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INVARIANT STABLY COMPLEX STRUCTURES ON TOPOLOGICAL TORIC MANIFOLDS

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Abstract

We show that any $(\mathbb{C}^*)^n$-invariant stably complex structure on a topological toric manifold of dimension $2n$ is integrable. We also show that such a manifold is weakly $(\mathbb{C}^*)^n$-equivariantly isomorphic to a toric manifold.

1. Introduction

A *toric manifold* is a nonsingular complete toric variety. As a topological analogue of a toric manifold, the notion of *topological toric manifold* has been introduced by the author, Y. Fukukawa and M. Masuda [2]. A topological toric manifold of dimension $2n$ is a smooth closed manifold endowed with an effective $(\mathbb{C}^*)^n$-action having an open dense orbit, and locally equivariantly diffeomorphic to a *smooth* representation space of $(\mathbb{C}^*)^n$. We note that a topological toric manifold is locally equivariantly diffeomorphic to an *algebraic* representation space if and only if it is a toric manifold.

A *quasitoric manifold* introduced by M. Davis and T. Januskiewicz [1] of dimension $2n$ is a smooth closed manifold endowed with a locally standard $(S^1)^n$-action, whose orbit space is a simple polytope. In [2], it is shown that any quasitoric manifold is a topological toric manifold with the restricted compact torus action. Conversely, it is also shown that any topological toric manifold of dimension less than or equal to 6 with the restricted compact torus action is a quasitoric manifold. However, there are infinitely many topological toric manifolds with the restricted compact torus action which are not equivariantly diffeomorphic to any quasitoric manifold.

Among quasitoric manifolds, some admit invariant almost complex structures under the compact torus actions. M. Masuda provided examples of 4-dimensional quasitoric manifolds which admit $(S^1)^2$-invariant almost complex structures (see [4, Theorem 5.1]). A. Kustarev described a necessary and sufficient condition for a quasitoric manifold to admit a torus invariant almost complex structure for arbitrary dimension (see [3, Theorem 1]).

As we mentioned, any quasitoric manifold is a topological toric manifold with the restricted compact torus action. In this paper, we discuss on $(\mathbb{C}^*)^n$-invariant stably, or

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almost complex structures on topological toric manifolds of dimension $2n$. The followings are our results:

Theorem 1.1. *Let X be a topological toric manifold of dimension $2n$. Let $\underline{\mathbb{R}}^{2l}$ be the product bundle of rank $2l$ over X, TX the tangent bundle of X. If there exists a $(\mathbb{C}^*)^n$-invariant stably complex structure J on $TX \oplus \underline{\mathbb{R}}^{2l}$, then TX becomes a complex subbundle of $TX \oplus \underline{\mathbb{R}}^{2l}$. Namely, X has an invariant almost complex structure.*

Theorem 1.2. *Let X be a topological toric manifold of dimension $2n$, J a $(\mathbb{C}^*)^n$-invariant almost complex structure. Then, J is integrable and X is weakly equivariantly isomorphic to a toric manifold. Namely, there are a toric manifold Y, a biholomorphism $f\colon X \to Y$ and a smooth automorphism ρ of $(\mathbb{C}^*)^n$ such that $f \circ g = \rho(g) \cdot f$ for all $g \in (\mathbb{C}^*)^n$.*

If we replace the condition "$(\mathbb{C}^*)^n$-invariant" by "$(S^1)^n$-invariant" on the almost complex structure J, then Theorem 1.2 does not hold. For example, $\mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$ with an effective $(S^1)^2$-action is a topological toric manifold with the restricted $(S^1)^2$-action. One can show that there exists an $(S^1)^2$-invariant almost complex structure on $\mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$ (see [4, Theorem 5.1]). However, $\mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$ carries no complex structure because $\mathbb{C}P^2 \# \mathbb{C}P^2 \# \mathbb{C}P^2$ does not fit Kodaira's classification of complex surfaces. Namely, the almost complex structure is not integrable.

For a topological toric manifold X of dimension $2n$, there is a canonical short exact sequence of $(\mathbb{C}^*)^n$-bundles

$$0 \to \underline{\mathbb{C}}^{m-n} \to \bigoplus_{i=1}^{m} L_i \to TX \to 0,$$

where L_i's are complex line bundles. (see [2, Theorem 6.1]). Theorems 1.1 and 1.2 say that the short exact sequence above does not split as $(\mathbb{C}^*)^n$-bundles unless X is a toric manifold.

