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TRANSLATION HYPERSURFACES WITH CONSTANT CURVATURE IN SPACE FORMS

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Abstract

We give a classification of the translation hypersurfaces with constant mean curvature or constant Gauss–Kronecker curvature in Euclidean space or Lorentz–Minkowski space. We also characterize the minimal translation hypersurfaces in the upper half-space model of hyperbolic space.

1. Introduction

In $\mathbb{R}^3$, a surface is called a *translation surface* if it is given by an immersion

$$X\colon U \subset \mathbb{R}^2 \to \mathbb{R}^3\colon \quad (x, y) \mapsto (x, y, f(x) + g(y)),$$

where $z = f(x) + g(y)$ and f and g are smooth functions. One of the famous examples of minimal surfaces in 3-dimensional Euclidean space is a Scherk's minimal translation surface. In fact, Scherk [10] showed in 1835 that except the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{c}\log\left|\frac{\cos cy}{\cos cx}\right|,$$

where c is a nonzero constant. This surface is called a Scherk's minimal translation surface. In 1991, Dillen et al. [3] generalized this result to higher-dimensional Euclidean space. (See also [11].) A hypersurface $M \subset \mathbb{R}^{n+1}$ is called a *translation hypersurface* if M is a graph of a function

$$f\colon \mathbb{R}^n \to \mathbb{R}\colon \quad (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n) = f_1(x_1) + \cdots + f_n(x_n),$$

where f_i is a smooth function of one real variable for $i = 1, 2, \dots, n$. More precisely, they proved

Theorem ([3]). *Let M be a minimal translation hypersurface in $\mathbb{R}^{n+1}$. Then M is either a hyperplane or $M = \Sigma \times \mathbb{R}^{n-2}$, where Σ is a Scherk's minimal translation surface in $\mathbb{R}^3$.*

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Scherk's minimal translation surface in 3-dimensional Euclidean space $\mathbb{R}^3$ was generalized to translation surfaces with constant mean curvature or constant Gaussian curvature in $\mathbb{R}^3$ by Liu [7]. In particular, he proved

Theorem ([7]). *Let M be a translation surface with constant Gaussian curvature K in $\mathbb{R}^3$. Then M is congruent to a cylinder, and hence $K \equiv 0$.*

In Section 2 we generalize these previous results to translation hypersurfaces with constant mean curvature or constant Gauss–Kronecker curvature in Euclidean space and Lorentz–Minkowski space. In particular, we prove the following theorems.

Theorem 1.1. *Let M be a translation hypersurface with constant mean curvature H in $\mathbb{R}^{n+1}$. Then M is congruent to a cylinder $\Sigma \times \mathbb{R}^{n-2}$, where Σ is a constant mean curvature surface in $\mathbb{R}^3$. In particular, if $H = 0$, then M is either a hyperplane or $\Sigma \times \mathbb{R}^{n-2}$, where Σ is a Scherk's minimal translation surface in $\mathbb{R}^3$.*

Theorem 1.2. *Let M be a translation hypersurface with constant Gauss–Kronecker curvature GK in $\mathbb{R}^{n+1}$. Then M is congruent to a cylinder, and hence $GK \equiv 0$.*

One may ask the similar problems for translation hypersurfaces in the upper half-space model of hyperbolic space $\mathbb{H}^{n+1}$. Recently López [8] proved that there is no minimal translation surface of type I in $\mathbb{H}^3$. (See Section 3 for the definition of translation hypersurface of type I or type II in hyperbolic space.) In Section 3, we prove an analogue of López's result for higher-dimensional cases in hyperbolic space as follows:

Theorem 1.3. *There is no minimal translation hypersurface of type* I *in $\mathbb{H}^{n+1}$.*

Furthermore we characterize the minimal translation surfaces of type II in $\mathbb{H}^3$. (See Theorem 3.3.)

