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ON THE DISTRIBUTION OF k-TH POWER FREE INTEGERS, II

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Abstract

The indicator function of the set of k-th power free integers is naturally extended to a random variable $X^{(k)}(\,\cdot\,)$ on $(\hat{\mathbb{Z}}, \lambda)$, where $\hat{\mathbb{Z}}$ is the ring of finite integral adeles and λ is the Haar probability measure. In the previous paper, the first author noted the strong law of large numbers for $\{X^{(k)}(\,\cdot\,+n)\}_{n=1}^{\infty}$, and showed the asymptotics: $E^{\lambda}[(Y_N^{(k)})^2] \asymp 1$ as $N \to \infty$, where $Y_N^{(k)}(x) := N^{-1/2k} \sum_{n=1}^{N} \left(X^{(k)}(x+n) - 1/\zeta(k)\right)$. In the present paper, we prove the convergence of $E^{\lambda}[(Y_N^{(k)})^2]$. For this, we present a general proposition of analytic number theory, and give a proof to this.

1. Introduction

Let $\hat{\mathbb{Z}}$ be the ring of finite integral adeles; $\mathcal{B}$ the Borel σ-field of $\hat{\mathbb{Z}}$; λ the Haar probability measure on $(\hat{\mathbb{Z}}, \mathcal{B})$. In [4, 1], the triplet $(\hat{\mathbb{Z}}, \mathcal{B}, \lambda)$ is introduced in the following way: For a prime number p, the p-adic metric d_p on $\mathbb{Z}$ is defined by

$$d_p(x, y) := \inf\{p^{-l};\ p^l \mid (x-y)\}, \quad x, y \in \mathbb{Z}.$$

The completion of $\mathbb{Z}$ by d_p is denoted by $\mathbb{Z}_p$. By extending the algebraic operations '$+$' and '$\times$' in $\mathbb{Z}$ continuously to those in $\mathbb{Z}_p$, the compact metric space $(\mathbb{Z}_p, d_p)$ becomes a ring. In particular, $(\mathbb{Z}_p, d_p)$ is a compact abelian group with respect to '$+$'. Thus, there is a unique Haar probability measure λ_p with respect to '$+$' on $(\mathbb{Z}_p, \mathcal{B}(\mathbb{Z}_p))$, where $\mathcal{B}(\mathbb{Z}_p)$ is the Borel σ-field of $\mathbb{Z}_p$.

Putting $p_i = i$-th prime number ($i = 1, 2, \ldots$), we set

$$\hat{\mathbb{Z}} := \prod_{i=1}^{\infty} \mathbb{Z}_{p_i}, \quad \lambda := \prod_{i=1}^{\infty} \lambda_{p_i}.$$

For $x = (x_i)$, $y = (y_i) \in \hat{\mathbb{Z}}$, we define

$$d(x, y) := \sum_{i=1}^{\infty} \frac{1}{2^i} d_{p_i}(x_i, y_i),$$

$$x + y := (x_i + y_i), \quad xy := (x_i y_i).$$

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By these definitions, $\hat{\mathbb{Z}}$ becomes a ring, which is just the ring of finite integral adeles stated above. $(\hat{\mathbb{Z}}, d)$ is again a compact metric space, and both '$+$' and '$\times$' are continuous. In particular, this is a compact abelian group with respect to '$+$', and its Haar probability measure is nothing but λ. By identifying $\mathbb{Z}$ with the diagonal set $\{(n, n, \dots) \in \mathbb{Z} \times \mathbb{Z} \times \cdots ;\ n \in \mathbb{Z}\} \subset \hat{\mathbb{Z}}$, it is seen that $\mathbb{Z}$ is a dense subring of $\hat{\mathbb{Z}}$. Thus $\hat{\mathbb{Z}}$ is a compactification of $\mathbb{Z}$.

Let k be an integer, ≥ 2. Let $B^{(k)}$ be the set of all elements in $\hat{\mathbb{Z}}$ having no k-th power factors, i.e.,

$$B^{(k)} := \{x \in \hat{\mathbb{Z}};\ p^k \nmid x\ \ (\forall p\colon \text{prime})\},$$

where $d \mid x \Leftrightarrow x \in d\hat{\mathbb{Z}}$ (, so $d \nmid x \Leftrightarrow x \in \hat{\mathbb{Z}} \setminus d\hat{\mathbb{Z}}$), and $X^{(k)} := \mathbf{1}_{B^{(k)}}$ (= the indicator function of $B^{(k)}$).

The following are results of Duy [1]:

Fact 1 (Strong law of large numbers). *$\lim_{N\to\infty}(1/N)\sum_{n=1}^{N} X^{(k)}(x+n) = 1/\zeta(k)$, λ-a.e. x. Here $\zeta(\cdot)$ is the Riemann zeta function.*

For each $N \in \mathbb{N}$, we set

$$Y_N^{(k)}(x) := \frac{1}{N^{1/(2k)}} \sum_{n=1}^{N} \left(X^{(k)}(x+n) - \frac{1}{\zeta(k)} \right). \tag{1}$$

Fact 2. *$E^{\lambda}[(Y_N^{(k)})^2] \asymp 1$ as $N \to \infty$.*

Fact 3. *A sequence $\{Y_N^{(k)}\}_{N=1}^{\infty}$ in $L^2(\hat{\mathbb{Z}}, \mathcal{B}, \lambda)$ has no limit point. Namely, for any subsequence $\{N_i\}_{i=1}^{\infty}$, $\{Y_{N_i}^{(k)}\}_{i=1}^{\infty}$ is not convergent in L^2 as $i \to \infty$.*

Fact 1 follows at once from the ergodicity of the shift $x \mapsto x+1$ and $E^{\lambda}[X^{(k)}] = 1/\zeta(k)$[1]. From this fact, we have the following question: *When $\sum_{n=1}^{N}(X^{(k)}(x+n) - 1/\zeta(k))$ is normalized appropriately, is its distribution weakly convergent as $N \to \infty$?* Fact 2 tells us that a normalizing constant must be $N^{1/(2k)}$, and that a sequence $\{\lambda(Y_N^{(k)} \in *)\}_{N=1}^{\infty}$ of distributions on $\mathbb{R}$ is tight. Fact 3 is a functional analytical result and brings no news for the behavior of $Y_N^{(k)}$ as $N \to \infty$. But, for this, we expect to have a limit theorem in probability theory. (Unfortunately, we still have no information on this limit theorem.)

In this paper, we make some remark about Fact 2 and Fact 3.

[1] Cf. 1° in the proof of Claim 1.

Theorem 1.

$$\lim_{N\to\infty} E^{\lambda}[(Y_N^{(k)})^2] = \left(\prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}-\frac{2}{p^k}\right)\right)\frac{\zeta(2-1/k)}{(2\pi)^{1-1/k}\Gamma(1/k)\sin(\pi/(2k))}.$$

Theorem 2. (i) $\lim_{N\to\infty}\lim_{M\to\infty} E^{\lambda}[(Y_M^{(k)}-Y_N^{(k)})^2] = 2\big(\prod_p(1-1/p)(1+1/p-2/p^k)\big)\zeta(2-1/k)/((2\pi)^{1-1/k}\Gamma(1/k)\sin(\pi/2k)) > 0.$ Fact 3 *above is a consequence of this.*
(ii) *But, a whole sequence* $\{Y_N^{(k)}\}_{N=1}^{\infty}$ *in* $L^2(\hat{\mathbb{Z}},\mathcal{B},\lambda)$ *is weakly convergent to* 0 *as* $N\to\infty$.

Throughout this paper, the letter p denotes a prime number, and the symbols $\prod_p$ and $\sum_p$ are a product and a summation extended over all prime numbers, respectively.
Theorems above will be proved in Section 4. In Section 2, an another computation of $E^{\lambda}[Y_M^{(k)}Y_N^{(k)}]$, which is different from one in Duy [1], is given. And, in Section 3, to prove Theorem 1, we prepare Proposition 1. This is a general proposition of analytic number theory, and will be proved in Section 5.

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2. Computation of $E^{\lambda}[Y_M^{(k)}Y_N^{(k)}]$

By a different approach[2] from Duy [1], we compute $E^{\lambda}[Y_M^{(k)}Y_N^{(k)}]$.

Claim 1. *For* $M\ge N\ge 1$,

$$E^{\lambda}[Y_M^{(k)}Y_N^{(k)}]$$
$$= \frac{1}{M^{1/(2k)}}\frac{1}{N^{1/(2k)}}\sum_{c=1}^{\infty}|\mu(c)|\left(\prod_{p\nmid c}\left(1-\frac{2}{p^k}\right)\right)\left\{\frac{M}{c^k}\right\}\wedge\left\{\frac{N}{c^k}\right\}\left(1-\left\{\frac{M}{c^k}\right\}\vee\left\{\frac{N}{c^k}\right\}\right).$$

Here $\mu(\cdot)$ *is the Möbius function and* $\{a\}$ *is the fractional part of the real number* a.

