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ALGEBRAIC VERSUS TOPOLOGICAL ENTROPY FOR SURFACES OVER FINITE FIELDS

HÉLÈNE ESNAULT and VASUDEVAN SRINIVAS

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Abstract

We show that, as in de Rham cohomology over the complex numbers, the value of the entropy of an automorphism of the surface over a finite field $\mathbb{F}_q$ is taken on the span of the Néron–Severi group inside of l-adic cohomology.

1. Introduction

If X is a smooth proper surface (any smooth proper surface over a field is projective) over the field of complex numbers, and $\varphi\colon X \to X$ is an automorphism, then a notion of *topological entropy* of φ has been defined on the underlying topological manifold $X(\mathbb{C})$ and shown to be the same as the following cohomological definition ([7], [17], see also [6] and [5, Theorem 2.1]): let $H^{2\bullet}(X(\mathbb{C})) = H^0(X(\mathbb{C})) \oplus H^2(X(\mathbb{C})) \oplus H^4(X(\mathbb{C}))$ be the even degree de Rham cohomology. The automorphism φ acts linearly on $H^{2\bullet}(X(\mathbb{C}))$ via contravariance, and as the identity on $H^0(X(\mathbb{C})) \oplus H^4(X(\mathbb{C}))$. Thus, the maximum absolute value of the eigenvalues of φ is ≥ 1. One defines the entropy $h(\varphi)$ to be the maximum of the natural logarithm of those absolute values. It is then ≥ 0 and of interest are the cases when it is > 0. Clearly, it can only happen when φ is not of finite order on $H^{2\bullet}(X(\mathbb{C}))$, so a fortiori when φ does not have finite order as an automorphism of X.

Keiji Oguiso observed (private communication) that Hodge theory implies that, *this maximum is taken on the span of the Néron–Severi group inside of de Rham cohomology*, in fact on the transcendental part of de Rham cohomology, φ has finite order (see Proposition 5.1 for a slightly more precise statement). On the other hand, the definition of the entropy stated above is clearly algebraic. One can replace de Rham cohomology by l-adic étale cohomology in the definition. Taking then a ring $R \subset \mathbb{C}$ of finite type over $\mathbb{Z}$ over which (X, φ) has a model (X_R, φ_R) such that X_R has good reduction at all closed points $s \in \operatorname{Spec} R$, one sees that the value of the entropy of $\varphi_s = \varphi_R \otimes_R \kappa(s)$ on $H^{2\bullet}(X_{\bar{s}}, \mathbb{Q}_l)$ is taken on the $\mathbb{Q}_l$-span of the Néron–Severi group inside of l-adic cohomology, where $X_s = X_R \times_R s$, $X_{\bar{s}} = X_s \otimes_{\kappa(s)} \bar{\kappa}(s)$.

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We ask whether this property comes from the fact that over the finite field $\kappa(s)$, (X_s, φ_s) is the reduction mod p of (X, φ) or whether it is true in general as a property of eigenvalues of automorphisms acting on l-adic cohomology over finite fields.

Our main result says:

Theorem 1.1. *Let X be a smooth, projective surface over a finite field $\mathbb{F}_q$, let Θ be a polarization, and let $\varphi \in \mathrm{Aut}(X)$ be an automorphism of the underlying surface. Let $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_p$ be the corresponding surface over an algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_q$ (where p is the characteristic).*

Let $l \neq p$ be a prime, and let

$$V = V(X, [\Theta], \varphi) \subset [\Theta]^{\perp} \subset H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1))$$

be the largest φ-stable subspace, which is contained in the orthogonal complement of $[\Theta] \in H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1))$ with respect to the cup product pairing

$$H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1)) \otimes_{\mathbb{Q}_l} H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1)) \to H^4_{\text{ét}}(\bar{X}, \mathbb{Q}_l(2)) \cong \mathbb{Q}_l.$$

Then, φ has finite order on V.

We note here that from the Hodge index theorem for divisors, the intersection form on the orthogonal complement of $[\Theta]$ within the $\mathbb{Q}$-span of the Néron–Severi group is a negative definite $\mathbb{Q}$-valued bilinear form. Hence there is an orthogonal direct sum decomposition, compatible with φ,

$$H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1)) = \langle \varphi^n \Theta \rangle \perp V,$$

where $\langle \varphi^n[\Theta] \rangle$ is the $\mathbb{Q}_l$-span of the images of $[\Theta]$ under iterates of φ and φ^{-1}. Thus V is well-defined, and the intersection form restricted to V is non-degenenerate.

We note that the formulation of Theorem 1.1 does not involve directly the $\mathbb{Q}_l$-span of the Néron–Severi, which is not always liftable to characteristic 0 even if X, defined over the finite field, is so liftable. Hence, one sees that one can reverse the classical argument sketched above, to get the following corollary:

Corollary 1.2. *Let (Y, Θ) be a polarized surface over an algebraically closed field k, and let $\varphi \colon Y \to Y$ be an algebraic automorphism of Y (with φ not necessarily preserving the polarization).*

Let l be a prime, invertible in k, and let

$$V = V(Y, [\Theta], \varphi) \subset [\Theta]^{\perp} \subset H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1))$$

be the largest φ-stable subspace, which is contained in the orthogonal complement of

$[\Theta] \in H^2_{\acute{e}t}(Y, \mathbb{Q}_l(1))$ *with respect to the cup product pairing*

$$H^2_{\text{ét}}(Y, \mathbb{Q}_l(1)) \otimes_{\mathbb{Q}_l} H^2_{\text{ét}}(Y, \mathbb{Q}_l(1)) \to H^4_{\text{ét}}(Y, \mathbb{Q}_l(2)) \cong \mathbb{Q}_l.$$

Then φ has finite order on V.

While the Hodge theoretic argument is purely abstract (i.e., depends only on the properties of Hodge structures, and not on geometric arguments), the arguments we present in this note for proving Theorem 1.1 rely on the classification of smooth projective surfaces, on the fact that surfaces of general type (over a finite field) have a finite group of automorphisms, on the Tate conjecture for abelian surfaces, and, unfortunately, on one argument involving lifting $K3$ surfaces to characteristic 0. So, due to this one $K3$ case, we can't say that we have a purely arithmetic proof of Corollary 1.2 over $\mathbb{C}$. On the other hand, Theorem 1.1 should follow from the standard conjectures (see Section 6.1). So, aside from its interest for entropy questions, it can also be viewed as a motivic statement. To reinforce this viewpoint, we show in section Theorem 6.1

Theorem 1.3. *In the situation of* Theorem 1.1, *the maximum of the absolute values of the eigenvalues of φ on $\bigoplus_{i=0}^4 H^i_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ (with respect to any complex embedding of $\mathbb{Q}_l$) is achieved on the $\mathbb{Q}_l$-span of $\langle \varphi^n[\Theta], n \in \mathbb{Z}\rangle$, in $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$.*

We deduce of course the same theorem over any field:

Corollary 1.4. *In the situation of* Corollary 1.2, *the maximum of the absolute values of the eigenvalues of φ on $\bigoplus_{i=0}^4 H^i_{\text{ét}}(\bar{Y}, \mathbb{Q}_l)$ (with respect to any complex embedding of $\mathbb{Q}_l$) is achieved on the $\mathbb{Q}_l$-span of $\langle \varphi^n[\Theta], n \in \mathbb{Z}\rangle$, in $H^2_{\text{ét}}(\bar{Y}, \mathbb{Q}_l)$.*

2. Some preliminaries and general reduction steps to prove Theorem 1.1

As was already done in the formulation of the theorem and its corollary, we write φ for the contravariant action of φ on cohomology; it should be clear from the context if φ denotes the automorphism, or the linear automorphism obtained from it in a specific linear representation.

