POINTWISE ESTIMATES FOR GROUND STATES OF SOME CLASSES OF POSITIVITY PRESERVING OPERATORS

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Abstract

We establish pointwise estimates for the ground states of some classes of positivity preserving operators. The considered operators are negatively perturbed (by measures) strongly local Dirichlet operators. These estimates will be written in terms of the Green's kernel of the considered operators, whose existence will be proved. In many circumstances our estimates are even sharp so that they recover known results about the subject. The results will deserve to obtain large time heat kernel estimates for the related operators.

1. Introduction

Let Ω be an open smooth connected and bounded subset of the Euclidean space $\mathbb{R}^d$ and $(-\Delta_\Omega)$ be the Dirichlet–Laplacian on Ω. It is well known that the ground state of $(-\Delta_\Omega)$, which we denote by φ_0, enjoys the property of being comparable to the function $(-\Delta_\Omega)^{-1}1$. In other words, if we designate by G_Ω the Green's kernel of $-\Delta_\Omega$, then

$$(1.1) \qquad \varphi_0 \sim \int_\Omega G_\Omega(\,\cdot\,, y)\,dy \quad \text{on} \quad \Omega.$$

This result was extended to negative perturbations of $-\Delta_\Omega$ satisfying Kato condition, namely to the ground state φ_0^V of the operators $-\Delta_\Omega - V$ where V is a positive measurable function in the Kato-class and under some regularity assumptions imposed on the domain Ω (see for instance the papers of Bañuelos [3], Davies [7] and Davies' book [8]).

Actually, Bañuelos proved (among others) in [3, Theorem 2] that if Ω is a non-tangentially accessible (NTA) bounded domain and V is in the Kato-class, such that

$$(1.2) \qquad \lambda_0^V := \inf_{f \in C_c^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2\,dx - \int_\Omega f^2 V\,dx}{\int_\Omega f^2\,dx} > 0,$$

is nondegenerate and has a strictly positive eigenfunction, denoted by φ_0^V, then

$$(1.3) \qquad \varphi_0 \sim \varphi_0^V.$$

However, for (NTA) domains the conditional gauge theorem (which is one of the main ingredients in Bañuelos' proof) holds true and the Green's functions of $-\Delta_\Omega$ and that of $-\Delta_\Omega - V$ (G_Ω and G_Ω^V) are comparable. So that the latter comparison can be written as

$$(1.4) \qquad \varphi_0^V \sim \int_\Omega G_\Omega^V(\,\cdot\,, y)\, dy.$$

Observing that the function $\int_\Omega G_\Omega^V(\,\cdot\,, y)\, dy$ is nothing else but the $W_0^{1,2}$-solution of the equation

$$(1.5) \qquad -\Delta u - V u = 1 \quad \text{on} \quad \Omega,$$

one can write estimate (1.4) in the form

$$(1.6) \qquad \varphi_0^V \sim u.$$

In [9], Dávila–Dupaigne improved the result to more general V that do not necessary belong to the Kato class, including for instance

$$(1.7) \qquad V(x) = \left(\frac{d-2}{2}\right)^2 |x|^{-2} \quad \text{and} \quad V(x) = \frac{1}{4}\,\text{dist}^{-2}(x, \partial\Omega),$$

where $d \geq 3$ and Ω is regular.

Those V should satisfy the conditions that $V \in L_{\text{loc}}^1$ and there is $p > 2$ such that

$$(1.8) \qquad \inf_{f \in C_c^1(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla f|^2\, dx - \int_\Omega f^2 V\, dx}{\left(\int_\Omega |f|^p\, dx\right)^{2/p}} > 0.$$

Obviously, condition (1.8) is equivalent to an improved Sobolev type inequality, whose relevance for intrinsic ultracontractivity property as well as for the compactness of the resolvent of the operator $-\Delta_\Omega - V$ was recognized in [9].

Being inspired by the latter observation, we shall consider, in this paper, the same problem in a more general framework. Precisely we shall replace the gradient energy form by a Dirichlet form, $\mathcal{E}$ with associated positive selfadjoint operator H, having the strong local property whose domain lies in some $L^2(X, m)$-space. The potential function V will be however replaced by a positive measure, μ charging no set having zero capacity.

We shall prove that under some realistic assumptions, and especially under the assumptions that some improved Sobolev–Orlicz and Hardy-type inequalities hold true,

then the positivity preserving operator related to the semi-Dirichlet form $\mathcal{E} - \mu$ still shares many interesting features as for the classical case. In particular they have compact resolvent and nondegenerate ground state energy. Furthermore the ground state is comparable to the solution, $\xi^{(\mu)}$ of the equation $H_\mu \xi^{(\mu)} = 1$ (i.e., comparable to $H_\mu^{-1} 1$), where H_μ is the nonnegative selfadjoint operator related to $\mathcal{E} - \mu$.

Our method is based on a transformation argument (Doob's transformation) that leads first, to construct the operator $H_\mu := H - \mu$ and to the fact that it has compact resolvent and second to some ultracontractive semigroups (in the particular case where the transformation is done by means of the ground state, if one already knows about its existence, this leads to the intrinsic ultracontractivity of the operator H_μ).

As an intermediate step, we shall prove that the positivity preserving operators under considerations can be approximated, in the norm resolvent sense by a sequence of operators whose ground states can be estimated in a sharp way. This will induce convergence of ground state energies and ground states and enables us to carry over the comparison for the approximating operators to the limit operator.

To get the estimates for the ground states of the approximating operator we shall use on one side the intrinsic ultracontractivity property and on the other side Moser's iteration technique as in [9].

2. The framework

We first shortly describe the framework in which we shall state our results.

Let X be a separable locally compact metric space, m a positive finite Radon measure on Borel subsets of X such that $m(U) > 0$, $\forall \emptyset \neq U \subset X$. All integrals of the type $\int \cdots$ are assumed to be over X. The space of real-valued continuous functions having compact support on X will be denoted by $C_c(X)$.

Let $\mathcal{E}$ be a regular symmetric transient Dirichlet form, with domain $\mathcal{F} := D(\mathcal{E})$ w.r.t. the space $L^2 := L^2(X, m)$. Along the paper we assume that $\mathcal{E}$ is strongly local, i.e., $\mathcal{E}(f, g) = 0$, whenever $f, g \in \mathcal{F}$ and f is constant on the support of g.

The local Dirichlet space related to $\mathcal{E}$ will be denoted by $\mathcal{F}_{\mathrm{loc}}$. A function f belongs to $\mathcal{F}_{\mathrm{loc}}$ if for every open bounded subset $\Omega \subset X$ there is $\tilde{f} \in \mathcal{F}$ such that $f = \tilde{f}$-a.e. on Ω.

We recall the known fact $\mathcal{E}$ induces a positive-valued sets function called capacity. If a property holds true up to a set having zero capacity we shall say that it holds quasi-everywhere and we shall write 'q.e.'.

It is well known (see [12]) that every element from $\mathcal{F}_{\mathrm{loc}}$ has a quasi-continuous (q.c. for short) modification. We shall always implicitly assume that elements from $\mathcal{F}_{\mathrm{loc}}$ has been modified so as to become quasi-continuous.

We also designate by $\mathcal{F}_b := \mathcal{F} \cap L^\infty(X, m)$ and $\mathcal{F}_{b,\mathrm{loc}} := \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(X, m)$. From the very definition we derive that both $\mathcal{F}_b$ and $\mathcal{F}_{b,\mathrm{loc}}$ are algebras.

Given $f, g \in \mathcal{F}$, we set $\Gamma[f]$ the *energy measure* of f and $\Gamma(f, g)$ the *mutual energy measure* of f, g (see [12, pp. 110–114]).

We recall the known fact that every strongly local Dirichlet form, $\mathcal{E}$ possesses the following representation

$$(2.1) \qquad \mathcal{E}[f] := \mathcal{E}(f, f) = \int_X d\Gamma[f], \quad \forall f \in \mathcal{F}.$$

The representation goes as follows: for $f \in \mathcal{F}_b$ its energy measure is defined by

$$(2.2) \qquad \int \phi \, d\Gamma[f] = \mathcal{E}(f, \phi f) - \frac{1}{2}\mathcal{E}(f^2, \phi), \quad \forall 0 \le \phi \in \mathcal{F} \cap C_c(X).$$

Truncation and monotone convergence allow then to define $\Gamma[f]$ for every $f \in \mathcal{F}$.

Furthermore with the help of the strong locality property, i.e.,

$$(2.3) \qquad \int_{\{f=c\}} d\Gamma[f] = 0, \quad \forall f \in \mathcal{F},$$

it is possible to define $\Gamma[f]$ for every $f \in \mathcal{F}_{\mathrm{loc}}$ as follows: for every open bounded subset $\Omega \subset X$

$$(2.4) \qquad 1_\Omega d\Gamma[f] = 1_\Omega d\Gamma[\tilde{f}],$$

where $\tilde{f} \in \mathcal{F}$ and $f = \tilde{f}$-q.e. on Ω.

By polarization and regularity we can thereby define a Radon-measure-valued bilinear form on $\mathcal{F}_{\mathrm{loc}}$ denoted by $\Gamma(f, g)$, so that

$$(2.5) \quad \mathcal{E}(f, g) = \int d\Gamma(f, g), \quad \forall f, g \in \mathcal{F}_{\mathrm{loc}}, \quad \text{either } f \text{ or } g \text{ has compact support.}$$

The truncation property for $\mathcal{E}$ reads as follows: For every $a \in \mathbb{R}$, every $f \in \mathcal{F}_{\mathrm{loc}}$, having compact support and every $g \in \mathcal{F}_{b,\mathrm{loc}}$ we have

$$(2.6) \qquad \mathcal{E}((f - a)_+, g) = \int_{\{f>a\}} d\Gamma(f, g) \quad \text{and} \quad \mathcal{E}[(f - a)_+] = \int_{\{f>a\}} d\Gamma[f].$$

Furthermore the following product formula holds true

$$(2.7) \qquad d\Gamma(fh, g) = f d\Gamma(h, g) + h d\Gamma(f, g), \quad \forall f, g, h \in \mathcal{F}_{b,\mathrm{loc}}.$$

By the regularity assumption the latter formula extends to every $f, g, h \in \mathcal{F}_{\mathrm{loc}}$.