Proof. Fix $M\ge N\ge 1$. We divide the proof into three steps:
1° First

$$E^{\lambda}[Y_M^{(k)}Y_N^{(k)}]$$
$$= E^{\lambda}\left[\frac{1}{M^{1/(2k)}}\frac{1}{N^{1/(2k)}}\sum_{m=1}^{M}\sum_{n=1}^{N}\left(X^{(k)}(x+m)-\frac{1}{\zeta(k)}\right)\left(X^{(k)}(x+n)-\frac{1}{\zeta(k)}\right)\right]$$

[2] Duy's method is originally due to [4]. The same kind of computation in the proof of Claim 1 appears in early study of [4]. So, a phrase 'different approach' may be too much to say.

$$= \frac{1}{M^{1/(2k)}} \frac{1}{N^{1/(2k)}} \sum_{\substack{1 \le m \le M, \\ 1 \le n \le N}} \Bigg(E^{\lambda}[X^{(k)}(x+m) X^{(k)}(x+n)] - \frac{1}{\zeta(k)} (E^{\lambda}[X^{(k)}(x+m)] + E^{\lambda}[X^{(k)}(x+n)]) + \left(\frac{1}{\zeta(k)}\right)^2 \Bigg).$$

Noting that

$$X^{(k)}(y) = \prod_p (1 - \rho_{p^k}(y)), \tag{2}$$

where, for $d \in \mathbb{N}$, $\rho_d(y) := \begin{cases} 1, & d \mid y \ (\Leftrightarrow y \in d\hat{\mathbb{Z}}), \\ 0, & \text{otherwise}, \end{cases}$

$$\{\rho_{p^k}\}_p \quad \text{is independent}, \tag{3}$$

$$\lambda(\rho_d = 1) = \frac{1}{d}, \quad \lambda(\rho_d = 0) = 1 - \frac{1}{d}, \tag{4}$$

we have

$$\begin{aligned} E^{\lambda}[X^{(k)}(x+m)] &= E^{\lambda}[X^{(k)}(x)] \quad \text{(by the shift invariance of } \lambda) \\ &= \prod_p \left(1 - \frac{1}{p^k}\right) \\ &= \frac{1}{\zeta(k)} \quad \text{(by Euler's product of } \zeta(\,\cdot\,)), \end{aligned}$$

and thus

$$\begin{aligned} &E^{\lambda}[Y_M^{(k)} Y_N^{(k)}] \\ &= \frac{1}{M^{1/(2k)}} \frac{1}{N^{1/(2k)}} \sum_{\substack{1 \le m \le M, \\ 1 \le n \le N}} \left(E^{\lambda}[X^{(k)}(x+m) X^{(k)}(x+n)] - \left(\frac{1}{\zeta(k)}\right)^2 \right). \end{aligned}$$

Since, by (2)

$$\begin{aligned} &X^{(k)}(x+m) X^{(k)}(x+n) \\ &= \prod_p (1 - \rho_{p^k}(x+m)) \cdot (1 - \rho_{p^k}(x+n)) \\ &= \prod_p (1 - \rho_{p^k}(x+n) - \rho_{p^k}(x+m) + \rho_{p^k}(x+m) \rho_{p^k}(x+n)) \\ &= \prod_p (1 - \rho_{p^k}(x+n) - \rho_{p^k}(x+m) + \rho_{p^k}(m-n) \rho_{p^k}(x+n)) \\ &\qquad \text{(by an identity: } \rho_d(x+m)\rho_d(x+n) = \rho_d(m-n)\rho_d(x+n)), \end{aligned}$$

we see from (3) and (4) that

$$\begin{aligned} &E^\lambda[Y_M^{(k)} Y_N^{(k)}] \\ &= \frac{1}{M^{1/(2k)}} \frac{1}{N^{1/(2k)}} \sum_{\substack{1\le m\le M,\\ 1\le n\le N}} \left(\prod_p \left(1 - \frac{2}{p^k} + \rho_{p^k}(m-n)\frac{1}{p^k}\right) - \left(\frac{1}{\zeta(k)}\right)^2 \right). \end{aligned}$$

<u>2°</u> By Euler's product of $\zeta(\cdot)$

$$\begin{aligned} \left(\frac{1}{\zeta(k)}\right)^2 &= \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{2k}}\right) \\ &= \prod_p \left(1 + \frac{1}{p^k}\left(-2 + \frac{1}{p^k}\right)\right) \\ &= \sum_d \frac{|\mu(d)|}{d^k} \prod_{p|d} \left(-2 + \frac{1}{p^k}\right) \\ &= \sum_d \frac{|\mu(d)|}{d^k} \sum_{c|d} (-2)^{\omega(d/c)} \prod_{p|c} \frac{1}{p^k} \end{aligned}$$

(where $\omega(n) := \#\{p;\, p \mid n\}$ = the number of different prime factors of n)

$$\begin{aligned} &= \sum_d \frac{|\mu(d)|}{d^k} \sum_{c|d} (-2)^{\omega(d/c)} \frac{1}{c^k} \\ &= \sum_{c_1,d_1} \frac{|\mu(c_1 d_1)|}{(c_1 d_1)^k} (-2)^{\omega(d_1)} \frac{1}{c_1^k} \end{aligned}$$

(there exists a one-to-one correspondence between the set $\{(c, d);\, d$ is square free and $c \mid d\}$ and the set $\{(c_1, d_1);\, c_1 d_1$ is square free$\}$; a correspondence from the former to the latter is $(c, d) \mapsto (c, d/c)$ and one from the latter to the former is $(c_1, d_1) \mapsto (c_1, c_1 d_1)$. Here (c, d) and (c_1, d_1) denote a pair of c and d, and that of c_1 and d_1, respectively)

$$= \sum_{c_1,d_1} \frac{|\mu(c_1 d_1)|}{c_1^{2k}} \mathbf{1}_{(c_1,d_1)=1} \frac{(-2)^{\omega(d_1)}}{d_1^k}$$

(where (c_1, d_1) is the greatest common divisor of c_1 and d_1. Note that $\mu(c_1 d_1) = 0$ if $(c_1, d_1) > 1$)

$$= \sum_{c_1,d_1} \frac{|\mu(c_1)|\, |\mu(d_1)|}{c_1^{2k}} \mathbf{1}_{(c_1,d_1)=1} \frac{(-2)^{\omega(d_1)}}{d_1^k}$$

(by the multiplicativity of μ)

$$= \sum_{c_1} \frac{|\mu(c_1)|}{c_1^{2k}} \sum_{d_1} \frac{|\mu(d_1)|}{d_1^k} \mathbf{1}_{(c_1,d_1)=1}(-2)^{\omega(d_1)}$$
$$= \sum_{c_1} \frac{|\mu(c_1)|}{c_1^{2k}} \prod_p \left(1 + \frac{|\mu(p)|}{p^k} \mathbf{1}_{(c_1,p)=1}(-2)^{\omega(p)}\right)$$

(by the multiplicativity of $d_1 \mapsto |\mu(d_1)|\mathbf{1}_{(c_1,d_1)=1}(-2)^{\omega(d_1)}$)

$$= \sum_{c_1} \frac{|\mu(c_1)|}{c_1^{2k}} \prod_{p \nmid c_1} \left(1 - \frac{2}{p^k}\right).$$

Similarly, since

$$\prod_p \left(1 - \frac{2}{p^k} + \rho_{p^k}(m-n)\frac{1}{p^k}\right)$$
$$= \sum_{c_1} \frac{|\mu(c_1)|}{c_1^k} \prod_{p \nmid c_1} \left(1 - \frac{2}{p^k}\right) \rho_{c_1^k}(m-n),$$

we have

$$\prod_p \left(1 - \frac{2}{p^k} + \rho_{p^k}(m-n)\frac{1}{p^k}\right) - \left(\frac{1}{\zeta(k)}\right)^2$$
$$= \sum_c \frac{|\mu(c)|}{c^k} \prod_{p \nmid c} \left(1 - \frac{2}{p^k}\right)\left(\rho_{c^k}(m-n) - \frac{1}{c^k}\right).$$

Thus, by 1° and 3° below

$$E^\lambda[Y_M^{(k)} Y_N^{(k)}]$$
$$= \frac{1}{M^{1/(2k)} N^{1/(2k)}} \sum_{\substack{1 \le m \le M, \\ 1 \le n \le N}} \sum_c \frac{|\mu(c)|}{c^k} \prod_{p \nmid c} \left(1 - \frac{2}{p^k}\right)\left(\rho_{c^k}(m-n) - \frac{1}{c^k}\right)$$
$$= \frac{1}{M^{1/(2k)} N^{1/(2k)}} \sum_c |\mu(c)| \prod_{p \nmid c} \left(1 - \frac{2}{p^k}\right) \frac{1}{c^k} \sum_{m=1}^M \sum_{n=1}^N \left(\rho_{c^k}(m-n) - \frac{1}{c^k}\right)$$
$$= \frac{1}{M^{1/(2k)} N^{1/(2k)}} \sum_c |\mu(c)| \prod_{p \nmid c} \left(1 - \frac{2}{p^k}\right) \left\{\frac{M}{c^k}\right\} \wedge \left\{\frac{N}{c^k}\right\} \left(1 - \left\{\frac{M}{c^k}\right\} \vee \left\{\frac{N}{c^k}\right\}\right).$$

This is the assertion of the claim.