As in Theorem 1.1, one considers the action of φ on $H^i(\bar{X}, \mathbb{Q}_l)$, we may as well replace $\mathbb{F}_q$ by a finite extension, and thus we will always assume that the Néron–Severi group $NS(\bar{X})$ is defined over $\mathbb{F}_q$.

We may also replace φ by any power φ^n, $n \neq 0$, without loss of generality. In particular, as already observed, Theorem 1.1 has content only when φ acts on $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1))$ through a linear automorphism of infinite order.

Lemma 2.1. *Suppose $[\Theta] \in NS(\bar{X})$ has a finite orbit under φ. Let $k \supset \mathbb{F}_q$ be a finite extension on which Θ is defined as a line bundle. Then φ itself has finite order as an automorphism on $X \otimes_{\mathbb{F}_q} k$, so a fortiori it has finite order on the whole cohomology $H^*_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1))$, and* Theorem 1.1 *is trivially valid.*

Proof. Replacing φ by a suitable power, we may assume that
(i) the algebraic equivalence class $[\Theta] \in NS(\bar{X})$ is fixed by φ
(ii) there is a very ample line bundle $\mathcal{L}$ on X, which satisfies $\varphi^*\mathcal{L} \cong \mathcal{L}$, whose class in $NS(\bar{X})$ is $m[\Theta]$ for some positive integer m.
Here, (i) is clear. For (ii), first choose a very ample $\mathcal{L}$ on X with class $m[\Theta]$ for some positive integer m. Now the orbit of $[\mathcal{L}]$ in $\mathrm{Pic}(\bar{X})$ under the group of automorphisms generated by φ is contained in a fixed coset of $\mathrm{Pic}^\tau(X)(\bar{\mathbb{F}}_p)$, consisting of $\mathbb{F}_q$-rational points in $\mathrm{Pic}(\bar{X})$, and this is a finite set. Now replacing φ by a suitable positive power, we may assume that the class of $\mathcal{L}$ in $\mathrm{Pic}(\bar{X})$ is fixed. Then (ii) holds, since $\varphi^*\mathcal{L}$ and $\mathcal{L}$ are line bundles on X, which become isomorphic on $\bar{X}$, so that they are isomorphic on X.

In particular, from (ii), if we fix an isomorphism $\varphi^*\mathcal{L} \cong \mathcal{L}$, then the automorphism φ of X yields a graded automorphism of the ring $A = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$. Conversely, φ is the induced automorphism on $X = \mathrm{Proj}\, A$, obtained from the graded ring automorphism of A. Now Lemma 2.2 below finishes the argument. □

Lemma 2.2. *Let k be a finite field, and $A = \bigoplus_{n \geq 0} A_n$ a finitely generated graded algebra over $A_0 = k$. Then any graded automorphism of A has finite order.*

Proof. Since A is finitely generated over k, it is generated by $W = \bigoplus_{i=0}^n A_i$ for some n, where we note that W is a finite vector space. Any graded automorphism of A restricts to a k-linear automorphism of W, and this restriction uniquely determines the graded automorphism. Thus we may identify the group of graded automorphisms with a subgroup of the finite group $GL(W)$. □

Proposition 2.3. *Let the notation be as in* Theorem 1.1. *Let $i \geq 0$, $j \in \mathbb{Z}$.*
(i) *The eigenvalues of φ acting on any étale cohomology group $H^i_{\text{ét}}(\bar{X}, \mathbb{Q}_l(j))$ are algebraic integers, which are units* (*that is, invertible elements in the ring of algebraic integers*).
(ii) *The characteristic polynomial of φ on $H^i_{\text{ét}}(\bar{X}, \mathbb{Q}_l(j))$ has integer coefficients, and is a monic polynomial with constant term ± 1. This polyomial is independent of j, and of the chosen prime $l \neq p$.*
(iii) *A similar conclusion holds for the characteristic polynomial of φ acting on any φ-stable subspace of $H^i_{\text{ét}}(\bar{X}, \mathbb{Q}_l(j))$, which can be defined using a projector in the ring of self-correspondences of X* (*i.e., corresponds to a direct summand of the Chow motive of X which, on base change to $\bar{\mathbb{F}}_p$, has only one non-zero étale cohomology group*). *In particular, this holds for the characteristic polynomial of φ on V.*

Proof. This is a standard consequence of the Grothendieck–Lefschetz fixed point formula, applied to powers of φ ([9, Proposition 2.7]), combined with Deligne's results on Weil's conjectures ([4]). In (ii) and (iii), the point is that the characteristic polynomial in question has rational coefficients, since it is determined by the action on cohomology of a Chow Motive, while its roots are algebraic integers (this argument appears in a paper of Katz and Messing [8]). Over finite fields, Deligne's theorem implies that the individual étale cohomology groups do correspond to Chow motives (see [8, Theorem 2.1], and the proof in [8, Theorem 2.1] applies equally well to any Chow motive which is a summand of the motive of X, whose étale cohomology is concentrated in one degree. $\square$

Next, we consider the subspace of V spanned by algebraic cycles.

Proposition 2.4. *Assume we are in the situation of* Theorem 1.1. *Let*

$$V_{\mathrm{alg}} = (NS(\bar{X}) \otimes \mathbb{Q}_l) \cap V.$$

Then V_{alg} is stable under φ, and φ on V_{alg} has finite order.

Proof. The $\mathbb{Q}_l$-vector space V_{alg} has a natural $\mathbb{Z}$-structure NV defined by the maximal φ-stable subgroup

$$NV \subset [\Theta]^{\perp} \subset NS(\bar{X}),$$

where $\perp$ is the orthogonal complement with respect to the intersection poduct on $NS(\bar{X})$. One has $V_{\mathrm{alg}} = NV \otimes \mathbb{Q}_l$, and this identification is φ-equivariant.

Now NV comes equipped with the intersection product $NV \otimes NV \to \mathbb{Z}$, which is non-degenerate after $\otimes \mathbb{R}$, and is *negative definite*, by the Hodge index theorem for divisors. This pairing is clearly φ-stable as well, so that φ can be considered as an orthogonal transformation for a Euclidean space structure on $NV \otimes \mathbb{R}$. In particular it is semi-simple. Moreover all eigenvalues of φ on $NV \otimes \mathbb{C}$ are of absolute value 1. Since these eigenvalues are algebraic integers (in fact units), and the characteristic polynomial of φ on $NV \otimes \mathbb{Q}$ has rational coefficients, the eigenvalues are in fact algebraic integers, all of whose conjugates have absolute value 1; thus they are roots of unity, by a well-known theorem of Kronecker. This finishes the proof. $\square$

One obtains the immediate corollary:

Corollary 2.5. Theorem 1.1 *holds whenever*

$$V = V_{\mathrm{alg}},$$

or equivalently, whenever

$$NS(\bar{X}) \otimes \mathbb{Q}_l = H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1)).$$

More generally, let

$$V_{\text{tr}} = V \cap NS(\bar{X})^{\perp},$$

where the orthogonal $\perp$ *is taken in* $H^2(\bar{X}, \mathbb{Q}_l)(1)$. *This is a* φ*-stable subspace of* V, *which may be defined as the cohomology of a suitable Chow motive, and the conclusion of* Theorem 1.1 *is equivalent to a similar statement about the eigenvalues of* φ *on the subspace* V_{tr}.

REMARK 2.6. The cup product also induces a non-degenerate symmetric bilinear form on V_{tr} with values in $\mathbb{Q}_l$. Since φ is an automorphism of X, φ acts as an orthogonal transformation of V_{tr} with respect to this bilinear form.