Another rule that we shall occasionally use is the *chain rule* (See [12, pp. 111–117]): For every function $\phi \colon \mathbb{R} \to \mathbb{R}$ of class C^1 with bounded derivative ($\phi \in C_b^1(\mathbb{R})$), every $f \in \mathcal{F}_{\mathrm{loc}}$ and every $g \in \mathcal{F}_{b,\mathrm{loc}}$ the function $\phi(f)$ belongs to $\mathcal{F}_{\mathrm{loc}}$ and

$$(2.8) \qquad d\Gamma(\phi(f), g) = \phi'(f) \, d\Gamma(f, g).$$

Formula (2.8) is still valid for $\phi(t) = |t|^{p/2}$ when restricted to locally quasi-bounded f.

As long as we are concerned with Sobolev–Orlicz inequalities, we will give some material related to the underlying spaces. From now on we shall denote the Lebesgue–Orlicz spaces $L^{\Phi}(\Omega, \nu)$ simply by $L^{\Phi}(\nu)$, whereas in the case $\nu = m$ they will be denoted by L^{Φ}. We also fix an N-function $\Phi \colon [0, \infty) \to [0, \infty)$, i.e., a convex function such that

$$(2.9) \qquad \Phi(t) = 0 \iff t = 0,$$

$$(2.10) \qquad \lim_{t \to 0} \frac{\Phi(t)}{t} = 0, \quad \lim_{t \to \infty} \Phi(t) = \lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty,$$

and denote by Ψ its *complementary* function and set

$$(2.11) \qquad \Lambda(s) := \frac{1}{s\,\Phi^{-1}(1/s)}, \quad s > 0.$$

An N-function Φ is said to be an *admissible*, if the following integrability condition near zero is satisfied

$$(2.12) \qquad \int_0^{\alpha} (s\Lambda(s))^{-1}\, ds < \infty \quad \text{for some} \quad \alpha > 0.$$

We quote that a necessary and sufficient condition for a N-function to be admissible is that the function $\Phi^{-1}(t)/t^2$ is integrable at infinity.

Among functions that are admissible we cite N-functions Φ satisfying the ∇_2-condition ($\Phi \in \nabla_2$ for short), i.e., there is $l > 1$ and $t_0 > 0$ such that

$$(2.13) \qquad \Phi(t) \leq \frac{1}{lt}\Phi(lt), \quad \forall t \geq t_0,$$

are admissible. Indeed, by [15, Corollary 5, p. 26], if $\Phi \in \nabla_2$ then there is a finite constant $C > 0$, $\epsilon > 0$ and $t_0 > 0$ such that

$$(2.14) \qquad \Phi(t) \geq Ct^{1+\epsilon}, \quad \forall t > t_0.$$

Yielding therefore $\Phi^{-1}(t) \leq C't^{1/1+\epsilon}$, for large t.

From now on we fix a positive Radon measure, μ on Borel subsets of X, which does not charge sets having zero capacity and shall adopt the following assumptions.

The first assumption that we shall adopt, along the paper, is the following: there is a function $s \in \mathcal{F}_{\text{loc}} \cap L^2$, $s > 0$-q.e. such that

$$(\text{SUP})\colon \mathcal{E}(s, f) - \int_X sf\, d\mu \geq 0, \quad \forall 0 \leq f \in \mathcal{F} \cap C_c(X).$$

This condition deserves some comments. First the additional assumption $s \in L^2$ is automatically satisfied if either X is relatively compact or s is bounded.

Second, assumption (SUP) asserts the existence of a nonnegative supersolution of the operator $H - \mu$ and is, according to [10, 4], almost equivalent to the occurrence of the following Hardy's inequality

$$(2.15) \qquad \int f^2 \, d\mu \leq \mathcal{E}[f], \quad \forall f \in \mathcal{F}.$$

By 'almost equivalent' we mean that if (SUP) holds true then inequality (2.15) holds true as well. However, if (2.15) occurs then for every $\delta \in (0,1)$ there is $s \in \mathcal{F}$ such that

$$(2.16) \qquad \mathcal{E}(s, f) - \delta \int_X sf \, d\mu \geq 0, \quad \forall 0 \leq f \in \mathcal{F} \cap C_c(X).$$

We shall maintain, throughout the paper, that the following improved Sobolev–Orlicz inequality holds true: there is a finite constant $C_S > 0$ such that

$$(\text{ISO}): \ \|f^2\|_{L^\Phi} \leq C_S \left(\mathcal{E}[f] - \int f^2 \, d\mu \right), \quad \forall f \in \mathcal{F}.$$

For discussions about connections between (ISO) (especially in the case where $\mu = 0$) and various types of Logarithmic–Sobolev inequalities we refer the reader to [6, 14].

In conjunction with Φ, there is another function which will play a decisive role in the paper and which we denote by $\phi_1 := \phi_1(t) = t \Psi^{-1}(t)$, $\forall t \geq 0$. We assume from now on that the function ϕ_1 is *admissible*.

The following lemma indicates that the latter condition is fulfilled in many situations, in particular for $\Phi(t) = t^p/p$, $t \geq 0$ and $1 < p < \infty$.

Lemma 2.1. *Assume that $\Phi \in \nabla_2$ and that ϕ_1 is convex. Then ϕ_1 is admissible.*

Proof.   From the fact that Φ is an N-function we deduce

$$(2.17) \qquad \phi_1(t) = 0 \iff t = 0,$$

$$(2.18) \qquad \lim_{t \to 0} \frac{\phi_1(t)}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{\phi_1(t)}{t} = \infty,$$

which together with the convexity assumption yields that ϕ_1 is an N-function.

The integrability condition: From the known inequality for conjugate Young functions

$$(2.19) \qquad t \leq \Phi^{-1}(t)\Psi^{-1}(t) \leq 2t, \quad \forall t \geq 0,$$

in conjunction with the fact that $\Phi \in \nabla_2$, we obtain that there is $a > 0$, $\epsilon > 0$ and $t_0 > 0$ such that

$$(2.20) \qquad \phi_1(t) \geq at^{2-1/(1+\epsilon)}, \quad \forall t > t_0.$$

Thus for large t we have $t^{-2}\phi_1^{-1}(t) \leq a^{-1}t^{(1+\epsilon)/(1+2\epsilon)-2}$, and the latter function is integrable at infinity, yielding the admissibility of ϕ_1. $\qquad\square$

We also have an inclusion relation between the spaces L^Φ and L^{ϕ_1}.

Lemma 2.2. *The space L^Φ is embedded continuously into L^{ϕ_1}.*

Proof. From Young's inequality

$$(2.21) \qquad tr \leq \Phi(t) + \Psi(r), \quad \forall r, t \geq 0,$$

we get

$$(2.22) \qquad t\Psi^{-1}(t) = \phi_1(t) \leq \Phi(t) + t, \quad \forall t \geq 0.$$

Taking the behavior of Φ at infinity into account: $\lim_{t\to\infty} \Phi(t)/t = \infty$, we conclude that there is $T > 0$ such that

$$(2.23) \qquad \phi_1(t) \leq 2\Phi(t), \quad \forall t > T.$$

Since $m(X) < \infty$, we conclude that $L^{2\Phi} \subset L^{\phi_1}$, with continuous inclusion. The result follows by observing that the spaces $L^{2\Phi}$ and L^Φ have equivalent norms. $\qquad\square$

3. Preparing results

Set H the positive selfadjoint operator associated with $\mathcal{E}$ via Kato's representation theorem. For every $t > 0$ we designate by $T_t := e^{-tH}$ the semigroup related to H.

In the next theorem we will collect some spectral properties of the operator H on the light of the improved Sobolev–Orlicz inequality.

Theorem 3.1. *For every $t > 0$, the operator T_t is ultracontractive. It follows that*
i) *The operator H has compact resolvent.*
ii) *Set λ_0 the smallest eigenvalue of H. Then λ_0 is nondegenerate, i.e., there is ψ_0 (the ground state) such that $\psi_0 > 0$-q.e. and $\ker(H - \lambda_0) = \mathbb{R}\psi_0$. Furthermore ψ_0 is quasi-bounded.*

Proof. Since ϕ_1 is admissible, and $L^\Phi \subset L^{\phi_1}$, continuously, with the help of [2, Theorem 3.4], we derive that $T_t := e^{-tH}$ is ultracontractive. Thus it has a nonnegative absolutely continuous essentially bounded kernel p_t, $\forall t > 0$. Hence since $m(X) < \infty$, we conclude that T_t is a Hilbert–Schmidt operator, yielding that H has compact resolvent.

On the other hand owing to [8, Proposition 1.4.3, p. 24], the Dirichlet form $\mathcal{E}$ is irreducible, which implies that the smallest eigenvalue of H, which we denote by λ_0, is simple and has a q.e. nonnegative normalized eigenfunction ψ_0. The quasi-boundedness of ψ_0 follows from the ultracontractivity property of T_t and the proof is finished. $\qquad\square$

REMARK 3.1.   We have already mentioned in the proof of Theorem 3.1 that the Dirichlet form $\mathcal{E}$ is irreducible, which implies together with the fact that $\mathcal{E}$ is strongly local, that X is connected (see [17]).

From the fact that T_t is a Hilbert–Schmidt operator, we also derive that the inverse operator H^{-1} possesses a Green kernel G_X which is positive, symmetric and measurable.