2. Preliminaries

In this section, we review the quotient construction of topological toric manifolds and the correspondence between topological toric manifolds and nonsingular complete topological fans (see [2] for details).

A *nonsingular complete topological fan* is a pair $\Delta = (\Sigma, \beta)$ such that
(1) Σ is an abstract simplicial complex on $[m] = \{1, \ldots, m\}$,
(2) $\beta\colon [m] \to (\mathbb{C} \times \mathbb{Z})^n$ is a function which satisfies the following:

(a) Let Re be the composition of two natural projections $(\mathbb{C} \times \mathbb{Z})^n \to \mathbb{C}^n$ and $\mathbb{C}^n \to \mathbb{R}^n$. We assign a cone

$$\left\{ \sum_{i \in I} a_i (\mathrm{Re} \circ \beta)(i) \;\middle|\; a_i \geq 0 \right\}$$

to each simplex I in Σ. Then, we have a collection of cones in $\mathbb{R}^n$.

(i) Each pair of two cones does not overlap on their relative interiors. Namely, the real part $\mathrm{Re} \circ \beta$ of β together with Σ forms an ordinary fan.

(ii) The union of all cones coincides with $\mathbb{R}^n$. Namely, the fan is complete.

(b) The integer part of β together with Σ forms a nonsingular multi-fan (see [4, p. 249]).

It follows from (2a) that Σ must be a simplicial $(n-1)$-sphere with m vertices. If we regard integers $\mathbb{Z}$ as a subset of $\mathbb{C} \times \mathbb{Z}$ via $a \mapsto (a, a)$ for $a \in \mathbb{Z}$, then any nonsingular complete fan can be regarded as a special case of a nonsingular complete topological fan. Conversely, if the image of β is contained in the diagonal subgroup $\mathbb{Z}^n$ of $(\mathbb{C} \times \mathbb{Z})^n$, then Δ becomes a nonsingular complete fan.

We express $\beta(i)$ as $\beta_i = (\beta_i^1, \dots, \beta_i^n) \in (\mathbb{C} \times \mathbb{Z})^n$ and $\beta_i^j = (b_i^j + \sqrt{-1} c_i^j, v_i^j) \in \mathbb{C} \times \mathbb{Z}$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. For a nonsingular complete topological fan $\Delta = (\Sigma, \beta)$, we can construct a topological toric manifold as follows. We set

$$U(I) := \{(z_1, \dots, z_m) \in \mathbb{C}^m \mid z_i \neq 0 \text{ for } i \notin I\}$$

for $I \in [m]$, and

$$U(\Sigma) := \bigcup_{I \in \Sigma} U(I).$$

We define a group homomorphism $\lambda_{\beta_i} \colon \mathbb{C}^* \to (\mathbb{C}^*)^n$ by

$$\lambda_{\beta_i}(h_i) := (h_i^{\beta_i^1}, \dots, h_i^{\beta_i^n}), \tag{2.1}$$

where

$$h_i^{\beta_i^j} := |h_i|^{b_i^j + \sqrt{-1} c_i^j} \left(\frac{h_i}{|h_i|} \right)^{v_i^j}. \tag{2.2}$$

We define a group homomorphism $\lambda \colon (\mathbb{C}^*)^m \to (\mathbb{C}^*)^n$ by the component-wise multiplication

$$\lambda(h_1, \dots, h_m) := \prod_{i=1}^m \lambda_{\beta_i}(h_i).$$

Then, the homomorphism λ is a surjective map. To see this, we consider the polar coordinate of $\mathbb{C}^* \cong \mathbb{R}_{>0} \times S^1$ and the matrix representation of the differential of λ

at the unit of $(\mathbb{C}^*)^m$. The matrix representation of the differential of λ at the unit is written as

$$\begin{pmatrix} b_1^1 & b_2^1 & \cdots & b_m^1 & 0 & 0 & \cdots & 0 \\ b_1^2 & b_2^2 & \cdots & b_m^2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1^n & b_2^n & \cdots & b_m^n & 0 & 0 & \cdots & 0 \\ c_1^1 & c_2^1 & \cdots & c_m^1 & v_1^1 & v_2^1 & \cdots & v_m^1 \\ c_1^2 & c_2^2 & \cdots & c_m^2 & v_1^2 & v_2^2 & \cdots & v_m^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1^n & c_2^n & \cdots & c_m^n & v_1^n & v_2^n & \cdots & v_m^n \end{pmatrix}.$$

It follows from the assumptions on Δ that the square matrices $(b_i^j)_{i\in I,\, j=1,\dots,n}$ and $(v_i^j)_{i\in I,\, j=1,\dots,n}$ are nonsingular for any $(n-1)$-dimensional simplex $I \in \Sigma$. This implies that the matrix above is of full-rank, and hence λ is a submersion. Since $(\mathbb{C}^*)^n$ and $(\mathbb{C}^*)^m$ are connected and commutative, it follows that λ is a surjective homomorphism.

