Z}^l \mid t_0 \leq \cdots \leq t_{l-1} \text{ and } t_{l-1} - t_0 \leq e-1\}.$$

We now describe an elementary procedure which permits to associate to any l-tuple $\mathbf{s} \in \mathbb{Z}^l$ an element $\mathbf{t} \in \mathcal{T}_{l,e}$ such that $\mathfrak{B}(\boldsymbol{\emptyset}, \mathbf{t})$ is obtained from $\mathfrak{B}(\boldsymbol{\emptyset}, \mathbf{s})$ by deleting e-periods.

If $\mathbf{s} = \mathbf{s}^{(0)} \notin \mathcal{T}_{l,e}$, we set $\mathbf{s}^{(1)} = \mathbf{s}'$ where $\mathfrak{B}(\boldsymbol{\emptyset}, \mathbf{s}')$ is obtained from $\mathfrak{B}(\boldsymbol{\emptyset}, \mathbf{s})$ by deleting its e-period. More generally we define $\mathbf{s}^{(p+1)}$ from $\mathbf{s}^{(p)} \notin \mathcal{T}_{l,e}$ such that $\mathbf{s}^{(p+1)} = (\mathbf{s}^{(p)})'$.

Lemma 2.5. *For any $\mathbf{s} \in \mathbb{Z}^l$, there exists $p \geq 0$ such that $\mathbf{s}^{(p)} \in \mathcal{T}_{l,e}$.*

Proof. First observe that for any $i = 0, \dots, l-2$ such that $s_{i+1} - s_i < 0$, we have $s'_{i+1} - s'_i \geq s_{i+1} - s_i$ with equality if and only if $s'_i = s_i$ and $s'_{i+1} = s_{i+1}$. For any $\mathbf{s} \in \mathbb{Z}^l$, set

$$f(\mathbf{s}) = \sum_{i=0}^{l-2} \min(0, s_{i+1} - s_i).$$

For any $i = 0, \dots, l-2$ with $s_{i+1} - s_i < 0$, there is an integer p such that $s_i^{(p)} < s_i$ (the i-th coordinates of the l-tuples $\mathbf{s}^{(p)}$, $p > 0$ cannot be left all untouched by the iteration of our procedure). Therefore, for such a p, we have $f(\mathbf{s}^{(p)}) > f(\mathbf{s})$. Since $f(\mathbf{s}) \leq 0$ for any $\mathbf{s} \in \mathbb{Z}^l$, we deduce there exists an integer p_0 such that $f(\mathbf{s}^{(p_0)}) = 0$ and thus such that $s_{i+1}^{(p_0)} - s_i^{(p_0)} \geq 0$ for any $i = 0, \dots, l-2$. We can thus assume that the coordinates of $\mathbf{s} \in \mathbb{Z}^l$ satisfy $s_{i+1} - s_i \geq 0$ for any $i = 0, \dots, l-2$. One then easily verifies that for any $p \geq 0$, the coordinates of $\mathbf{s}^{(p)}$ also weakly increase. Observe that for any $i = 0, \dots, l-2$ such that $s_{i+1} - s_i \geq e$, we have $s'_{i+1} - s'_i \leq s_{i+1} - s_i$ with equality if and only if $s'_i = s_i$ and $s'_{i+1} = s_{i+1}$. Set

$$g(\mathbf{s}) = \sum_{i=0}^{l-2} \min(0, e - 1 - (s_{i+1} - s_i)).$$

Assume $\mathbf{s} \notin \mathcal{T}_{l,e}$. Since a pair (s_i, s_{i+1}) with $s_{i+1} - s_i \geq e$ cannot remain untouched by the iteration of our procedure, there exists an integer p such that $g(\mathbf{s}^{(p)}) > g(\mathbf{s})$. So we have an integer p_0 such that $g(\mathbf{s}^{(p_0)}) = 0$ and since the coordinates of $\mathbf{s}^{(p_0)}$ weakly increase, one has $\mathbf{s}^{(p_0)} \in \mathcal{T}_{l,e}$ as desired. $\square$

EXAMPLE 2.6. Consider $\mathbf{s} = (5, 3, 5, 0, 1)$ for $e = 3$. We obtain

$$\mathbf{s}^{(0)} = (5, 3, 5, 0, 1), \quad \mathbf{s}^{(1)} = (2, 3, 5, 0, 1), \quad \mathbf{s}^{(2)} = (2, 2, 3, 0, 1),$$
$$\mathbf{s}^{(3)} = (0, 2, 2, 0, 1), \quad \mathbf{s}^{(4)} = (-1, 0, 2, 0, 1) \quad \text{and} \quad \mathbf{s}^{(5)} = (-1, -1, 0, 0, 1),$$

and we have $\mathbf{s}^{(5)} \in \mathcal{T}_{5,3}$.

3. Module structures on the Fock space

We now introduce quantum group modules structures on the Fock space of level l and describe the associated crystal graphs.

3.1. Roots and weights. Let $e \in \mathbb{Z}_{>1} \cup \{\infty\}$. Let $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$ (resp. $\mathcal{U}_q(\mathfrak{sl}_\infty)$) be the quantum group of affine type $A_{e-1}^{(1)}$ (resp. of type A_∞). This is an associative $\mathbb{Q}(q)$-algebra with generators e_i, f_i, t_i, t_i^{-1} with $i = 0, \dots, e-1$ (resp. $i \in \mathbb{Z}$). We refer to [4, Chapter 6] for the complete description of the relations between these generators since we do not use them in the sequel. To avoid repetition, we will attach a label e

to the notions we define. When e is finite, they are associated with $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$ whereas the case $e=\infty$ corresponds to $\mathcal{U}_q(\mathfrak{sl}_\infty)$.

We write $\Lambda_{i,e}$, $i=0,\dots,e-1$ for the fundamental weights. The simple roots are then given by:

$$\alpha_{i,e}=-\Lambda_{i-1,e}+2\Lambda_{i,e}-\Lambda_{i+1,e}$$

for $i=0,\dots,e-1$. As usual the indices are taken modulo e. For $\mathbf{s}\in\mathbb{Z}^l$, we also write $\Lambda_{\mathbf{s},e}:=\sum_{0\leq c\leq l-1}\Lambda_{s_c,e}$.

There is an action of the extended affine symmetric group $\hat{S}_l$ on $\mathbb{Z}^l$ (see [6, §5.1]). This group is generated by the elements $\sigma_1,\dots,\sigma_{l-1}$ and $y_0,\dots,y_{l-1}$ together with the relations

$$\begin{gathered}\sigma_c\sigma_{c+1}\sigma_c=\sigma_{c+1}\sigma_c\sigma_{c+1},\quad \sigma_c\sigma_d=\sigma_d\sigma_c\quad\text{for}\quad |c-d|>1,\quad \sigma_c^2=1,\\ y_cy_d=y_dy_c,\quad \sigma_cy_d=y_d\sigma_c\quad\text{for}\quad d\neq c,c+1,\quad \sigma_cy_c\sigma_c=y_{c+1},\end{gathered}$$

for relevant indices. Then we obtain a faithful action of $\hat{S}_l$ on $\mathbb{Z}^l$ by setting for any $\mathbf{s}=(s_0,\dots,s_{l-1})\in\mathbb{Z}^l$

$$\sigma_c(\mathbf{s})=(s_0,\dots,s_c,s_{c-1},\dots,s_{l-1})\quad\text{and}\quad y_c(\mathbf{s})=(s_0,\dots,s_{c-1},s_c+e,\dots,s_{l-1}).$$

Given $\mathbf{s},\mathbf{s}'\in\mathbb{Z}^l$, we have $\Lambda_{\mathbf{s},e}=\Lambda_{\mathbf{s}',e}$ if and only if $\mathbf{s}$ and $\mathbf{s}'$ are in the same orbit modulo the action of $\hat{S}_l$. In this case, we denote $\mathbf{s}\equiv_e\mathbf{s}'$. Set

$$\mathcal{V}_{l,e}=\{\mathbf{v}=(v_0,\dots,v_{l-1})\in\mathbb{Z}^l\mid 0\leq v_0\leq\cdots\leq v_{l-1}\leq e-1\}.\tag{1}$$

Given any $\mathbf{s}\in\mathbb{Z}^l$ there exists a unique $\mathbf{v}$ in $\mathcal{V}_l$ such that $\mathbf{s}\equiv_e\mathbf{v}$.

3.2. Module structures. We fix $\mathbf{s}\in\mathbb{Z}^l$. The *Fock space* $\mathcal{F}_{\mathbf{s}}$ is the $\mathbb{Q}(q)$-vector space defined as follows:

$$\mathcal{F}_{\mathbf{s}}=\bigoplus_{n\in\mathbb{Z}_{\geq 0}}\bigoplus_{\boldsymbol{\lambda}\vdash_l n}\mathbb{Q}(q)\boldsymbol{\lambda}.$$

According to [13, §2.1], there is an action of $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$ on the Fock space (see [4, §6.2]). This action depends on e and we will denote by $\mathcal{F}_{\mathbf{s},e}$ the $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-module so obtained. In $\mathcal{F}_{\mathbf{s},e}$, each partition is a weight vector (with respect to a multicharge $\mathbf{s}$) with weight given by (see [13, §4.2])

$$\mathrm{wt}(\boldsymbol{\lambda},\mathbf{s})_e=\Lambda_{\mathbf{s},e}-\sum_{0\leq i\leq e-1}N_i(\boldsymbol{\lambda},\mathbf{s})\alpha_{i,e},$$

where $N_i(\boldsymbol{\lambda},\mathbf{s})$ denotes the number of i-nodes in $\boldsymbol{\lambda}$ (where the residues are computed with respect to $\mathbf{s}$). For any $e\in\mathbb{Z}_{>1}\cup\{\infty\}$, the empty multipartition is always a highest weight vector of weight $\Lambda_{\mathbf{s},e}$. We write $V_e(\mathbf{s})$ for the associated $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-module. We clearly have $V_e(\mathbf{s}) \simeq V_e(\mathbf{s}')$ if and only if $\mathbf{s} \equiv_e \mathbf{s}'$.