We shall assume, throughout the paper, that the following Hardy-type inequality holds true: There is a constant $0 < C_H < \infty$ such that

$$\text{(HI)}: \int \frac{f^2}{\psi_0^2} \, dm \leq C_H \mathcal{E}[f], \quad \forall f \in \mathcal{F}.$$

Proposition 3.1.   *There exists a finite constant $C_G > 0$ such that*

$$\text{(3.1)} \qquad\qquad G_X(x, y) \geq C_G \psi_0(x) \psi_0(y), \quad a.e.$$

Proof.   Set $\mathcal{E}^{\psi_0}$ the quadratic form defined on $L^2(\psi_0^2 \, dm)$ by

$$\text{(3.2)} \qquad D(\mathcal{E}^{\psi_0}) = \{f : \psi_0 f \in \mathcal{F}\}, \quad \mathcal{E}^{\psi_0}[f] = \mathcal{E}[\psi_0 f], \quad \forall f \in D(\mathcal{E}^{\psi_0}).$$

Then $\mathcal{E}^{\psi_0}$ is a Dirichlet form. Indeed, $\mathcal{E}^{\psi_0}$ is related (via Kato's representation theorem) to the operator $H^{\psi_0} := \psi_0^{-1} H \psi_0$, so that $e^{-tH^{\psi_0}} = \psi_0^{-1} e^{-tH} \psi_0$, which is Markovian.

In this step we will prove that $D(\mathcal{E}^{\psi_0})$ is embedded continuously into the space L^{ϕ_1}. We claim that

$$\text{(3.3)} \qquad\qquad \|f^2\|_{L^{\phi_1}(\psi_0^2 \, dm)} \leq 2(C_S + C_H)\mathcal{E}^{\psi_0}[f], \quad \forall f \in D(\mathcal{E}^{\psi_0}).$$

Indeed, by Hölder's inequality we find

$$
\begin{aligned}
\text{(3.4)} \quad &\int \psi_0^2 \phi_1\left(\frac{f^2}{(C_S + C_H)\mathcal{E}^{\psi_0}[f]}\right) dm \\
&= \int \frac{\psi_0^2 f^2}{(C_S + C_H)\mathcal{E}^{\psi_0}[f]} \Psi^{-1}\left(\frac{f^2}{(C_S + C_H)\mathcal{E}^{\psi_0}[f]}\right) dm \\
&\leq 2\left\|\frac{\psi_0^2 f^2}{(C_S + C_H)\mathcal{E}^{\psi_0}[f]}\right\|_{L^\Phi} \cdot \left\|\Psi^{-1}\left(\frac{f^2}{(C_S + C_H)\mathcal{E}^{\psi_0}[f]}\right)\right\|_{L^\Psi}.
\end{aligned}
$$

By (ISO), we have

$$\left\|\frac{\psi_0^2 f^2}{(C_S + C_H)\mathcal{E}^{\psi_0}[f]}\right\|_{L^\Phi} \leq \frac{C_S}{C_S + C_H}.$$

On the other hand, by inequality (HI), we get

$$\text{(3.5)} \quad \int \Psi\left(\Psi^{-1}\left(\frac{f^2}{(C_S + C_H)\mathcal{E}^{\psi_0}[f]}\right)\right) dm = \int \frac{f^2}{(C_S + C_H)\mathcal{E}^{\psi_0}[f]} \, dm \leq \frac{C_S}{C_S + C_H} \\ \leq 1,$$

yielding

$$\text{(3.6)} \quad \left\|\Psi^{-1}\left(\frac{f^2}{(C_S + C_H)\mathcal{E}^{\psi_0}[f]}\right)\right\|_{L^\Psi} \leq 1.$$

Finally, from the definition of the Luxemburg's norm we achieve

$$\text{(3.7)} \quad \|f^2\|_{L^{\phi_1}(\psi_0^2 \, dm)} \leq 2(C_S + C_H)\mathcal{E}^{\psi_0}[f], \quad \forall f \in D(\mathcal{E}^{\psi_0}[f]),$$

and the claim is proved.

Now since ϕ_1 is admissible, using another time [2, Theorem 3.4], we derive that the semigroup $S_t := e^{-tH^{\psi_0}}$, $t > 0$ is ultracontractive and has an absolutely continuous essentially bounded kernel k_t, furthermore

$$\text{(3.8)} \quad k_t(x, y) = \frac{p_t(x, y)}{\psi_0(x)\psi_0(y)}, \quad \text{a.e.}$$

By standard way (see [8, p.112]), we conclude that there is $T > 0$ such that, $\forall t > T$,

$$\text{(3.9)} \quad \frac{1}{2}e^{-\lambda_0 t}\psi_0(x)\psi_0(y) \leq p_t(x, y), \quad \text{a.e.}$$

Hence

$$\text{(3.10)} \quad G_X(x, y) = \int_0^\infty p_t(x, y) \, dt \geq \int_T^\infty p_t(x, y) \, dt \\ \geq \frac{1}{2}\psi_0(x)\psi_0(y)\int_T^\infty e^{-\lambda_0 t} \, dt = \frac{e^{-\lambda_0 T}}{2\lambda_0}\psi_0(x)\psi_0(y), \quad \text{a.e.},$$

which finishes the proof. $\square$

Through the proof of Proposition 3.1, we have proved that the operator H is in fact intrinsically ultracontractive.

From now on we set $\dot{\mathcal{E}}_\mu$ the form defined by

$$D(\dot{\mathcal{E}}_\mu) = \mathcal{F}, \quad \dot{\mathcal{E}}_\mu[f] = \mathcal{E}[f] - \int f^2 \, d\mu, \quad \forall f \in \mathcal{F}.$$

Since $\mathcal{E}$ is a Dirichlet from, then $\dot{\mathcal{E}}_\mu$ is a *semi-Dirichlet form*, i.e.,

$$(3.11) \qquad \forall f \in D(\dot{\mathcal{E}}_\mu) \to |f| \in D(\dot{\mathcal{E}}_\mu) \quad \text{and} \quad \mathcal{E}[|f|] \leq \mathcal{E}[f].$$

We will prove in the following lines that the form $\dot{\mathcal{E}}_\mu$ is closable.

Let us stress that since the measure μ is not assumed to be a small perturbation we can not conclude directly the closability of $\dot{\mathcal{E}}_\mu$ by means of the KLMN theorem. To overcome this difficulty we give first some auxiliary results.

We say that a function $u \in \mathcal{F}_{\text{loc}}$ is a supersolution of $H - \mu$ if

$$(3.12) \qquad \mathcal{E}(u, f) - \int uf \, d\mu \geq 0, \quad \forall 0 \leq f \in \mathcal{F}_{\text{loc}} \cap C_c(X).$$

Lemma 3.1. *Let $s \geq 0$ q.e. be a positive supersolution of $H - \mu$. Then*

$$(3.13) \qquad s(x) \geq C_G \psi_0(x) \int \psi_0(y) s(y) \, d\mu(y), \quad q.e.$$

Proof. Let $f \in \mathcal{F} \cap C_c(X)$ be nonnegative. Set $U = \text{supp}\, f$ and let $u \in \mathcal{F}$ be such that $u = s$ q.e. on U (such u exists because $s \in \mathcal{F}_{\text{loc}}$). Since $|u| \in \mathcal{F}$ and $|u| = u = s$ q.e. on U ($s \geq 0$ q.e.), we may and do suppose that $u \geq 0$ q.e. Owing to the definition of s we derive

$$(3.14) \qquad 0 \leq \mathcal{E}(s, f) - \int sf \, d\mu = \mathcal{E}(u, f) - \int uf \, d\mu = \mathcal{E}(u, f) - \mathcal{E}(K^\mu u, f),$$

where

$$(3.15) \qquad K^\mu u := \int G_X(\cdot, y) u(y) \, d\mu(y),$$

is the potential of the measure $u\mu$. Thus $u - K^\mu u$ is a potential, obtaining thereby that $u - K^\mu u \geq 0$ q.e. Since $u = s$ q.e. on U and u is positive q.e., and whence μ a.e., we get with the help of the lower bound for the Green function in term of the ground state ψ_0 (see Proposition 3.1)

$$(3.16) \qquad s(x) \geq C_G \psi_0(x) \int \psi_0(y) u(y) \, d\mu(y), \quad \text{q.e. on} \quad U.$$

Now let (U_k) be a sequence of compact sets exhausting X and $(u_k) \subset \mathcal{F}$ such that $u_k \geq 0$ q.e. and $u_k = s$ q.e. on U_k for every integer k. Since $U_k \subset U_l$, $\forall l \geq k$, we get $u_k = u_l$ q.e. on U_k, $\forall l \geq k$. Furthermore $u_k \uparrow s$ q.e. So that the estimate established above yields

$$(3.17) \qquad s(x) \geq C_G \psi_0(x) \int \psi_0(y) u_l(y) \, d\mu(y), \quad \text{q.e. on } U_k \quad \forall l \geq k.$$

Passing to the limit w.r.t. l yields

$$(3.18) \qquad s(x) \geq C_G \psi_0(x) \int \psi_0(y) s(y) \, d\mu(y), \quad \text{q.e. on } U_k \quad \forall k.$$

Regarding (U_k) exhausts X, the lemma is proved. $\qquad\square$

Let $s > 0$ q.e. be a supersolution of $H - \mu$ (such an s exists by assumption (SUP)). As a second step toward proving the closability of the form $\dot{\mathcal{E}}_\mu$ we will prove that the s-transform of $\dot{\mathcal{E}}_\mu$ is in fact a pre-Dirichlet form.

We designate by $\dot{\mathcal{E}}_\mu^s$ (the s-transform of $\dot{\mathcal{E}}_\mu$) the form defined by

$$(3.19) \quad D(\dot{\mathcal{E}}_\mu^s) := \mathcal{F}^s = \{f : sf \in \mathcal{F}\} \subset L^2(s^2 \, dm), \quad \dot{\mathcal{E}}_\mu^s[f] = \dot{\mathcal{E}}_\mu[sf], \quad \forall f \in \mathcal{F}^s.$$

The following result was mentioned in [10] with a probabilistic proof. For the convenience of the reader we will give an alternative analytic proof.