3° Fix $u \in \mathbb{N}$. Let Q and s be a quotient and a remainder of N divided by u, respectively. Thus $N = Qu + s$, where $Q = \lfloor N/u \rfloor$[3], $s = \{N/u\}u \in \{0, 1, \ldots, u-1\}$. Then

$$\begin{aligned}
&\sum_{n=1}^{N} \rho_u(m-n) \\
&= \sum_{q=1}^{Q} \sum_{j=1}^{u} \rho_u(m-((q-1)u+j)) + \sum_{j=1}^{s} \rho_u(m-(Qu+j)) \\
&= \sum_{q=1}^{Q} \sum_{j=1}^{u} \rho_u(m-j-(q-1)u) + \sum_{j=1}^{s} \rho_u(m-j-Qu) \\
&= \sum_{q=1}^{Q} \sum_{j=1}^{u} \rho_u(m-j) + \sum_{j=1}^{s} \rho_u(m-j) \quad \text{(by an identity: } \rho_u(y+u) = \rho_u(y)) \\
&= Q \sum_{j=1}^{u} \rho_u(m-j) + \sum_{j=1}^{s} \rho_u(m-j) \\
&= Q + \sum_{j=1}^{s} \rho_u(m-j)
\end{aligned}$$

(first $\sum_{j=1}^{u} \rho_u(m-j) = \sum_{0 \le j < u} \rho_u(m-j) = \sum_{0 \le j < u} \rho_u(m \bmod u - j)$, where $m \bmod u :=$ the remainder of m divided by u. Secondly, noting that for $0 \le j < u$, $\rho_u(m \bmod u - j) = 1 \Leftrightarrow j \equiv m \bmod u \pmod u \Leftrightarrow j = m \pmod u$, we see $\sum_{j=1}^{u} \rho_u(m-j) = 1$)

$$= \left\lfloor \frac{N}{u} \right\rfloor + \sum_{j=1}^{s} \rho_u(m-j).$$

Therefore

$$\begin{aligned}
&\frac{1}{u} \sum_{m=1}^{M} \sum_{n=1}^{N} \left(\rho_u(m-n) - \frac{1}{u} \right) \\
&= \frac{1}{u} \sum_{m=1}^{M} \sum_{n=1}^{N} \rho_u(m-n) - \frac{MN}{u^2} \\
&= \frac{1}{u} \sum_{m=1}^{M} \left(\left\lfloor \frac{N}{u} \right\rfloor + \sum_{j=1}^{s} \rho_u(m-j) \right) - \frac{MN}{u^2}
\end{aligned}$$

[3]For $a \in \mathbb{R}$, $\lfloor a \rfloor := \max\{n \in \mathbb{Z};\ n \le a\}$ and $\lceil a \rceil := \min\{n \in \mathbb{Z};\ a \le n\}$. We call $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ and $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ the floor function and the ceiling function, respectively. Note that $\{a\} = a - \lfloor a \rfloor \in [0, 1)$.

$$
\begin{aligned}
&= \frac{1}{u}\left(M\left\lfloor \frac{N}{u} \right\rfloor + \sum_{j=1}^{s}\sum_{m=1}^{M} \rho_u(j-m) \right) - \frac{MN}{u^2} \\
&= \frac{1}{u}\left(M\left\lfloor \frac{N}{u} \right\rfloor + \sum_{j=1}^{s}\left(\left\lfloor \frac{M}{u} \right\rfloor + \sum_{i=1}^{r} \rho_u(j-i) \right) \right) - \frac{MN}{u^2} \quad \text{(where } r = \{M/u\}u\text{)} \\
&= \frac{1}{u}\left(M\left\lfloor \frac{N}{u} \right\rfloor + s\left\lfloor \frac{M}{u} \right\rfloor + \sum_{i=1}^{r}\sum_{j=1}^{s} \rho_u(i-j) \right) - \frac{MN}{u^2} \\
&= \frac{1}{u}\left(M\left\lfloor \frac{N}{u} \right\rfloor + s\left\lfloor \frac{M}{u} \right\rfloor + r \wedge s \right) - \frac{MN}{u^2}
\end{aligned}
$$

(for $0 < i, j < u$, $-u < i-j < u$. Also $\rho_u(i-j) = 1 \Leftrightarrow i-j \equiv 0 \pmod u$. Thus $\rho_u(i-j) = 1 \Leftrightarrow i = j$)

$$
\begin{aligned}
&= \frac{r}{u} \wedge \frac{s}{u} - \left(\frac{M}{u}\frac{N}{u} - \frac{M}{u}\left\lfloor \frac{N}{u} \right\rfloor - \frac{s}{u}\left\lfloor \frac{M}{u} \right\rfloor \right) \\
&= \frac{r}{u} \wedge \frac{s}{u} - \frac{r}{u} \cdot \frac{s}{u} \quad \text{(because } \{M/u\} = r/u,\ \{N/u\} = s/u\text{)} \\
&= \left\{ \frac{M}{u} \right\} \wedge \left\{ \frac{N}{u} \right\} \left(1 - \left\{ \frac{M}{u} \right\} \vee \left\{ \frac{N}{u} \right\} \right) \quad \text{(by an identity: } ab = (a \wedge b)(a \vee b)\text{).} \qquad \square
\end{aligned}
$$

Claim 2. *For each $N \in \mathbb{N}$, $\lim_{M\to\infty} E^{\lambda}[Y_M^{(k)} Y_N^{(k)}] = 0$.*

Proof. Let $M \geq N \geq 1$. Since $0 \leq \{M/c^k\}, \{N/c^k\} < 1$,

$$
\begin{aligned}
0 &\leq \left\{ \frac{M}{c^k} \right\} \wedge \left\{ \frac{N}{c^k} \right\} \left(1 - \left\{ \frac{M}{c^k} \right\} \vee \left\{ \frac{N}{c^k} \right\} \right) \\
&\leq \left\{ \frac{N}{c^k} \right\} \left(1 - \left\{ \frac{N}{c^k} \right\} \right).
\end{aligned}
$$

Multiplying both sides by $(1/M^{1/(2k)})(1/N^{1/(2k)})|\mu(c)| \prod_{p\nmid c}(1-2/p^k)$, and then adding them over $c \in \mathbb{N}$ yield that

$$
\begin{aligned}
0 &\leq E^{\lambda}[Y_M^{(k)} Y_N^{(k)}] \\
&\leq \frac{1}{M^{1/(2k)}} \frac{1}{N^{1/(2k)}} \sum_{c=1}^{\infty} |\mu(c)| \left(\prod_{p\nmid c} \left(1 - \frac{2}{p^k} \right) \right) \left\{ \frac{N}{c^k} \right\} \left(1 - \left\{ \frac{N}{c^k} \right\} \right) \\
&= \left(\frac{N}{M} \right)^{1/(2k)} E^{\lambda}[(Y_N^{(k)})^2].
\end{aligned}
$$

From this, the assertion of the claim follows. $\square$

3. Presentation of Proposition 1

By Claim 1

$$
(5)\qquad
\begin{aligned}
E^{\lambda}[(Y_N^{(k)})^2] &= \left(\prod_p \left(1-\frac{2}{p^k}\right)\right)\frac{1}{N^{1/k}}\sum_{c=1}^{\infty}\frac{|\mu(c)|}{\prod_{p|c}(1-2/p^k)}\left\{\frac{N}{c^k}\right\}\left(1-\left\{\frac{N}{c^k}\right\}\right)\\
&= \left(\prod_p \left(1-\frac{2}{p^k}\right)\right)\frac{1}{N^{1/k}}\sum_{c=1}^{\infty} f(c)\left\{\frac{N}{c^k}\right\}\left(1-\left\{\frac{N}{c^k}\right\}\right),
\end{aligned}
$$

where

$$
(6)\qquad f(n) := \frac{|\mu(n)|}{\prod_{p|n}(1-2/p^k)}, \quad n \in \mathbb{N}.
$$

To show the convergence of $E^{\lambda}[(Y_N^{(k)})^2]$ as $N \to \infty$ and to find the value of this limit, we present a general proposition:

Proposition 1. *Let an arithmetic function f, i.e., $f\colon \mathbb{N} \to \mathbb{C}$ satisfy the following condition* (7) *or* (8):

$$
(7)\qquad \sum_{n=1}^{\infty}\frac{1}{n}\left|\sum_{d|n}\mu(d) f\left(\frac{n}{d}\right)\right| < \infty,
$$

(8)
- $\sup_{n\in\mathbb{N}}|f(n)| < \infty$,
- *f has the mean-value $M(f)$, i.e., $\lim_{x\to\infty}(1/x)\sum_{n\le x} f(n)$ is convergent to a finite limit $M(f)$.*

Then, it holds that for $\forall k \in (1, \infty)$[4] *and $\forall h \in C^1[0, 1]$ with $h(0) = 0$*

$$
(9)\qquad \lim_{N\to\infty} N^{-1/k}\sum_{n=1}^{\infty} f(n)h\left(\left\{\frac{N}{n^k}\right\}\right) = M(f)\frac{1}{k}\int_0^{\infty}\frac{h(\{x\})}{x^{1/k+1}}\,dx.
$$

Before proving this proposition, we give some comments on the conditions (7) and (8):

Claim 3. *If $f : \mathbb{N} \to \mathbb{C}$ satisfies the condition* (7), *then f has the mean-value*

$$
M(f) = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{d|n}\mu(d) f\left(\frac{n}{d}\right).
$$

[4] Here k may be a real number, > 1, though k was an integer, ≥ 2 at the beginning of this paper.

Proof. For simplicity, we define $f' : \mathbb{N} \to \mathbb{C}$ by

$$f'(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right), \quad n \in \mathbb{N}. \tag{10}$$

Since, by the Möbius inversion formula

$$f(n) = \sum_{d|n} f'(d), \tag{11}$$

we have for $x, y \in [1, \infty)$

$$\begin{aligned} \frac{1}{x}\sum_{n\le x} f(n) &= \frac{1}{x}\sum_{d\le x}\left(\sum_{n\le x; d|n} 1\right) f'(d) \\ &= \frac{1}{x}\sum_{d\le x}\left\lfloor \frac{x}{d}\right\rfloor f'(d) \\ &= \sum_{d\le x}\frac{f'(d)}{d} - \frac{1}{x}\sum_{d\le x}\left\{\frac{x}{d}\right\} f'(d) \\ &= \sum_{d=1}^{\infty}\frac{f'(d)}{d} - \sum_{d>x}\frac{f'(d)}{d} - \sum_{d\le x/y}\frac{1}{x/d}\left\{\frac{x}{d}\right\}\frac{f'(d)}{d} \\ &\quad - \sum_{x/y<d\le x}\frac{1}{x/d}\left\{\frac{x}{d}\right\}\frac{f'(d)}{d}. \end{aligned}$$

Transposing the first term of the last right-hand side, and then taking the absolute value, we see that

$$\begin{aligned} &\left|\frac{1}{x}\sum_{n\le x} f(n) - \sum_{d=1}^{\infty}\frac{f'(d)}{d}\right| \\ &\le \sum_{d>x}\frac{|f'(d)|}{d} + \sum_{d\le x/y}\frac{1}{x/d}\left\{\frac{x}{d}\right\}\frac{|f'(d)|}{d} + \sum_{x/y<d\le x}\frac{1}{x/d}\left\{\frac{x}{d}\right\}\frac{|f'(d)|}{d} \\ &\le \sum_{d>x}\frac{|f'(d)|}{d} + \frac{1}{y}\sum_{d\le x/y}\frac{|f'(d)|}{d} + \sum_{x/y<d\le x}\frac{|f'(d)|}{d} \\ &\le 2\sum_{d>x/y}\frac{|f'(d)|}{d} + \frac{1}{y}\sum_{d\le x/y}\frac{|f'(d)|}{d}. \end{aligned} \tag{12}$$

By letting $x \to \infty$ and $y \to \infty$, the assertion of the claim follows. □

REMARK 1. Schwarz–Spilker [3] calls Claim 3 Wintner's theorem.