2. Translation hypersurface with constant curvature in Euclidean space and Lorentz–Minkowski space

Theorem 2.1. *Let M be a translation hypersurface with constant mean curvature H in $\mathbb{R}^{n+1}$. Then M is congruent to a cylinder $\Sigma \times \mathbb{R}^{n-2}$, where Σ is a constant mean curvature surface in $\mathbb{R}^3$. In particular, if $H = 0$, then M is either a hyperplane or $M = \Sigma \times \mathbb{R}^{n-2}$, where Σ is a Scherk's minimal translation surface in $\mathbb{R}^3$.*

Proof. Let a translation hypersurface M be an immersion given by

$$X : \mathbb{R}^n \to \mathbb{R}^{n+1} : \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, f(x_1, \ldots, x_n)),$$

where $f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$ and each f_i is a smooth function for $i = 1, \ldots, n$. One can easily see that the unit normal vector N and the mean curvature H are given by

$$N = \frac{(-f_1', \ldots, -f_n', 1)}{\sqrt{1 + \sum_{i=1}^n f_i'^2}}$$

and

$$H = \frac{\sum_{i=1}^n \left(1 + \sum_{j \neq i}^n f_j'^2\right) f_i''}{n\left(1 + \sum_{i=1}^n f_i'^2\right)^{3/2}}, \tag{2.1}$$

respectively. Since M has constant mean curvature H, differentiating the equation (2.1) with respect to x_1, we get

$$\frac{\left(1 + \sum_{j=2}^n f_j'^2\right) f_1''' + 2 f_1' f_1'' \left(\sum_{j=2}^n f_j''\right)}{n\left(1 + \sum_{i=1}^n f_i'^2\right)^{1/2}} = 3nH f_1' f_1''. \tag{2.2}$$

Differentiate the equation (2.2) with respect to x_2, and we have

$$(2 f_1' f_1'' f_2''' + 2 f_2' f_2'' f_1''')\left(1 + \sum_{i=1}^n f_i'^2\right)^{1/2} = 3nH f_1' f_1'' f_2' f_2''. \tag{2.3}$$

Now suppose that $f_1' f_1'' f_2' f_2'' \neq 0$. Then the equation (2.3) implies

$$2\left(\frac{f_1'''}{f_1' f_1''} + \frac{f_2'''}{f_2' f_2''}\right)\left(1 + \sum_{i=1}^n f_i'^2\right)^{1/2} = 3nH. \tag{2.4}$$

Note that $1 + \sum_{i=1}^n f_i'^2$ is a nonconstant function of a variable x_1 or x_2 from the assumption that $f_1' f_1'' f_2' f_2'' \neq 0$. If each f_i' is constant for $i = 3, 4, \ldots, n$, then

$$f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n) = f_1(x_1) + f_2(x_2) + a_3 x_3 + \cdots + a_n x_n,$$

where each a_i is constant for $i = 3, 4, \ldots, n$. This implies that M is congruent to a cylinder $\Sigma \times \mathbb{R}^{n-2}$, where Σ is a constant mean curvature surface in $\mathbb{R}^3$. If f_k' is not a constant function for some $k = 3, 4, \ldots, n$, then one sees that H must vanish from the above equation (2.4). As mentioned in the introduction, by the result of Dillen et al. [3], one sees that a minimal translation hypersurface $M \subset \mathbb{R}^{n+1}$ is either a hyperplane or $M = \Sigma \times \mathbb{R}^{n-2}$, where Σ is a Scherk's minimal translation surface in $\mathbb{R}^3$. Hence we conclude that M is a hyperplane or $M = \Sigma \times \mathbb{R}^{n-2}$, where Σ is a Scherk's minimal translation surface in $\mathbb{R}^3$. Otherwise, if $f_1' f_1'' f_2' f_2'' = 0$, then it follows that either f_1 or f_2 is linear, that is,

$$f_1 = a_1 x_1 + b_1 \quad \text{or} \quad f_2 = a_2 x_2 + b_2,$$

where a_i and b_i are constants for $i = 1, 2$. Without loss of generality, we may assume that $f_1 = a_1 x_1 + b_1$. It immediately follows that

$$\begin{aligned} X(x_1, \dots, x_n) &= (x_1, \dots, x_n, f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n)) \\ &= (x_1, \dots, x_n, a_1 x_1 + b_1 + f_2(x_2) + \cdots + f_n(x_n)) \\ &= x_1(1, 0, \dots, 0, a_1) + (0, x_2, \dots, x_n, b_1 + f_2(x_2) + \cdots + f_n(x_n)), \end{aligned}$$

which implies that M is a cylinder. This completes the proof of Theorem 2.1. □

Let $\mathbb{L}^{n+1}$ be the $(n+1)$-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentz–Minkowski metric