In general, the modules structures on $\mathcal{F}_{\mathbf{s}}$ are not compatible when we consider distinct values of e. Nevertheless, we have the following proposition stated in [2, §2.1].

Proposition 3.1. *Let $e \in \mathbb{N}_{>0}$.*

1. *Any $\mathcal{U}_q(\mathfrak{sl}_\infty)$-irreducible component V_∞ of $\mathcal{F}_{\mathbf{s},\infty}$ is stable under the action of the $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-Chevalley generators e_i, f_i, t_i, $i \in \mathbb{Z}/e\mathbb{Z}$. Therefore V_∞ has also the structure of a $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-module.*
2. *In particular, the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module $V_\infty(\mathbf{s})$ is endowed with the structure of a $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-module. Moreover $V_e(\mathbf{s})$ then coincides with the $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-irreducible component of $V_\infty(\mathbf{s})$ with highest weight vector the empty l-partition $\boldsymbol{\emptyset}$.*

REMARK 3.2. The algebras $\mathfrak{sl}_\infty$ and $\widehat{\mathfrak{sl}}_e$ can be realized as algebras of infinite matrices (see [8]). Then $\widehat{\mathfrak{sl}}_e$ is regarded as a subalgebra of $\mathfrak{sl}_\infty$. In particular, the irreducible $\mathfrak{sl}_\infty$-module of highest weight $\Lambda_{\mathbf{s},\infty}$ admits the structure of a $\widehat{\mathfrak{sl}}_e$-module by restriction. The highest $\widehat{\mathfrak{sl}}_e$-weights involved in its decomposition into irreducible then coincide with those appearing in the decomposition of $V_\infty(\mathbf{s})$ into its irreducible $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-components.

3.3. Crystal bases and crystal graphs. We now recall some results on the crystal bases of $\mathcal{F}_{\mathbf{s},e}$ established in [7] and [13]. Let $\mathbb{A}(q)$ be the ring of rational functions without pole at $q = 0$. Set

$$\mathcal{L} := \bigoplus_{n \geq 0} \bigoplus_{\boldsymbol{\lambda} \vdash_l n} \mathbb{A}(q)\boldsymbol{\lambda}$$

and

$$\mathcal{G} := \{\boldsymbol{\lambda} \pmod{q}\ \mathcal{L}) \mid \boldsymbol{\lambda} \text{ is an } l\text{-partition}\}.$$

Theorem 3.3 (Jimbo–Misra–Miwa–Okado, Uglov)**.** *The pair $(\mathcal{L}, \mathcal{G})$ is a crystal basis for $\mathcal{F}_{\mathbf{s},e}$ and $\mathcal{F}_{\mathbf{s},\infty}$.*

Observe that the crystal basis of the Fock space is the same for $\mathcal{F}_{\mathbf{s},e}$ and $\mathcal{F}_{\mathbf{s},\infty}$. Nevertheless, the crystal structures $\mathcal{G}_{e,\mathbf{s}}$ and $\mathcal{G}_{\infty,\mathbf{s}}$ on $\mathcal{G}$ do not coincide for $\mathcal{F}_{\mathbf{s},e}$ and $\mathcal{F}_{\mathbf{s},\infty}$. To describe these crystal structures we begin by defining a total order on the removable or addable i-nodes. Let γ, γ' be two removable or addable i-nodes of $\boldsymbol{\lambda}$. We set

$$\gamma \prec_{\mathbf{s}} \gamma' \overset{\text{def}}{\iff} \begin{cases} \text{either} & b - a + s_c < b' - a' + s_{c'}, \\ \text{or} & b - a + s_c = b' - a' + s_{c'} \quad \text{and} \quad c > c'. \end{cases}$$

Let $\boldsymbol{\lambda}$ be an l-partition. We can consider its set of addable and removable i-nodes. Let $w_i(\boldsymbol{\lambda})$ be the word obtained first by writing the addable and removable i-nodes of $\boldsymbol{\lambda}$ in increasing order with respect to $\prec_{\mathbf{s}}$ next by encoding each addable i-node by the letter A and each removable i-node by the letter R. Write $\tilde{w}_i(\boldsymbol{\lambda}) = A^p R^q$ for the word derived from w_i by deleting as many subwords of type RA as possible. The word $w_i(\boldsymbol{\lambda})$ is called the i-word of $\boldsymbol{\lambda}$ and $\tilde{w}_i(\boldsymbol{\lambda})$ the reduced i-word of $\boldsymbol{\lambda}$. The addable i-nodes in $\tilde{w}_i(\boldsymbol{\lambda})$ are called the normal addable i-nodes. The removable i-nodes in $\tilde{w}_i(\boldsymbol{\lambda})$ are called the normal removable i-nodes. If $p > 0$, let γ be the rightmost addable i-node in $\tilde{w}_i$. The node γ is called the good addable i-node. If $q > 0$, the leftmost removable i-node in $\tilde{w}_i$ is called the good removable i-node. We set

$$\varphi_i(\boldsymbol{\lambda}) = p \quad \text{and} \quad \varepsilon_i(\boldsymbol{\lambda}) = q. \tag{2}$$

By Kashiwara's crystal basis theory [9, §4.2] we have another useful expression for $\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_e$

$$\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_e = \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (\varphi_i(\boldsymbol{\lambda}) - \varepsilon_i(\boldsymbol{\lambda}))\Lambda_{i,e}. \tag{3}$$

We denote by $\mathcal{G}_{e,\mathbf{s}}$ the crystal of the Fock space computed using the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$. By [7], this is the graph with

- vertices: the l-partitions $\boldsymbol{\lambda} \vdash_l n$ with $n \in \mathbb{Z}_{\geq 0}$
- arrows: $\boldsymbol{\lambda} \xrightarrow{i} \boldsymbol{\mu}$ that is $\tilde{e}_i \boldsymbol{\mu} = \boldsymbol{\lambda}$ if and only if $\boldsymbol{\mu}$ is obtained by adding to $\boldsymbol{\lambda}$ a good addable i-node, or equivalently, $\boldsymbol{\lambda}$ is obtained from $\boldsymbol{\mu}$ by removing a good removable i-node.

Note that the order induced by $\prec_{\mathbf{s}}$ does not change if we translate each component of the multicharge by a common multiple of e (nor does the associated $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-weight). Thus, if there exists $k \in \mathbb{Z}$ such that $\mathbf{s} = (s'_0, s'_1, \ldots, s'_{l-1}) = (s_0 + k.e, s_1 + k.e, \ldots, s_{l-1} + k.e)$ then the crystal $\mathcal{G}_{e,\mathbf{s}}$ and $\mathcal{G}_{e,\mathbf{s}'}$ are identical.

The crystal $\mathcal{G}_{e,\mathbf{s}}$ has several connected components. They are parametrized by its highest weight vertices which are the l-partitions $\boldsymbol{\lambda}$ with no good removable node (that is such that $\varepsilon_i(\boldsymbol{\lambda}) = 0$). Given such an l-partition $\boldsymbol{\lambda}$, we denote by $\mathcal{G}_{e,\mathbf{s}}(\boldsymbol{\lambda})$ its associated connected component. One easily verifies that $\mathrm{wt}(\boldsymbol{\emptyset}, \mathbf{s})_e = \Lambda_{\mathbf{s}\,(\mathrm{mod}\, e)}$. So the crystal $\mathcal{G}_{e,\mathbf{s}}(\boldsymbol{\emptyset})$ is isomorphic to the abstract crystal $\mathcal{G}_e(\Lambda_{\mathbf{s}\,(\mathrm{mod}\, e)})$. In general, for any highest weight vertex $\boldsymbol{\lambda}$, $\mathcal{G}_{e,\mathbf{s}}(\boldsymbol{\lambda})$ is isomorphic to the abstract crystal $\mathcal{G}_e(\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_e)$. By setting $\Lambda_{\mathbf{v}\,(\mathrm{mod}\, e)} = \mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_e$, we thus obtain a crystal isomorphism $f^{e,\boldsymbol{\lambda}}_{\mathbf{s},\mathbf{v}} : \mathcal{G}_{e,\mathbf{s}}(\boldsymbol{\lambda}) \to \mathcal{G}_{e,\mathbf{v}}(\boldsymbol{\emptyset})$.

3.4. Crystal graphs and symbols. Consider $i \in \mathbb{Z}/e\mathbb{Z}$. The reduced i-word $\tilde{w}_i$ of a multipartition $\boldsymbol{\lambda}$ may be easily computed from its symbol. Let $j_{\mathrm{low}} \in \mathbb{Z}$ be the greatest integer such that $j_{\mathrm{low}} \equiv i \pmod e$ and such that each row of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ contains all the integers lowest or equal to j_{low}. Such an integer exists since the rows of our symbols are infinite. For any $j \in \mathbb{Z}$ such that $j \equiv i \pmod e$ and $j \geq j_{\mathrm{low}}$ let u_j be the word obtained by reading in the rows of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ the entries j or $j+1$ from top to bottom and right to left. Write

$$u_i = \prod_{t=0}^{\infty} u_{j_0+te}$$

for the concatenation of the words u_j. Here all but a finite number of words u_{j_0+te} are empty. We then encode in u_i each letter j by A and each letter $j+1$ by R and delete recursively the factors RA. Write $\tilde{u}_i$ for the resulting word.