We give an example of f satisfying the condition (7):

EXAMPLE 1. Let $f\colon \mathbb{N} \to \mathbb{C}$ be multiplicative, i.e., $f \neq 0$ and $f(mn) = f(m)f(n)$ provided that $(m, n) = 1$. If, in addition,

$$\sum_p \frac{|f(p) - 1|}{p} < \infty, \quad \sum_p \sum_{l \geq 2} \frac{|f(p^l)|}{p^l} < \infty, \tag{13}$$

then f satisfies the condition (7).

Proof. Multiplicativity of μ and f is inherited to f', and so $|f'|$. In general, multiplicativity of an arithmetic function implies a product representation of Dirichlet series associated with the function. Thus

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n}|f'(n)| &= \prod_p \left(1 + \frac{|f'(p)|}{p} + \frac{|f'(p^2)|}{p^2} + \cdots\right) \\ &\leq \exp\left\{\sum_p \frac{|f'(p)|}{p} + \sum_p \sum_{l \geq 2} \frac{|f'(p^l)|}{p^l}\right\} \\ &\qquad \text{(by an inequality: } 1 + x \leq e^x \ (\forall x \in \mathbb{R})).\end{aligned}$$

Since, by (10)

$$\begin{aligned} f'(p) &= \mu(1)f(p) + \mu(p)f(1)f\left(\frac{p}{d}\right) \\ &= f(p) - 1 \quad \text{(note that } f(1) = 1), \end{aligned} \tag{14}$$

$$\begin{aligned} f'(p^l) &= \mu(1)f(p^l) + \mu(p)f(p^{l-1}) \quad \text{(note that } \mu(p^j) = 0 \ (j \geq 2)) \\ &= f(p^l) - f(p^{l-1}) \quad (l \geq 2), \end{aligned} \tag{15}$$

we have

$$\begin{aligned} &\sum_p \frac{|f'(p)|}{p} + \sum_p \sum_{l \geq 2} \frac{|f'(p^l)|}{p^l} \\ &= \sum_p \frac{|f(p) - 1|}{p} + \sum_p \sum_{l \geq 2} \frac{|f(p^l) - f(p^{l-1})|}{p^l} \\ &\leq \sum_p \frac{|f(p) - 1|}{p} + \sum_p \sum_{l \geq 2} \frac{|f(p^l)|}{p^l} + \sum_p \sum_{l \geq 2} \frac{|f(p^l)|}{p^{l+1}} + \sum_p \frac{|f(p)|}{p^2} \\ &\leq \sum_p \left(1 + \frac{1}{p}\right)\frac{|f(p) - 1|}{p} + \sum_p \frac{1}{p^2} + \sum_p \sum_{l \geq 2} \left(1 + \frac{1}{p}\right)\frac{|f(p^l)|}{p^l} \end{aligned}$$

$$\le \frac{3}{2}\left(\sum_p \frac{|f(p)-1|}{p} + \sum_p \sum_{l\ge 2} \frac{|f(p^l)|}{p^l}\right) + \sum_p \frac{1}{p^2}$$
$$< \infty \quad \text{(by (13))}.$$

Therefore f satisfies the condition (7). □

The condition (7) does not always imply the condition (8).

EXAMPLE 2. Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative, and satisfy

$$f(p) = 1 + \frac{1}{p^\alpha}, \quad f(p^l) = 0 \quad (l \ge 2)$$

for each prime p, where $\alpha \in (0, \infty)$. Since

$$\sum_p \frac{|f(p)-1|}{p} = \sum_p \frac{1}{p^{\alpha+1}} < \infty,$$

$f(\cdot)$ satisfies the condition (7) from Example 1. Also, since

$$f(p_1 \cdots p_m) = f(p_1)\cdots f(p_m)$$
$$= \prod_{i=1}^m \left(1 + \frac{1}{p_i^\alpha}\right) \begin{cases} \le \prod_{i=1}^m e^{1/p_i^\alpha} = e^{\sum_{i=1}^m 1/p_i^\alpha}, \\ \ge \prod_{i=1}^m e^{(1/p_i^\alpha)/(1+1/p_i^\alpha)} \\ \text{(by an inequality: } \log(1+x) \ge x/(1+x)\ (x \ge 0)) \\ \qquad = e^{\sum_{i=1}^m (1/p_i^\alpha)/(1+1/p_i^\alpha)} \\ \qquad \ge e^{(2^\alpha/(2^\alpha+1)) \sum_{i=1}^m 1/p_i^\alpha}, \end{cases}$$

we see that

$$\lim_{m\to\infty} f(p_1 \cdots p_m) \begin{cases} < \infty & \text{if } \alpha > 1, \\ = \infty & \text{if } 0 < \alpha \le 1. \end{cases}$$

This implies that

$$\sup_{n \ge 1} |f(n)| \begin{cases} < \infty & \text{if } \alpha > 1, \\ = \infty & \text{if } 0 < \alpha \le 1. \end{cases}$$

4. Proof of two theorems

Proof of Theorem 1. $f : \mathbb{N} \to \mathbb{C}$, defined by (6), is clearly multiplicative, and satisfies

$$\sum_p \frac{|f(p)-1|}{p} = \sum_p \frac{1}{p^{k+1}} \frac{2}{1-2/p^k} \le \frac{2^k}{2^{k-1}-1} \sum_p \frac{1}{p^{k+1}} < \infty,$$

$$\sum_p \sum_{l \ge 2} \frac{|f(p^l)|}{p^l} = 0 < \infty.$$

Also, note that

$$0 \le f(n) \le e^{\sum_p 4/p^k} \quad (n \in \mathbb{N})$$

(because, by $1/(1-2/p^k) \le 1+4/p^k$, $\prod_{p|n} 1/(1-2/p^k) \le \prod_{p|n}(1+4/p^k) \le \prod_{p|n} e^{4/p^k} = e^{\sum_{p|n} 4/p^k} \le e^{\sum_p 4/p^k}$). Hence, this $f(\cdot)$ satisfies both the condition (7) and the condition (8), so that applying Proposition 1, we see

$$\lim_{N\to\infty} \frac{1}{N^{1/k}} \sum_{c=1}^{\infty} f(c)\left\{\frac{N}{c^k}\right\}\left(1-\left\{\frac{N}{c^k}\right\}\right) = M(f)\frac{1}{k}\int_0^\infty \frac{\{x\}(1-\{x\})}{x^{1/k+1}}\,dx. \tag{16}$$

Let f' be a multiplicative function defined by (10). By (14) and (15)

$$f'(p) = \frac{2/p^k}{1-2/p^k},$$

$$f'(p^l) = \begin{cases} -\dfrac{1}{1-2/p^k}, & l = 2, \\ 0, & l \ge 3 \end{cases}$$

for prime p and integer $l, \ge 2$. Claim 3 then implies that

$$\begin{aligned} M(f) &= \sum_{n=1}^{\infty} \frac{f'(n)}{n} \\ &= \prod_p \left(1 + \frac{f'(p)}{p} + \sum_{l\ge 2} \frac{f'(p^l)}{p^l}\right) \\ &= \prod_p \left(1 + \frac{1}{p}\frac{2/p^k}{1-2/p^k} - \frac{1}{p^2}\frac{1}{1-2/p^k}\right) \\ &= \prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\frac{1}{1-2/p^k}\right). \end{aligned}$$

Collecting (5), (16) and this, we have

$$\lim_{N\to\infty} E^{\lambda}[(Y_N^{(k)})^2] = \left(\prod_p \left(1-\frac{1}{p}\right)\left(1-\frac{2}{p^k}+\frac{1}{p}\right)\right)\frac{1}{k}\int_0^\infty \frac{\{x\}(1-\{x\})}{x^{1/k+1}}\,dx.$$

Let us find the value of an integral on the right-hand side. The Fourier expansion of a function $\{x\}(1-\{x\})$ is as follows:

$$\begin{aligned}\{x\}(1-\{x\}) &= \frac{1}{6}-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^2}\\ &= \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1-\cos 2n\pi x}{n^2} \qquad \left(\text{because } \textstyle\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6\right)\\ &= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2 n\pi x}{n^2}.\end{aligned}$$

Termwise integration yields that

$$\begin{aligned}\frac{1}{k}\int_0^\infty \frac{\{x\}(1-\{x\})}{x^{1/k+1}}\,dx &= \frac{1}{k}\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^\infty \frac{\sin^2 n\pi x}{x^{1/k+1}}\,dx\\ &= \frac{1}{k}\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^\infty \frac{\sin^2 y}{(y/n\pi)^{1/k+1}}\frac{dy}{n\pi}\\ &= \frac{1}{k}\frac{2}{\pi^{2-1/k}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2-1/k}}\right)\int_0^\infty \frac{\sin^2 y}{y^{1/k+1}}\,dy\\ &= \frac{2}{\pi^{2-1/k}}\left(\frac{1}{k}\int_0^\infty \frac{\sin^2 y}{y^{1/k+1}}\,dy\right)\zeta\left(2-\frac{1}{k}\right).\end{aligned}$$