$$ds^2 = dx_1{}^2 + \cdots + dx_n{}^2 - dx_{n+1}{}^2$$

and $x_1, \dots, x_{n+1}$ are the canonical coordinates of $\mathbb{R}^{n+1}$. We say that a vector $v \in \mathbb{L}^{n+1} \setminus \{0\}$ is *spacelike, timelike or lightlike* if $|v|^2 = \langle v, v\rangle$ is positive, negative or zero, respectively. The zero vector 0 is spacelike by convention. A hyperplane in $\mathbb{L}^{n+1}$ is said to be *spacelike*, *timelike or lightlike* if the normal vector of the hyperplane is timelike, spacelike, or lightlike, respectively. An immersed hypersurface $M \subset \mathbb{L}^{n+1}$ is called *spacelike* if every tangent hyperplane of M is a spacelike. We define a spacelike translation hypersurface $M \subset \mathbb{L}^{n+1}$ as follows:

DEFINITION 2.2. A spacelike hypersurface $M \subset \mathbb{L}^{n+1}$ is called a *spacelike translation hypersurface* if it is given by an immersion

$$X : \mathbb{R}^n \to \mathbb{L}^{n+1} : \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, f(x_1, \dots, x_n))$$

where $f(x_1, \dots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$ and each f_i is a smooth function for $i = 1, \dots, n$.

In the above definition, a function f should satisfy that $|\nabla f| < 1$ since M is a spacelike hypersurface in $\mathbb{L}^{n+1}$. Applying the similar arguments as in the proof of Theorem 2.1 we can also obtain a similar result in the Lorentz–Minkowski space as follows:

Theorem 2.3. *Let M be a spacelike translation hypersurface with constant mean curvature H in $\mathbb{L}^{n+1}$. Then M is congruent to a cylinder $\Sigma \times \mathbb{R}^{n-2}$, where Σ is a constant mean curvature surface in $\mathbb{L}^3$. In particular, if $H = 0$, then M is either a hyperplane or $M = \Sigma \times \mathbb{R}^{n-2}$, where Σ is a Scherk's maximal spacelike translation surface in $\mathbb{L}^3$.*

REMARK 2.4. A spacelike hypersurface with vanishing mean curvature is called a *maximal spacelike hypersurface*. Kobayashi [5] gave various examples of maximal spacelike surfaces in $\mathbb{L}^3$ including Scherk's maximal spacelike translation surface. In 1976, Cheng and Yau [2] proved that the only entire solutions to the maximal spacelike hypersurface equation are linear. Even though there is no entire maximal spacelike graph by the result of Cheng and Yau, one has many kinds of maximal spacelike graphs locally. However Theorem 2.3 implies that the only nontrivial maximal spacelike translation hypersurface is locally $M = \Sigma \times \mathbb{R}^{n-2}$, where Σ is a Scherk's maximal spacelike translation surface in $\mathbb{L}^3$.

Scherk's minimal translation surface in $\mathbb{R}^3$ was generalized to translation surfaces with constant Gaussian curvature in $\mathbb{R}^3$ by Liu [7]. In the following, we generalize his result to higher-dimensional Euclidean space.

Theorem 2.5. *Let M be a translation hypersurface with constant Gauss–Kronecker curvature GK in $\mathbb{R}^{n+1}$. Then M is congruent to a cylinder, and hence $GK \equiv 0$.*

Proof. Let a translation hypersurface M be an immersion given by

$$X : \mathbb{R}^n \to \mathbb{R}^{n+1} : \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$$

where $f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$ and each f_i is a smooth function for $i = 1, \ldots, n$. Then it follows that the unit normal vector N and the Gauss–Kronecker curvature GK are given by

$$N = \frac{(-f_1', \ldots, -f_n', 1)}{\sqrt{1 + \sum_{i=1}^n f_i'^2}}$$

and

$$GK = \frac{f_1'' f_2'' \cdots f_n''}{\left(1 + \sum_{i=1}^n f_i'^2\right)^{(n+2)/2}}, \tag{2.5}$$

respectively. Differentiating the equation (2.5) with respect to x_1 and using the assumption that the Gauss–Kronecker curvature GK is constant, we get

$$0 = f_2'' \cdots f_n'' \left[f_1''' \left(1 + \sum_{i=1}^n f_i'^2 \right) - (n+2) f_1' f_1''^2 \right].$$