We note that the $(\mathbb{C}^*)^m$-action on $U(\Sigma)$ given by coordinatewise multiplications induces the action of $(\mathbb{C}^*)^m/\ker\lambda$ on the quotient space $X(\Delta) := U(\Sigma)/\ker\lambda$. Since λ is surjective, we can identify $(\mathbb{C}^*)^m/\ker\lambda$ with $(\mathbb{C}^*)^n$ through λ. Hence $X(\Delta)$ is equipped with the $(\mathbb{C}^*)^n$-action. One can show that $X(\Delta)$ is a topological toric manifold (see [2, Corollary 6.3]).

We shall remember the equivariant charts and transition functions of $X(\Delta)$ described in [2] for later use. We set

$$\mathcal{R} := \left\{ \begin{pmatrix} b & 0 \\ c & v \end{pmatrix} \;\middle|\; b, c \in \mathbb{R},\ v \in \mathbb{Z} \right\}.$$

We regard $\beta_i^j = (b_i^j + \sqrt{-1}c_i^j, v_i^j)$ as the following matrix:

$$\beta_i^j = \begin{pmatrix} b_i^j & 0 \\ c_i^j & v_i^j \end{pmatrix} \in \mathcal{R}.$$

And we also regard β_i as an n-tuple $(\beta_i^1, \dots, \beta_i^n)$ of elements in $\mathcal{R}$. Let $\Sigma^{(n)}$ denote the set of $(n-1)$-dimensional simplices in Σ. For $I \in \Sigma^{(n)}$, the dual $\{\alpha_i^I\}_{i\in I}$ of $\{\beta_i\}_{i\in I}$ is defined to be

$$\langle \alpha_h^I, \beta_i \rangle = \delta_h^i \mathbf{1}, \tag{2.3}$$

where δ denotes the the Kronecker delta, and $\langle\ ,\ \rangle$ is given by

$$\langle \alpha, \beta \rangle = \sum_{j=1}^{n} \alpha^j \beta^j$$

for n-tuples $\alpha = (\alpha^1, \dots, \alpha^n) \in \mathcal{R}^n$ and $\beta = (\beta^1, \dots, \beta^n) \in \mathcal{R}^n$. The dual $\{\alpha_i^I\}_{i\in I}$ of $\{\beta_i\}_{i\in I}$ exists for all $I \in \Sigma^{(n)}$ (see [2, Lemma 2.4]). The equivariant charts are given as follows. For $\alpha = (\alpha^1, \dots, \alpha^n) \in \mathcal{R}^n$, we define a representation $\chi^\alpha \colon (\mathbb{C}^*)^n \to \mathbb{C}^*$ by

$$\chi^\alpha(g_1, \dots, g_n) := \prod_{j=1}^n g_j^{\alpha^j},$$

where we regard each $\alpha^j \in \mathcal{R}$ as an element in $\mathbb{C} \times \mathbb{Z}$ as well as β_i^j. Let $V(\chi^\alpha)$ denote the representation space of χ^α. For $I \in \Sigma^{(n)}$, the equivariant chart $\varphi_I \colon U(I)/\ker\lambda \to \bigoplus_{i\in I} V(\chi^{\alpha_i^I})$ is defined by

$$\varphi_I([z_1, \dots, z_m]) := \left(\prod_{j=1}^m z_j^{\langle \alpha_i^I, \beta_j\rangle}\right)_{i\in I},$$

where $[z_1, \dots, z_m]$ denotes the equivalence class of $(z_1, \dots, z_m) \in U(\Sigma)$. The collection $\left\{\varphi_I \colon U(I)/\ker\lambda \to \bigoplus_{i\in I} V(\chi^{\alpha_i^I})\right\}_{I\in\Sigma^{(n)}}$ is an equivariant coordinate system of $X(\Delta)$. The i-th component of φ_I is given as