We here note that from a formula: $\int_0^\infty (\sin vx)/x^u\,dx = \pi v^{u-1}/(2\Gamma(u)\sin(u\pi/2))$ $(0 < u < 2,\ v > 0)$

$$\begin{aligned}\frac{1}{k}\int_0^\infty \frac{\sin^2 y}{y^{1/k+1}}\,dy &= \int_0^\infty (-y^{-1/k})'\sin^2 y\,dy\\ &= [-y^{-1/k}\sin^2 y]_0^\infty - \int_0^\infty (-y^{-1/k})2\sin y\cos y\,dy\\ &= \int_0^\infty \frac{\sin 2y}{y^{1/k}}\,dy\end{aligned}$$

(because $\lim_{y\to 0} y^{-1/k}\sin^2 y = \lim_{y\to 0} y^{2-1/k}((\sin y)/y)^2 = 0$, $\lim_{y\to\infty} y^{-1/k}\sin^2 y = \lim_{y\to\infty}\sin^2 y/y^{1/k} = 0$)

$$= \frac{\pi 2^{1/k-1}}{2\Gamma(1/k)\sin(\pi/2k)}.$$

Substituting this into the above, we have

$$\frac{1}{k}\int_0^\infty \frac{\{x\}(1-\{x\})}{x^{1/k+1}}\,dx = \frac{2}{\pi^{2-1/k}}\frac{\pi 2^{1/k-1}}{2\Gamma(1/k)\sin(\pi/2k)}\zeta\left(2-\frac{1}{k}\right)$$
$$= \frac{\zeta(2-1/k)}{(2\pi)^{1-1/k}\Gamma(1/k)\sin(\pi/2k)}.$$

Consequently, the assertion of the theorem follows at once. □

REMARK 2. Since, by the functional equation

$$\zeta(s) = 2\Gamma(1-s)\sin\left(\frac{\pi}{2}s\right)(2\pi)^{s-1}\zeta(1-s)$$

of the Riemann zeta function,

$$\zeta\left(2-\frac{1}{k}\right) = 2\Gamma\left(\frac{1}{k}-1\right)\left(\sin\frac{\pi}{2k}\right)(2\pi)^{1-1/k}\zeta\left(\frac{1}{k}-1\right),$$

we see

$$\frac{\zeta(2-1/k)}{(2\pi)^{1-1/k}\Gamma(1/k)\sin(\pi/(2k))} = \frac{2\Gamma(1/k-1)(\sin(\pi/(2k)))(2\pi)^{1-1/k}\zeta(1/k-1)}{(2\pi)^{1-1/k}\Gamma(1/k)\sin(\pi/(2k))}$$
$$= 2\frac{\zeta(1/k-1)}{1/k-1}.$$

Then the appearance of Theorem 1 becomes good as

$$\lim_{N\to\infty} E^\lambda[(Y_N^{(k)})^2] = \left(\prod_p\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}-\frac{2}{p^k}\right)\right)2\frac{\zeta(1/k-1)}{1/k-1}.$$

Proof of Theorem 2. (i) For $M \geq N \geq 1$

$$E^\lambda[(Y_M^{(k)}-Y_N^{(k)})^2] = E^\lambda[(Y_M^{(k)})^2] - 2E^\lambda[Y_M^{(k)}Y_N^{(k)}] + E^\lambda[(Y_N^{(k)})^2].$$

The assertion of (i) is obvious from Claim 2 and Theorem 1.

(ii) By Theorem 1, $\{Y_N^{(k)}\}_{N=1}^\infty$ is L^2-bounded, and thus for any subsequence $\{N_i\}_{i=1}^\infty$

$$\exists\{i_m\}_{m=1}^\infty\text{: subsequence, } \exists Y\in L^2(\hat{\mathbb{Z}},\mathcal{B},\lambda) \quad \text{s.t.} \quad \text{w-lim}_{m\to\infty} Y_{N_{i_m}}^{(k)} = Y.$$

Then

$$\lim_{m\to\infty} E^{\lambda}[Y^{(k)}_{N_{i_m}} Y^{(k)}_{N_{i_n}}] = E^{\lambda}[Y Y^{(k)}_{N_{i_n}}], \quad \forall n \in \mathbb{N}.$$

But, by Claim 2

$$E^{\lambda}[Y Y^{(k)}_{N_{i_n}}] = 0 \quad (\forall n \in \mathbb{N}).$$

Letting $n \to \infty$ yields that $E^{\lambda}[Y^2] = 0$. This implies that $\text{w-lim}_{N\to\infty} Y^{(k)}_N = 0$. □

5. Proof of Proposition 1

We now take up the proof of Proposition 1.

Suppose $f(\cdot)$ satisfies the condition (7) or (8). Fix $k \in (1, \infty)$ and $h \in C^1[0, 1]$ with $h(0) = 0$. We divide $N^{-1/k} \sum_{n=1}^{\infty} f(n)h(\{N/n^k\})$ into three terms as

$$\begin{aligned} N^{-1/k} \sum_{n=1}^{\infty} f(n) h\left(\left\{\frac{N}{n^k}\right\}\right) &= M(f) N^{-1/k} \sum_{n\le N^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right) \\ &\quad + N^{-1/k} \sum_{n\le N^{1/k}} (f(n) - M(f)) h\left(\left\{\frac{N}{n^k}\right\}\right) \\ &\quad + N^{-1/k} \sum_{n> N^{1/k}} f(n) h\left(\frac{N}{n^k}\right). \end{aligned} \tag{17}$$

To find a limit of each term as $N \to \infty$, we present the following lemma:

Lemma 1. *Let $1 \le a < b < \infty$ and $\varphi \in C^1[a, b]$.*

(i) *Given a sequence $\{a_n\}_{n=1}^{\infty}$, set $S(t) = \sum_{n\le t} a_n$ $(t \in \mathbb{R})$. Then, for $a \le \forall x < \forall y \le b$*

$$\sum_{x<n\le y} a_n \varphi(n) = -\int_x^y S(t)\varphi'(t)\,dt + S(y)\varphi(y) - S(x)\varphi(x).$$

(ii) *For $a \le \forall x < \forall y \le b$*

$$\begin{aligned} \sum_{x<n\le y} \varphi(n) &= \int_x^y \varphi(t)\,dt \\ &\quad - \left(\left(\{y\} - \frac{1}{2}\right)\varphi(y) - \left(\{x\} - \frac{1}{2}\right)\varphi(x)\right) + \int_x^y \left(\{t\} - \frac{1}{2}\right)\varphi'(t)\,dt. \end{aligned}$$

Proof. Let $1 \le a < b < \infty$, $\varphi \in C^1[a, b]$ and $a \le x < y \le b$.

(i) In case $\lfloor x \rfloor < \lfloor y \rfloor$, noting that $a \le x < \lfloor x \rfloor + 1 \le \lfloor y \rfloor \le y \le b$, we have

$$\begin{aligned}
&\text{the left-hand side} \\
&= \sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor} a_n \varphi(n) \\
&= \sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor} (S(n) - S(n-1))\varphi(n) \\
&= \sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor} S(n)\varphi(n) - \sum_{\lfloor x \rfloor \le n \le \lfloor y \rfloor - 1} S(n)\varphi(n+1) \\
&= \sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor - 1} S(n)(\varphi(n) - \varphi(n+1)) + S(\lfloor y \rfloor)\varphi(\lfloor y \rfloor) - S(\lfloor x \rfloor)\varphi(\lfloor x \rfloor + 1) \\
&= -\sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor - 1} \int_n^{n+1} S(n)\varphi'(t)\,dt + S(\lfloor y \rfloor)\varphi(\lfloor y \rfloor) - S(\lfloor x \rfloor)\varphi(\lfloor x \rfloor + 1) \\
&= -\sum_{\lfloor x \rfloor < n \le \lfloor y \rfloor - 1} \int_n^{n+1} S(t)\varphi'(t)\,dt + S(\lfloor y \rfloor)\varphi(\lfloor y \rfloor) - S(\lfloor x \rfloor)\varphi(\lfloor x \rfloor + 1)
\end{aligned}$$

(because $S(t) = S(\lfloor t \rfloor)$)

$$\begin{aligned}
&= -\int_{\lfloor x \rfloor + 1}^{\lfloor y \rfloor} S(t)\varphi'(t)\,dt + S(\lfloor y \rfloor)\varphi(\lfloor y \rfloor) - S(\lfloor x \rfloor)\varphi(\lfloor x \rfloor + 1) \\
&= -\int_x^y S(t)\varphi'(t)\,dt + \int_x^{\lfloor x \rfloor + 1} S(t)\varphi'(t)\,dt + \int_{\lfloor y \rfloor}^y S(t)\varphi'(t)\,dt \\
&\quad + S(\lfloor y \rfloor)\varphi(\lfloor y \rfloor) - S(\lfloor x \rfloor)\varphi(\lfloor x \rfloor + 1) \\
&= -\int_x^y S(t)\varphi'(t)\,dt + S(y)\varphi(y) - S(x)\varphi(x) \\
&= \text{the right-hand side.}
\end{aligned}$$

In case $\lfloor x \rfloor = \lfloor y \rfloor$, since $\lfloor x \rfloor \le x < y < \lfloor y \rfloor + 1 = \lfloor x \rfloor + 1$,

$$\text{the left-hand side} = \sum_{\lfloor x \rfloor < n \le \lfloor x \rfloor} a_n \varphi(n) = 0,$$

$$\text{the right-hand side} = -S(\lfloor x \rfloor)(\varphi(y) - \varphi(x)) + S(\lfloor x \rfloor)(\varphi(y) - \varphi(x)) = 0.$$

Thus, we obtain the assertion of (i).