Lemma 3.4. *We have* $\tilde{w}_i = \tilde{u}_i$.

Proof. For any $j \equiv i \pmod e$, write w_j for the word obtained by reading the addable or removable nodes with content j (with respect to $\mathbf{s}$) successively in the partitions λ^c, $c = l-1, \ldots, 0$. Observe there is no ambiguity since each partition λ^c contains at most one node with content j which is addable or removable. By definition of the order $\prec_{\mathbf{s}}$, we have

$$w_i = \prod_{t=0}^{\infty} w_{j_0+te} \tag{4}$$

where all but a finite set of the words w_i are empty. Now we come back to the word u_j. The contribution to the c-th row of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})^c$ of u_j is one of the factors $(j+1)j$, $j+1$, j or $\emptyset$. The factors $(j+1)j$ will be encoded RA so they will disappear during the cancellation process and we can neglect their contribution. Write u'_j for the word obtained by deleting in u_j the factors $(j+1)j$ corresponding to entries in the same row. There is a bijection between the letters of u'_j and w_j which associates to each letter $j+1$ (resp. j) in u'_j appearing in the row c a node R (resp. A) of w_j. This easily implies that $\tilde{u}_i = \tilde{w}_i$. $\square$

4. Compatibility of crystal bases and weight lattices

4.1. Crystal basis of the $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module $V_\infty(\mathbf{s})$. Consider $e \in \mathbb{Z}_{>1} \cup \{+\infty\}$. The general theory of crystal bases (see [9]) permits to define the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$, $i \in \mathbb{Z}/e\mathbb{Z}$ on the whole Fock space $\mathcal{F}_{\mathbf{s},e}$ by decomposing, for any $i \in \mathbb{Z}/e\mathbb{Z}$, $\mathcal{F}_{\mathbf{s},e}$ in irreducible $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)_i$ components. These operators do not depend on the decomposition considered (see [9, §4.2]). This implies that the Kashiwara operators associated with any $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-submodule M_e of $\mathcal{F}_{\mathbf{s},e}$ are obtained by restriction of the Kashiwara operators defined on $\mathcal{F}_{\mathbf{s},e}$.

Set $\mathbf{s} \in \mathbb{Z}^l$. By Proposition 3.1, we know that $V_\infty(\mathbf{s})$ has the structure of a $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-module. Set $L_\infty(\mathbf{s}) = \mathcal{L} \cap V_\infty(\mathbf{s})$ and $B_\infty(\mathbf{s}) = L_\infty(\mathbf{s})/qL_\infty(\mathbf{s})$. It immediately follows from crystal basis theory that the pair $(L_\infty(\mathbf{s}), B_\infty(\mathbf{s}))$ is a crystal basis for $V_\infty(\mathbf{s})$ regarded as a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module. In fact this is also true when $V_\infty(\mathbf{s})$ is regarded as an $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module.

Proposition 4.1. *The pair $(L_\infty(\mathbf{s}), B_\infty(\mathbf{s}))$ is a $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-crystal basis of the $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module $V_\infty(\mathbf{s})$.*

Proof. Observe first that we have the weight spaces decompositions

$$L_\infty(\mathbf{s}) = \bigoplus_{\mu\in P_e} \mathcal{L}_\mu \cap V_\infty(\mathbf{s}) \quad \text{and} \quad B_\infty(\mathbf{s}) = \bigoplus_{\mu\in P_e} (\mathcal{L}_\mu/q\mathcal{L}_\mu) \cap B_\infty(\mathbf{s})$$

where P_e is the weight space of the affine root system of type $A^{(1)}_{e-1}$. By Theorem 3.3, for any $i \in \mathbb{Z}/e\mathbb{Z}$, $\tilde{e}_i$ and $\tilde{f}_i$ stabilize $\mathcal{L}$. They also stabilize the $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-submodule $V_\infty(\mathbf{s})$ by the previous discussion. Therefore, they stabilize $L_\infty(\mathbf{s})$ and $B_\infty(\mathbf{s})$. Moreover, we have for any $b_1, b_2 \in B_\infty(\mathbf{s})$, $\tilde{f}_i(b_1) = b_2$ if and only if $\tilde{e}_i(b_2) = b_1$ since this is true in $\mathcal{B}$. This shows that the pair $(L_\infty(\mathbf{s}), B_\infty(\mathbf{s}))$ satisfies the general definition of a crystal basis for the $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-module $V_\infty(\mathbf{s})$. □

Since $(L_\infty(\mathbf{s}), B_\infty(\mathbf{s}))$ is a crystal basis for $V_\infty(\mathbf{s})$ regarded as a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module, $B_\infty(\mathbf{s})$ has the structure of a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal that we have denoted by $\mathcal{G}_{\infty,\mathbf{s}}(\emptyset)$. By the previous proposition, $B_\infty(\mathbf{s})$ (which can be regarded as the set of vertices of $\mathcal{G}_{\infty,\mathbf{s}}(\emptyset)$) has also the structure of a $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-crystal that we denote by $\mathcal{G}^e_{\infty,\mathbf{s}}(\emptyset)$. This crystal is also a subcrystal of $\mathcal{G}_{e,\mathbf{s}}$ since the actions of the Kashiwara operators on $\mathcal{G}^e_{\infty,\mathbf{s}}(\emptyset)$ are obtained by restriction from $\mathcal{G}_{e,\mathbf{s}}$. Let us now recall the following result obtained in [6, Theorem 4.2.2] which shows that $\mathcal{G}_{e,\mathbf{s}}$ is in fact a subgraph of $\mathcal{G}_{\infty,\mathbf{s}}$

Proposition 4.2. *Consider $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ two l-partitions such that there is an arrow $\boldsymbol{\lambda} \xrightarrow{i} \boldsymbol{\mu}$ in $\mathcal{G}_{e,\mathbf{s}}$. Let $j \in \mathbb{Z}$ be the content of the node $\boldsymbol{\mu} \setminus \boldsymbol{\lambda}$. Then, we have the arrow $\boldsymbol{\lambda} \xrightarrow{j} \boldsymbol{\mu}$ in $\mathcal{G}_{\infty,\mathbf{s}}$.*

By combining the two previous propositions, we thus obtain the following corollary.

Corollary 4.3. *The $\mathcal{U}_q(\mathfrak{sl}_e)$-crystal $\mathcal{G}^e_{\infty,\mathbf{s}}(\emptyset)$ is a subgraph of the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-crystal $\mathcal{G}_{\infty,\mathbf{s}}(\emptyset)$. It decomposes into $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-connected components. This decomposition gives the decomposition of $V_\infty(\mathbf{s})$ into its irreducible $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-components.*

4.2. Weights lattices. Let P_e and P_∞ be the weight lattices of $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_q(\mathfrak{sl}_\infty)$. We have a natural projection defined by

$$\pi : \begin{cases} P_\infty \to P_e, \\ \Lambda_{j,\infty} \mapsto \Lambda_{j \bmod e, e}. \end{cases} \tag{5}$$

Consider $\mathbf{s} \in \mathbb{Z}^l$ and $\boldsymbol{\lambda}$ an l-partition.

Lemma 4.4. *We have* $\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_e = \pi(\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_\infty)$.

Proof. By (3), for any $e \in \mathbb{Z}_{>1}$, the coordinate of $\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_e$ on $\Lambda_{i,e}$ is also equal to the number of letters A in u_i minus the number of letters R. This is equal to the sum over the integer j such that $j \equiv i \pmod e$ of the number of letters A in u_j minus the number of letters R. The coordinate of $\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_e$ on $\Lambda_{i,e}$ is thus equal to the sum of the coordinates of $\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_\infty$ on the $\Lambda_{j,\infty}$ with $j \equiv i \pmod e$ as desired. $\square$

One easily verifies that the kernel of π is generated by the $\omega_k := \Lambda_{k+1,\infty} - \Lambda_{k-e+1,\infty}$, $k \in \mathbb{Z}$. The weight ω_k have level 0. In fact level 0 weights for $\mathcal{U}_q(\mathfrak{sl}_\infty)$ are the $\mathbb{Z}$-linear combinations of the elementary weights $\varepsilon_j = \Lambda_{j+1,\infty} - \Lambda_{j,\infty}$, $j \in \mathbb{Z}$. The contribution of an entry $j \in \mathbb{Z}$ of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ to the weight $\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_\infty$ is exactly ε_j. We also have $\omega_k = \varepsilon_k + \cdots + \varepsilon_{k-e+1}$.

5. A combinatorial characterization of the highest weight vertices

Our aim is now to give a combinatorial description of the highest weights vertices of $\mathcal{G}_{e,\mathbf{s}}$, the crystal of the Fock space $\mathcal{F}_{e,\mathbf{s}}$. Such a vertex is an l-partition without good removable i-node for any $i \in \mathbb{Z}/e\mathbb{Z}$.