$$w_i := \prod_{j=1}^m z_j^{\langle \alpha_i^I, \beta_j\rangle}.$$

An *omniorientation* of a topological toric manifold X is a choice of orientations of normal bundles of *characteristic* submanifolds of X. Here, a characteristic submanifold of X is a connected $(\mathbb{C}^*)^n$-invariant submanifold of codimension 2. Since the action of $(\mathbb{C}^*)^n$ on X locally looks like a smooth faithful representation of $(\mathbb{C}^*)^n$, characteristic submanifold is point-wise fixed by a $\mathbb{C}^*$-subgroup of $(\mathbb{C}^*)^n$. By the construction of $X(\Delta)$, β allows us to decide an omniorientation of $X(\Delta)$ as follows. Let $q \colon U(\Sigma) \to X(\Delta) = U(\Sigma)/\ker\lambda$ be the quotient map. The preimage of a characteristic submanifold of $X(\Delta)$ by q is a $(\mathbb{C}^*)^m$-invariant submanifold of codimension 2. Hence there are m characteristic submanifolds

$$X_i := \{[z_1, \dots, z_m] \in X(\Delta) \mid z_i = 0\}, \quad i = 1, \dots, m, \tag{2.4}$$

where $[z_1, \dots, z_m]$ denotes the equivalence class of $(z_1, \dots, z_m) \in U(\Sigma)$. It is easy to see that each characteristic submanifold X_i is point-wise fixed by the $\mathbb{C}^*$-subgroup $\lambda_{\beta_i}(\mathbb{C}^*)$ of $(\mathbb{C}^*)^n$. We choose the orientation of the normal bundle of X_i so that $(\xi, (\lambda_{\beta_i}(\sqrt{-1}))_*(\xi))$ is a positive basis, where ξ is a nonzero normal vector at a point in X_i and $(\lambda_{\beta_i}(\sqrt{-1}))_*$ is the differential of the action $\lambda_{\beta_i}(\sqrt{-1})$.

The correspondence $\Delta \mapsto X(\Delta)$ is bijective between nonsingular complete topological fans and omnioriented topological toric manifolds (see [2, Theorem 8.1]). We shall see the inverse correspondence. For a topological toric manifold X of dimension $2n$ with an omniorientation, let us denote characteristic submanifolds of X by $X_1, \dots, X_m$. Define

$$\Sigma = \left\{ I \in [m] \;\middle|\; \bigcap_{i \in I} X_i \neq \emptyset \right\}.$$

For an orientation on normal bundle of X_i, we can find a unique complex structure J_i such that

- the orientation coincides with the orientation which comes from J_i,
- the $\mathbb{C}^*$-subgroup of $(\mathbb{C}^*)^n$ which fixes each point of X_i acts on the normal bundle as $\mathbb{C}$-linear with respect to J_i.

For J_i, we can find a unique $\beta_i \in (\mathbb{C} \times \mathbb{Z})^n$ such that

$$(\lambda_{\beta_i}(h))_*(\xi) = h\xi$$

for any normal vector ξ and $h \in \mathbb{C}^*$, where the right hand side is the multiplication with complex number. For an omniorientation of X, define $\beta \colon [m] \to (\mathbb{C} \times \mathbb{Z})^n$ as $\beta(i) := \beta_i$. Then, the pair $\Delta(X) = (\Sigma, \beta)$ becomes a nonsingular complete topological fan and the correspondence $X \mapsto \Delta(X)$ is the inverse correspondence of $\Delta \mapsto X(\Delta)$. Namely, there exists an equivariant diffeomorphism $X \to X(\Delta(X))$ which preserves the omniorientations.

The transition functions of $X(\Delta)$ are given as follows. Let K be another element in $\Sigma^{(n)}$. By direct computation, k-component of $\varphi_K(\varphi_I^{-1}(w_i)_{i \in I})$ for $k \in K$ is given as

$$\prod_{i \in I} w_i^{\langle \alpha_k^K, \beta_i \rangle} \tag{2.5}$$

(see [2, Lemma 5.2]). We remark that

$$\frac{\partial}{\partial \bar{w}_j} \left(\prod_{i \in I} w_i^{\langle \alpha_k^K, \beta_i \rangle} \right) = 0$$

if and only if $\langle \alpha_k^K, \beta_j \rangle = \gamma_{k,j}^K \mathbf{1}$ for some integer $\gamma_{k,j}^K$ (see (2.2)). This implies that all transition functions are holomorphic if and only if there is an integer $\gamma_{k,j}^K$ such that $\langle \alpha_k^K, \beta_i \rangle = \gamma_{k,j}^K \mathbf{1}$ for all $i \in [m]$, $k \in K$ and $K \in \Sigma^{(n)}$. In this case, each transition function is a Laurent monomial and hence $X(\Delta)$ is weakly equivariantly diffeomorphic to a toric manifold.