3. Using classification of surfaces

Now we consider the possibilites for the surface X, from the perspective of the Enriques–Bombieri–Mumford classification of surfaces in arbitrary characteristic (a convenient reference for most of what we need is the book [1]). Since we need only consider surfaces where $V \neq V_{\text{alg}}$, *we may assume that the Kodaira dimension of* $\bar{X}$ *is* ≥ 0.

As a consequence, from [1, Corollary 10.22], since X has Kodaira dimension ≥ 0, the birational equivalence class of $\bar{X}$ has a unique non-singular minimal model, say X_0. Increasing the finite field $\mathbb{F}_q$, we may assume the model X_0, and the morphism $\bar{X} \to X_0$, are defined over $\mathbb{F}_q$. Since φ acts on X, it acts on its function field, and thus on this unique minimal model ([1, Theorem 10.21]). So the automorphism φ of $\bar{X}$ descends to X_0, and the spaces V_{tr} of $\bar{X}$ and X_0 are naturally identified. Hence *we are reduced to the case when* $\bar{X}$ *is itself minimal*, i.e., $\bar{X}$ does not contain any exceptional curves of the first kind.

We may also assume X *is not of general type*. Indeed, φ yields a graded automorphism of the canonical model $\text{Proj}\big(\bigoplus H^0(X, \omega_X^{\otimes n})\big)$ of X, which (by Lemma 2.2) has finite order. Hence, some power of φ is an automorphism which acts trivially on the function field of X, since it is trivial on the canonical model, which (if $\bar{X}$ is of general type) is birational to X. Thus a power of φ agrees with the identity on a Zariski dense subset, and hence equals the identity.

Thus, we need only focus on the cases when the Kodaira dimension of X is 0 or 1.

Proposition 3.1. *In the situation of* Theorem 1.1, *suppose* X *has Kodaira dimension* 1. *Then the conclusion of* Theorem 1.1 *holds.*

Proof. Let $C = \mathrm{Proj}\big(\bigoplus_{n\geq 0} H^0(X,\omega_X^{\otimes n})\big)$ (this graded ring is finitely generated; see [1, Theorem 9.9]). Then from classification (same result in [1]) there is a morphism

$$f : X \to C$$

which gives rise to an elliptic or quasi-elliptic fibration, i.e., the generic fiber is a regular projective curve which has arithmetic genus 1, and the geometric generic fiber is either an elliptic curve, or is an irreducible rational curve with an ordinary cusp (this can occur only in characteristics 2 and 3, [1, Theorem 7.18]).

In the quasi-elliptic case, the Leray spectral sequence for étale cohomology implies that

$$NS(\bar{X}) \otimes \mathbb{Q}_l = H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1)),$$

since the stalks of $R^1 f_* \mathbb{Q}_l(1)$ at all geometric points of C vanish. (Alternately, as remarked by the referee, one could argue that, after a purely inseparable base-change $C' \to C$, one obtains a surface birational to a $\mathbb{P}^1$-bundle over C', which implies that X is uniruled; in particular, all cycles on X are algebraic.)

Hence we may assume without loss of generality that *f is an elliptic fibration*. The proof of [1, Theorem 9.9]) implies that some power of the canonical line bundle of $\bar{X}$ is the pullback of a line bundle from C.

Now we note that φ induces a graded automorphism of the canonical ring $\bigoplus_{n\geq 0} H^0(X, \omega_X^{\otimes n})$, and thus an automorphism of C, which we may also denote by φ, such that $f : X \to C$ is φ-equivariant.

As usual, after replacing φ by a power, we may assume (from Lemma 2.2) that the induced automorphism of the canonical ring (and thus of the base curve C) is trivial.

Now for any morphism $D \to C$, φ acts in a canonical way on the total space of the base changed morphism $X \times_C D \to D$, preserving the fibers; we denote this induced automorphism also by φ. Hence, if $D \to C$ is a finite morphism of nonsingular curves, so that $X \times_C D$ is an integral projective surface, φ also acts on the normalization of $X \times_C D$, which is a normal projective surface, denoted by X_D. Making a suitable such base change g: $D \to C$, and normalizing, we may arrange that the resulting elliptic fibration $X_D \to D$ has a section. Clearly the singular locus of the normal surface X_D is stabilized (as a set) by the automorphism. Consider the minimal resolution of singularities $\tilde{X} \to X_D$. If we write it as the blow-up of some ideal sheaf whose radical defines the singular locus, we may assume (after replacing φ by a suitable power) that this ideal sheaf is stabilized by φ, so that φ lifts canonically to an automorphism of the blow-up $\tilde{X}$ (since φ clearly determines an automorphism of the Rees algebra sheaf). Note also that a power of the canonical sheaf of $\tilde{X}$ is the pullback of a line bundle from D, the surface $\tilde{X}$ is also an elliptic surface of Kodaira dimension 1, and $\tilde{X} \to D$ is the morphism determined by the canonical ring of $\tilde{X}$.

The morphism $\tilde{X} \to X$ is a generically finite proper morphism between smooth projective surfaces, which is φ-equivariant. We may choose a polarization $[\tilde{\Theta}]$ for $\tilde{X}$

which is the sum of the pullback of $[\Theta]$ and a divisor class with support in the exceptional divisor of $\tilde{X} \to X_D$. (This is a consequence of the negative definiteness of the intersection pairing on the exceptional curves, and the Nakai–Moishezon ampleness criterion.) Then the resulting space $V(\tilde{X}, [\tilde{\Theta}], \varphi)$ contains $V = V(X, [\Theta], \varphi)$ as a φ-stable subspace.

Thus, we are further reduced to considering the situation where the map $f\colon X \to C$ determined by the canonical divisor of X, is an elliptic fibration which has a section, and φ is an automorphism of X preserving the fibers.

Let $U \subset C$ be the maximal open subset over which f is smooth, so that $f_U\colon f^{-1}(U) \to U$ is an abelian scheme of relative dimension 1. Let $\bar{C}$, $\bar{U}$, $f^{-1}(\bar{U})$ be the corresponding schemes over $\bar{\mathbb{F}}_p$. The localisation sequence

$$\bigoplus_{s\in\Sigma} H^2_{X_{\bar{s}}}(\bar{X}, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(f^{-1}(\bar{U}), \mathbb{Q}_l(1))$$

is exact and φ-equivariant, where Σ is the discriminant of f. On one hand, each summand $H^2_{X_{\bar{s}}}(\bar{X}, \mathbb{Q}_l)$ is (up to a Tate twist) the free abelian group on the irreducible components of the geometric fiber $X_{\bar{s}}$, and φ acts via a permutation on the classes of these components. Thus φ has finite order on $\bigoplus_{s\in\Sigma} H^2_{X_{\bar{s}}}(\bar{X}, \mathbb{Q}_l)$. Further, the map from V_{tr} to $H^2_{\text{ét}}(f^{-1}(\bar{U}), \mathbb{Q}_l(1))$ is injective.

On the other hand, any automorphism of (the total space of) the abelian scheme $f^{-1}(U)$, which is compatible with the structure morphism f_U, is the composition of a group-scheme automorphism (which has finite order) and a translation, since this is true on the elliptic curve which forms the geometric generic fiber. Replacing φ by a power, we may assume further that φ acts on $f^{-1}(U)$ as a translation by a section, with respect to the abelian scheme structure.