5.1. Removing a period in a symbol. Let $\boldsymbol{\lambda}$ be an l-partition. We define the l-partition $\boldsymbol{\lambda}^-$ and a multicharge $\mathbf{s}^-$ as follows:

- If $\boldsymbol{\lambda}$ is not e-periodic then $\boldsymbol{\lambda}^- := \boldsymbol{\lambda}$ and $\mathbf{s}^- := \mathbf{s}$.
- Otherwise, delete the elements of the e-period in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$. This gives a new symbol $\mathfrak{B}(\boldsymbol{\mu}, \mathbf{s}')$ which is the symbol of an l-partition associated with another multicharge $\mathbf{s}'$. We then set $\boldsymbol{\lambda}^- := \boldsymbol{\mu}$ and $\mathbf{s}^- := \mathbf{s}'$.

Proposition 5.1. *Let $\boldsymbol{\lambda}$ be an e-periodic multipartition. For any $i \in \mathbb{Z}/e\mathbb{Z}$, write $\tilde{u}_i$ and $\tilde{u}_i^-$ for the reduced words obtained from the symbols $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^-$ as in* §3.4.

1. $\tilde{u}_i = \tilde{u}_i^-$.
2. $\varphi_i(\boldsymbol{\lambda}^-) = \varphi_i(\boldsymbol{\lambda})$ *and* $\varepsilon_i(\boldsymbol{\lambda}^-) = \varepsilon_i(\boldsymbol{\lambda})$.

Proof. 1: Write $(j_a, \lambda_{j_a}^{c_a}, c_a)$, $a = 1, \ldots, e$ for the e-period in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$. Recall we have by convention $c_1 \geq \cdots \geq c_e$. Consider $i \in \mathbb{Z}/e\mathbb{Z}$. Let u_i be the word constructed in §3.4. By definition, there exists a unique $a \in \{1, \dots, e\}$ such that $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{j_a}^{c_a} \equiv i \pmod{e}$. Assume first $a > 1$. Write x_{a-1} and x_a for the letters of u_i associated to $(j_a, \lambda_{j_{a-1}}^{c_{a-1}}, c_{a-1})$ and $(j_a, \lambda_{j_a}^{c_a}, c_a)$. We have $x_{a-1} = x_a + 1$. Set $u_i = u_i' x_{a-1} v x_a u_i''$ where u_i', v, u_i'' are words with letters in $\mathbb{Z}$. By definition of the e-period, v is empty or contains only letters equal to x_a. Indeed, x_{a-1} should be the rightmost occurrence of the integer x_{a-1} in u_i. Therefore the contribution of x_{a-1} and x_a can be neglected in the computation of $\tilde{u}_i$ since they are encoded by symbols R and A, respectively. Now assume $a = 1$. Write y_1 and y_e the letters of u_i associated with $(j_1, \lambda_{j_1}^{c_1}, c_1)$ and $(j_e, \lambda_{j_e}^{c_e}, c_e)$. We have $y_e = y_1 - e + 1$. By definition of u_i, we can write $u_i = u_i' y_e v y_1 u_i''$ where u_i', v, u_i'' are words with letters in $\mathbb{Z}$. By definition of the e-period, v is empty or contains only letters y_1. Indeed, y_e should be the rightmost occurrence of the integer y_e in u_i. Therefore the contribution of y_e and y_1 can be neglected in the computation of $\tilde{u}_i$ since they are encoded by symbols R and A, respectively. By the previous arguments, we see that the contribution of the e-period in u_i can be neglected when we compute $\tilde{u}_i$. This shows that $\tilde{u}_i = \tilde{u}_i'$. Assertion 2 follows immediately from 1, (2) and Lemma 3.4. □

5.2. The peeling procedure. Given $\boldsymbol{\lambda}$ an arbitrary l-partition and $\mathbf{s}$ a multicharge, we define recursively the l-partition $\boldsymbol{\lambda}^\circ$ and the multicharge $\mathbf{s}^\circ$ as follows:

- If $\boldsymbol{\lambda}$ is not e-periodic, or $\boldsymbol{\lambda}$ is empty with $\mathbf{s} \in \mathcal{T}_{l,e}$, then we set $\boldsymbol{\lambda}^\circ := \boldsymbol{\lambda}$ and $\mathbf{s}^\circ := \mathbf{s}$.
- Otherwise we set $\boldsymbol{\lambda}^\circ := (\boldsymbol{\lambda}^-)^\circ$ and $\mathbf{s}^\circ := (\mathbf{s}^-)^\circ$.

REMARK 5.2. When $\lambda = \boldsymbol{\emptyset}$, we have $\mathbf{s}^\circ := \mathbf{s}$ only if $\mathbf{s} \in \mathcal{T}_{l,e}$.

Lemma 5.3. *The previous procedure terminates, that is the pair* $(\boldsymbol{\lambda}^\circ, \mathbf{s}^\circ)$ *is well-defined. Moreover we have* $\mathbf{s}^\circ \in \mathcal{T}_{l,e}$ *if* $\boldsymbol{\lambda}^\circ = \boldsymbol{\emptyset}$.

Proof. If $\boldsymbol{\lambda}$ is not empty and $\boldsymbol{\lambda}^- \neq \boldsymbol{\lambda}$, then $|\boldsymbol{\lambda}^-| < |\boldsymbol{\lambda}|$. So when we apply the previous procedure to $(\boldsymbol{\lambda}, \mathbf{s})$, we obtain after a finite number of steps an aperiodic pair $(\boldsymbol{\lambda}', \mathbf{s}')$ or a pair $(\emptyset, \mathbf{u})$. In the first case, we have $(\boldsymbol{\lambda}', \mathbf{s}') = (\boldsymbol{\lambda}^\circ, \mathbf{s}^\circ)$ and the procedure terminates. In the second case, we have already noticed in Remark 5.10 that $(\emptyset, s')$ admits an e-period. The lemma then follows from Lemma 2.5. □

DEFINITION 5.4. The pair $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ is said to be *totally periodic* when $\boldsymbol{\lambda}^\circ = \boldsymbol{\emptyset}$ and $\mathbf{s}^\circ \in \mathcal{T}_{l,e}$.

EXAMPLE 5.5. Here are a couple of examples.

1. First, assume that $e = 3$, let $\mathbf{s} = (1, 1)$ and let $\boldsymbol{\lambda} = (3.3, 4.4.3)$. We have

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) = \begin{pmatrix} \cdots & -2 & 2 & 4 & \mathbf{5} \\ \cdots & -2 & -1 & \mathbf{3} & \mathbf{4} \end{pmatrix}.$$

If we delete the 3-period we obtain the symbol:

$$\mathfrak{B}(\boldsymbol{\mu}, \mathbf{s}') = \begin{pmatrix} \cdots & -2 & 2 & 4 \\ \cdots & -2 & -1 & \end{pmatrix}$$

which is the symbol of the bipartition $\boldsymbol{\mu} = \boldsymbol{\lambda}^- = (\emptyset, 4.3)$ with multicharge $\mathbf{s}^- = (-1, 0)$. We don't have any 3-period so $\boldsymbol{\lambda}^\circ = (1, 3.2)$ and $\mathbf{s}^\circ = (-1, 0)$. Note that we have $(-1, 0) \equiv_e (0, 2)$.

2. Now take $e = 4$, let $\mathbf{s} = (4,5)$ and $\boldsymbol{\lambda} = (2.2.2.1.1, 2)$. We obtain the following symbol

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) = \begin{pmatrix} \cdots & -1 & 0 & 1 & 2 & 3 & 4 & \mathbf{7} \\ \cdots & -1 & 1 & 2 & \mathbf{4} & \mathbf{5} & \mathbf{6} & \end{pmatrix}.$$

By deleting the 4-period, we obtain:

$$\mathfrak{B}(\boldsymbol{\lambda}^-, \mathbf{s}^-) = \begin{pmatrix} \cdots & -1 & 0 & 1 & 2 & \mathbf{3} & \mathbf{4} \\ \cdots & -1 & \mathbf{1} & \mathbf{2} & & & \end{pmatrix}.$$

Thus, we get $\boldsymbol{\lambda}^- = (1.1, \emptyset)$ and $\mathbf{s}^- = (1, 4)$. Now deleting the 4-period, we have:

$$\mathfrak{B}((\boldsymbol{\lambda}^-)^-, (\mathbf{s}^-)^-) = \begin{pmatrix} \cdots & -1 & \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \cdots & \mathbf{-1} & & & \end{pmatrix}$$

and we derive $(\boldsymbol{\lambda}^-)^- = (\emptyset, \emptyset)$ and $(\mathbf{s}^-)^- = (-1, 2)$. Finally, we can delete the 4-period $2, 1, 0, -1$ in the last symbol, this gives

$$\mathfrak{B}(\boldsymbol{\lambda}^\circ, \mathbf{s}^\circ) = \begin{pmatrix} \cdots & -1 & & & \\ \cdots & -1 & & & \end{pmatrix}$$

$\boldsymbol{\lambda}^\circ = (\boldsymbol{\lambda}^-)^- = (\emptyset, \emptyset)$ and $\mathbf{s}^\circ = (\mathbf{s}^-)^- = (-1, -1)$.