3. Proof of Theorem 1.1

Let X be a $2n$-dimensional topological toric manifold, TX the tangent bundle of X, J a $(\mathbb{C}^*)^n$-invariant complex structure on $TX \oplus \underline{\mathbb{R}}^{2l}$. We take an omniorientation of

X and consider the topological fan $\Delta = (\Sigma, \beta)$ associated to $X = X(\Delta)$ with the given omniorientation. We define a cross section $\underline{e}_h \colon X \to \underline{\mathbb{R}}^{2l} = X \times \mathbb{R}^{2l}$ for $h = 1, \dots, 2l$ by $x \mapsto (x, e_h)$ for all $x \in X$, where e_h denotes the h-th standard basis vector of $\mathbb{R}^{2l}$. We will compute the matrix representation of the complex structure J on $TX \oplus \underline{\mathbb{R}}^{2l}$ with respect to the local coordinates. And we will see that the vector subbundle TX of $TX \oplus \underline{\mathbb{R}}^{2l}$ is stable under J. There is a natural inclusion $(\mathbb{C}^*)^n \hookrightarrow X$ given by $g \mapsto g \cdot [1, \dots, 1]$ where $[1, \dots, 1]$ denotes the equivalence class of $(1, \dots, 1)$ in $U(\Sigma)$. For $I \in \Sigma^{(n)}$, the inclusion is of the form

$$\bigoplus_{i \in I} \chi^{\alpha_i^I} \colon g = (g_j)_{j=1,\dots,n} \mapsto (\chi^{\alpha_i^I}(g))_{i \in I} \tag{3.1}$$

via the equivariant local chart $\varphi_I \colon U_I/\ker\lambda \to \bigoplus_{i \in I} V(\chi^{\alpha_i^I})$. We identify $\bigoplus_{i \in I} V(\chi^{\alpha_i^I})$ with $\mathbb{R}^{2n}$ by

$$w_i = x_i + \sqrt{-1} y_i \tag{3.2}$$

for $i \in I$, where $(w_i)_{i \in I}$ denote the coordinates of $\bigoplus_{i \in I} V(\chi^{\alpha_i^I})$. We also identify $(\mathbb{C}^*)^n$ with $(\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})^n$ by

$$\psi \colon (g_j)_{j=1,\dots,n} \mapsto \left(\log|g_j|, -\sqrt{-1} \log\left(\frac{g_j}{|g_j|} \right) \right)_{j=1,\dots,n}. \tag{3.3}$$

Let $(\tau_j, \theta_j)_{j=1,\dots,n}$ be the coordinates of $(\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})^n$. Since J is $(\mathbb{C}^*)^n$-invariant, the matrix representation, denoted J_0, of J on $(\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})^n$ with respect to the coordinates $(\tau_j, \theta_j)_{j=1,\dots,n}$ and sections $\underline{e}_h$'s is constant.

Let $\Psi_I \colon (\mathbb{R}^2 \setminus \{0\})^n \to (\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})^n$ be the composition of the identification (3.2), the inverse of (3.1), and ψ. Namely,

$$\Psi_I((x_i, y_i)_{i \in I}) := \psi \circ \left(\bigoplus_{i \in I} \chi^{\alpha_i^I} \right)^{-1} ((x_i + \sqrt{-1} y_i)_{i \in I}). \tag{3.4}$$