(ii) Let $a_n = 1$ $(n \in \mathbb{N})$. In this case, $S(t) = \lfloor t \rfloor$ $(t \geq 0)$, so by (i)

$$\begin{aligned}
\sum_{x<n\leq y} \varphi(n) &= -\int_x^y \lfloor t \rfloor \varphi'(t)\,dt + \lfloor y \rfloor \varphi(y) - \lfloor x \rfloor \varphi(x) \\
&= -\int_x^y t\varphi'(t)\,dt + \int_x^y \{t\}\varphi'(t)\,dt \\
&\quad + y\varphi(y) - x\varphi(x) - \{y\}\varphi(y) + \{x\}\varphi(x) \\
&= -[t\varphi(t)]_x^y + \int_x^y \varphi(t)\,dt + \int_x^y \left(\{t\} - \frac{1}{2}\right)\varphi'(t)\,dt \\
&\quad + \frac{1}{2}(\varphi(y) - \varphi(x)) + [t\varphi(t)]_x^y - \{y\}\varphi(y) + \{x\}\varphi(x) \\
&= \int_x^y \varphi(t)\,dt - \left(\left(\{y\} - \frac{1}{2}\right)\varphi(y) - \left(\{x\} - \frac{1}{2}\right)\varphi(x)\right) \\
&\quad + \int_x^y \left(\{t\} - \frac{1}{2}\right)\varphi'(t)\,dt.
\end{aligned}$$

□

REMARK 3. This identity is called the Euler summation formula (cf. [3, Theorem 1.2 in Chapter I]) or the Euler–Maclaurin summation formula (cf. [2, Lemma 2.1]).

Proof of Proposition 1 under the condition (7).

<u>1°</u> The first term of (17).

<u>1°-1</u> For $L \in \mathbb{N}$ with $L + 1 \leq N$,

$$\begin{aligned}
&\sum_{(N/(L+1))^{1/k} < n \leq N^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right) \\
&= \sum_{l=1}^{L} \sum_{n;\lfloor N/n^k \rfloor = l} h\left(\left\{\frac{N}{n^k}\right\}\right)
\end{aligned}$$

(note that $(N/(L+1))^{1/k} < n \leq N^{1/k} \Leftrightarrow 1 \leq \lfloor N/n^k \rfloor \leq L$)

$$= \sum_{l=1}^{L} \sum_{l \leq N/n^k < l+1} h\left(\frac{N}{n^k} - l\right)$$

(when $\lfloor N/n^k \rfloor = l$, $\{N/n^k\} = N/n^k - l$. Also $\lfloor N/n^k \rfloor = l \Leftrightarrow l \leq N/n^k < l + 1$)

$$= \sum_{l=1}^{L} \sum_{(N/(l+1))^{1/k} < n \leq (N/l)^{1/k}} h\left(\frac{N}{n^k} - l\right)$$

$$= \sum_{l=1}^{L}\left(\int_{(N/(l+1))^{1/k}}^{(N/l)^{1/k}} h\left(\frac{N}{t^k}-l\right)dt + h(1)\left(\left\{\left(\frac{N}{l+1}\right)^{1/k}\right\}-\frac{1}{2}\right)\right.$$
$$\left. - k\int_{(N/(l+1))^{1/k}}^{(N/l)^{1/k}} h'\left(\frac{N}{t^k}-l\right)\frac{\{t\}-1/2}{t^{k+1}}N\,dt\right)$$

(apply Lemma 1 (ii) for $\varphi(t)=h(N/t^k-l)$ $((N/(l+1))^{1/k}\le t\le (N/l)^{1/k})$)

$$= \sum_{l=1}^{L}\left(\int_{(N/(l+1))^{1/k}}^{(N/l)^{1/k}} h\left(\left\{\frac{N}{t^k}\right\}\right)dt + h(1)\left(\left\{\left(\frac{N}{l+1}\right)^{1/k}\right\}-\frac{1}{2}\right)\right.$$
$$\left. - k\int_{(N/(l+1))^{1/k}}^{(N/l)^{1/k}} h'\left(\left\{\frac{N}{t^k}\right\}\right)\frac{\{t\}-1/2}{t^{k+1}}N\,dt\right)$$

(note that $(N/(l+1))^{1/k}<t\le (N/l)^{1/k}\Leftrightarrow \lfloor N/t^k\rfloor = l$)

$$= \int_{(N/(L+1))^{1/k}}^{N^{1/k}} h\left(\left\{\frac{N}{t^k}\right\}\right)dt + h(1)\sum_{l=1}^{L}\left(\left\{\left(\frac{N}{l+1}\right)^{1/k}\right\}-\frac{1}{2}\right)$$
$$- k\int_{(N/(L+1))^{1/k}}^{N^{1/k}} h'\left(\left\{\frac{N}{t^k}\right\}\right)\frac{\{t\}-1/2}{t^{k+1}}N\,dt$$
$$= N^{1/k}\frac{1}{k}\int_1^{L+1}\frac{h(\{x\})}{x^{1/k+1}}dx + h(1)\sum_{l=1}^{L}\left(\left\{\left(\frac{N}{l+1}\right)^{1/k}\right\}-\frac{1}{2}\right)$$
$$- \int_1^{L+1} h'(\{x\})\left(\left\{\left(\frac{N}{x}\right)^{1/k}\right\}-\frac{1}{2}\right)dx \quad \text{(by change of variable: } x=N/t^k\text{).}$$

<u>1°-2</u> Let $N\gg 1$ and $L=L(N)=\lfloor N^{1/(k+1)}\rfloor$. Then $L(N)\le N^{1/(k+1)}<L(N)+1$, and so $L(N)+1<N$, $1/(L(N)+1)<(1/N)^{1/(k+1)}$. Since, by 1°-1

$$N^{-1/k}\sum_{n\le N^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right)$$
$$= N^{-1/k}\sum_{n\le (N/(L(N)+1))^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right) + N^{-1/k}\sum_{(N/(L(N)+1))^{1/k}<n\le N^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right)$$
$$= N^{-1/k}\sum_{n\le (N/(L(N)+1))^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right)$$
$$+ \frac{1}{k}\int_1^{L(N)+1}\frac{h(\{x\})}{x^{1/k+1}}dx + h(1)N^{-1/k}\sum_{l=1}^{L(N)}\left(\left\{\left(\frac{N}{l+1}\right)^{1/k}\right\}-\frac{1}{2}\right)$$
$$- N^{-1/k}\int_1^{L(N)+1} h'(\{x\})\left(\left\{\left(\frac{N}{x}\right)^{1/k}\right\}-\frac{1}{2}\right)dx,$$

we see

$$
\begin{aligned}
&\left| N^{-1/k} \sum_{n \le N^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right) - \frac{1}{k}\int_1^{L(N)+1} \frac{h(\{x\})}{x^{1/k+1}}\,dx \right| \\
&\le N^{-1/k} \sum_{n \le (N/(L(N)+1))^{1/k}} \left| h\left(\left\{\frac{N}{n^k}\right\}\right)\right| \\
&\quad + |h(1)| N^{-1/k} \sum_{l=1}^{L(N)} \left| \left\{\left(\frac{N}{l+1}\right)^{1/k}\right\} - \frac{1}{2}\right| \\
&\quad + N^{-1/k} \int_1^{L(N)+1} |h'(\{x\})| \left| \left\{\left(\frac{N}{x}\right)^{1/k}\right\} - \frac{1}{2}\right| dx \\
&\le N^{-1/k}\left(\frac{N}{L(N)+1}\right)^{1/k}\left(\max_{0\le x\le 1}|h(x)|\right) + |h(1)| N^{-1/k} L(N)\cdot\frac{1}{2} \\
&\quad + N^{-1/k} L(N)\left(\max_{0\le x\le 1}|h'(x)|\right)\cdot\frac{1}{2} \\
&\le \left(\left(\frac{1}{L(N)+1}\right)^{1/k} + \left(\frac{1}{N}\right)^{1/k} L(N)\right)\left(\max_{0\le x\le 1}|h'(x)|\right)
\end{aligned}
$$

(note that $\max_{0\le x\le 1}|h(x)| \le \max_{0\le x\le 1}|h'(x)|$)

$$
\begin{aligned}
&\le \left(\left(\frac{1}{N}\right)^{(1/k)\cdot(1/(k+1))} + \left(\frac{1}{N}\right)^{1/k-1/(k+1)}\right)\left(\max_{0\le x\le 1}|h'(x)|\right) \\
&= 2\left(\frac{1}{N}\right)^{1/(k(k+1))}\left(\max_{0\le x\le 1}|h'(x)|\right) \to 0 \quad \text{as} \quad N \to \infty.
\end{aligned}
$$

This shows that

$$
\text{(18)} \qquad \text{the first term of (17)} \to M(f)\frac{1}{k}\int_1^{\infty} \frac{h(\{x\})}{x^{1/k+1}}\,dx \quad \text{as} \quad N \to \infty.
$$

<u>2°</u> The second term of (17).

For simplicity, set $a_n = f(n) - M(f)$, $S(x) = \sum_{n\le x} a_n$.