5.3. Crystal properties of periods.

Proposition 5.6. *Let $\mathbf{s} \in \mathbb{Z}^l$ and let $\boldsymbol{\lambda} \vdash_l n$. Then for $i \in \{0, 1, \ldots, e-1\}$, we have $\tilde{e}_i(\boldsymbol{\lambda}) = 0$ if and only if $\tilde{e}_i(\boldsymbol{\lambda}^-) = 0$*

Proof. If $\boldsymbol{\lambda}^-$ is $\boldsymbol{\lambda}$ or the empty l-partition, the lemma is immediate. Otherwise it follows from Lemma 5.1. □

Proposition 5.7. *Let $\boldsymbol{\lambda} \vdash_l n$ be such that $\boldsymbol{\lambda} \neq \boldsymbol{\emptyset}$ and assume that $\tilde{e}_i(\boldsymbol{\lambda}) = 0$ for any $i \in \mathbb{Z}e/\mathbb{Z}$. Then $\boldsymbol{\lambda}$ admits an e-period.*

Proof. Consider c_1 minimal such that $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_1^{c_1} = M$ is the largest entry of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$. Let $i \in \mathbb{Z}/e\mathbb{Z}$ be such that $M \equiv i + 1 \pmod{e}$. Then, in the encoding of the letters of

u_i by symbols A or R, the contribution of $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_1^{c_1}$ is the rightmost symbol R of u_i. Since $\tilde{e}_i(\boldsymbol{\lambda})=0$, there exists in u_i an entry $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_{a_2}^{c_2}$ encoded by A immediately to the right of $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_1^{c_1}$ (to have a cancellation RA). By maximality of M and definition of u_i, we must have $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_{a_2}^{c_2}=M-1$ and $c_2\leq c_1$. We can also choose c_2 minimal such that $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_{a_2}^{c_2}=M-1$ (or equivalently, the contribution of $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_{a_2}^{c_2}$ is the rightmost A in u_i). Then, the entries in any row c with $c<c_2$ are less than $M-1$. If we use $\tilde{e}_{i-1}(\boldsymbol{\lambda})=0$, we obtain similarly an entry $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_{a_3}^{c_3}$ with $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_{a_3}^{c_3}=M-2$, $c_3\leq c_2$ such that the entries in any rows c with $c<c_3$ are less than $M-2$. By induction, this gives a sequence of entries $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_{a_m}^{c_m}=M-m+1$, for $m=1,\dots,e$, $c_1\geq\cdots\geq c_e$ and the entries in any row $c<c_m$ are less than $M-m+1$, that is the desired e-period. $\square$

Proposition 5.8. *Let $\mathbf{s}\in\mathbb{Z}^l$ and let $\boldsymbol{\lambda}\vdash_l n$ be such that $\boldsymbol{\lambda}\neq\boldsymbol{\emptyset}$. Assume that $(\boldsymbol{\lambda},\mathbf{s})$ admits an e-period of the form $M, M-1,\dots,M-e+1$. We have*
1. $\mathrm{wt}(\boldsymbol{\lambda},\mathbf{s})_e=\mathrm{wt}(\boldsymbol{\lambda}^-,\mathbf{s}^-)_e$.
2. $\mathrm{wt}(\boldsymbol{\lambda},\mathbf{s})_\infty=\mathrm{wt}(\boldsymbol{\lambda}^-,\mathbf{s}^-)_\infty+\omega_M$.

Proof. 1: Recall that for any $(\boldsymbol{\lambda},\mathbf{s})$, we have by §3.3

$$\mathrm{wt}(\boldsymbol{\lambda},\mathbf{s})=\sum_{i\in\mathbb{Z}/e\mathbb{Z}}(\varphi_i(\boldsymbol{\lambda})-\varepsilon_i(\boldsymbol{\lambda}))\Lambda_i.$$

By assertion 2 of Proposition 5.1, we have $\varepsilon_i(\boldsymbol{\lambda})=\varepsilon_i(\boldsymbol{\lambda}^-)$ and $\varphi_i(\boldsymbol{\lambda})=\varphi_i(\boldsymbol{\lambda}^-)$ for all $i\in\mathbb{Z}/e\mathbb{Z}$. Therefore $\mathrm{wt}(\boldsymbol{\lambda},\mathbf{s})_e=\mathrm{wt}(\boldsymbol{\lambda}^-,\mathbf{s}^-)_e$. Assertion 2 follows from the fact that the contribution to each entry $j\in\mathbb{Z}$ in $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})$ to $\mathrm{wt}(\boldsymbol{\lambda},\mathbf{s})_\infty$ is ε_j. So

$$\mathrm{wt}(\boldsymbol{\lambda},\mathbf{s})_\infty=\mathrm{wt}(\boldsymbol{\lambda}^-,\mathbf{s}^-)_\infty+\varepsilon_M+\cdots+\varepsilon_{M-e+1}=\mathrm{wt}(\boldsymbol{\lambda}^-,\mathbf{s}^-)_\infty+\omega_M. \qquad\square$$

5.4. A combinatorial description of the highest weight vertices.

Theorem 5.9. *Let $\mathbf{s}\in\mathbb{Z}^l$ and let $\boldsymbol{\lambda}\vdash_l n$ then $(\boldsymbol{\lambda},\mathbf{s})$ is a $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-highest weight vertex if and only if it is totally periodic.*

Proof. First assume that $(\boldsymbol{\lambda},\mathbf{s})$ is totally periodic, that is $\boldsymbol{\lambda}^\circ$ is the empty l-partition and $\mathbf{s}^\circ\in\mathcal{T}_{l,e}$. An easy induction and Proposition 5.6 show that $(\boldsymbol{\lambda},\mathbf{s})$ is a $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-highest weight vertex. In addition, the weight of $(\boldsymbol{\lambda},\mathbf{s})$ is equal to the weight of $(\boldsymbol{\lambda}^\circ,\mathbf{s}^\circ)$ by Proposition 5.8. Conversely, if $\boldsymbol{\lambda}$ is a $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-highest weight vertex, we know by Proposition 5.7 that it admits a period and by Proposition 5.6 that $\boldsymbol{\lambda}^-$ is also a highest weight vertex. Moreover, for any $\mathbf{s}\notin\mathcal{T}_{l,e}$, we have seen in Remark 2.4 that $\mathfrak{B}(\emptyset,\mathbf{s}^\circ)$ contains an e-period. By Lemma 5.3, this implies that $\boldsymbol{\lambda}^\circ$ is empty with $\mathbf{s}^\circ\in\mathcal{T}_{l,e}$. $\square$

REMARK 5.10. 1. We can obtain the highest weight vertices of $\mathcal{G}_{\mathbf{s},\infty}$ by adapting the previous theorem. It suffices to interpret $\mathcal{G}_{\mathbf{s},\infty}$ as the limit when e tends to infinity of the crystals $\mathcal{G}_{\mathbf{s},e}$. Then $(\boldsymbol{\lambda},\mathbf{s})$ is a highest weight vertex if and only if $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})$

is totally periodic for $e = \infty$. A period for $e = \infty$ is defined as the natural limit of an e-period when e tends to infinity. This is an infinite sequence of the form $M, M-1, M-2, \ldots$ in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ where M is the maximal entry of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$. We say that $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ is totally periodic for $e = \infty$ when it reduces to the empty symbol after deletion of its periods following the procedure described in §5.2. In this case, since these periods are infinite, a row of the symbol disappears at each deletion of a period. In particular, there are l infinite periods.

2. Recall that a word w with letters in $\mathbb{Z}$ is a reverse lattice (or Yamanouchi) word if it can be decomposed into subwords of the form $a(a-1)\cdots \min(w)$ where $\min(w)$ is the minimal letter of w. Let m be the maximal integer in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ such that each row of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ contains all the integer $k < m$. One easily verify that the periodicity of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ for $e = \infty$ is equivalent to say that the word w obtained by reading successively the entries greater or equal to m in the rows of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ from left to right and top to bottom is a reverse lattice word. Indeed, we always dispose in the symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ of integers less than m to complete any decreasing sequence $a, a-1, \ldots, m$ into an infinite sequence. Observe that this imposes in particular that $M \leq \sum_{c=0}^{l-1}(s_c - m + 1)$. We will see in §6.2 that this easily gives the decomposition of $\mathcal{F}_{\mathbf{s},\infty}$into its $\mathcal{U}_q(\mathfrak{sl}_\infty)$-irreducible components.

EXAMPLE 5.11. Take $e = 4$, $l = 3$, $\mathbf{s} = (3, 4, 6)$ and $\boldsymbol{\lambda} = (\emptyset, 2.2, 2.2.1.1.1.1)$. Consider the symbol

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) = \begin{pmatrix} \cdots & -2 & -1 & 0 & 2 & 3 & 4 & 5 & \mathbf{7} & \mathbf{8} \\ \cdots & -2 & -1 & 0 & 1 & 2 & \mathbf{5} & \mathbf{6} & & \\ \cdots & -2 & -1 & 0 & 1 & 2 & 3 & & & \end{pmatrix}.$$

By deleting successively the 4-periods (pictured in bold), we obtain

$$\mathfrak{B}(\boldsymbol{\lambda}^-, \mathbf{s}^-) = \begin{pmatrix} \cdots & -2 & -1 & 0 & 2 & 3 & \mathbf{4} & \mathbf{5} \\ \cdots & -2 & -1 & 0 & 1 & 2 & & \\ \cdots & -2 & -1 & 0 & 1 & \mathbf{2} & \mathbf{3} & \end{pmatrix},$$

$$\begin{pmatrix} \cdots & -2 & -1 & 0 & 2 & \mathbf{3} \\ \cdots & -2 & -1 & 0 & 1 & \mathbf{2} \\ \cdots & -2 & -1 & \mathbf{0} & \mathbf{1} & \end{pmatrix}, \begin{pmatrix} \cdots & -2 & -1 & 0 & \mathbf{2} \\ \cdots & -2 & -1 & \mathbf{0} & \mathbf{1} \\ \cdots & -2 & \mathbf{-1} & & \end{pmatrix}, \begin{pmatrix} \cdots & -2 & -1 & 0 \\ \cdots & -2 & -1 & \\ \cdots & -2 & & \end{pmatrix}.$$

Finally we obtain the empty 3-partition and $\mathbf{s}^\circ = (-2, -1 - 0) \in \mathcal{T}_{3,4}$. So $(\boldsymbol{\lambda}, \mathbf{s})$ is a highest weight vertex.