Since $\left(\bigoplus_{i \in I} \chi^{\alpha_i^I}\right)^{-1}$ coincides with $\prod_{i \in I} \lambda_{\beta_i}$ (see [2, Lemma 2.3]), it follows from (2.1) and (2.2) that the coordinates $(\tau_j, \theta_j)_{j=1,\dots,n}$ are represented as

$$\begin{aligned} \tau_j &= \log\left(\prod_{i \in I} |(x_i + \sqrt{-1} y_i)^{\beta_i^j}| \right) \\ &= \frac{1}{2} \sum_{i \in I} b_i^j \log(x_i^2 + y_i^2) \end{aligned}$$

and

$$
\begin{aligned}
\theta_j &= -\sqrt{-1}\log\left(\prod_{i\in I}\frac{(x_i+\sqrt{-1}y_i)^{\beta_i^j}}{|(x_i+\sqrt{-1}y_i)^{\beta_i^j}|}\right)\\
&= -\sqrt{-1}\log\left(\prod_{i\in I}|x_i+\sqrt{-1}y_i|^{\sqrt{-1}c_i^j}\left(\frac{x_i+\sqrt{-1}y_i}{|x_i+\sqrt{-1}y_i|}\right)^{v_i^j}\right)\\
&= \sum_{i\in I}\left(\frac{c_i^j+\sqrt{-1}v_i^j}{2}\log(x_i^2+y_i^2)-\sqrt{-1}v_i^j\log(x_i+\sqrt{-1}y_i)\right).
\end{aligned}
$$

Then, by direct computation, we have

$$
\begin{aligned}
\frac{\partial\tau_j}{\partial x_i} &= \frac{b_i^j x_i}{x_i^2+y_i^2},\quad \frac{\partial\tau_j}{\partial y_i}=\frac{b_i^j y_i}{x_i^2+y_i^2},\\
\frac{\partial\theta_j}{\partial x_i} &= \frac{(c_i^j+\sqrt{-1}v_i^j)x_i}{x_i^2+y_i^2}-\sqrt{-1}v_i^j\frac{1}{x_i+\sqrt{-1}y_i}\\
&= \frac{c_i^j x_i - v_i^j y_i}{x_i^2+y_i^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial\theta_j}{\partial y_i} &= \frac{(c_i^j+\sqrt{-1}v_i^j)y_i}{x_i^2+y_i^2}+v_i^j\frac{1}{x_i+\sqrt{-1}y_i}\\
&= \frac{c_i^j y_i + v_i^j x_i}{x_i^2+y_i^2}.
\end{aligned}
$$

Therefore,

$$
\begin{pmatrix} \dfrac{\partial\tau_j}{\partial x_i} & \dfrac{\partial\tau_j}{\partial y_i}\\[2ex] \dfrac{\partial\theta_j}{\partial x_i} & \dfrac{\partial\theta_j}{\partial y_i}\end{pmatrix}
=\begin{pmatrix} b_i^j & 0\\ c_i^j & v_i^j\end{pmatrix}
\begin{pmatrix} \dfrac{x_i}{x_i^2+y_i^2} & \dfrac{y_i}{x_i^2+y_i^2}\\[2ex] \dfrac{-y_i}{x_i^2+y_i^2} & \dfrac{x_i}{x_i^2+y_i^2}\end{pmatrix}
=\beta_i^j t_i,
$$

where

$$
t_i = \frac{1}{x_i^2+y_i^2}\begin{pmatrix} x_i & y_i\\ -y_i & x_i\end{pmatrix}\in \mathrm{GL}(2,\mathbb{R}).
$$

We set two square matrices

$$
B=(\beta_i^j)_{j=1,\dots,n,\ i\in I}\quad\text{and}\quad T=\mathrm{diag}(t_i;\, i\in I)
$$

of size n whose entries are square matrices of size 2. Then, the differential $T_{(x,y)}\Psi_I$ of Ψ_I at (x,y) is represented as BT with respect to the coordinates $(x_i,y_i)_{i\in I}$ and

$(\tau_j,\ \theta_j)_{j=1,\dots,n}$. Hence the complex structure J of $TX \oplus \underline{\mathbb{R}}^{2l}$ is represented on $\bigoplus_{i\in I} V(\chi^{\alpha_i^I}) = \mathbb{R}^{2n}$ as the following square matrix

$$J_I := \begin{pmatrix} (BT)^{-1} & \\ & I_{2l} \end{pmatrix} J_0 \begin{pmatrix} BT & \\ & I_{2l} \end{pmatrix} \tag{3.5}$$

of size $2n + 2l$ with respect to the coordinates $(x_i, y_i)_{i\in I}$ and sections $\underline{e}_h$, where I_{2l} denote the identity matrix of size $2l$. We set

$$J_0 =: \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

where $J_{11}, J_{12}, J_{21}, J_{22}$ are matrices of $2n \times 2n$, $2n \times 2l$, $2l \times 2n$, $2l \times 2l$, respectively. Then,

$$J_I = \begin{pmatrix} T^{-1}(B^{-1}J_{11}B)T & T^{-1}B^{-1}J_{12} \\ J_{21}BT & J_{22} \end{pmatrix}. \tag{3.6}$$