We claim that, in this situation, φ is unipotent on $H^2_{\text{ét}}(f^{-1}(\bar{U}), \mathbb{Q}_l(1))$. Indeed, the Leray spectral sequence yields a φ-equivariant exact sequence

$$0 \to H^1_{\text{ét}}(\bar{U}, R^1 f_*\mathbb{Q}_l(1)) \to H^2_{\text{ét}}(f^{-1}(\bar{U}), \mathbb{Q}_l(1)) \to H^0_{\text{ét}}(\bar{U}, R^2 f_*\mathbb{Q}_l(1)) \to 0.$$

The action of φ on $H^0_{\text{ét}}(\bar{U}, R^2 f_*\mathbb{Q}_l(1))$ is the identity, while the action of φ in $R^1(f_U)_*\mathbb{Q}_l$ is trivial.

On the other hand, the composition

$$H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(f^{-1}(\bar{U}), \mathbb{Q}_l(1)) \to H^0_{\text{ét}}(\bar{U}, R^2 f_*\mathbb{Q}_l(1)) \cong \mathbb{Q}_l$$

is identified with the intersection product with the cohomology class of a geometric fiber, from the projection formula. In particular, V_{tr} is contained in the kernel of the restriction map

$$H^2_{\text{ét}}(f^{-1}(\bar{U}), \mathbb{Q}_l(1)) \to H^0_{\text{ét}}(\bar{U}, R^2 f_*\mathbb{Q}_l(1)),$$

that is,

$$V_{\mathrm{tr}} \subset H^1_{\text{ét}}(\bar{U},\, R^1 f_* \mathbb{Q}_l(1))$$

as a φ-stable subspace on which φ acts *trivially*. We conclude that the action φ on V_{tr} is trivial as it is on $R^1 f_* \mathbb{Q}_l(1)$.

Since, in the discussion above, we had possibly replaced φ by a power, we conclude that the original automorphism φ has finite order on V_{tr}. Furthermore, we also conclude that the eigenvalues of φ on the whole group $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1))$ are roots of unity. □

REMARK 3.2. Our proof shows that if X is elliptic, the eigenvalues of φ are roots of unity on the whole $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$, that is φ acts quasi-unipotently on it.

Proposition 3.3. *Suppose that, in the sitution of* Theorem 1.1, *the surface* $\bar{X}$ *is minimal of Kodaira dimension* 0, *and* $\bar{X}$ *is not a* $K3$ *or abelian surface. Then* Theorem 1.1 *holds for* X.

Proof. As stated in [3, p. 1], the minimal surfaces with Kodaira dimension 0 fall into 4 classes: $K3$ surfaces, Enriques surfaces (both of "classical" and "non-classical" type), abelian surfaces and surfaces fibered over their Albanese, which is an elliptic curve (and the fibrations are either elliptic or quasi-elliptic).

In case the Albanese variety of $\bar{X}$ is an elliptic curve, we may assume (after increasing $\mathbb{F}_q$ if needed) that X has an $\mathbb{F}_q$-rational φ-fixed point. Then $\mathrm{Alb}(\bar{X})$ and the Albanese mapping are defined over $\mathbb{F}_q$, and φ induces a unique (group-scheme) automorphism of the elliptic curve $\mathrm{Alb}(\bar{X})$ making the Albanese mapping φ-equivariant. Since the automorphism group of an elliptic curve is finite, replacing φ by a power, we reduce to the situation where the action on $\mathrm{Alb}(\bar{X})$ is trivial.

Now we may argue just as in the proof of Proposition 3.1, using the Albanese mapping instead of the mapping deduced from the canonical ring. Again, the case when V_{tr} is possibly nontrivial is for an elliptic fibration, and a similar Leray spectral sequence argument goes through.

In the case of Enriques surfaces, including the non-classical ones, in fact one has $V = V_{\mathrm{alg}}$ (see [3, Theorem 4]), by an argument of Artin involving the Brauer group, so we conclude by Corollary 2.5. □

4. The case of an abelian surface

Any automorphism of the abelian surface $\bar{X}$ is the composition of a group automorphism and a translation by a closed point, where the translation has finite order. Hence, increasing the finite field $\mathbb{F}_q$ and replacing φ by a power, if necessary, we may assume X is an abelian surface over $\mathbb{F}_q$, and φ is a group-scheme automorphism of

X. We may assume that all the endomorphisms of $\bar{X}$ are defined over $\mathbb{F}_q$ (since the endomorphism ring is finitely generated).

We may also deduce (by again increasing $\mathbb{F}_q$ and replacing φ by a power, if necessary) that the validity of Theorem 1.1 for X depends only on the isogeny class of $\bar{X}$. This follows because for any $n > 1$, φ acts as an automoprhism of finite order on the n-torsion $X(\bar{\mathbb{F}}_p)[n]$, and thus for any isogeny $\bar{X} \to X'$, some power of φ acts trivially on its kernel, and so a suitable power of φ descends to a compatible automorphism of X'.

Let $F\colon \bar{X} \to \bar{X}$ be the *geometric Frobenius morphism* associated to X, considered as an $\mathbb{F}_q$-scheme; thus

$$F\colon \bar{X} \to \bar{X}$$

is an $\bar{\mathbb{F}}_p$-morphism of degree q^2, which acts on each $H^i_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$, with (by Deligne's theorem) a characteristic polynomial with $\mathbb{Z}$-coefficients, whose (algebraic integer) roots all have complex absolute value $(\sqrt{q})^i$. We may assume without loss of generality that q is an even power of p, so that these absolute values are integers.

We now define $P(t) \in \mathbb{Z}[t]$ to be the (monic) *minimal polynomial* of F viewed as an element of the finite rank torsion-free $\mathbb{Z}$-module $\operatorname{End}(X)$. Thus $P(t) \in \mathbb{Q}[t]$ is the minimal polynomial of F as an element in $\operatorname{End}(X) \otimes \mathbb{Q}$, and $P(t) \in \mathbb{Q}_l[t]$ is the minimal polynomial of F as an element in

$$\operatorname{End}(X) \otimes \mathbb{Q}_l \subset \operatorname{End}(H^1(\bar{X}, \mathbb{Q}_l)) = \operatorname{End}(H^1(\bar{X}, \mathbb{Q}_l(j))) \quad \text{for all} \quad j \in \mathbb{Z}.$$

From Tate's theorems [16] (see in particular Theorem 2; see also [12, Appendix 1, Theorem 3]), proving the Tate conjecture for endomorphisms of abelian varieties over finite fields, we know in particular that $P(t)$ *has no multiple roots*, and is thus a product of distinct monic irreducible polynomials which are pairwise relatively prime. Equivalently, F acts semisimply on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$.

We consider also the *characteristic polynomial* $\in \mathbb{Z}[t]$ of F, as defined in [12], §19, Theorem 4, which is the same, viewed in $\mathbb{Q}_l[t]$, as the characteristic polynomial of F as an element in $\operatorname{End}(H^1(\bar{X}, \mathbb{Q}_l))$.

Since $\dim_{\mathbb{Q}_l} H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l) = 4$, it has degree 4.

In case the minimal polynomial is irreducible over $\mathbb{Q}$, the characteristic polynomial must be a power of this minimal polynomial $P(t)$, and so the degree of the minimal polynomial $P(t)$ must divide 4.

We now distinguish between several cases.

CASE 1: *The minimal polynomial $P(t)$ is reducible over $\mathbb{Q}$.*

In this case, $\bar{X}$ is not simple, since its endomorphism algebra has zero divisors, and in fact $\bar{X}$ must then be isogenous to a product of two mutually non-isogeneous elliptic curves (this is the only way to have two mutually coprime factors of $P(t)$). But then $\bar{X}$ has a finite group of automorphisms as an abelian variety, since this is the case for an elliptic curve, and any automorphism of a product of two non-isogeneous elliptic curves is a product of automorphisms on each of the two factors. Hence, in this situation, φ cannot have infinite order, and we have nothing to prove.