Lemma 3.2. *The form $\dot{\mathcal{E}}_\mu^s$ is a pre-Dirichlet form in $L^2(s^2 \, dm)$. It follows in particular that $\dot{\mathcal{E}}_\mu$ is closable and its closure is a semi-Dirichlet form.*

Proof. Following Fitzsimmons [11], we set

$$(3.20) \ \ \mathcal{C}^s := \{f : f \in \mathcal{F}_b, \ f \in L^2(s^2 \, dm), \ f \in L^2(\Gamma[s]), \ s \in L^2(\Gamma[f])\} \subset L^2(s^2 \, dm)$$

and Q the form defined by

$$(3.21) \qquad D(Q) = \mathcal{C}^s, \quad Q[f] = \int s^2 d\Gamma[f], \quad \forall f \in D(Q).$$

We claim first, that for every $f \in \mathcal{C}^s$, $sf \in \mathcal{F}$ (so that $f \in \mathcal{F}^s$) and

$$(3.22) \qquad \dot{\mathcal{E}}_\mu^s[f] = Q[f] + 2\int sf \, d\Gamma(s, f) + \int f^2 \, d\Gamma[s] - \int f^2 s^2 \, d\mu.$$

Indeed, let $f \in \mathcal{C}^s$. Then $sf \in \mathcal{F}_{\text{loc}}$ and by the chain rule we get for every open bounded subset $U \subset X$,

$$(3.23) \qquad \int_U d\Gamma[sf] = \int_U s^2 \, d\Gamma[f] + 2\int_U sf \, d\Gamma(s, f) + \int_U f^2 \, d\Gamma[s].$$

Owing to the properties of f, and exhausting X by open subsets, we get by Schwarz's

inequality together with monotone convergence theorem

$$\mathcal{E}[sf] = \int d\Gamma[sf] = Q[f] + 2\int sf\, d\Gamma(s, f) + \int f^2\, d\Gamma[s]$$

(3.24)

$$\le Q[f] + 2\left(\int f^2\, d\Gamma[s]\right)^{1/2}\left(\int s^2\, d\Gamma[f]\right)^{1/2} + \int f^2\, d\Gamma[s] < \infty,$$

yielding that $sf \in \mathcal{F}$ and the corresponding formula for $\dot{\mathcal{E}}_\mu^s[f]$.

As a second step we define another form, which we denote by q, as follows

(3.25) $D(q) = \mathcal{C}^s, \quad q[f] = 2\int sf\, d\Gamma(s, f) + \int f^2\, d\Gamma[s] - \int f^2 s^2\, d\mu, \quad \forall f \in D(q).$

Then q is well defined. Since for every $f \in \mathcal{C}^s$ also $f^2 \in \mathcal{C}^s$, we get by the preceding step that $sf^2 \in \mathcal{F}$. Thus, owing to the fact that s is a supersolution we obtain

(3.26) $q[f] = \mathcal{E}(s, sf^2) - \int s(sf^2)\, d\mu \ge 0, \quad \forall f \in \mathcal{C}^s.$

We shall prove that there is a positive measure, $\tilde{\mu}$ charging no set having zero capacity such that

$$q[f] = \int f^2\, d\tilde{\mu}, \quad \forall f \in \mathcal{C}^s.$$

Let $f \in \mathcal{C}^s$, having compact support and $f \ge 0$ a.e. Set

(3.27) $L(f) := \mathcal{E}(s, sf) - \int s(sf)\, d\mu = \int d\Gamma(s, sf) - \int s(sf)\, d\mu.$

Since s is a supersolution we conclude that $L(f) \ge 0$, $\forall f \in \mathcal{C}^s$ having compact support. On the other hand observing that the map $f \mapsto d\Gamma(s, sf)$ is a Radon measure charging no set having zero capacity, we derive that L is actually a positive Radon measure charging no set having zero capacity: There is a positive Radon measure $\tilde{\mu}$, charging no set having zero capacity such that

(3.28) $L(f) = \int f\, d\tilde{\mu}.$

Noting that $L(f^2) = q[f]$ we get $q[f] = \int f^2\, d\tilde{\mu}$, for every $f \in \mathcal{C}^s$ having compact support and whence for every $f \in \mathcal{C}^s$.

Now set

(3.29) $S := Q + q.$

Then S coincides with $\dot{\mathcal{E}}_\mu^s$ restricted to $\mathcal{C}^s$.

On one hand according to [11, Theorem 3.10], the form Q is closable and its closure $\bar{Q}$ is a Dirichlet form having the strong local property. On the other hand since the measure q is positive and absolutely continuous w.r.t. the capacity, then according to [16], the form S is closable, yielding the closability of $\dot{\mathcal{E}}_\mu^s$ and whence of $\dot{\mathcal{E}}_\mu$. The fact that the closure of $\dot{\mathcal{E}}_\mu$ is a semi-Dirichlet form is derived from the fact that $\mathcal{E}_\mu$ is itself a semi-Dirichlet form.

Let us denote by $\bar{S}$, respectively $\bar{Q}$ the closure of S, respectively of Q and by L_S, respectively L_Q the selfadjoint operator associated with S, respectively Q. Then since $\bar{S} \geq \bar{Q} \geq 0$ we derive that

$$(3.30) \qquad 0 \leq e^{-tL_S} \leq e^{-tL_Q}, \quad \forall t > 0.$$

Owing to the fact that $\bar{Q}$ is a Dirichlet form we get that the operator e^{-tL_Q} is Markovian for every $t > 0$, and whence e^{-tL_S}, $t > 0$ is Markovian as well or equivalently $\bar{S}$ is a Dirichlet form. Clearly $\bar{S}$ is local and the proof is finished. $\qquad \square$

We quote that the improved Sobolev–Orlicz inequality (ISO) has no relevance for the closability of the form $\dot{\mathcal{E}}_\mu$.

From now on we denote by $\mathcal{E}_\mu^s$, respectively $\mathcal{E}_\mu$, the closure of $\dot{\mathcal{E}}_\mu^s$, respectively of $\dot{\mathcal{E}}_\mu$. Actually, we deduce from the last proof that since $\mathcal{C}^s$ is a common core for both $\bar{S}$ and $\mathcal{E}_\mu^s$, then $\bar{S} = \mathcal{E}_\mu^s$.

The form $\mathcal{E}_\mu$ is a densely defined nonnegative form, and is even a semi-Dirichlet form. Let H_μ be the self-adjoint operator associated with $\mathcal{E}_\mu$. Then H_μ is positivity preserving and by inequality (ISO) is invertible with bounded inverse, which we denote by H_μ^{-1}. Henceforth we denote by H_μ^s the operator related to the form $\mathcal{E}_\mu^s$ and by e^{-tH_μ}, $t > 0$, respectively $T_t^s := e^{-tH_\mu^s}$, $t > 0$ the semigroup of operators related to H_μ, respectively H_μ^s.

Theorem 3.2. *Let s be a function satisfying assumption (SUP). Then for every $t > 0$, the operator T_t^s is a Hilbert–Schmidt operator. It follows, in particular that e^{-tH_μ}, $t > 0$ is a Hilbert–Schmidt operator as well and the operator H_μ has compact resolvent.*

Let us emphasize that the latter theorem is the only place where we used the supplementary assumption $s \in L^2$.

Proof of Theorem 3.2. By similar arguments to those used in the proof of Proposition 3.1, we derive that there is a finite constant $C > 0$ such that

$$(3.31) \qquad \|f^2\|_{L^{\phi_1}(s^2\, dm)} \leq C\mathcal{E}_\mu^s[f], \quad \forall f \in \mathcal{C}^s.$$

Having in mind that $\mathcal{C}^s$ is a core for $\mathcal{E}_\mu^s$, the latter inequality extends to every element from the space $\mathcal{F}^s$. Since ϕ_1 is admissible, $\mathcal{E}_\mu^s$ is a Dirichlet form (by Lemma 3.2) and $s \in L^2$, we get according to [2, Theorem 3.4] that T_t^s is a Hilbert–Schmidt operator for every $t > 0$. Now the rest of the proof follows directly by realizing that $e^{-tH_\mu} = sT_t^s s^{-1}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

From now on we denote by $\lambda_0^{(\mu)}$ the smallest eigenvalue of the operator H_μ. We proceed to prove that $\lambda_0^{(\mu)}$ is nondegenerate, i.e. its eigenspace has dimension one and may be generated by a nonnegative eigenfunction. To that end we shall approximate the operator H_μ, in the norm resolvent sense, by a sequence of operators having the mentioned property.

Since μ charges no set having zero capacity, by [12, Lemma 2.2.5, p. 78] and the remark after the proof on p. 79, there is a sequence (μ_k) of finite positive measures charging no set having zero capacity such that

$$(3.32) \qquad \mu_k \uparrow \mu \quad \text{and} \quad K^{\mu_k}1 := \int G_X(\,\cdot\,, y)\, d\mu_k(y) \in L^\infty, \quad \forall k.$$

Multiplying μ_k by $1-1/k$, if necessary, we shall and do assume that there are constants $0 < \kappa_k < 1$ such that for every $k \in \mathbb{N}$ we have

$$(3.33) \qquad \int f^2\, d\mu_k \le \kappa_k \mathcal{E}[f], \quad \forall f \in \mathcal{F}.$$

By the assumption $0 < \kappa_k < 1$, we conclude that the following forms

$$D(\mathcal{E}_{\mu_k}) = \mathcal{F}, \quad \mathcal{E}_{\mu_k}[f] = \mathcal{E}[f] - \int_\Omega f^2\, d\mu_k, \quad \forall f \in \mathcal{F},$$

are closed in L^2. For every integer k, we shall designate by H_k the self-adjoint operator related to $\mathcal{E}_{\mu_k}$.