6. Decomposition of the Fock space

Consider $\mathbf{s} = (s_0, \ldots, s_l) \in \mathbb{Z}^l$. We can assume without loss of generality that $\mathbf{s} \in \mathcal{T}_{l,\infty}$, that is $s_0 \leq \cdots \leq s_{l-1}$. The aim of this section is to provide the decomposition of $\mathcal{G}_{\mathbf{s},e}$ into its connected $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-components. The multiplicity of an irreducible module in $\mathcal{F}_{\mathbf{s},e}$ can be infinite. Nevertheless, we have a filtration of the highest weight vertices in $\mathcal{G}_{\mathbf{s},\infty}$ by their $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weights. We are going to see that the number of totally periodic symbols of fixed $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weight is finite and can be counted by simple combinatorial objects. We proceed in two steps. First, we give the decomposition of $\mathcal{G}_{\mathbf{s},\infty}$ into its $\mathcal{U}_q(\mathfrak{sl}_\infty)$-connected components, next we give the decomposition of each crystal $\mathcal{G}_\infty(\mathbf{v})$, $\mathbf{v} \in \mathcal{T}_{l,\infty}$ into its $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-connected components.

6.1. Totally periodic tableaux. Let $\mathbf{t} \in \mathcal{T}_{l,e}$ such that $t_i \leq s_i$ for any $i = 0, \dots, l-1$. We denote by $\mathbf{s} \setminus \mathbf{t}$ the skew Young diagram with rows of length $s_c - t_c$, $c = 0, \dots, l-1$. By a *skew* (*semistandard*) *tableau* of shape $\mathbf{s} \setminus \mathbf{t}$, we mean a filling τ of $\mathbf{s} \setminus \mathbf{t}$ by integers such that the rows of τ strictly increase from left to right and its column weakly increase from top to bottom. The weight of τ is the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weight

$$\mathrm{wt}(\tau)_\infty = \sum_{b \in \tau} \varepsilon_{c(b)}$$

of level 0. Here b runs over the boxes of $\mathbf{s} \setminus \mathbf{t}$ and $c(b)$ is the entry of the box b in τ. The trivial tableau of shape $\mathbf{s} \setminus \mathbf{t}$ denoted $\tau_{\mathbf{s}\setminus\mathbf{t}}$ is the one in which the c-th row contains exactly the letters $t_c + 1, \dots, s_c$.

A tableau is a skew tableau of shape $\mathbf{s} \setminus \mathbf{t}$ where $\mathbf{t}$ is such that $t_0 = \cdots = t_{l-1}$. In that case $\lambda = \mathbf{s} \setminus \mathbf{t}$ is an ordinary Young diagram. Given a level 0 weight $\mu = \sum_{j \in \mathbb{Z}} \mu_j \varepsilon_j$ (where all but a finite number of μ_j are equal to zero), we then denote by $K_{\lambda,\mu}$ the Kostka number associated to λ and μ. Recall that $K_{\lambda,\mu}$ is the number of tableaux of shape λ and $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weight μ.

EXAMPLE 6.1. Take $e = 2$, $\mathbf{s} = (2, 3, 6)$ and $\mathbf{t} = (0, 0, 1)$. Then

$$\tau = \begin{pmatrix} & 2 & 4 & 5 & 7 & 8 \\ 1 & 3 & 6 & & & \\ 2 & 5 & & & & \end{pmatrix}$$

is a tableau of shape $\mathbf{s} \setminus \mathbf{t}$ and weight $\mu = \omega_8 + \omega_6 + \omega_5 + \omega_3 + \omega_2$.

The peeling procedure described in §5.2 can be adapted to the skew tableaux by successively removing their periods. For a skew tableau τ, denote by $\mathrm{w}(\tau)$ the word obtained by reading the entries in the rows of τ from right to left and from top to bottom. When it exists, the e-period of τ is the subword u of $w(\tau)$ of the form $u = u_0 \cdots u_{e-1}$ where for any $k = 0, \dots, e-1$

- $u_k = M - k$ with M the largest entry in $w(\tau)$,
- u_k is the rightmost letter of $w(\tau)$ equal to $M - k$.

When τ is e-periodic, we write τ^- for the skew tableau obtained by deleting its period. By condition on the rows and the columns of τ, τ^- is also a skew tableau. Its shape can be written on the form $\mathbf{s}' \setminus \mathbf{t}$ with $\mathbf{s}' \in \mathcal{T}_{l,\infty}$.

More generally, given a skew tableau τ of shape $\mathbf{s} \setminus \mathbf{t}$, define the skew tableau τ° of shape $\mathbf{s}^{\circ} \setminus \mathbf{t}$ as the result of the following peeling procedure:

- If τ is not periodic or $\tau = \tau_{\mathbf{s} \setminus \mathbf{t}}$ with $\mathbf{s} \in \mathcal{T}_{l,e}$, then $\tau^{\circ} = \tau$ and $\mathbf{s}^{\circ} = \mathbf{s}$.
- Otherwise, $\tau^{\circ} = (\tau^{\circ})'$ and $\mathbf{s}^{\circ} = (\mathbf{s}^{\circ})'$.

When $\tau^{\circ} = \emptyset$ is the empty tableau, we have

$$\mathrm{wt}(\tau)_{\infty} = \sum_{T} \omega_{M(T)}$$

where T runs over the e-periods of τ and for any period, $M(T)$ is the largest integer in T. Write π_e^+ for the set of $\mathcal{U}_q(\mathfrak{sl}_{\infty})$-weights which are linear combinations of the ω_j, $j \in \mathbb{Z}$ with nonnegative integer coefficients. When $\tau^{\circ} = \emptyset$, we have $\mathrm{wt}(\tau)_{\infty} \in \pi_e^+$.

DEFINITION 6.2. A *totally periodic skew tableau* of shape $\mathbf{s} \setminus \mathbf{t}$ is a skew tableau τ of shape $\mathbf{s} \setminus \mathbf{t}$ such that

1. Each row $c = 0, \ldots, l-1$ contains integers greater than t_c.
2. We have $\tau^{\circ} = \emptyset$.

We denote by $\mathrm{Tab}^{e}_{\mathbf{s} \setminus \mathbf{t}}$ the set of totally e-periodic skew tableaux of shape $\mathbf{s} \setminus \mathbf{t}$. For any $\gamma \in \pi_e^+$, let $\mathrm{Tab}^{e}_{\mathbf{s} \setminus \mathbf{t}, \gamma}$ be the subset of $\mathrm{Tab}^{e}_{\mathbf{s} \setminus \mathbf{t}}$ of tableaux with $\mathcal{U}_q(\mathfrak{sl}_{\infty})$-weight γ.

EXAMPLE 6.3. By applying the peeling procedure to the tableau τ of Example 6.1, we first obtain the sequence of tableaux

$$\tau = \begin{pmatrix} & 2 & 4 & 5 & 7 & 8 \\ 1 & 3 & 6 & & & \\ 2 & 5 & & & & \end{pmatrix}, \quad \tau^{(1)} = \begin{pmatrix} & 2 & 4 & 5 \\ 1 & 3 & 6 & \\ 2 & 5 & & \end{pmatrix}, \quad \tau^{(2)} = \begin{pmatrix} & 2 & 4 & 5 \\ 1 & 3 & & \\ 2 & & & \end{pmatrix}$$

and

$$\tau^{(3)} = \begin{pmatrix} & 2 \\ 1 & 3 \\ 2 & \end{pmatrix}.$$

The tableau $\tau^{(3)}$ has shape $\mathbf{s}^{(3)} \setminus \mathbf{t}$ with $\mathbf{t} = (0, 0, 1)$ and $\mathbf{s}^{(3)} = (1, 2, 2)$. Since $\tau^{(3)} \neq \tau_{\mathbf{s} \setminus \mathbf{t}}$, the peeling procedure goes on. We obtain

$$\tau^{(4)} = \begin{pmatrix} & 2 \\ 1 & \\ & \end{pmatrix}$$

which has shape $\mathbf{s}^{(4)} \setminus \mathbf{t}$ with $\mathbf{s}^{(4)} = (0, 1, 2)$. Now $\mathbf{s}^{(4)} \notin \mathcal{T}_{l,e}$, so the procedure finally yields $\tau^{(5)} = \tau^{\circ} = \emptyset$. Therefore, τ is totally 2-periodic.

6.2. Decomposition of $\mathcal{G}_{\mathbf{s},\infty}$. In the sequel we assume $\mathbf{s} \in \mathcal{T}_{l,\infty}$ is fixed. By a slight abuse of notation, we will identify each vertex $(\boldsymbol{\lambda}, \mathbf{s})$ of $\mathcal{G}_{\mathbf{s},\infty}$ with its symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$. For any $\mathbf{v} \in \mathcal{T}_{l,\infty}$, let $\mathcal{H}^{\mathbf{v}}_{\mathbf{s},\infty}$ be the set of highest weight vertices in $\mathcal{G}_{\mathbf{s},\infty}$ of highest weight $\Lambda_{\mathbf{v},\infty}$.