Since J is a smooth cross section of the vector bundle $(TX \oplus \underline{\mathbb{R}}^{2l}) \oplus (TX \oplus \underline{\mathbb{R}}^{2l})^* \to X$ where $(TX \oplus \underline{\mathbb{R}}^{2l})^*$ is the dual vector bundle of $TX \oplus \underline{\mathbb{R}}^{2l}$, each entry of J_I must be a smooth function on $\mathbb{R}^{2n}$, in particular, at the origin. By the definitions of B and T, each entry of $J_{21}BT$ is a linear combination of $x_i/(x_i^2 + y_i^2)$ and $y_i/(x_i^2 + y_i^2)$, $i = 1, \dots, n$. Hence each entry of $J_{21}BT$ must be 0. Otherwise J_I can not be defined at the origin. It follows from $J_{21}BT = 0$ that the tangent space at any point of X is stable under J. Thus, TX is a complex subbundle of $TX \oplus \underline{\mathbb{R}}^{2l}$ with respect to J. The theorem is proved. □

4. Proof of Theorem 1.2

Let X be a topological toric manifold of dimension $2n$ with a $(\mathbb{C}^*)^n$-invariant almost complex structure J. Then, each characteristic submanifold of X becomes an almost complex submanifold. In fact, a characteristic submanifold X_i is a connected component of the fixed points of a $\mathbb{C}^*$-subgroup G_i of $(\mathbb{C}^*)^n$. The tangent space T_xX at a point $x \in X_i$ of X is a complex representation space of the $\mathbb{C}^*$-subgroup. The vector subspace of T_xX fixed by G_i coincides with the tangent space T_xX_i at the point $x \in X_i$ of the characteristic submanifold X_i. Thus T_xX_i is a complex subspace of T_xX with respect to J.

Since any characteristic submanifold of X and X itself are almost complex submanifolds, the normal bundles of characteristic submanifolds of X become complex line bundles. Hence, we have a topological fan $\Delta = (\Sigma, \beta)$ associated to X. Namely, for each characteristic submanifold X_i of X, we choose the unique $\beta_i \in (\mathbb{C} \times \mathbb{Z})^n$ so that $\lambda_{\beta_i}(\mathbb{C}^*)$ fixes all points in X_i, and

$$(\lambda_{\beta_i}(h))_*(\xi) = h\xi$$

for any $h \in \mathbb{C}^*$ and any normal vector ξ of X_i.

According to the proof of Theorem 1.1, we identify the dense orbit with $(\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})^n$ and $\bigoplus_{i\in I} V(\chi^{\alpha_i^I})$ with $\mathbb{R}^{2n}$ for $I \in \Sigma^{(n)}$. Since J is $(\mathbb{C}^*)^n$-invariant, the matrix representation, denoted J_0, of J on $(\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})^n$ with respect to the coordinate $(\tau_j, \theta_j)_{j=1,\dots,n}$ of $(\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})^n$ is constant. Let us remind $\Psi_I\colon (\mathbb{R}^2 \setminus \{0\})^n \to (\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z})^n$ (see (3.4)). Then, the almost complex structure J is represented on $\bigoplus_{i\in I} V(\alpha_i^I) = \mathbb{R}^{2n}$ as the following square matrix

$$J_I := (BT)^{-1}J_0(BT) = T^{-1}(B^{-1}J_0B)T \tag{4.1}$$

(this is the case when $l = 0$ in (3.5) and (3.6)). Since B and J_0 are constant, each entry of J_I is a linear combination of $x_hx_i/(x_i^2+y_i^2)$, $x_hy_i/(x_i^2+y_i^2)$ and $y_hy_i/(x_i^2+y_i^2)$, $h, i \in I$.