According to general results about convergence of sequences of monotone quadratic forms (see [13]), one can realize that $H_k \to H_\mu$, in the strong resolvent sense as $k \to \infty$. We shall improve this observation in the following way:

Lemma 3.3. *The operators H_k have compact resolvents and*

$$(3.34) \qquad\qquad\qquad\qquad \lim_{k\to\infty} \|H_k^{-1} - H_\mu^{-1}\| = 0.$$

Proof.   Observe that $0 \le H_k^{-1} \le H_\mu^{-1}$. Now the first statement follows from the fact that H_μ^{-1} is compact and the second one follows from the known fact that H_μ^{-1} is compact together with the norm resolvent convergence [13, Theorem 3.5, p. 453]. $\quad\square$

The latter lemma will have a great influence on the strategy that we shall follow. This is illustrated through the following:

Corollary 3.1. i) *Let $\lambda_0^{(k)}$, respectively $\lambda_0^{(\mu)}$ be the smallest eigenvalue of the operator H_k respectively H_μ. Then $\lim_{k\to\infty}|\lambda_0^{(k)} - \lambda_0^{(\mu)}| = 0$.*
ii) *Let $P^{(k)}$, respectively $P^{(\mu)}$ be the eigenprojection of the eigenvalue $\lambda_0^{(k)}$, respectively of the eigenvalue $\lambda_0^{(\mu)}$. Then*

$$(3.35) \qquad \lim_{k\to\infty} \| P^{(k)} - P^{(\mu)} \| = 0.$$

It follows, in particular, that if $\lambda_0^{(k)}$ is nondegenerate for large k, then so is $\lambda_0^{(\mu)}$ and conversely.

Proof. i): Follows from the inequality $|1/\lambda_0^{(k)} - 1/\lambda_0^{(\mu)}| \leq \|H_k^{-1} - H_\mu^{-1}\|$ and Lemma 3.3.

ii): Follows from Lemma 3.3 and the fact that if P and Q are two orthogonal projections such that $\|P - Q\| < 1$, then their respective ranges have the same dimension [13, Theorem 6.32, p. 56]. $\qquad\square$

Lemma 3.4. *Let ν be a positive Radon measure on Borel subset of X such that there is a constant $0 < C_\nu < 1$ with*

$$(3.36) \qquad \int f^2\, d\nu \leq C_\nu \mathcal{E}[f], \quad \forall f \in \mathcal{F}.$$

Let $\mathcal{E}_\nu$ be the form defined by

$$D(\mathcal{E}_\nu) = \mathcal{F}, \quad \mathcal{E}_\nu[f] = \mathcal{E}[f] - \int f^2\, d\nu, \quad \forall f \in \mathcal{F},$$

and $\lambda_0^{(\nu)}$ be the smallest eigenvalue of $\mathcal{E}_\nu$.
i) *Let $\varphi \geq 0$ q.e. be an eigenfunction associated with $\lambda_0^{(\nu)}$. Then*

$$(3.37) \qquad \varphi(x) \geq \left(C_G \lambda_0^{(\nu)} \int \psi_0(y)\varphi(y)\, dm(y) \right) \psi_0(x), \quad q.e.$$

It follows that $\varphi > 0$ q.e.
ii) *The eigenvalue $\lambda_0^{(\nu)}$ is nondegenerate and has a positive normalized ground state which we shall denote by $\varphi_0^{(\nu)}$.*

Proof. i): Let $\varphi \geq 0$ q.e. be any eigenfunction associated with $\lambda_0^{(\nu)}$. Set

$$(3.38) \quad K^\nu \varphi = \int G_X(\cdot, y)\varphi(y)\, d\nu, \quad K\varphi = \int G_X(\cdot, y)\varphi(y)\, dm, \quad u = \varphi - K^\nu \varphi - \lambda_0^{(\nu)} K\varphi.$$

Owing to the fact that φ lies in $\mathcal{F}$ and hence lies in $L^2(\nu)$, we obtain that the measure $\varphi\nu$ has finite energy integral with respect to the Dirichlet form $\mathcal{E}$, i.e.,

$$\tag{3.39} \int |f\varphi|\, d\nu \le \alpha (\mathcal{E}[f])^{1/2}, \quad \forall f \in \mathcal{F} \cap C_c(X),$$

and therefore $K^\nu\varphi \in \mathcal{F}$. Thus $u \in \mathcal{F}$ and satisfies the identity

$$\tag{3.40} \begin{aligned} \mathcal{E}(u, g) &= \mathcal{E}(\varphi, g) - \int \varphi g\, d\nu - \lambda_0^{(\nu)} \int \varphi g\, dm \\ &= \mathcal{E}_\nu(\varphi, g) - \lambda_0^{(\nu)} \int \varphi g\, dm = 0, \quad \forall g \in \mathcal{F}. \end{aligned}$$

Since $\mathcal{E}$ is positive definite we conclude that $u = 0$ a.e. (and hence q.e.), which yields

$$\tag{3.41} \begin{aligned} \varphi &= K^\nu\varphi + \lambda_0^{(\nu)} K\varphi \ge \lambda_0^{(\nu)} K\varphi = \lambda_0^{(\nu)} \int G_X(\cdot, y)\varphi(y)\, dm(y) \\ &\ge \left(C_G \lambda_0^{(\nu)} \int \psi_0(y)\varphi(y)\, dm(y) \right) \psi_0, \quad \text{q.e.,} \end{aligned}$$

where the latter inequality is obtained from Proposition 3.1.

ii):   Let φ be an eigenfunction associated with $\lambda_0^{(\nu)}$. Since $\mathcal{E}_\nu$ is a semi-Dirichlet form, then $|\varphi| \in \mathcal{F}$ and minimizes the ratio

$$\left\{ \frac{\mathcal{E}_\nu[f]}{\int f^2\, dm} : f \in \mathcal{F} \setminus \{0\} \right\}.$$

Thus $|\varphi|$ is an eigenfunction associated with $\lambda_0^{(\nu)}$ as well and by assertion (i), $|\varphi| > 0$ q.e.

Set $\tilde{\varphi} := |\varphi| - \varphi$. Then $\tilde{\varphi}$ satisfies $H_\nu\tilde{\varphi} = \lambda_0^{(\nu)}\tilde{\varphi}$. Now, either $\tilde{\varphi} = 0$ a.e. which would imply that $\varphi = |\varphi|$ a.e. or $\tilde{\varphi}$ is a non-negative eigenfunction associated with $\lambda_0^{(\nu)}$. In the latter case we derive from assertion (i) that $\tilde{\varphi} > 0$ q.e. or equivalently $|\varphi| > \varphi$ q.e. We have thereby proved that every eigenfunction associated with $\lambda_0^{(\nu)}$ has a constant sign, from which (ii) follows. $\square$

On the light of Corollary 3.1 together with Lemma 3.4, we conclude that $\lambda_0^{(\mu)}$ is nondegenerate as well and we can get even more:

Lemma 3.5. *Let $\varphi_0^{(k)}$ be the normalized a.e. positive eigenfunction associated with $\lambda_0^{(k)}$. Then there is a subsequence $(\varphi_0^{(k_j)})$ such that*

$$\lim_{j\to\infty} \|\varphi_0^{(k_j)} - \varphi_0^{(\mu)}\|_{L^2} = 0,$$

where $\varphi_0^{(\mu)}$ is the normalized a.e. positive eigenfunction associated with $\lambda_0^{(\mu)}$.

Proof. Since the sequence $(\varphi_0^{(k)})$ is bounded in L^2, there is a subsequence, which we still denote by $(\varphi_0^{(k)})$, and $h \in L^2$ such that $\varphi_0^{(k)} \to h$, weakly in L^2. Let P be the eigenprojection associated with $\lambda_0^{(\mu)}$. Since P is a rank one operator, we get $P\varphi_0^{(k)} \to Ph$ in L^2. Thus

$$(3.42) \qquad P_k \varphi_0^{(k)} = \varphi_0^{(k)} = (P_k - P)\varphi_0^{(k)} + P\varphi_0^{(k)} \to Ph,$$

and $\|Ph\|_{L^2} = 1$.

On the other hand we may and shall suppose that $Ph \geq 0$ a.e. (by mean of a subsequence if necessary). Now Setting $\varphi_0^{(\mu)} := Ph$, and recalling that $\operatorname{Ran} P = \ker(H_\mu - \lambda_0^{(\mu)})$ (by the fact that $\dim \operatorname{Ran} P = 1$) we get that $\varphi_0^{(\mu)}$ is an eigenfunction corresponding to $\lambda_0^{(\mu)}$ and $\varphi_0^{(\mu)} \geq 0$ a.e. Finally Corollary 3.1 together with the lower bound for $\varphi_0^{(k)}$ given by inequality Lemma 3.4, lead to

$$(3.43) \qquad \varphi_0^{(\mu)} \geq \left(C_G \lambda_0^{(\mu)} \int_\Omega \psi_0(y) \varphi_0^{(\mu)}(y) \, dm(y) \right) \psi_0, \quad \text{a.e.,}$$

yielding $\varphi_0^{(\mu)} > 0$ a.e., which completes the proof. □

At the end of this section we resume our strategy. Define

$$(3.44) \qquad \xi^{(k)} := H_k^{-1} 1, \quad \xi^{(\mu)} := H_\mu^{-1} 1.$$

Theorem 3.3. *Let $\varphi_0^{(k)}$, $\xi^{(k)}$, $\lambda_0^{(\mu)}$, $\varphi_0^{(\mu)}$, $\xi^{(\mu)}$ be as above. Assume that for every $k \in \mathbb{N}$ there is a constant $0 < \Gamma_k < \infty$ such that $\lim_{k \to \infty} \Gamma_k = \Gamma \in (0, \infty)$ and*

$$(3.45) \qquad \Gamma_k^{-1} \xi^{(k)} \leq \varphi_0^{(k)} \leq \Gamma_k \xi^{(k)}, \ a.e. \quad \forall k \ large.$$

Then $\Gamma^{-1} \xi^{(\mu)} \leq \varphi_0^{(\mu)} \leq \Gamma \xi^{(\mu)}$, a.e.

Proof. By the norm resolvent convergence of H_k towards H_μ (Lemma 3.3), we obtain $\xi^{(k)} = H_k^{-1} 1 \to H_\mu^{-1} 1 = \xi^{(\mu)}$ in $L^2(\Omega)$ and we can assume that $\lim_{k \to \infty} \xi^{(k)} = \xi^{(\mu)}$, a.e. Now the result follows from the assumptions of the theorem together with Lemma 3.5. □

Our main task in the next section is to establish estimate (3.45).

4. Estimating the ground state

Towards proving the estimates of the ground states for the approximating forms we shall need some inequalities which we shall state and prove and which have an independent interest.