CASE 2: *$P(t)$ is a linear polynomial.*

Then the characteristic polynomial is a power of a linear polynomial, and from the Tate conjecture, this implies that $\bar{X}$ is isogenous to $E \times E$ for a supersingular elliptic curve E. But in this case, since the endomorphism algebra of E is a quaternion division algebra, which has dimension 4 over $\mathbb{Q}$, the Picard number of $E \times E$ is 6, which is also the second Betti number; one thus has that $V = V_{\text{alg}}$, and $V_{\text{tr}} = 0$, so we conclude with Corollary 2.5.

CASE 3: *$P(t)$ is an irreducible quadratic polynomial over $\mathbb{Z}$.*

Let λ be a complex root of $P(t)$. Then λ is a non-real complex number, with $|\lambda|^2 = \lambda\bar{\lambda} = q$. Indeed, if λ is a real root, it must be an integer, since q is an even power of p; however $P(t)$ is irreducible.

Hence we must have that $P(t) = (t-\lambda)(t-\bar{\lambda})$, where $\mathbb{Q}(F) \cong \mathbb{Q}(\lambda)$ is an imaginary quadratic field. Clearly the characteristic polynomial of F on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ is just $P(t)^2$ (see [12, Appendix 1, Theorem 3 (e)]).

Since X is an abelian surface, the cup product gives an isomorphism

$$H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l) = \left(\bigwedge^2 H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)\right).$$

Thus we see that the characteristic polynomial of F on $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ has to be

$$(t-\lambda^2)(t-\bar{\lambda}^2)(t-|\lambda|^2)^4 = (t-\lambda^2)(t-\bar{\lambda}^2)(t-q)^4.$$

From the Tate conjecture for divisors on $\bar{X}$, we conclude that V_{tr} is 2-dimensional, and the characteristic polynomial of F on V_{tr} is the quadratic polynomial

$$(t-\lambda^2)(t-\bar{\lambda}^2).$$

As noted before, the cup product on $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1))$ gives rise to a non-degenerate symmetric bilinear form on V_{tr} with values in $\mathbb{Q}_l$, and φ and $(1/q)F$ are orthogonal transformations with respect to this form, which commute.

Now φ is a unit in the ring of endomorphisms of the abelian variety X, and from the Tate conjecture, $\mathrm{End}(\bar{X}) \otimes \mathbb{Q}$ is a central simple algebra of dimension 4 over its centre $K = \mathbb{Q}[F]$, the subalgebra generated by F (this central simple algebra is either a matrix algebra of size 2, or a quaternion divison algebra). Hence the reduced characteristic polynomial[1] of φ, considered as an element of this endomorphism algebra, is

[1]This is the characteristic polynomial of φ, considered as an element of $\mathrm{End}(\bar{X}) \otimes_K \bar{K}$, which is the algebra of 2×2 matrices over $\bar{K}$.

a quadratic polynomial with coefficients in K,

$$f(x) = x^2 - \mathrm{Trd}(\varphi)x + \mathrm{Nrd}(\varphi) \tag{4.1}$$

where $\mathrm{Trd}(\varphi) \in K$ and $\mathrm{Nrd}(\varphi) \in K$ are the values of the reduced norm and trace of the central simple algebra.

Now on

$$H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l) \otimes \bar{\mathbb{Q}}_l$$

we may diagonalize F. Fixing an embedding of $\mathbb{Q}_l$ into the complex number field $\mathbb{C}$, we can split the resulting complex vector space into its λ and $\bar{\lambda}$ eigenspaces for the action of F.

Clearly φ, considered as a complex linear transformation, stabilizes this decomposition, since φ commutes with F. Further, on each of the 2-dimensional F-eigenspaces, on which F acts as $\lambda \cdot \mathrm{Id}$ and $\bar{\lambda} \cdot \mathrm{Id}$, φ has the appropriate characteristic polynomial (with $\mathbb{C}$-coefficients) $\sigma(f)$ or $\bar{\sigma}(f)$, where $\sigma, \bar{\sigma}$ are the embeddings of K into $\mathbb{C}$ determined by $\sigma(F) = \lambda$, $\bar{\sigma}(F) = \bar{\lambda}$ (resulting in two conjugate embeddings $K[x] \hookrightarrow \mathbb{C}[x]$, denoted the same way).

The upshot is that, on the 2-dimensional complex vector space

$$V_{\mathrm{tr}} \otimes_{\mathbb{Q}_l} \mathbb{C}$$

φ is diagonalizable, and has eigenvalues $\sigma(\mathrm{Nrd}(\varphi))$ and $\bar{\sigma}(\mathrm{Nrd}(\varphi))$. But $\mathrm{Nrd}(\varphi) \in K$ is actually an algebraic integer, which is a unit. Since K is an imaginary quadratic field, $\mathrm{Nrd}(\varphi)$ must be a root of unity. Thus φ is semisimple on V_{tr}, with eigenvalues which are roots of unity, and this finishes the proof of Theorem 1.1 in this case.

CASE 4:

$P(t)$ is an irreducible polynomial over $\mathbb{Q}$ of degree 4.

In this case, F has 4 distinct algebraic non-real eigenvalues on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$, which (once we embed $\mathbb{Q}_l$ into $\mathbb{C}$) are of the form $\lambda, \bar{\lambda}, \mu, \bar{\mu}$, with $|\lambda|^2 = |\mu|^2 = q$.

In this case, on $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$, F has the eigenvalues $\lambda\mu, \bar{\lambda}\mu, \lambda\bar{\mu}, \bar{\lambda}\bar{\mu}$, which are again all distinct and non-real, as well as the eigenvalue q with multiplicity 2. From the Tate conjecture for $\bar{X}$, we see that V_{tr} is a 4-dimensional space, on which F acts with the above 4 distinct non-real eigenvalues.

Now the Tate conjecture implies (see [16, Theorem 2], or [12, Appendix 1, Theorem 3]) that the minimal and characteristic polynomials of F on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ coincide, and we have that $\mathrm{End}(\bar{X}) \otimes \mathbb{Q} = \mathbb{Q}(F) = K$. Hence for some polynomial $f(t) \in \mathbb{Q}[t]$, we have that $\varphi = f(F) \in K$.

Fixing an embedding of $\mathbb{Q}_l$ into $\mathbb{C}$, we may choose a basis of eigenvectors $\{v_\lambda, v_{\bar{\lambda}}, v_\mu, v_{\bar{\mu}}\}$ for F on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l) \otimes \mathbb{C}$, indexed by the corresponding eigenvalues. Then these are also eigenvectors for φ, with eigenvalues $f(\lambda)$, $f(\bar{\lambda})$, $f(\mu)$, $f(\bar{\mu})$ respectively.

Now $V_{\mathrm{tr}} \otimes \mathbb{C}$ has a resulting basis $\{v_\lambda \wedge v_\mu, v_\lambda \wedge v_{\bar\mu}, v_{\bar\lambda} \wedge v_\mu, v_{\bar\lambda} \wedge v_{\bar\mu}\}$. For this basis, it is then clear that φ acts diagonally, with eigenvalues $f(\lambda) f(\mu)$, $f(\lambda) f(\bar\mu)$, etc.

We now observe that K *is a CM field*, i.e., a totally non-real quadratic extension of a totally real number subfield. Indeed, the distinct embeddings of K into $\mathbb{C}$ are determined by $F \mapsto \lambda$, $F \mapsto \mu$, and their complex conjugate embeddings, so K is totally non-real. It is also clear that the subfield $L = \mathbb{Q}(F + q/F)$ is totally real, and K is a quadratic extension, since $|\lambda|^2 = |\mu|^2 = q$.