Consider $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{H}^{\mathbf{v}}_{\mathbf{s},\infty}$. For any fixed $k \in \mathbb{Z}$, the contribution of all the integers k in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ to $\mathrm{wt}(\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}))_\infty$ is equal to $d_k\varepsilon_k$ where d_k is the number of occurrences of k in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$. Each row contains at most a letter k, therefore $d_k \leq l$ and $d_k = l$ if and only if k appear in each row of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$. Since $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ has weight $\Lambda_{\mathbf{v},\infty}$, we must have $d_k = l$ for any $k \leq v_0$ and $d_k < l$ otherwise. This means that the maximal integer m such that $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ contains each integer $k < m$ defined in Remark 5.10 is equal to $v_0 + 1$. Let $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}$ be the truncated symbol obtained by deleting in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ the entries less or equal to v_0. By Remark 5.10 (2), the reading of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}$ is a reverse lattice word.

EXAMPLE 6.4. One verifies that

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) = \begin{pmatrix} \cdots & -1 & 0 & 1 & 2 & 3 & 5 & 6 \\ \cdots & -1 & 0 & 1 & 2 & 4 & & \\ \cdots & -1 & 1 & 2 & 3 & & & \\ \cdots & -1 & 0 & & & & & \end{pmatrix}$$

with $\mathbf{s} = (0, 2, 3, 5)$ is of highest weight $\Lambda_{\mathbf{v},\infty}$ with $\mathbf{v} = (-1, 2, 3, 6)$. Then the reading of

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 5 & 6 \\ 0 & 1 & 2 & 4 & & \\ 1 & 2 & 3 & & & \\ 0 & & & & & \end{pmatrix} \tag{6}$$

is the reverse lattice word

$$w = 65321042103210.$$

Set $\mathbf{t}(\mathbf{v}) = (v_0, \ldots, v_0) \in \mathbb{Z}^l$. Set $\lambda = \mathbf{v} \setminus \mathbf{t}(\mathbf{v})$. Then λ can be regarded as an ordinary Young diagram. We define λ^* has the conjugate diagram of λ. We now associate to $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}$ a tableau T of shape $\lambda(\mathbf{v}) = \lambda^*$ and weight

$$\mu(\mathbf{v}) = \sum_{c=0}^{l-1} \mu_c \varepsilon_c$$

where for any $c = 0, \ldots, l-1$, $\mu_c = s_c - v_0$ is the length of the c-th row of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}$. Observe that λ^* is simply the sequence recording the number of occurrences of each integer $k > v_0$ in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}$ (see the example below). Our procedure is a variant of the one-to-one correspondence (reflecting the Schur duality) described in [10] between the highest weight vertices of the $\mathcal{U}_q(\mathfrak{sl}_n)$-Fock spaces and the semi-standard tableaux.

First normalize $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}$ by translating its entries by $-v_0$. Write $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}^t$ for the resulting truncated symbol. It has entries in $\mathbb{Z}_{>0}$ and its reading is a reverse lattice word. Let $T^{(0)}$ be the tableau with one column containing μ_0 letters 1. Assume the sequence of tableaux $T^{(0)}, \ldots, T^{(c-1)}$, $c < l-1$ is defined. Then $T^{(c)}$ is obtained by adding in $T^{(c-1)}$ exactly μ_c letters $c+1$ at distance from the top row given by the nonnegative integers appearing in the c-th row of $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_{\mathbf{v}}^t$. Since the reading of $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})_{\mathbf{v}}^t$ is a reverse lattice word, $T^{(c)}$ is in fact a semi-standard tableau. We set $T = T^{(l-1)}$.

EXAMPLE 6.5. Let us compute T for $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}$ as in (6). We have $v_0 = -1$

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}^t = \begin{pmatrix} 1 & 2 & 3 & 4 & 6 & 7 & \\ 1 & 2 & 3 & 5 & & & \\ 2 & 3 & 4 & & & & \\ 1 & & & & & & \end{pmatrix}$$

and we successively obtain for the tableaux $T^{(c)}$

$$T^{(0)} = \begin{pmatrix} 1 \end{pmatrix}, \quad T^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \quad T^{(2)} = \begin{pmatrix} 1 & 3 \\ 2 & 3 \\ 2 & 3 \\ 2 & \\ 3 & \end{pmatrix} \quad \text{and} \quad T^{(3)} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 4 & \\ 3 & & \\ 4 & & \\ 4 & & \end{pmatrix}.$$

We verify that $T^{(3)} = T$ has shape $\lambda(\mathbf{v}) = (3,3,3,2,1,1,1)$ and weight $\mu(\mathbf{v}) = (1,3,4,6)$.

The previous procedure is reversible (for $\mathbf{s}, \mathbf{v} \in \mathcal{T}_{l,\infty}$ fixed). Starting from T a tableau of shape $\lambda(\mathbf{v})$ and weight $\mu(\mathbf{v})$, we can construct a truncated symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}^t$, next $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_{\mathbf{v}}$ by translating the entries by v_0. This proves that the cardinality of $\mathcal{H}_{\mathbf{s},\infty}^{\mathbf{v}}$ is finite and equal to $K_{\lambda(\mathbf{v}),\mu(\mathbf{v})}$ the number of tableaux of shape $\lambda(\mathbf{v})$ and weight $\mu(\mathbf{v})$. We thus obtain the following theorem.

Theorem 6.6. *Consider $\mathbf{s} \in \mathcal{T}_{l,\infty}$. As a $\mathcal{U}_q(\mathfrak{sl}_\infty)$-module, the Fock space $\mathcal{F}_{\mathbf{s},\infty}$ decomposes as*

$$\mathcal{F}_{\mathbf{s},\infty} = \bigoplus_{\mathbf{v} \in \mathcal{T}_{l,\infty}} V_\infty(\mathbf{v})^{\oplus K_{\lambda(\mathbf{v}),\mu(\mathbf{v})}}.$$

6.3. Branching rule for the restriction of $V_\infty(\mathbf{s})$ to $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$. Consider $\mathbf{s} \in \mathcal{T}_{l,\infty}$. We now give the decomposition of $\mathcal{G}_{\infty,\mathbf{s}}(\boldsymbol{\emptyset})$ into its $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$-connected components. By Corollary 4.3, this reflects the branching rule for the restriction of $V_\infty(\mathbf{s})$ from $\mathcal{U}_q(\mathfrak{sl}_\infty)$

to the $\mathcal{U}_q'(\widehat{\mathfrak{sl}}_e)$ action. By our assumption we have $s_0 \leq \cdots \leq s_{l-1}$. It is then easy to describe the symbols associated with the l-partitions appearing in $\mathcal{G}_{\infty,\mathbf{s}}(\emptyset)$. Indeed, $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s}) \in \mathcal{G}_{\infty,\mathbf{s}}(\emptyset)$ if and only if it is semistandard (see [6]). This means that its columns weakly increase from top to bottom.

Assume that $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ is totally periodic in $\mathcal{G}_{\infty,\mathbf{s}}(\emptyset)$. Set $\mathbf{s}^\circ = (s_0^\circ, \dots, s_{l-1}^\circ) \in \mathcal{T}_{l,e}$. We define the level l-part of the symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ as the symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_l = \mathfrak{B}(\emptyset, \mathbf{s}^\circ)$ which can be regarded as a subsymbol of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ in a natural sense. The level 0-part of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ is then

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0 = \mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) \setminus \mathfrak{B}(\emptyset, \mathbf{s}^\circ).$$

For any $\mathbf{t} \in \mathcal{T}_{l,e}$, we set

$$S_\mathbf{t} = \{\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) \in \mathcal{G}_{\infty,\mathbf{s}}(\emptyset) \text{ totally periodic} \mid \mathbf{s}^\circ = \mathbf{t}\}.$$

The following lemma is immediate from the definitions of the peeling procedures on symbols and tableaux.

Lemma 6.7. *Fix $\mathbf{t} \in \mathcal{T}_{l,e}$. The map $\psi : \mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) \mapsto \mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0$ is a one-to-one correspondence between the sets $S_\mathbf{t}$ and $\mathrm{Tab}^e_{\mathbf{s}\setminus\mathbf{t}}$. We have moreover*

$$\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_\infty = \Lambda_{\mathbf{t},\infty} + \mathrm{wt}(\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0). \tag{7}$$

EXAMPLE 6.8. Take $e = 2$, $\mathbf{s} = (2, 3, 6)$ and

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s}) = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 & 4 & 5 & 7 & 8 \\ -2 & -1 & 0 & 1 & 3 & 6 & & & \\ -2 & -1 & 0 & 2 & 5 & & & & \end{pmatrix}.$$

We obtain

$$\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0 = \begin{pmatrix} & 2 & 4 & 5 & 7 & 8 \\ 1 & 3 & 6 & & & \\ 2 & 5 & & & & \end{pmatrix} \in \mathrm{Tab}^2_{\mathbf{s}\setminus\mathbf{t}}$$

with $\mathbf{t} = (0, 0, 1)$. We have $\mathrm{wt}(\boldsymbol{\lambda}, \mathbf{s})_\infty = \Lambda_{\mathbf{t},\infty} + \omega_8 + \omega_6 + \omega_5 + \omega_3 + \omega_2$.