Suppose that $f_2'' \cdots f_n'' = 0$. Then one of f_i's is linear for $i = 2, \ldots, n$, which implies that M is congruent to a cylinder. Therefore one may assume that $f_2'' \cdots f_n'' \neq 0$. Thus one has

$$f_1''' \left(1 + \sum_{i=1}^n f_i'^2 \right) = (n+2) f_1' f_1''^2. \tag{2.6}$$

In the left-hand side of the equation (2.6), $1 + \sum_{i=1}^{n} f_i'^2$ is a nonconstant function of variables $x_2, \ldots, x_n$ since $f_2'' \cdots f_n'' \neq 0$. However the right-hand side is a function of variable x_1. Hence we obtain that $f_1''' \equiv 0$ and $f_1'' \equiv 0$, which means that f_1 is linear. Therefore we conclude that M is congruent to a cylinder. □

3. Minimal translation hypersurfaces in hyperbolic space

Anderson [1] gave many examples of minimal surfaces with various topological types in the hyperbolic space $\mathbb{H}^n$ using geometric measure theory. Later by solving the minimal surface equation in the hyperbolic space, many examples of minimal surfaces in the 3-dimensional hyperbolic space $\mathbb{H}^3$ have been found in [4, 6, 9]. In order to search Scherk's minimal translation hypersurfaces in the hyperbolic space, we consider the upper half-space model of the n-dimensional hyperbolic space $\mathbb{H}^n$, that is, $\mathbb{R}^n_+ = \{(x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n : x_n > 0\}$ equipped with the hyperbolic metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

Note that unlike in Euclidean space, the coordinates $x_1, \ldots, x_{n-1}$ are interchangeable, but not for the coordinate x_n in $\mathbb{H}^n$. Motivated by this observation, we give the following definition of translation hypersurfaces in $\mathbb{H}^{n+1}$. It should be mentioned that López [8] gave the same definition when $n = 2$.

DEFINITION 3.1. A hypersurface $M \subset \mathbb{H}^{n+1}$ is called a *translation hypersurface of type* I if it is given by an immersion $X : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}_+$ satisfying

$$X(x_1, \ldots, x_n) = (x_1, \ldots, x_n, f_1(x_1) + \cdots + f_n(x_n)),$$

where each f_i is a smooth function on $U \subset \mathbb{R}^n$ for $i = 1, \ldots, n$. Similarly a hypersurface $M \subset \mathbb{H}^{n+1}$ is called a *translation hypersurface of type* II if it is given by an immersion $X : U \subset \mathbb{R}^n \to \mathbb{R}^{n+1}_+$ satisfying

$$X(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, f_1(x_1) + \cdots + f_n(x_n), x_n).$$

Let M be a hypersurface in the upper half-space model of $\mathbb{H}^{n+1}$. If we denote by N_h a unit normal vector field on M with respect to the hyperbolic metric in $\mathbb{R}^{n+1}_+$, then a unit normal vector field N on M with respect to the Euclidean metric is given by

$$N = \frac{N_h}{x_{n+1}}.$$

Moreover, if we denote by H_h and H_e the hyperbolic and Euclidean mean curvature on M respectively, then it is well-known that

$$H_h = x_{n+1} H_e + N_{n+1}, \tag{3.1}$$

where N_{n+1} is the $(n+1)$-th component of the unit normal vector N.

Contrary to the Euclidean case, it was proved that there is no minimal translation surface of type I in $\mathbb{H}^3$ by López [8]. We prove an analogue for higher-dimensional cases in the following.

Theorem 3.2. *There is no minimal translation hypersurface of type* I *in* $\mathbb{H}^{n+1}$.

Proof. It suffices to prove this theorem for $n \geq 3$ because the case when $n = 2$ was done by López [8]. Let M be a translation hypersurface of type I which is given by an immersion

$$X : \mathbb{R}^n \to \mathbb{R}^{n+1}_+ : \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$$

where $f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$ and each f_i is a smooth function for $i = 1, \ldots, n$. Then it follows from the equation (3.1) that

$$H_h = \left(\sum_{i=1}^{n} f_i\right) \frac{\sum_{k=1}^{n}\left(1 + \sum_{j\neq k}^{n} f_j'^2\right) f_k''}{\left(1 + \sum_{i=1}^{n} f_i'^2\right)^{3/2}} + \frac{n}{\left(1 + \sum_{i=1}^{n} f_i'^2\right)^{1/2}}.$$