In this section we fix:

i)   A positive measure ν satisfying assumptions of Lemma 3.4 and such that

$$(4.1) \qquad K^\nu 1 := \int G_X(\,\cdot\,, y)\, d\nu(y) \in L^\infty.$$

ii)   Two real-valued, measurable a.e. positive and essentially bounded functions V and F on X such that either $V \neq 0$ or $F \neq 0$.

Let $w \in \mathcal{F}$. We say that w is a solution of the equation

$$(4.2) \qquad H_\nu w = V w + F,$$

if

$$(4.3) \qquad \mathcal{E}_\nu(w, f) = \int f V w\, dm + \int f F\, dm, \quad \forall f \in \mathcal{F}.$$

Let $w > 0$ q.e. be a solution (if any) of the equation $H_\nu w = V w + F$. Define Q^w the form:

$$(4.4) \quad D(Q^w) = \{f\colon wf \in \mathcal{F}\}, \quad Q^w[f] = \mathcal{E}_\nu^w[f] - \int f^2 w^2 V\, dm, \quad \forall f \in D(Q^w).$$

Then by the same arguments used in the proof of Lemma 3.2, we deduce that Q^w is a Dirichlet form on $L^2(w^2\, dm)$ having the local property. Moreover since $w \in L^2$, then the vector space

$$(4.5) \qquad \mathcal{C}^w := \{f\colon f \in \mathcal{F}_b,\ w \in L^2(d\Gamma[f])\},$$

is a core for Q^w.

We claim that

$$(4.6) \qquad Q^w[f] = \int w^2\, d\Gamma[f] + \int f^2 F w\, dm, \quad \forall f \in \mathcal{C}^w.$$

Indeed, from the product formula for the energy measure, we derive

$$(4.7) \quad Q^w[f] = \int w^2\, d\Gamma[f] - \int f^2 w^2\, d\nu - \int f^2 w^2 V\, dm + \int d\Gamma(w, wf^2), \quad \forall f \in \mathcal{C}^w.$$

Using the fact that w is a solution of equation (4.1), we get for every $f \in \mathcal{C}^w$, $wf^2 \in \mathcal{C}^w$ and

$$(4.8) \quad \mathcal{E}_\nu(w, wf^2) = \int V f^2 w^2\, dm + \int F w f^2\, dm = \int d\Gamma(w, wf^2) - \int f^2 w^2\, d\nu,$$

and substituting in Eq. (4.7) we get the claim.

We also note that the operator $w^{-1}(H_\nu - V)w$ is the self-adjoint operator in $L^2(w^2\, dm)$ associated with the Dirichlet form Q^w.

Henceforth, we define

$$(4.9) \qquad C' := C_G^{-2}\left(\int \psi_0(y)w(y)V(y)\, dm + \int \psi_0(y)F(y)\, dm\right)^{-2},$$

and

$$(4.10) \qquad C := \max(C_H C', C_H C' \lambda_0).$$

Theorem 4.1. *Let V, F be as in the beginning of this section. Let $w \in \mathcal{F}$, $w > 0$ q.e. be a solution of the equation*

$$(4.11) \qquad H_\nu w = Vw + F,$$

Set

$$A := (C + 2C_s)(1 + 2C_S\|1\|_{L^\Psi}).$$

Then

$$(ISO1) \quad \|f^2\|_{L^{\phi_1(w^2\, dm)}} \le A\left(Q^w[f] + \int Vf^2 w^2\, dm\right), \quad \forall f \in D(Q^w).$$

The proof of Theorem 4.1 relies upon auxiliary results which we shall state in three lemmata.

Lemma 4.1. *Let w be as in* Theorem 4.1. *Then the following inequality holds true*

$$(4.12) \qquad w \ge C_G\psi_0\left(\int \psi_0(y)V(y)w(y)\, dm + \int \psi_0(y)F(y)\, dm\right) \quad q.e.$$

Proof. As in the proof of Lemma 3.4 we show that w satisfies

$$w - K^\nu w = KVw + KF.$$

Whence, with the help of the lower estimate Proposition 3.1 for the Green function G_X we achieve,

$$w \ge KVw + KF \ge C_G\psi_0\int \psi_0(y)V(y)w(y)\, dm + C_G\psi_0\int \psi_0(y)F(y)\, dm \quad \text{q.e.} \quad \square$$

Lemma 4.2. *Let w be as in* Theorem 4.1. *Then*

$$(4.13) \qquad \int f^2 \, dm \leq C \int w^2 d\Gamma[f] + C \int w^2 f^2 \, dm, \quad \forall f \in \mathcal{C}^w.$$

Proof.   At this stage we use Hardy's inequality (HI), which states that there is a constant $C_H > 0$ depending only on X such that

$$(4.14) \qquad \int \frac{u^2}{\psi_0^2} \, dm \leq C_H \int d\Gamma[u], \quad \forall u \in \mathcal{F}.$$

Let $f \in \mathcal{C}^w$. Taking $u = f\psi_0$ in inequality (4.14) yields

$$
\begin{aligned}
\int f^2 \, dm = \int \frac{f^2 \psi_0^2}{\psi_0^2} \, dm &\leq C_H \int d\Gamma[f\psi_0] \\
(4.15) \qquad &= C_H \int \psi_0^2 \, d\Gamma[f] + 2C_H \int \psi_0 f \, d\Gamma(\psi_0, f) + C_H \int f^2 d\Gamma[\psi_0] \\
&= C_H \int \psi_0^2 \, d\Gamma[f] + C_H \int d\Gamma(\psi_0, \psi_0 f^2).
\end{aligned}
$$

Thanks to the fact that ψ_0 is an eigenfunction associated with λ_0, we achieve

$$(4.16) \qquad \int d\Gamma(\psi_0, \psi_0 f^2) = \lambda_0 \int f^2 \psi_0^2 \, dm.$$

Combining (4.16) with (4.15) we obtain

$$(4.17) \qquad \int f^2 \, dm \leq C_H \int {\psi_0}^2 d\Gamma[f] + C_H \lambda_0 \int {\psi_0}^2 f^2 \, dm, \quad \forall f \in \mathcal{C}^w.$$

Having the lower bound for w given by Lemma 4.1 in hand, we establish

$$(4.18) \qquad \int f^2 \, dm \leq C_H C' \int w^2 \, d\Gamma[f] + C_H C' \lambda_0 \int w^2 f^2 \, dm, \quad \forall f \in \mathcal{C}^w. \qquad \square$$

Lemma 4.3. *Let w be as in* Theorem 4.1. *Set*

$$(4.19) \qquad \Lambda_1 = 1 + \frac{C_H C'}{2}, \quad \Lambda_2 = \frac{\|F\|_\infty^2}{2} + \frac{C_H C' \lambda_0}{2},$$

C' being the constant appearing Lemma 4.2. *Then*

$$(4.20) \qquad Q^w[f] \leq \Lambda_1 \int w^2 \, d\Gamma[f] + \Lambda_2 \int w^2 f^2 \, dm, \quad \forall f \in \mathcal{C}^w.$$

Proof.   We have already established that

$$(4.21) \qquad Q^w[f] = \int w^2 \, d\Gamma[f] + \int f^2 F w \, dm, \quad \forall f \in \mathcal{C}^w.$$

Making use of Hölder's and Young's inequality together with inequality (4.13) we obtain

$$Q^w[f] \leq \int w^2 \, d\Gamma[f] + \left(\int f^2 \, dm \right)^{1/2} \left(\int f^2 F^2 w^2 \, dm \right)^{1/2}$$

$$\leq \Lambda_1 \int w^2 \, d\Gamma[f] + \Lambda_2 \int f^2 w^2 \, dm, \quad \forall f \in \mathcal{C}^w,$$

which finishes the proof.   □

Proof of Theorem 4.1.   We observe first that

$$(4.22) \qquad Q^w[f] + \int V f^2 w^2 \, dm = \mathcal{E}_\nu^w[f] := \mathcal{E}_\nu[wf], \quad \forall f \in \mathcal{C}^w.$$

So that due to the fact $\mathcal{C}^w$ is a core for the form Q^w it suffices to prove inequality (ISO1) on $\mathcal{C}^w$.

For $f \in \mathcal{C}^w$, set $\lambda := A\mathcal{E}_\nu^w[f]$. By Hölder's inequality for Orlicz norms, we get for every $f \in \mathcal{C}^w$,

$$\int w^2 \phi_1 \left(\frac{f^2}{\lambda} \right) dm = \int \frac{w^2 f^2}{\lambda} \Psi^{-1} \left(\frac{f^2}{\lambda} \right) dm$$

$$\leq 2 \left\| \frac{w^2 f^2}{\lambda} \right\|_{L^\Phi} \left\| \Psi^{-1} \left(\frac{f^2}{\lambda} \right) \right\|_{L^\Psi}.$$

By (ISO), we have

$$(4.23) \qquad 2 \left\| \frac{w^2 f^2}{\lambda} \right\|_{L^\Phi} \leq \frac{2}{\lambda} C_s \mathcal{E}_\nu^w[f] \leq 1.$$

On the other hand we have, according to Lemma 4.2

$$(4.24) \qquad \int \Psi \left(\Psi^{-1} \left(\frac{f^2}{\lambda} \right) \right) dm = \int \frac{f^2}{\lambda} \, dm$$

$$\leq \frac{C}{\lambda} \left(\int w^2 \, d\Gamma(f, f) \, dm + \int w^2 f^2 \, dm \right).$$

Applying another time Hölder's inequality we get

$$(4.25) \qquad \int (fw)^2 \, dm \leq \|1\|_{L^\Psi} \|(fw)^2\|_{L^\Phi} \leq C_S \|1\|_{L^\Psi} \mathcal{E}_\nu^w[f], \quad \forall f \in \mathcal{C}^w.$$

Recalling that $\mathcal{E}_\nu^w[f] \geq \int w^2 \, d\Gamma[f]$, we achieve

$$(4.26) \qquad \int \frac{f^2}{\lambda} \, dm \leq \frac{C}{\lambda}(1 + 2C_S\|1\|_{L^\Psi})\mathcal{E}_\nu^w[f] \leq 1, \quad \forall f \in \mathcal{C}^w.$$

Thus

$$(4.27) \qquad \left\|\Psi^{-1}\left(\frac{f^2}{\lambda}\right)\right\|_{L^\Psi} \leq 1,$$

and whence

$$(4.28) \qquad \int w^2 \phi_1\left(\frac{f^2}{\lambda}\right) dm \leq 1, \quad \forall f \in \mathcal{C}^w,$$

and the theorem is proved, according to the definition of the Orlicz norm. $\qquad\square$

For every $t > 0$ we designate by T_t^w the semigroup associated with the form Q^w in the space $L^2(w^2 \, dm)$. We are yet ready to prove the ultracontractivity of T_t^w.