Lemma. *Let g and h be homogeneous polynomial functions on $\mathbb{R}^n$. Assume that g and h have the same degrees. Then, the rational function $f = g/h$ is a smooth function on $\mathbb{R}^n$ if and only if f is constant.*

Proof. The "if" part is obvious. We shall show the "only if" part. Let $l\colon \mathbb{R} \to \mathbb{R}^n$ be any linear map. If f is a smooth function on $\mathbb{R}^n$, the composition $f \circ l$ is also a smooth function on $\mathbb{R}$. Moreover, the composition $f \circ l\colon \mathbb{R} \to \mathbb{R}$ is also a homogeneous polynomial function. Thus, $f \circ l$ is a constant function. Since $f \circ l$ is constant for any l, it follows that all partial derivatives of f at the origin vanish. Therefore f is constant. The lemma is proved. $\square$

It follows from the lemma above that J_I must be constant. We will think of $\mathbb{C}^*$ as a subgroup of $\mathrm{GL}(2, \mathbb{R})$ via the injective homomorphism defined by

$$a + \sqrt{-1}b \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad a, b \in \mathbb{R} \text{ and } a^2 + b^2 \neq 0.$$

Accordingly, we will think of $(\mathbb{C}^*)^n$ as a subgroup of $\mathrm{GL}(2n, \mathbb{R})$, that is, each element in $(\mathbb{C}^*)^n$ will be regarded as a square matrix of size $2n$ whose entries are real numbers. Since $J_I = T^{-1}(B^{-1}J_0B)T$ by (4.1), $T^{-1}(B^{-1}J_0B)T$ is also constant. Since T can take any element, in particular the unit, in $(\mathbb{C}^*)^n$, $J_I = B^{-1}J_0B$. So $T^{-1}J_IT = J_I$, that is, J_I and T commute. Since J_I and T commute and T can take any element in $(\mathbb{C}^*)^n \subset \mathrm{GL}(2n, \mathbb{R})$, J_I should be a matrix of the form

$$J_I = \mathrm{diag}(J_i; i \in I) \tag{4.2}$$

where J_i is a square matrix of size 2 and of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathbb{C}^* \subset \mathrm{GL}(2, \mathbb{R}), \quad a, b \in \mathbb{R}.$$

Moreover, J_I^2 is the minus identity matrix because J_I is the matrix representing the almost complex structure J. It follows from $J_I^2 = -1$ and $J_I \in (\mathbb{C}^*)^n$ that J_I must be of the form

$$\begin{pmatrix} s_{i_1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & s_{i_n}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix},$$

where $s_{i_1}, \ldots, s_{i_n} = \pm 1$ and $\{i_1, \ldots, i_n\} = I$. We recall that each β_i is taken so that $(\xi, (\lambda_{\beta_i}(\sqrt{-1}))_*(\xi))$ is a positive basis for any nonzero normal vector ξ of X_i. We will see that $s_{i_1}, \ldots, s_{i_n}$ are equal to 1 from the choice of $\beta_{i_1}, \ldots \beta_{i_n}$. It follows from (2.4) and the definition of φ_I that the characteristic submanifold X_i for $i \in I$ is represented as the set

$$\left\{ (w_h)_{h\in I} \in \bigoplus_{h\in I} V(\chi^{\alpha_h^I}) \;\middle|\; w_i = 0 \right\}$$

on $U(I)/\ker\lambda$. For any point $p \in X_i \cap U(I)/\ker\lambda$,

$$\left(\left(\frac{\partial}{\partial x_i}\right)_p, s_i \left(\frac{\partial}{\partial y_i}\right)_p \right)$$

is a positive basis of the normal vector space at $p \in X_i$ because we chose the orientation of the normal bundle of X_i to be compatible with the almost complex structure J. However, it follows from a direct computation,

$$(\lambda_{\beta_i}(\sqrt{-1}))_* \left(\left(\frac{\partial}{\partial x_i}\right)_p \right) = \left(\frac{\partial}{\partial y_i}\right)_p.$$

Thus, we have $s_i = 1$ for all $i \in I$ and hence we have

$$J_I = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}.$$

Clearly, the complex structure J_I on $\mathbb{R}^{2n}$ comes from the identification (3.2). Therefore, J is integrable and the local chart $\varphi_I \colon U_I/\ker\lambda \to \bigoplus_{i\in I} V(\chi^{\alpha_i^I})$ is a holomorphic chart for all $I \in \Sigma^{(n)}$. This implies that for another simplex $K \in \Sigma^{(n)}$, k-component of $\varphi_K(\varphi_I^{-1}(w_i)_{i \in I})$ given as (2.5) for $k \in K$ must be holomorphic. Thus, the transition functions must be Laurent monomials as remarked at the end of Section 2 and hence $X(\Delta)$ is weakly equivariantly isomorphic to a toric manifold. The theorem is proved. $\square$

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