Since $\varphi \in K$ is an automorphism of X, it is a unit in the ring of integers $\mathcal{O}_K$. From the *Dirichlet unit theorem* (see for example [2, Chapter 2, Theorem 5]), the unit groups of $\mathcal{O}_K$ and of the integers $\mathcal{O}_L$ in the totally real subfield L have the same rank. This means that, after replacing φ by some power, we may assume φ lies in L, and all of its eigenvalues on $H^1_{\text{ét}}(\bar X, \mathbb{Q}_l) \otimes \mathbb{C}$ are *real algebraic numbers*.

Hence on $H^1_{\text{ét}}(\bar X, \mathbb{Q}_l) \otimes \mathbb{C}$, φ has two distinct eigenvalues $f(\lambda) = f(\bar\lambda)$ and $f(\mu) = f(\bar\mu)$, each with multiplicity 2 (since φ has infinite order, and determinant 1 (as the degree of φ is 1), these two real numbers must be distinct, and satisfy $f(\lambda)^2 f(\mu)^2 = 1$). But this implies φ acts on $V_{\mathrm{tr}} \otimes \bar{\mathbb{Q}}_l$ as the real scalar $f(\lambda) f(\mu)$, which must be ± 1. □

5. The case of a $K3$ surface

In the proof of Theorem 1.1, there is one case remaining: the case when X is a $K3$ surface. As in [3], this means $\bar X$ is a smooth, projective minimal surface, and we have the properties

$$\omega_X \cong \mathcal{O}_X, \quad H^1(X, \mathcal{O}_X) = 0, \quad \dim_{\mathbb{Q}_l} H^2_{\text{ét}}(\bar X, \mathbb{Q}_l(1)) = 22, \quad \operatorname{Pic}^\tau(\bar X) = 0.$$

We first treat the case of a supersingular $K3$ surface in the sense of Shioda. Then by definition of supersingularity (in this sense) $H^2(\bar X, \mathbb{Q}_l(1))$ is algebraic and we can apply Corollary 2.5.

We now rely on the crutch of lifting to characteristic 0. From a recent paper [10] (see in particular Theorem 6.1 and the bottom of p. 8), it follows that if X is not a Shioda-supersingular $K3$ surface, we can find

(i) a complete discrete valuation ring R, with residue field $\bar{\mathbb{F}}_p$, and quotient field of characteristic 0,

(ii) an R-scheme $\pi : \mathcal{X} \to \operatorname{Spec} R$, such that π is projective and smooth, of relative dimension 2, with closed fiber $\bar X$,

(iii) if $Y := \mathcal{X}_{\bar\eta}$ is the geometric generic fiber of π, then the specialization homomorphism $\operatorname{Pic}(Y) \to \operatorname{Pic}(\bar X)$ is an isomorphism, which induces an isomorphism between the respective cones of effective cycles,

(iv) there is an *injective* specialization homomorphism $\operatorname{Aut}(Y) \to \operatorname{Aut}(\bar X)$, whose image has finite index.

The specialization map on automorphisms in (iv), which is important for us here, is defined as follows. If ψ is an automorphism of Y, then (after making a base change if needed), the authors of [10] prove that it is induced by an automorphism of the generic fiber $\mathcal{X}_\eta$, which extends to an R-automorphism $\psi_{\mathcal{X}}$ of $\mathcal{X} \setminus S$ for some finite set $S \subset \bar{X} \subset \mathcal{X}$ of closed points; the induced automorphism of $\bar{X} \setminus S$ then extends to an automorphism of $\bar{X}$, which is defined to be the specialization of ψ.

Granting this, we see that, after replacing φ by a power, if necessary, we may assume φ is the specialization of an automorphism of Y, in the above sense. It then follows that, under the specialization isomorphism

$$H^2_{\text{ét}}(Y, \mathbb{Q}_l(1)) \cong H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l(1))$$

the respective actions of ψ and φ are compatible. Further, the polarization $[\Theta]$ of $\bar{X}$ determines uniquely a polarization of Y, which we may also denote by $[\Theta]$, compatibly with the specialization isomorphism. The specialization isomorphism above is of course one component of an isomorphism between cohomology rings, and so respects the corresponding cup products, thus inducing also an isomorphism of l-adic vector spaces

$$V(Y, [\Theta], \psi) \cong V(\bar{X}, [\Theta], \varphi),$$

again compatible with the respective automorphisms ψ, φ. It thus suffices to prove that the ψ has finite order on $V(Y, [\Theta], \psi)$.

We may identify the algebraic closure of the quotient field of the DVR R with the complex number field $\mathbb{C}$, and thus also consider ψ as an automorphism of the complex projective $K3$ surface Y.

In fact, one has the following more general assertion; this observation is, in a sense, the motivation for Theorem 1.1 proved in this paper, and was explained to us by K. Oguiso (in the shape that φ on $H^2_{\text{tr}}(Y, \mathbb{C})$ has finite order):

Proposition 5.1. *Suppose ψ is an automorphism of a projective smooth surface Y over $\mathbb{C}$, with a polarization Θ (not necessarily invariant under ψ). Then ψ has finite order on $V(Y, [\Theta], \psi)$.*

Proof. By the comparison theorem between étale and singular cohomology, we reduce to proving a similar assertion for the action of ψ on $H^2(Y, \mathbb{Q})$. In other words, it suffices to show that the eigenvalues of ψ acting on the similarly defined $\mathbb{Q}$ vector space

$$V(Y, [\Theta], \psi) \subset H^2(Y, \mathbb{Q})$$

are roots of unity. Since ψ is also compatible with the cup product, it defines an orthogonal transformation with respect to the non-degenerate bilinear form on V defined by the cup product.

Consider now the associated non-degenerate real bilinear form on $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$. From the Hodge decomposition, we may write $V_{\mathbb{R}}$ as an orthogonal direct sum

$$V_{\mathbb{R}} = V_{\mathbb{R}}^{(1,1)} \perp V_{\mathbb{R},\mathrm{tr}},$$

where

$$V_{\mathbb{R}}^{(1,1)} = V_{\mathbb{R}} \cap H^{(1,1)} \subset H^2(Y, \mathbb{C}),$$

and the other summand is its orthogonal complement. This does give an orthogonal direct sum decomposition of $V_{\mathbb{R}}$, since by the Hodge index theorem (the Hodge theoretic version), the cup product pairing on $V_{\mathbb{R}}$ is negative definite on $V_{\mathbb{R}}^{(1,1)}$, and positive definite on its orthogonal complement.

Since the Hodge decomposition on $H^2(Y,\mathbb{C})$ is also preserved by ψ, it follows that ψ preserves the above orthogonal direct sum decomposition of $V_{\mathbb{R}}$. Hence, after changing the sign of the inner product on $V_{\mathbb{R}}^{(1,1)}$, we see that ψ preserves a non-degenerate Euclidean form on $V_{\mathbb{R}}$. Hence the ψ is semi-simple and all its eigenvalues are complex numbers of absolute value 1.

However, we also know that the eigenvalues of ψ are algebraic integers, which are invertible, and the characteristic polynomial of ψ has integer coefficients (since it obviously has rational coefficients). Thus, by Kronecker's theorem, these eigenvalues are roots of unity. $\square$

6. Some further remarks

6.1. Standard conjectures and Theorem 1.1. P. Deligne explained to us that our Theorem 1.1 would be a consequence of the standard conjectures, were they available. We reproduce his argument.