To that end we collect some preparing notations. We recall the expression of the constant A

$$(4.29) \qquad A := (C + 2C_s)(1 + 2C_S\|1\|_{L^\Psi}).$$

Let Λ be the function defined by

$$(4.30) \qquad \Lambda(s) := \frac{1}{s\phi_1^{-1}(1/s)}, \quad \forall s > 0,$$

and γ be the solution of the equation

$$(4.31) \qquad t := 8A \int_0^{\gamma(t)} \frac{1}{s\Lambda(s)} \, ds.$$

We finally denote by

$$(4.32) \qquad \beta(t) := \frac{4}{\gamma(t)}.$$

Theorem 4.2. *Let V, F and w be as in* Theorem 4.1. *Then T_t^w is ultracontractive for every $t > 0$ and*

$$(4.33) \qquad \|T_t^w\|_{L^1(w^2 \, dm), L^\infty} \leq \beta\left(\frac{t}{2}\right)e^{\|V\|_\infty t}, \quad \forall t > 0.$$

Proof.   From Theorem 4.1, we derive

$$(4.34) \qquad \|f^2\|_{L^{\phi_1}(w^2\,dm)} \leq A\left(Q^w[f] + \|V\|_\infty \int f^2 w^2\,dm\right), \quad \forall f \in D(Q^w).$$

Since ϕ_1 is admissible, we get according to [2], that the semigroup T_t^w is ultracontractive for every $t > 0$ and

$$(4.35) \qquad \|T_t^w\|_{L^1(\Omega, w^2\,dm), L^\infty} \leq \beta\left(\frac{t}{2}\right)e^{\|V\|_\infty t}, \quad \forall t > 0. \qquad \square$$

We shall apply Theorem 4.1, to the special cases $V = 0$, $F = 1$ which corresponds to $w = \xi^{(v)}$.

Theorem 4.3. *Let v be as in the beginning of this section. Then the following pointwise upper bound for $\varphi_0^{(v)}$ holds true*

$$(4.36) \qquad \varphi_0^{(v)} \leq \left(\beta\left(\frac{t}{2}\right)e^{t\lambda_0^{(v)}}\right)\|\xi^{(v)}\|_{L^2}\xi^{(v)}, \ a.e. \quad \forall t > 0.$$

Proof.   Applying Theorem 4.2 to the case $V = 0$, $F = 1$, so that we may and do choose $w = \xi^{(v)}$, yields that the semi-group $T_t^{\xi^{(v)}}$ is ultracontractive and $\varphi_0^{(v)}/\xi^{(v)}$ is an eigenfunction for $T_t^{\xi^{(v)}}$ associated with the eigenvalue $e^{-t\lambda_0^{(v)}}$, $\forall t > 0$. Thus

$$(4.37) \qquad \begin{aligned} \left\|\frac{\varphi_0^{(v)}}{\xi^{(v)}}\right\|_\infty &\leq e^{t\lambda_0^{(v)}}\|T_t^{\xi^{(v)}}\|_{L^1((\xi^{(v)})^2\,dm), L^\infty}\left\|\frac{\varphi_0^{(v)}}{\xi^{(v)}}\right\|_{L^1((\xi^{(v)})^2\,dm)} \\ &\leq \beta\left(\frac{t}{2}\right)e^{t\lambda_0^{(v)}}\int \varphi_0^{(v)}\xi^{(v)}\,dm \leq \beta\left(\frac{t}{2}\right)e^{t\lambda_0^{(v)}}\|\xi^{(v)}\|_{L^2}, \quad \forall t > 0, \end{aligned}$$

and

$$(4.38) \qquad \varphi_0^{(v)} \leq \xi^{(v)}\beta\left(\frac{t}{2}\right)e^{t\lambda_0^{(v)}}\|\xi^{(v)}\|_{L^2}, \ a.e. \quad \forall t > 0,$$

which was to be proved. $\square$

While for the upper pointwise estimate we exploited the idea of intrinsic ultra-contractivity, for the reversed estimate we shall however, make use of Moser's iteration technique as utilized in [9]. To that end and being inspired by Dávila–Dupaigne [9], we shall further assume that the function ϕ_1 satisfies the following growth condition: there is $\epsilon > 0$, and a finite constant $a > 0$ such that

$$(4.39) \qquad \phi_1(t) \geq at^{1+\epsilon}, \quad \forall t \geq 0.$$

Regarding the equivalence between the norms of the Orlicz spaces L^{ϕ_1} and $L^{a^{-1}\phi_1}$ we may and shall assume that $a = 1$.

Before stating the result we need a short preparation. We denote by

$$(4.40) \qquad I_\nu := (\mathcal{F}, \mathcal{E}) \to L^2(\nu), \quad f \mapsto f, \quad K^\nu := I_\nu I_\nu^*, \quad \text{and} \quad K := H^{-1}.$$

We recall [1] that $C_\nu = \|K^\nu\|$.

An elementary computation yields that

$$(4.41) \qquad (I_\nu K)^* : L^2(\nu) \to L^2, \quad f \mapsto \int G_X(\,\cdot\,, y) f(y) \, d\nu,$$

$$(4.42) \qquad K^\nu f : L^2(\nu) \to L^2(\nu), \quad K^\nu f = \int G_X(\,\cdot\,, y) f(y) \, d\nu, \quad \forall f \in L^2(\nu).$$

Furthermore according to [5, formula (24)]

$$(4.43) \qquad H_\nu^{-1} = K + (I_\nu K)^* (1 - K^\nu)^{-1} I_\nu K.$$

Lemma 4.4. *The function $\xi^{(\nu)}$ belongs to L^∞.*

Proof.   Formula (4.43) learns that $\xi^{(\nu)} = K1 + (I_\nu K)^* (1 - K^\nu)^{-1} I_\nu K1$. Let $f \in L^\infty \subset L^2$. Since K is a Dirichlet operator, then $Kf \in L^\infty \cap \mathcal{F}$, and is quasicontinuous. Hence $Kf \leq \|f\|_{L^\infty} \|K1\|_{L^\infty}$ q.e. Using the smoothness of ν we get $Kf \leq \|f\|_{L^\infty} \|K1\|_{L^\infty} \nu$ a.e., yielding that $I_\nu K$ maps L^∞ into $L^\infty(\nu)$.

On the other side formulae (4.41) and (4.42) indicate that K^ν maps continuosly $L^\infty(\nu)$ into itself and $(I_\nu K)^*$ maps $L^\infty(\nu)$ into L^∞. Indeed, arguing as in the beginning of the proof and having assumption (4.1) on ν in hands, we derive

$$\|K^\nu f\|_{L^\infty(\nu)} \leq \|f\|_{L^\infty(\nu)} \|K^\nu 1\|_{L^\infty}, \quad \|(I_\nu K)^* f\|_{L^\infty} \leq \|f\|_{L^\infty(\nu)} \|K^\nu 1\|_{L^\infty}, \quad \forall f \in L^\infty(\nu),$$

yielding that $\xi^{(\nu)} \in L^\infty$. $\square$

Theorem 4.4. *For every $t > 0$, the following estimate holds true*

$$(4.44) \qquad \xi^{(\nu)} \leq (AC + 1)(C(\nu, t) + 1)\varphi_0^{(\nu)}, \quad a.e.,$$

where

$$(4.45) \qquad C(\nu, t) := \beta\left(\frac{t}{2}\right) e^{t\lambda_0^{(\nu)}}, \quad \forall t > 0.$$

Proof.   Consider the ratio

$$(4.46) \qquad \rho := \frac{\xi^{(\nu)}}{\varphi_0^{(\nu)}}.$$

By [4, Lemma 2.2, Lemma 2.1], the function $1/\varphi_0^{(\nu)}$ lies in the space $\mathcal{F}_{b,\text{loc}}$. Thus according to Lemma 4.4, $\rho \in \mathcal{F}_{b,\text{loc}}$. Now using the chain rule together with the equations satisfied by the ground state $\varphi_0^{(\nu)}$ and the function $\xi^{(\nu)}$, we find, for every $f \in \mathcal{F}_{\text{loc}}$ having compact support,

$$
\begin{aligned}
&\int (\varphi_0^{(\nu)})^2 \, d\Gamma(f, \rho) \\
&= \int d\Gamma(\varphi_0^{(\nu)} f, \xi^{(\nu)}) - \int f \, d\Gamma(\varphi_0^{(\nu)}, \xi^{(\nu)}) - \int \xi^{(\nu)} \, d\Gamma(\varphi_0^{(\nu)}, f) \\
(4.47) \quad &= \int f \varphi_0^{(\nu)} \, dm + \int \varphi_0^{(\nu)} f \xi^{(\nu)} \, d\nu - \int d\Gamma(\varphi_0^{(\nu)}, f \xi^{(\nu)}) \\
&= \int f \varphi_0^{(\nu)} \, dm + \int \varphi_0^{(\nu)} f \xi^{(\nu)} \, d\nu - \left(\lambda_0^{\nu} \int \varphi_0^{(\nu)} f \xi^{(\nu)} \, dm + \int \varphi_0^{(\nu)} f \xi^{(\nu)} \, d\nu \right) \\
&= \int \varphi_0^{(\nu)} f \, dm - \lambda_0^{(\nu} \int \varphi_0^{(\nu)} f \xi^{(\nu)} \, dm.
\end{aligned}
$$