<u>2°-1</u> First

$$
\begin{aligned}
\frac{1}{y}|S(y)| &= \frac{1}{y}\left|\sum_{n\le y} f(n) - \lfloor y\rfloor M(f)\right| \\
&= \left|\frac{1}{y}\sum_{n\le y} f(n) - M(f) + \frac{\{y\}}{y}M(f)\right|
\end{aligned}
$$

$$\leq \left| \frac{1}{y} \sum_{n \leq y} f(n) - M(f) \right| + \frac{1}{y} |M(f)|$$
$$\to 0 \quad \text{as} \quad y \to \infty.$$

2°-2 In the same way as in 1°-1, we have that for $L \in \mathbb{N}$ with $L + 1 \leq N$

$$\sum_{(N/(L+1))^{1/k} < n \leq N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right)$$
$$= \sum_{l=1}^{L} \sum_{(N/(l+1))^{1/k} < n \leq (N/l)^{1/k}} a_n h\left(\frac{N}{n^k} - l\right)$$
$$= \sum_{l=1}^{L} \left(k \int_{(N/(l+1))^{1/k}}^{(N/l)^{1/k}} h'\left(\frac{N}{t^k} - l\right) \frac{S(t)}{t^{k+1}} N\, dt - h(1) S\left(\left(\frac{N}{l+1}\right)^{1/k}\right)\right)$$

(apply Lemma 1 (i) for $\varphi(t) = h(N/t^k - l)$ $((N/(l+1))^{1/k} \leq t \leq (N/l)^{1/k})$)

$$= k \int_{(N/(L+1))^{1/k}}^{N^{1/k}} h'\left(\left\{\frac{N}{t^k}\right\}\right) \frac{S(t)}{t^{k+1}} N\, dt - \sum_{l=1}^{L} h(1) S\left(\left(\frac{N}{l+1}\right)^{1/k}\right)$$
$$= \int_1^{L+1} h'(\{x\}) S\left(\left(\frac{N}{x}\right)^{1/k}\right) dx - \sum_{l=1}^{L} \int_l^{l+1} h(1) S\left(\left(\frac{N}{l+1}\right)^{1/k}\right) dx$$
$$= \int_1^{L+1} \left(h'(\{x\}) S\left(\left(\frac{N}{x}\right)^{1/k}\right) - h(1) S\left(\left(\frac{N}{\lceil x \rceil}\right)^{1/k}\right)\right) dx.$$

2°-3 Fix $L \in \mathbb{N}$ with $L + 1 \leq N$. By 2°-2

$$\left| N^{-1/k} \sum_{(N/(L+1))^{1/k} < n \leq N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \right|$$
$$\leq N^{-1/k} \int_1^{L+1} \left(|h'(\{x\})| \left| S\left(\left(\frac{N}{x}\right)^{1/k}\right) \right| + |h(1)| \left| S\left(\left(\frac{N}{\lceil x \rceil}\right)^{1/k}\right) \right| \right) dx$$
$$\leq \left(\max_{0 \leq x \leq 1} |h'(x)|\right) N^{-1/k} \int_1^{L+1} \left(\frac{|S((N/x)^{1/k})|}{(N/x)^{1/k}} \left(\frac{N}{x}\right)^{1/k} + \frac{|S((N/\lceil x \rceil)^{1/k})|}{(N/\lceil x \rceil)^{1/k}} \left(\frac{N}{\lceil x \rceil}\right)^{1/k}\right) dx$$
$$\leq \left(\max_{0 \leq x \leq 1} |h'(x)|\right) N^{-1/k} \int_1^{L+1} \left(\frac{|S((N/x)^{1/k})|}{(N/x)^{1/k}} + \frac{|S((N/\lceil x \rceil)^{1/k})|}{(N/\lceil x \rceil)^{1/k}}\right) \left(\frac{N}{x}\right)^{1/k} dx$$

(note that $1 \leq x \leq L+1 \Rightarrow 1 \leq x \leq \lceil x \rceil \leq L+1 \Rightarrow (N/(L+1))^{1/k} \leq (N/\lceil x \rceil)^{1/k} \leq (N/x)^{1/k}$)

$$\leq \left(\max_{0 \leq x \leq 1} |h'(x)|\right) \left(2 \sup_{y \geq (N/(L+1))^{1/k}} \frac{|S(y)|}{y}\right) \int_1^{L+1} x^{-1/k}\, dx$$

$$= \left(\max_{0\le x\le 1}|h'(x)|\right)\left(2\sup_{y\ge(N/(L+1))^{1/k}}\frac{|S(y)|}{y}\right)\left[\frac{x^{1-1/k}}{1-1/k}\right]_1^{L+1}$$
$$\le \left(\max_{0\le x\le 1}|h'(x)|\right)\frac{k}{k-1}\left(2\sup_{y\ge(N/(L+1))^{1/k}}\frac{|S(y)|}{y}\right)(L+1)^{1-1/k}$$
$$\le \left(\left(\max_{0\le x\le 1}|h'(x)|\right)\frac{2k}{k-1}+1\right)\left(\sup_{y\ge(N/(L+1))^{1/k}}\frac{|S(y)|}{y}\right)(L+1)^{1-1/k}.$$

Also

$$\left|N^{-1/k}\sum_{n\le(N/(L+1))^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right)\right|$$
$$= \left|N^{-1/k}\sum_{n\le(N/(L+1))^{1/k}} f(n)h\left(\left\{\frac{N}{n^k}\right\}\right) - M(f)N^{-1/k}\sum_{n\le(N/(L+1))^{1/k}} h\left(\left\{\frac{N}{n^k}\right\}\right)\right|$$
$$\le \left(\max_{0\le x\le 1}|h(x)|\right)\left(N^{-1/k}\sum_{n\le(N/(L+1))^{1/k}}|f(n)| + |M(f)|N^{-1/k}\sum_{n\le(N/(L+1))^{1/k}} 1\right)$$
$$\le \left(\max_{0\le x\le 1}|h(x)|\right)\left(N^{-1/k}\sum_{n\le(N/(L+1))^{1/k}}\sum_{d|n}|f'(d)| + |M(f)|N^{-1/k}\left(\frac{N}{L+1}\right)^{1/k}\right)$$
$$= \left(\max_{0\le x\le 1}|h(x)|\right)\left(N^{-1/k}\sum_{d\le(N/(L+1))^{1/k}}\left\lfloor\frac{1}{d}\left(\frac{N}{L+1}\right)^{1/k}\right\rfloor|f'(d)| + |M(f)|\left(\frac{1}{L+1}\right)^{1/k}\right)$$
$$\le \left(\max_{0\le x\le 1}|h(x)|\right)\left(N^{-1/k}\left(\frac{N}{L+1}\right)^{1/k}\sum_{d\le(N/(L+1))^{1/k}}\frac{|f'(d)|}{d} + |M(f)|\left(\frac{1}{L+1}\right)^{1/k}\right)$$
$$\le \left(\max_{0\le x\le 1}|h(x)|\right)\left(\frac{1}{L+1}\right)^{1/k} 2\sum_{d=1}^{\infty}\frac{|f'(d)|}{d}$$
$$\le \left(2\left(\max_{0\le x\le 1}|h(x)|\right)\sum_{d=1}^{\infty}\frac{|f'(d)|}{d}+1\right)(L+1)^{-1/k}.$$

Combining two estimates above, we have

$$
\begin{aligned}
&\left| N^{-1/k} \sum_{n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \right| \\
&\le \left(\left(\max_{0\le x\le 1} |h'(x)|\right)\frac{2k}{k-1} + 1\right)\left(\sup_{y \ge (N/(L+1))^{1/k}} \frac{|S(y)|}{y}\right)(L+1)^{1-1/k} \\
&\quad + \left(2\left(\max_{0\le x\le 1}|h(x)|\right)\sum_{d=1}^{\infty}\frac{|f'(d)|}{d} + 1\right)(L+1)^{-1/k} \\
&=: A\left(\sup_{y\ge (N/(L+1))^{1/k}} \frac{|S(y)|}{y}\right)(L+1)^{1-1/k} + B(L+1)^{-1/k}.
\end{aligned} \tag{19}
$$

2°-4 Take $\varepsilon > 0$ so that $B/((k-1)A\varepsilon) > 2$. By 2°-1

$$\exists y_0 > 1 \text{ s.t. } \frac{|S(y)|}{y} < \varepsilon \quad (\forall y \ge y_0).$$

Let $L = \lfloor B/((k-1)A\varepsilon)\rfloor - 1 \in \mathbb{N}$. For $N \ge y_0^k B/((k-1)A\varepsilon)$,

$$\frac{N}{y_0^k} \ge \left\lfloor \frac{N}{y_0^k} \right\rfloor \ge \left\lfloor \frac{B}{(k-1)A\varepsilon} \right\rfloor = L+1.$$

Since $(N/(L+1))^{1/k} \ge y_0$, $\sup_{y\ge(N/(L+1))^{1/k}} |S(y)|/y \le \varepsilon$. Using this in (19), we have

$$
\begin{aligned}
&\left| N^{-1/k} \sum_{n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \right| \\
&\le A\varepsilon(L+1)^{1-1/k} + B(L+1)^{-1/k} \\
&= A\varepsilon \left\lfloor \frac{B}{(k-1)A\varepsilon} \right\rfloor^{1-1/k} + B\left\lfloor \frac{B}{(k-1)A\varepsilon}\right\rfloor^{-1/k} \\
&\le A\varepsilon\left(\frac{B}{(k-1)A\varepsilon}\right)^{1-1/k} + B\left(\frac{B}{(k-1)A\varepsilon} - 1\right)^{-1/k} \\
&\le A\varepsilon\left(\frac{B}{(k-1)A\varepsilon}\right)^{1-1/k} + B\left(\frac{1}{2}\frac{B}{(k-1)A\varepsilon}\right)^{-1/k} \\
&= (k-1)^{1/k-1}A^{1/k}B^{1-1/k}\varepsilon^{1/k} + 2^{1/k}(k-1)^{1/k}A^{1/k}B^{1-1/k}\varepsilon^{1/k} \\
&= ((k-1)^{1/k-1} + 2^{1/k}(k-1)^{1/k})A^{1/k}B^{1-1/k}\varepsilon^{1/k}.
\end{aligned}
$$

Letting $N \to \infty$ yields

$$\begin{aligned}
&\overline{\lim_{N\to\infty}} \left| N^{-1/k} \sum_{n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right)\right| \\
&\le ((k-1)^{1/k-1} + 2^{1/k}(k-1)^{1/k}) A^{1/k} B^{1-1/k} \varepsilon^{1/k} \to 0 \quad \text{as} \quad \varepsilon \searrow 0.
\end{aligned}$$

This shows that

$$\text{the second term of (17)} \to 0 \quad \text{as} \quad N \to \infty. \tag{20}$$

<u>3°</u> The third term of (17).