As explained in Section 2, we have to show that φ has finite order on transcendental cohomology $H^2_{\mathrm{tr}}(\bar{X}, \mathbb{Q}_l(1))$, where X is a smooth projective surface over a finite field $\mathbb{F}_q$. We denote by M the underlying Chow motive with $\mathbb{Q}$ coefficients, which is endowed with a quadratic form $b\colon M \otimes M \to \mathbb{Q}$, which induces the cup-product $H^2_{\mathrm{tr}}(\bar{X}, \mathbb{Q}_l(1)) \otimes H^2_{\mathrm{tr}}(\bar{X}, \mathbb{Q}_l(1)) \to H^4(\bar{X}, \mathbb{Q}_l(2))$ in l-adic realization. The automorphism φ induces an orthogonal automorphism of M. Its characteristic polynomial lies in $\mathbb{Q}[t]$. But the l-adic realization of the characterisitc polynomial lies in $\mathbb{Z}[t]$ ([8]), thus in fact, it lies on $\mathbb{Z}[t]$. On the other hand, there should exist ([14, V 2.4.5.1 (iv)]) a fiber functor ω over $\mathbb{R}$ on the category of Chow motives of weight 0, with the extra property that $b(\omega)$ is a positive definite form. Thus this implies already that φ is semi-simple and that its eigenvalues on M have absolute value 1. On the other hand, they are algebraic integers again by [8]. We conclude by Kronecker's theorem that the eigenvalues are roots of unity.

6.2. Entropy, even and odd degree cohomology. Recall that the *entropy* of a homeomorphism $\varphi\colon M \to M$ of a compact, orientable manifold M is defined to be the natural logarithm of the spectral radius of the linear tranformation induced by φ on the rational cohomology algebra $H^\bullet(M,\mathbb{Q})$. Since φ is a homeomorphism, it induces a ($\mathbb{Z}$-linear) automorphism of the integral cohomology algebra, so that the characteristic polynomial of φ acting on cohomology has integer coefficients, and the eigenvalues of φ on cohomology are algebraic integers which are invertible, that is, are units in the ring of algebraic integers.

If M is a complex smooth projective variety, and φ is an algebraic automorphism, the value of the entropy is taken on the *even* degree cohomology $H^{2\bullet}(M,\mathbb{Q})$ (see [5, Theorem 2.1]).

We can now go through our proof of Theorem 1.1 from which we deduce:

Theorem 6.1. *In the situation of* Theorem 1.1, *the maximum of the absolute values of the eigenvalues of φ on $\bigoplus_{i=0}^4 H^i_{\text{ét}}(\bar{X},\mathbb{Q}_l)$ with respect to any complex embedding is achieved on the $\mathbb{Q}_l$-span of $\langle\varphi^n[\Theta],\, n\in\mathbb{Z}\rangle$, in $H^2_{\text{ét}}(\bar{X},\mathbb{Q}_l)$.*

Proof. The automorphism φ acts as the identity on $H^i_{\text{ét}}(\bar{X},\mathbb{Q}_l)$, for $i=0$ and $i=4$. Since φ respects the cup-product $H^1_{\text{ét}}(\bar{X},\mathbb{Q}_l)\times H^3_{\text{ét}}(\bar{X},\mathbb{Q}_l)\to H^4_{\text{ét}}(\bar{X},\mathbb{Q}_l)$, its eigenvalues on $H^3_{\text{ét}}(\bar{X},\mathbb{Q}_l)$ are the inverse of its eigenvalues on $H^1_{\text{ét}}(\bar{X},\mathbb{Q}_l)$. On the other hand, the characteristic polynomial of φ on any $H^i_{\text{ét}}(\bar{X},\mathbb{Q}_l)$ has $\mathbb{Z}$-coefficients, and the eigenvalues lie in $\bar{\mathbb{Z}}$. Thus the constant term of this polynomial is ±1 and, fixing a complex embedding of a number field containing all the roots, at least one eigenvalue has absolute value ≥ 1. Thus the maximum of the absolute values is always achieved on $\bigoplus_{i=1}^3 H^i_{\text{ét}}(\bar{X},\mathbb{Q}_l)$.

By Theorem 1.1, we just have to see that the absolute values of the eigenvalues on $H^1_{\text{ét}}(\bar{X},\mathbb{Q}_l)$ and $H^3_{\text{ét}}(\bar{X},\mathbb{Q}_l)$ are at most those on $H^2_{\text{ét}}(\bar{X},\mathbb{Q}_l)$.

Again we may assume (after replacing $\mathbb{F}_q$ by a finite extension and φ by a power) that X has a rational fixed point under φ, which we take to define the Albanese mapping $\text{alb}\colon X\to \text{Alb}(X)$. Then the action of φ extends so as to make alb a φ-equivariant map.

If the image of alb is 0, this means $H^1_{\text{ét}}(\bar{X},\mathbb{Q}_l)=H^3_{\text{ét}}(\bar{X},\mathbb{Q}_l)=0$, there is nothing to prove.

If the image of alb is a curve C, then φ acts on C, thus on its normalization $\tilde{C}$. Since the genus of $\tilde{C}$ is ≥ 1, the action of φ on $\tilde{C}$, thus on C has finite order. Thus via the surjective pull-back map $\text{alb}^*\colon H^1_{\text{ét}}(\bar{C},\mathbb{Q}_l)\to H^1_{\text{ét}}(\bar{X},\mathbb{Q}_l)$, and its injective push-down dual map $\text{alb}_*\colon H^3_{\text{ét}}(\bar{X},\mathbb{Q}_l)\to H^1_{\text{ét}}(\bar{C},\mathbb{Q}_l)$ the action of φ on $H^i_{\text{ét}}(\bar{X},\mathbb{Q}_l)$, $i=1,3$ is finite as well.

If the image of alb is 2-dimensional, then either X is of general type, in which case φ has finite order and there is nothing to prove, or else X is an abelian surface. In this case, we have a more general Proposition 6.2 below on H^1. But for an abelian surface, $H^i_{\text{ét}}(\bar{X},\mathbb{Q}_l)=H^j_{\text{ét}}(\bar{X}^\vee,\mathbb{Q}_l)$ for $(i,j)=(3,1)$ and $(2,2)$, where $X^\vee$ is the dual abelian surface. Since the eigenvalues of the induced autmorphism $\varphi^\vee$ on $H^2_{\text{ét}}(\bar{X}^\vee,\mathbb{Q}_l)$ are those of φ on $H^2_{\text{ét}}(\bar{X},\mathbb{Q}_l)$, Proposition 6.2 concludes the proof of Theorem 6.1. □

We now show that for automorphisms of abelian varieties, the spectral radius of the induced linear automorphism on H^1 is at most that for the similar linear automorphism of H^2.

Proposition 6.2. *Let X be an abelian variety over a field k, and φ an automorphism of X. Let $\bar{X} = X \otimes_k \bar{k}$ be the corresponding variety over an algebraic closure $\bar{k}$, and let l be a prime invertible in k.*

Then the complex absolute values of the eigenvalues of φ on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ are bounded above by the maximum of the complex absolute values of the eigenvalues of φ on $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$.