Since M is minimal, we get

$$\left(\sum_{i=1}^{n} f_i\right)\left[\sum_{k=1}^{n}\left(1 + \sum_{j\neq k}^{n} f_j'^2\right) f_k''\right] = -n\left(1 + \sum_{i=1}^{n} f_i'^2\right). \tag{3.2}$$

We claim that $f_i'' \neq 0$ for each $i = 1, \ldots, n$. To see this, suppose first that $f_i'' \equiv 0$ for each $i = 1, \ldots, n$. Then while the left-hand side of the equation (3.2) vanishes, the right-hand side cannot be zero. Thus $f_i'' \neq 0$ for some $1 \leq i \leq n$. Now suppose that $f_j'' \equiv 0$ for some $1 \leq j \leq n$. Then $f_j = ax_j + b$, where a and b are constants. Note that $a \neq 0$ since M is a graph. While the right-hand side of the equation (3.2) has a degree 0 in the variable x_j, the left-hand side has a degree 1 in the variable x_j, which is a contradiction. Hence this proves our claim.

We now have three possibilities as follows:

CASE (1): $f_i''' \equiv 0$ for all $i = 1, \ldots, n$.

CASE (2): $f_i''' \equiv 0$ for some i and $f_j''' \neq 0$ for some j.

CASE (3): $f_i''' \neq 0$ for all $i = 1, \ldots, n$.

For Case (1), $f_i'' = \text{constant} \neq 0$ for each $i = 1, \ldots, n$, by our above claim. So each f_i is a quadratic polynomial. The right-hand side of the equation (3.2) has a degree 2 in the variable x_i. Because the left-hand side has the same degree, we have

$$\sum_{l\neq i}^{n} f_l'' \equiv 0. \tag{3.3}$$

Since the equation (3.3) holds for each $i = 1, \dots, n$, one can obtain that

$$f_1'' = \cdots = f_n'' \equiv 0,$$

which is impossible. For Case (2), we may assume that $f_1''' \equiv 0$ and $f_2''' \neq 0$ without loss of generality. Since f_1 is a quadratic polynomial, the right-hand side of the equation (3.2) has a degree 2 in the variable x_1. Hence we see that

$$f_2'' + \cdots + f_n'' \equiv 0.$$

Therefore we get

$$f_2''' = \cdots = f_n''' \equiv 0,$$

which is a contradiction to our assumption that $f_2''' \neq 0$. For Case (3), differentiating the equation (3.2) with respect to x_1, x_2 and x_3, we get

$$f_1'(f_2' f_2'' f_3''' + f_3' f_3'' f_2''') + f_2'(f_3' f_3'' f_1''' + f_1' f_1'' f_3''') + f_3'(f_1' f_1'' f_2''' + f_2' f_2'' f_1''') = 0.$$

Since $f_i'' \neq 0$ by the assumption, dividing both sides of the above equation by $f_1' f_1'' f_2' f_2'' f_3' f_3''$, we have

$$\frac{1}{f_1''}\left(\frac{f_2'''}{f_2' f_2''} + \frac{f_3'''}{f_3' f_3''}\right) + \frac{1}{f_2''}\left(\frac{f_3'''}{f_3' f_3''} + \frac{f_1'''}{f_1' f_1''}\right) + \frac{1}{f_3''}\left(\frac{f_1'''}{f_1' f_1''} + \frac{f_2'''}{f_2' f_2''}\right) = 0. \tag{3.4}$$

Differentiation of the above equation with respect to x_1 gives

$$\frac{(f_1'''/(f_1' f_1''))'}{(1/f_1'')'} = -\frac{f_2'''/(f_2' f_2'') + f_3'''/(f_3' f_3'')}{1/f_2'' + 1/f_3''} = c_1, \tag{3.5}$$

where c_1 is a constant. Therefore

$$f_1''' = c_1 f_1' + d_1 f_1' f_1'', \tag{3.6}$$

where d_1 is a constant. Similarly one can get

$$\begin{aligned} f_2''' &= c_2 f_2' + d_2 f_2' f_2'', \\ f_3''' &= c_3 f_3' + d_3 f_3' f_3'', \end{aligned} \tag{3.7}$$

where c_i and d_i are constants for $i = 2,3$. Using the equations (3.5) and (3.7), we obtain

$$\frac{(f_1'''/(f_1' f_1''))'}{(1/f_1'')'} = -\frac{(d_2 + d_3) f_2'' f_3'' + c_2 f_3'' + c_3 f_2''}{f_2'' + f_3''} = c_1. \tag{3.8}$$