<u>3°-1</u> First, we check the convergence of a series $\sum_{n>N^{1/k}} f(n)h(N/n^k)$. Let $L, M \in \mathbb{N}$, $N \le L < M$. Lemma 1 (i) for $\varphi(t) = h(N/t^k)$ $(L^{1/k} \le t \le M^{1/k})$ tells us that

$$\begin{aligned}
&\sum_{L^{1/k} < n \le M^{1/k}} f(n) h\left(\frac{N}{n^k}\right) \\
&= \int_{L^{1/k}}^{M^{1/k}} \left(\frac{1}{t}\sum_{n\le t} f(n)\right) h'\left(\frac{N}{t^k}\right)\frac{Nk}{t^k}\,dt \\
&\quad + \left(\frac{1}{M^{1/k}}\sum_{n\le M^{1/k}} f(n)\right)\frac{h(N/M)}{N/M}\frac{N}{M^{1-1/k}} - \left(\frac{1}{L^{1/k}}\sum_{n\le L^{1/k}} f(n)\right)\frac{h(N/L)}{N/L}\frac{N}{L^{1-1/k}}.
\end{aligned}$$

Here, noting that since $k > 1$, $\int_1^\infty dt/t^k < \infty$, $\lim_{M\to\infty} 1/M^{1-1/k} = \lim_{L\to\infty} 1/L^{1-1/k} = 0$ and since $h(0) = 0$, $\lim_{x\to 0} h(x)/x = h'(0)$, we see the convergence of this series.

<u>3°-2</u> Next, letting $L = N$ and $M \to \infty$ in the above yields that

$$\begin{aligned}
&\sum_{n > N^{1/k}} f(n) h\left(\frac{N}{n^k}\right) \\
&= \int_{N^{1/k}}^{\infty} \left(\frac{1}{t}\sum_{n\le t} f(n)\right) h'\left(\frac{N}{t^k}\right)\frac{Nk}{t^k}\,dt - N^{1/k}\left(\frac{1}{N^{1/k}}\sum_{n\le N^{1/k}} f(n)\right) h(1) \\
&= \int_1^\infty \left(\frac{1}{N^{1/k}\tau}\sum_{n\le N^{1/k}\tau} f(n)\right) h'(\tau^{-k})\frac{k}{\tau^k} N^{1/k}\,d\tau \\
&\quad - N^{1/k}\left(\frac{1}{N^{1/k}}\sum_{n\le N^{1/k}} f(n)\right) h(1) \quad (\text{by change of variable: } \tau = t/N^{1/k}).
\end{aligned}$$

By multiplying both sides by $N^{-1/k}$, it turns out that

$$\begin{aligned}&\text{the third term of (17)}\\&=\int_1^\infty\left(\frac{1}{N^{1/k}\tau}\sum_{n\le N^{1/k}\tau}f(n)\right)h'(\tau^{-k})\frac{k}{\tau^k}\,d\tau-\left(\frac{1}{N^{1/k}}\sum_{n\le N^{1/k}}f(n)\right)h(1).\end{aligned}$$

Thus, by the Lebesgue convergence theorem,

(21)

$$\begin{aligned}\text{the third term of (17)}\to&\int_1^\infty M(f)h'(\tau^{-k})k\tau^{-k}\,d\tau-M(f)h(1)\\&=M(f)\left(\int_1^\infty(h'(\tau^{-k})k\tau^{-k-1})\tau\,d\tau-h(1)\right)\\&=M(f)\left(\int_1^\infty(-h(\tau^{-k}))'\tau\,d\tau-h(1)\right)\\&=M(f)\left([-h(\tau^{-k})\tau]_1^\infty+\int_1^\infty h(\tau^{-k})\,d\tau-h(1)\right)\\&=M(f)\int_1^\infty h(\tau^{-k})\,d\tau\end{aligned}$$

(because $h(\tau^{-k})\tau=(h(\tau^{-k})/\tau^{-k})(1/\tau^{k-1})\to 0$ as $\tau\to\infty$)

$$=M(f)\frac{1}{k}\int_0^1\frac{h(x)}{x^{1/k+1}}\,dx$$

(by change of variable: $x=\tau^{-k}$)

$$=M(f)\frac{1}{k}\int_0^1\frac{h(\{x\})}{x^{1/k+1}}\,dx\quad\text{as}\quad N\to\infty.$$

4° Collecting (18), (20) and (20), we have

$$\begin{aligned}&\lim_{N\to\infty}N^{-1/k}\sum_{n=1}^\infty f(n)h\left(\left\{\frac{N}{n^k}\right\}\right)\\&=M(f)\frac{1}{k}\int_1^\infty\frac{h(\{x\})}{x^{1/k+1}}\,dx+M(f)\frac{1}{k}\int_0^1\frac{h(\{x\})}{x^{1/k+1}}\,dx\\&=M(f)\frac{1}{k}\int_0^\infty\frac{h(\{x\})}{x^{1/k+1}}\,dx.\end{aligned}$$

□

Proof of Proposition 1 under the condition (8). The argument of 1° and 3° in the previous proof is also valid in this case. Thus we have the convergences (18) and (21).

In the following, we slightly modify the argument of 2° in the previous proof. Let $a_n = f(n) - M(f)$, $S(x) = \sum_{n \le x} a_n$.

2°-1 First

$$\frac{1}{y}|S(y)| \to 0 \quad \text{as} \quad y \to \infty.$$

2°-2 For $L \in \mathbb{N}$ with $L + 1 \le N$

$$\begin{aligned}
&\sum_{(N/(L+1))^{1/k} < n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \\
&= \int_1^{L+1} \left(h'(\{x\}) S\left(\left(\frac{N}{x}\right)^{1/k}\right) - h(1) S\left(\left(\frac{N}{\lceil x \rceil}\right)^{1/k}\right)\right) dx.
\end{aligned}$$

2°-3 Fix $L \in \mathbb{N}$ with $L + 1 \le N$. By 2°-2

$$\begin{aligned}
&\left| N^{-1/k} \sum_{(N/(L+1))^{1/k} < n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \right| \\
&\le \left(\left(\max_{0 \le x \le 1} |h'(x)|\right) \frac{2k}{k-1} + 1\right) \left(\sup_{y \ge (N/(L+1))^{1/k}} \frac{|S(y)|}{y}\right) (L+1)^{1-1/k}.
\end{aligned}$$

Clearly

$$\begin{aligned}
\left| N^{-1/k} \sum_{n \le (N/(L+1))^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \right| &\le N^{-1/k} \sum_{n \le (N/(L+1))^{1/k}} |a_n| \left| h\left(\left\{\frac{N}{n^k}\right\}\right) \right| \\
&\le \left(\sup_{n \ge 1} |a_n|\right) \left(\max_{0 \le x \le 1} |h(x)|\right) (L+1)^{-1/k} \\
&\le \left(\left(\sup_{n \ge 1} |a_n|\right) \left(\max_{0 \le x \le 1} |h(x)|\right) + 1\right) (L+1)^{-1/k}.
\end{aligned}$$

Combining these estimates, we have

$$\begin{aligned}
&\left| N^{-1/k} \sum_{n \le N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right) \right| \\
&\le \left(\left(\max_{0 \le x \le 1} |h'(x)|\right) \frac{2k}{k-1} + 1\right) \left(\sup_{y \ge (N/(L+1))^{1/k}} \frac{|S(y)|}{y}\right) (L+1)^{1-1/k} \\
&\quad + \left(\left(\sup_{n \ge 1} |a_n|\right) \left(\max_{0 \le x \le 1} |h(x)|\right) + 1\right) (L+1)^{-1/k} \qquad (22) \\
&=: A \left(\sup_{y \ge (N/(L+1))^{1/k}} \frac{|S(y)|}{y}\right) (L+1)^{1-1/k} + C (L+1)^{-1/k}.
\end{aligned}$$

2°-4 Take $\varepsilon > 0$ so that $C/((k-1)A\varepsilon) > 2$ and choose $y_0 > 1$ such that $|S(y)|/y < \varepsilon$ ($\forall y \geq y_0$). Let $L = \lfloor C/((k-1)A\varepsilon)\rfloor - 1 \in \mathbb{N}$. For $N \geq y_0^k C/((k-1)A\varepsilon)$, $\sup_{y\geq(N/(L+1))^{1/k}}|S(y)|/y \leq \varepsilon$. Using this in (22)

$$\begin{aligned}\left|N^{-1/k}\sum_{n\leq N^{1/k}} a_n h\left(\left\{\frac{N}{n^k}\right\}\right)\right| &\leq A\varepsilon(L+1)^{1-1/k} + C(L+1)^{-1/k}\\ &\leq ((k-1)^{1/k-1} + 2^{1/k}(k-1)^{1/k})A^{1/k}C^{1-1/k}\varepsilon^{1/k}.\end{aligned}$$

Letting $N \to \infty$, and then $\varepsilon \searrow 0$, we have the convergence (20).

Consequently, the assertion of Proposition 1 under the condition (8) follows. □

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