Proof. By the standard arguments involving the choice of a model over a finitely generated $\mathbb{Z}$-algebra, and specialization, we reduce to the case when $k = \mathbb{F}_q$ is a finite field. We also fix an embedding $\mathbb{Q}_l \hookrightarrow \mathbb{C}$, so that we may speak of the eigenvalues as complex numbers. Without loss of generality, we may also increase the size of the finite field $\mathbb{F}_q$, replace φ by a power, and replace X by an isogenous abelian variety. Thus, we may write $X = X_1 \times \cdots \times X_r$ where the X_i are powers of mutually non-isogenous absolutely simple abelian varieties, in which case φ must be a product $\varphi_1 \times \cdots \times \varphi_r$ with $\varphi_j \in \mathrm{Aut}(X_j)$. From the Künneth formula, it follows that it suffices to consider the case when $X = X_1$ is a power of an absolutely simple abelian variety. In this case, $\mathrm{End}(X) \otimes \mathbb{Q}$ is a central simple algebra over a number field.

We also make use of the fact that $H^2_{\text{ét}}(\bar{X}, \mathbb{Q}_l) = \bigwedge^2 H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ for an abelian variety. The automorphism φ has eigenvalues on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ which are invertible algebraic integers whose product is 1, and so the maximal absolute value of these eigenvalues is always ≥ 1.

Thus, if we consider the complex absolute values of the eigenvalues of φ on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$, counted with multiplicity, the proposition is clearly true, unless the largest such absolute value is > 1, and appears exactly once, while all the other absolute values are < 1. Since the set of eigenvalues is closed under complex conjugation (as the characteristic polynomial of φ has integer coefficients), this largest absolute value must correspond to a real eigenvalue, which we may take to be positive (replace φ by its square if needed).

In other words, we have to rule out the possibility that φ acting on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ has one real eigenvalue $\lambda > 1$, occuring with multiplicity 1, and all other eigenvalues of complex absolute value < 1 (in particular, λ must be a "Pisot–Vijayaraghavan number"). We do this by induction on the dimension of X. Let $P(t) \in \mathbb{Z}[t]$ be the monic minimal polynomial of φ as an element of $\mathrm{End}(\bar{X})$, and let $f(t) \in \mathbb{Z}[t]$ be the monic minimal polynomial over $\mathbb{Q}$ for the real algebraic integer λ. Then there is a factorization of polynomials $P(t) = f(t)g(t)$, since λ is an eigenvalue for φ, so that $P(\lambda) = 0$. Now λ must be a simple root of $P(t)$, so that $f(t)$, $g(t)$ are relatively prime polynomials in $\mathbb{Q}[t]$. If $g(t)$ is non-constant, then the identity component Y of the subgroup-scheme $\ker f(\varphi) \subset X$ is a subabelian variety of dimension ≥ 1 and $< \dim X$ which is φ-stable and such that $\lambda > 1$ is an eigenvalue of φ on $H^1_{\text{ét}}(\bar{Y}, \mathbb{Q}_l)$. Thus Y has dimension ≥ 2 and we can replace X by Y to show Proposition 6.2, that is we can assume that $g(t)$ is constant, so that $P(t) = f(t)$ (as they are both monic). Further, since λ occurs with multiplicity 1 as an eigenvalue, $P(t)$ is also the characteristic polynomial of φ on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$.

In particular, the subring $L \subset \text{End}(X) \otimes \mathbb{Q}$ generated by φ over $\mathbb{Q}$, is a subfield, isomorphic to $\mathbb{Q}(\lambda)$. We must also have

$$[L : \mathbb{Q}] = \deg P(t) = \dim_{\mathbb{Q}_l} H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l) = 2 \dim X.$$

Thus φ has distinct eigenvalues on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$, and is diagonalizable, and $\mathbb{Q}_l(\varphi)$ is a maximal commutative subring of $\text{End}_{\mathbb{Q}_l}(H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l))$. In particular $L \subset \text{End}(\bar{X})$ is also a maximal commutative subring. Thus the geometric Frobenius $F \in \text{End}(\bar{X})$, which commutes with φ, lies in L, and $F = Q(\varphi)$ for some polynomial $Q(t) \in \mathbb{Q}[t]$. We conclude that F has the eigenvalue $Q(\lambda) \in \mathbb{R}$ on $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$. This means, assuming, as we may, that q is an even power of p, that F has an integer eigenvalue. Since X is a power of an absolutely simple abelian variety, Tate's theorems imply that the minimal polynomial of F in $\text{End}(\bar{X}) \otimes \mathbb{Q}$ is irreducible, and so, having an integer root, must be a linear polynomial. This forces X to be isomorphic to a power of a supersingular elliptic curve, say $X \cong E^n$.

Now $\text{End}(\bar{X}) \otimes \mathbb{Q} \cong M_n(D)$, where $D = \text{End}(\bar{E}) \otimes \mathbb{Q}$ is the unique quaternion division algebra over $\mathbb{Q}$ which splits at all places apart from p and ∞. Since $L \subset M_n(D)$ is a maximal commutative subfield of the central simple algebra $M_n(D)$, we know that L is a splitting field for the algebra, i.e., $M_n(D) \otimes_{\mathbb{Q}} L \cong \text{End}_L(M_n(D)) \cong M_{2n}(L)$ as central simple algebras over L (where D is regarded as an L-vector space through right multiplication; the isomorphism is given by $M_n(D) \otimes L \ni a \otimes b \mapsto (x \mapsto axb) \in \text{End}_L(M_n(D))$). (We thank M.S. Raghunathan for a discussion on this point.). Since L has a real embedding, we conclude that $M_n(D) \otimes_{\mathbb{Q}} \mathbb{R} \cong M_{2n}(\mathbb{R})$, which contradicts that D is non-split at ∞. This concludes the proof. $\square$

6.3. Algebraic entropy. In general, if X is a smooth proper variety over a field k, and φ an algebraic automorphism of X, then we may associate to it two numerical invariants, as follows.

(1) Let l be a prime invertible in k, and let $\bar{X} = X \times_k \bar{k}$ be the corresponding (smooth, proper) variety over an algebraic closure $\bar{k}$. The characteristic polynomial of φ on $H^\bullet_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ is independent of l, and has integer coefficients, and algebraic integer roots (which are units); hence we may define the spectral radius of φ on $H^\bullet_{\text{ét}}(\bar{X}, \mathbb{Q}_l)$ as a real number ≥ 1, and define its natural logarithm to be the *entropy* of φ. When $k \subset \mathbb{C}$, so that we may associate to (X, φ) a compact complex manifold $X_{\mathbb{C}}$, and a holomorphic automorphism $\varphi_{\mathbb{C}}$, then our definition agrees with the usual one (given above) for $\varphi_{\mathbb{C}}$.

(2) We may instead define an invariant using algebraic cycles, as follows. Let $\bar{X}$ be as above, and $CH^\bullet_{\text{num}}(\bar{X})$ the ring of algebraic cycles on $\bar{X}$ modulo numerical equivalence. Then φ yields an automorphism of the ring $CH^\bullet_{\text{num}}(\bar{X})$, whose underlying abelian group

is known to be a finitely generated free abelian group; thus the characteristic polynomial of φ on this ring has integer coefficients, and eigenvalues which are algebraic integer units. We may now define the *algebraic entropy* of φ to be the natural logarithm of the spectral radius of φ acting on $CH^{\bullet}_{\mathrm{num}}(\bar{X})$.

Our main result, Theorem 1.1, and its Corollary 1.2, with Theorem 6.1, imply that for automorphisms of smooth projective algebraic surfaces, *the algebraic entropy coincides with the entropy*. One may ask whether this is true in arbitrary dimension. It would in particular imply that the value of the entropy on the whole l-adic cohomology is taken on even degree cohomology, which is true in characterisitc 0 (see Section 6.2).

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Hélène Esnault
Universität Duisburg-Essen
Mathematik, 45117 Essen
Germany
e-mail: esnault@uni-due.de

Vasudevan Srinivas
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba, Mumbai-400005
India
e-mail: srinivas@math.tifr.res.in