Using the above equation (3.8) and the assumption that $f_i''' \neq 0$ for all $i = 1, \dots, n$, we see that

$$c_2 = c_3 \quad \text{and} \quad d_2 + d_3 = 0.$$

From the similar arguments, it follows that

$$c_1 = c_2 \quad \text{and} \quad d_1 + d_2 = 0,$$
$$c_1 = c_3 \quad \text{and} \quad d_1 + d_3 = 0.$$

Combining these relations, we can conclude that

$$c_1 = c_2 = c_3 = c \quad \text{and} \quad d_1 = d_2 = d_3 = 0. \tag{3.9}$$

From the equations (3.4), (3.6), (3.7), and (3.9), one can get that

$$c(f_1'' + f_2'' + f_3'') = 0.$$

Since $f_i'' \neq 0$ for each $i = 1, \dots, n$ by the assumption, one sees that $c = 0$, that is,

$$f_1''' = f_2''' = f_3''' = 0,$$

by the equation (3.6), (3.7), and (3.9). However this is a contradiction. Therefore we obtain the desired conclusion. □

In [8], López proved that the only minimal translation surfaces of type II in $\mathbb{H}^3$ were totally geodesic planes. However there is a gap in his proof which leads to wrong conclusion. Nevertheless, using his arguments, we characterize the minimal translation surfaces of type II in $\mathbb{H}^3$ as follows:

Theorem 3.3. *Let $M \subset \mathbb{H}^3$ be a minimal translation surface of type* II *given by the parametrization $X(x, z) = (x, f(x) + g(z), z)$. Then the functions f and g are as follows*:

$$f(x) = ax + b,$$
$$g(z) = \sqrt{1 + a^2} \int \frac{cz^2}{\sqrt{1 - c^2 z^4}}\, dz,$$

where a, b, and c are constants.

Proof. Since the Euclidean mean curvature H_e on M and the third component N_3 of a unit normal vector field on M with respect to the Euclidean metric are given by

$$H_e = -\frac{1}{2} \frac{(1 + g'^2) f'' + (1 + f'^2) g''}{(1 + f'^2 + g'^2)^{3/2}}$$

and

$$N_3 = \frac{g'}{\sqrt{1 + f'^2 + g'^2}},$$

respectively, the hyperbolic mean curvature H_h on M is given by

$$H_h = -\frac{1}{2}\frac{(1+g'^2)f''+(1+f'^2)g''}{(1+f'^2+g'^2)^{3/2}}z + \frac{g'}{\sqrt{1+f'^2+g'^2}}$$

from the equation (3.1). Since M is minimal, we have

$$z\left(\frac{f''}{1+f'^2}+\frac{g''}{1+g'^2}\right) = 2g'\frac{1+f'^2+g'^2}{(1+f'^2)(1+g'^2)}. \tag{3.10}$$

Differentiating the above equation with respect to x, we get

$$z\left(\frac{f''}{1+f'^2}\right)' = -4\frac{f'f''}{(1+f'^2)^2}\frac{g'^3}{1+g'^2}. \tag{3.11}$$

First suppose that $f'f''=0$. Then $f(x)=ax+b$ for some constants a and b. The equation (3.10) says that

$$zg'' = \frac{2g'(1+a^2+g'^2)}{(1+a^2)}.$$

Solving this ordinary differential equation with respect to z, we obtain

$$g(z) = \sqrt{1+a^2}\int \frac{cz^2}{\sqrt{1-c^2z^4}}\,dz,$$

where c is a constant.

Now suppose that $f'f''\neq 0$. Then by the equation (3.11) one sees that

$$\frac{(f''/(1+f'^2))'}{-4f'f''/(1+f'^2)^2} = \frac{g'^3}{z(1+g'^2)} = d,$$

where d is a constant. If $d=0$, then $f=mx+n$ and $g=$ constant, which is impossible by our assumption that $f'f''\neq 0$. If $d\neq 0$, then one can obtain a contradiction by applying López's arguments as in [8]. □

REMARK 3.4. Note that if $c=0$ in the Theorem 3.3, then the minimal translation surface can be parametrized as

$$X(x,z) = (x, ax+b+m, z),$$

where a, b, and m are constants. This surface is a vertical Euclidean plane which is a totally geodesic plane in $\mathbb{H}^3$.

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