Let U be a compact subset of X. Testing the latter equation with $f = 1_U \rho^{2j-1}$, $j \geq 1$, ($f \in \mathcal{F}_{b,\text{loc}}$ by Lemma 4.4), we deduce

$$
(4.48) \qquad \int_U (\varphi_0^{(\nu)})^2 \, d\Gamma(\rho^{2j-1}, \rho) = \int_U \rho^{2j-1} (\varphi_0^{(\nu)} - \lambda_0^{(\nu)} \xi^{(\nu)} \varphi_0^{(\nu)}) \, dm,
$$

which yields, due to the positivity of both functions $\varphi_0^{(\nu)}$ and $\xi^{(\nu)}$

$$
(4.49) \qquad
\begin{aligned}
\frac{2j-1}{j^2} \int_U (\varphi_0^{(\nu)})^2 \, d\Gamma[\rho^j] &= \int_U \rho^{2j-1} (\varphi_0^{(\nu)} - \lambda_0^{(\nu)} \xi^{(\nu)} \varphi_0^{(\nu)}) \, dm \\
&\leq \int_U \rho^{2j-1} \varphi_0^{(\nu)} \, dm.
\end{aligned}
$$

According to Theorem 4.3, we obtain

$$
(4.50) \qquad \int_U (\varphi_0^{(\nu)})^2 \, d\Gamma[\rho^j] \leq C(\nu, t) j \int_U (\varphi_0^{(\nu)}) \rho^{2j} \, dm.
$$

Using Hölder inequality and Lemma 4.2 (with $V = \lambda_0^{(\nu)}$, $F = 0$, $w = \varphi_0^{(\nu)}$ and $f = 1_U \rho^j$), it follows from (4.50) that

$$
(4.51)
\begin{aligned}
&\int_U (\varphi_0^{(\nu)})^2 \, d\Gamma[\rho^j] \\
&\leq C(\nu, t) j \left(\int_U (\varphi_0^{(\nu)})^2 \rho^{2j} \, dm \right)^{1/2} \left(\int \rho^{2j} \, dm \right)^{1/2} \\
&\leq C^{1/2} C(\nu, t) j \left(\int_U (\varphi_0^{(\nu)})^2 \rho^{2j} \, dm \right)^{1/2} \left(\int_U (\varphi_0^{(\nu)})^2 \, d\Gamma[\rho^j] + \int_U (\varphi_0^{(\nu)})^2 \rho^{2j} \, dm \right)^{1/2}.
\end{aligned}
$$

By Young's inequality, we obtain

$$
\text{(4.52)} \quad
\begin{aligned}
\int_U (\varphi_0^{(\nu)})^2 \, d\Gamma[\rho^j] &\leq \frac{1}{2}(CC^2(\nu,t)j^2 + 1)\int_U (\varphi_0^{(\nu)})^2 \rho^{2j} \, dm \\
&\quad + \frac{1}{2}\int_U (\varphi_0^{(\nu)})^2 \, d\Gamma[\rho^j],
\end{aligned}
$$

so that

$$
\text{(4.53)} \quad \int_U (\varphi_0^{(\nu)})^2 \, d\Gamma[\rho^j] \leq (CC^2(\nu,t)j^2 + 1)\int_U (\varphi_0^{(\nu)})^2 \rho^{2j} \, dm.
$$

By (4.6), with $V = \lambda_0^{(\nu)}$, $F = 0$, $w = \varphi_0^{(\nu)}$ and $f = 1_U \rho^j$, we get from (ISO1)

$$
\text{(4.54)} \quad \|1_U \rho^{2j}\|_{L^{\phi_1}((\varphi_0^{(\nu)})^2 \, dm)} \leq A\left(\int_U (\varphi_0^{(\nu)})^2 \, d\Gamma[\rho^j] + \lambda_0^{(\nu)} \int_U \rho^{2j}(\varphi_0^{(\nu)})^2 \, dm\right),
$$

which yields

$$
\text{(4.55)} \quad
\begin{aligned}
\|1_U \rho^{2j}\|_{L^{\phi_1}((\varphi_0^{(\nu)})^2 \, dm)} &\leq A(CC^2(\nu,t)j^2 + 1 + \lambda_0^{(\nu)})\int_U (\varphi_0^{(\nu)})^2 \rho^{2j} \, dm \\
&\leq (ACC(\nu,t) + 1)(C(\nu,t) + 1)j^2 \int_U (\varphi_0^{(\nu)})^2 \rho^{2j} \, dm.
\end{aligned}
$$

Having the growth property (4.39) for the function ϕ_1 in hands, we achieve

$$
\text{(4.56)} \quad \left(\int_U \rho^{2j(1+\varepsilon)}(\varphi_0^{(\nu)})^2 \, dm\right)^{1/(1+\varepsilon)} \leq (ACC(\nu,t)+1)(C(\nu,t)+1)j^2 \int_U \rho^{2j}(\varphi_0^{(\nu)})^2 \, dm.
$$

We iterate (4.56). Define $j_k = 2(1+\varepsilon)^k$ for $k = 0, 1, \ldots$ and

$$
\text{(4.57)} \quad \Theta_k^U = \left(\int_U \rho^{j_k}(\varphi_0^{(\nu)})^2 \, dm\right)^{1/j_k} \quad \text{and} \quad M(\nu,t) := (ACC(\nu,t) + 1)(C(\nu,t) + 1).
$$

Then (4.56) can be written as

$$
\text{(4.58)} \quad \Theta_{k+1}^U \leq (M(\nu,t)(1+\varepsilon)^{2k})^{1/2(1+\varepsilon)^k} \Theta_k^U.
$$

Using this recursively yields

$$
\text{(4.59)} \quad \Theta_k^U \leq M(\nu,t)\Theta_0^U = M(\nu,t)\left(\int_U (\varphi_0^{(\nu)})^2 \, dm\right)^{1/2} \leq M(\nu,t),
$$

for all $k = 0, 1, \ldots$. Since the right-hand-side of the latter inequality is independent from U, we deduce

$$
\text{(4.60)} \quad \lim_{k\to\infty} \Theta_k^X = \sup_X \rho \leq M(\nu,t),
$$

and this shows that

(4.61) $$\xi^{(\nu)} \le M(\nu, t)\varphi_0^{(\nu)}, \quad \forall t > 0,$$

which was to be proved. $\qquad\square$

Theorem 4.5. *Let μ be a positive Radon measure on Borel subsets of X charging no set having zero capacity. Then under assumptions* (SUP), (ISO), (HI) *and the growth condition* (4.39), *the following sharp estimate for the ground state $\varphi_0^{(\mu)}$ holds true, a.e.*

$$\left(\left(AC \inf_{t>0} \beta\left(\frac{t}{2}\right)e^{t\lambda_0^{(\mu)}} + 1\right)\left(\inf_{t>0} \beta\left(\frac{t}{2}\right)e^{t\lambda_0^{(\mu)}} + 1\right)\right)^{-1}\xi^{(\mu)}$$

$$\le \varphi_0^{(\mu)} \le \xi^{(\mu)}\left(\inf_{t>0} \beta\left(\frac{t}{2}\right)e^{t\lambda_0^{(\mu)}}\right)\|\xi^{(\mu)}\|_{L^2}.$$

Proof.   Let $\mu_k \uparrow \mu$ (as specified in Section 2). By Theorem 4.3 it holds

(4.62) $$\varphi_0^{(k)} \le \xi_0^{(k)}\beta\left(\frac{t}{2}\right)e^{\lambda_0^{(k)}t}\|\xi_0^{(k)}\|_{L^2}, \quad \forall t > 0.$$

Now the right-hand-side inequality follows directly by letting $k \to \infty$ and using Lemma 3.3 together with Corollary 3.1.

The reversed inequality is obtained in the same manner by using Theorem 4.4 and Corollary 3.1. $\qquad\square$

Let us recall that according to Theorem 3.2, e^{-tH_μ} is a Hilbert–Schmidt operator for every $t > 0$. Thus e^{-tH_μ} has a $m \times m$ absolutely continuous kernel. For every $t > 0$, we designate by p_t^μ the heat kernel of e^{-tH_μ}.

By standard way, we deduce that the operator H_μ has a Green's kernel which we denote by G_X^μ. We can rephrase Theorem 4.5 in term of the Green's kernel.

Corollary 4.1.   *We have*

(4.63) $$\varphi_0^{(\mu)} \sim \int G_X^\mu(\,\cdot\,, y)\,dy, \quad a.e.$$

REMARK 4.1.   a)   We immediately derive from the latter corollary that if the Green's functions G_X and G_X^μ are comparable then the ground states of H and H_μ are comparable as well.

b)   If μ is such that there is $C^\mu \in (0, 1)$ with

$$\int_X f^2\, d\mu \le C^\mu \mathcal{E}[f], \quad \forall f \in \mathcal{F},$$

then by (2.16) (changing μ by $(1/C^\mu)\mu$), (SUP) is satisfied. Furthermore since $\mathcal{E}$ and $\mathcal{E}_\mu$ are equivalent, inequality (ISO) can be changed by the weaker Sobolev–Orlicz inequality

$$\|f^2\|_{L^\Phi} \le C_S \mathcal{E}[f], \quad f \in \mathcal{F}.$$

In this situation the compactness of H_μ^{-1} can be obtained directly from formula (4.43).

We also derive by standard way the following large time asymptotics for the heat kernel.

Corollary 4.2. *There is $T > 0$ such that for every $t > T$,*

$$(4.64) \qquad p_t^\mu(x, y) \sim e^{-\lambda_0^{(\mu)} t} \varphi_0^{(\mu)}(x) \varphi_0^{(\mu)}(y) \sim e^{-\lambda_0^{(\mu)} t} \xi^{(\mu)}(x) \xi^{(\mu)}(y), \quad m \times m \quad a.e.$$

It follows, *in particular that*

$$(4.65) \qquad -\lambda_0^{(\mu)} = \lim_{t \to \infty} \frac{1}{t} \ln\left(\frac{p_t^\mu(x, y)}{\xi^{(\mu)}(x) \xi^{(\mu)}(y)} \right).$$

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