Let $P_{e,\infty}$be the subset of P_∞ of weights ν which can be written on the form

$$\nu = \Lambda_{\mathbf{t}(\nu),\infty} + \gamma(\nu) \quad \text{with} \quad \mathbf{t}(\nu) \in \mathcal{T}_{l,e} \quad \text{and} \quad \gamma(\nu) = \sum_{k > t_0(\nu)+e} a_k \omega_k \in \pi_e^+ \tag{8}$$

where all but a finite number of the coefficients a_k are equal to 0. Observe that the previous decomposition is then unique. Indeed, for any $\mathbf{t} \in \mathcal{T}_{l,e}$ and any $k > t_0 + e$, the weight $\Lambda_{\mathbf{t},\infty} + \omega_k$ cannot be written on the form $\Lambda_{\mathbf{t}',\infty}$ with $\mathbf{t}' \in \mathcal{T}_{l,e}$. Let $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ a highest weight vertex of $\mathcal{G}_{e,\mathbf{s}}$ with weight ν.

Lemma 6.9. *The $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weight of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ belongs to $P_{e,\infty}$. Moreover, we have*

$$\mathbf{t}(\nu) = \mathbf{s}^\circ \quad \textit{and} \quad \gamma(\nu) = \mathrm{wt}(\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0)$$

where $\mathbf{s}^\circ$ and $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0$ are obtained by the peeling procedure as in (7).

Proof. In view to (7), the weight ν decomposes on the form

$$\nu = \Lambda_{\mathbf{s}^\circ,\infty} + \mathrm{wt}(\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0$$

where by Theorem 5.9 and Lemma 6.7, we have $\mathbf{s}^\circ \in \mathcal{T}_{l,e}$ and $\mathrm{wt}(\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0 \in \pi_e^+$. Set $\mathrm{wt}(\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0 = \sum_{k\in\mathbb{Z}} a_k\omega_k$. The entries of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0$ are those of the periods of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ and a_k is the number of periods $\{k, \dots, k-e+1\}$ in $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0$. Let k_0 be the minimal integer such that $a_{k_0} \neq 0$. By definition of the peeling procedure, the addition of the letters $\{k_0-e+1, \dots, k_0\}$ in the symbol $\mathfrak{B}(\boldsymbol{\emptyset}, \mathbf{s}^\circ)$, yields a symbol $\mathfrak{B}(\boldsymbol{\emptyset}, \mathbf{u})$ with $\mathbf{u} \in \mathcal{T}_{l,\infty}$ but $\mathbf{u} \notin \mathcal{T}_{l,e}$. Since $\mathbf{u} \in \mathcal{T}_{l,\infty}$, we must have $k_0 - e + 1 > s_0^\circ$, that is $k_0 \geq s_0^\circ + e$. We cannot have $k_0 = s_0^\circ + e$, otherwise $\mathbf{u} = (s_1^\circ, \dots, s_l^\circ, s_0^\circ + e) \in \mathcal{T}_{l,e}$. Thus $k_0 > s_0^\circ + e$. Since the decomposition (8) is unique, this imposes that $\mathbf{t}(\nu) = \mathbf{s}^\circ$ and $\gamma(\nu) = \mathrm{wt}(\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0)$ as desired. □

Proposition 6.10. *Consider a totally periodic symbol $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ in $\mathcal{G}_{\infty,\mathbf{s}}(\boldsymbol{\emptyset})$ of $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weight ν.*

1. *The successive symbols appearing during the peeling procedure of $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ of $\mathcal{G}_{\infty,\mathbf{s}}(\boldsymbol{\emptyset})$ remain semistandard.*
2. *The number of highest weight vertices in $\mathcal{G}_{\infty,\mathbf{s}}(\boldsymbol{\emptyset})$ with $\mathcal{U}_q(\mathfrak{sl}_\infty)$-highest weight $\nu \in P_{e,\infty}$ is finite equal to $m_{\mathbf{s},\nu}^e = |\mathrm{Tab}_{\mathbf{s}\setminus\mathbf{t}(\nu),\gamma(\nu)}|$.*

Proof. Assertion 1 follows from the fact that the columns of $\mathfrak{B}(\boldsymbol{\lambda},\mathbf{s})$ increase from top to bottom and each entry k in a period is the lowest possible occurrence of the integer k in the symbol considered. Consider $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ of highest weight ν. By Lemma 6.9, we have the decomposition $\nu = \mathbf{s}^\circ + \mathrm{wt}(\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0)$. Then the restriction of the bijection ψ defined in Lemma 6.7 to the symbols of weight ν yields a one-to-one correspondence between the symbols $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})$ of highest weight ν and the tableaux $\mathfrak{B}(\boldsymbol{\lambda}, \mathbf{s})_0$ of shape $\mathbf{s} \setminus \mathbf{s}^\circ$ and weight $\gamma(\nu)$. Assertion 2 follows. □

We thus obtain the following theorem.

Theorem 6.11. *Assume e is finite and consider $\mathbf{s} \in \mathcal{T}_{l,\infty}$.*

1. *The crystal $\mathcal{G}_{\infty,\mathbf{s}}(\boldsymbol{\emptyset})$ decomposes into irreducible $\mathcal{U}'_q(\widehat{\mathfrak{sl}_e})$-components whose highest weight vertices are also weight vertices for the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-structure.*
2. *The $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weight of such a vertex belongs to $P_{e,\infty}$.*
3. *The number of highest weight vertices in $\mathcal{G}_{\infty,\mathbf{s}}(\boldsymbol{\emptyset})$ with $\mathcal{U}_q(\mathfrak{sl}_\infty)$-highest weight $\nu \in P_{e,\infty}$ is finite equal to the cardinality $m_{\mathbf{s},\nu}^e = |\mathrm{Tab}_{\mathbf{s}\setminus\mathbf{t}(\nu),\gamma(\nu)}|$.*

By combining with Theorem 6.6, this yields the decomposition of the Fock space in its irreducible $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-components.

Theorem 6.12. *Assume e is finite and consider $\mathbf{s} \in \mathcal{T}_{l,\infty}$.*
1. *The crystal $\mathcal{G}_{\mathbf{s},e}$ decomposes into irreducible $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-components whose highest weight vertices are also weight vertices for the $\mathcal{U}_q(\mathfrak{sl}_\infty)$-structure $\mathcal{G}_{\mathbf{s},\infty}$.*
2. *The $\mathcal{U}_q(\mathfrak{sl}_\infty)$-weight of such a vertex belongs to $P_{e,\infty}$*
3. *The number of $\mathcal{U}'_q(\widehat{\mathfrak{sl}}_e)$-highest weight vertices in $\mathcal{G}_e$ with $\mathcal{U}_q(\mathfrak{sl}_\infty)$-highest weight $\nu \in P_{e,\infty}$ is finite equal to $M^e_{\mathbf{s},\nu} = \sum_{\mathbf{v}\in\mathcal{T}_{l,\infty}} K_{\lambda(\mathbf{v}),\mu(\mathbf{v})} m^e_{\mathbf{v},\nu}$.*

References

[1] S. Ariki: Representations of Quantum Algebras and Combinatorics of Young Tableaux, translated from the 2000 Japanese edition and revised by the author, University Lecture Series **26**, Amer. Math. Soc., Providence, RI, 2002.

[2] S. Ariki, N. Jacon and C. Lecouvey: *Factorization of the canonical bases for higher-level Fock spaces*, Proc. Edinb. Math. Soc. (2) **55** (2012), 23–51, arXiv:0909.2954.

[3] W. Fulton: Young Tableaux, London Mathematical Society Student Texts **35**, Cambridge Univ. Press, Cambridge, 1997.

[4] M. Geck and N. Jacon: Representations of Hecke Algebras at Roots of Unity, Algebra and Applications **15**, Springer, London, 2011.

[5] I. Gordon and I. Losev: *On category $\mathcal{O}$ for the cyclotomic rational Cherednik algebras*, preprint 2011, arXiv:1109.2315.

[6] N. Jacon and C. Lecouvey: *Crystal isomorphisms for irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules of higher level*, Algebr. Represent. Theory **13** (2010), 467–489.

[7] M. Jimbo, K.C. Misra, T. Miwa and M. Okado: *Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}}(n))$ at $q = 0$*, Comm. Math. Phys. **136** (1991), 543–566.

[8] V.G. Kac: Infinite-Dimensional Lie Algebras, third edition, Cambridge Univ. Press, Cambridge, 1990.

[9] M. Kashiwara: *On crystal bases*; in Representations of Groups (Banff, AB, 1994), CMS Conf. Proc. **16**, Amer. Math. Soc., Providence, RI, 1995, 155–197.

[10] A. Nakayashiki and Y. Yamada: *Kostka–Foulkes Polynomials and Energy Function in Sovable Lattice Models*, Selecta Mathematica New Series **3** (1997), 547–599.

[11] P. Shan: *Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras*, Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), 147–182.

[12] P. Shan and E. Vasserot: *Heisenberg algebras and rational double affine Hecke algebras*, J. Amer. Math. Soc. **25** (2012), 959–1031, arXiv:1011.6488.

[13] D. Uglov: *Canonical bases of higher-level q-deformed Fock spaces and Kazhdan–Lusztig polynomials*; in Physical Combinatorics (Kyoto, 1999), Progr. Math. **191**, Birkhäuser, Boston, Boston, MA, 2000, 249–299.

[14] X. Yvonne: *Bases canoniques d'espaces de Fock de niveau supérieur*, Thèse de doctorat de l'Université de Caen, 2005 http://tel.archives-ouvertes.fr/tel-00137705/fr.

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