

ON RELATIVE HEIGHT ZERO BRAUER CHARACTERS

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Abstract

Let $N \triangleleft G$ where G is a finite group and let B be a p -block of G , where p is a prime. A Brauer character $\psi \in \text{IBr}_p(B)$ is said to be of relative height zero with respect to N provided that the height of ψ is equal to that of an irreducible constituent of ψ_N . Now assume G is p -solvable. In this paper, we count the number of relative height zero irreducible Brauer characters of B with respect to N that lie over any given $\varphi \in \text{IBr}_p(N)$. As a consequence, we show that if D is a defect group of B and \hat{B} is the unique p -block of $NN_G(D)$ with defect group D such that $\hat{B}^G = B$, then B and \hat{B} have equal numbers of relative height zero irreducible Brauer characters with respect to N .

1. Introduction

Fix a prime p and let N be a normal subgroup of a finite group G . Let B be a p -block of G and let $\psi \in \text{IBr}(B)$, the set of irreducible Brauer characters belonging to B . Suppose θ is an irreducible constituent of ψ_N , and write b for the p -block of N to which θ belongs. As in [8], a defect group D of B is called an inertial defect group of B (with respect to b) if it is a defect group of the Fong–Reynolds correspondent of B in the inertial group T of b in G .

By [8, Lemma 3.2], we have $\text{ht}(\psi) \geq \text{ht}(\theta)$. If θ' is any other irreducible constituent of ψ_N , then θ' is G -conjugate to θ and belongs to a G -conjugate of b . Since G -conjugate p -blocks of N have equal defects, the difference $\text{ht}(\psi) - \text{ht}(\theta)$ does not depend on the choice of the constituent θ .

The Brauer character ψ is said to be of *relative height zero* with respect to N provided that $\text{ht}(\psi) = \text{ht}(\theta)$. We denote by $\text{IBr}_N^0(B)$, the set of irreducible Brauer characters belonging to B and having relative height zero with respect to N . If $\text{ht}(\psi) = 0$, then $\text{ht}(\theta) = 0$ as $\text{ht}(\psi) \geq \text{ht}(\theta)$. Hence every irreducible Brauer character in B of height zero lies in $\text{IBr}_N^0(B)$, and so in particular $\text{IBr}_N^0(B) \neq \emptyset$. Furthermore when $N = 1$, then $\text{IBr}_N^0(B)$ is precisely the set of irreducible Brauer characters in B of height zero. (See also Corollary 2.2 below.)

Let $\varphi \in \text{IBr}(b)$. We write $\text{IBr}_N^0(B \mid \varphi)$ for the set of all those Brauer characters in $\text{IBr}_N^0(B)$ that lie over φ . Theorem 3.3 in [8] implies that $\text{IBr}_N^0(B \mid \varphi) \neq \emptyset$ if and only if

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φ is D -invariant for some inertial defect group D of B . So in particular, $\text{IBr}_N^0(B \mid \varphi) \neq \emptyset$ when φ is G -invariant.

Assume now that G is p -solvable. In [6], we counted the number of height zero irreducible Brauer characters in B that lie over φ . If φ is not of height zero, then in view of the inequality above, there are no height zero irreducible Brauer characters in B over φ . So, in effect, the main result of [6] counts the number of elements in $\text{IBr}_N^0(B \mid \varphi)$ in case φ is of height zero. In this paper, we count the number of elements in $\text{IBr}_N^0(B \mid \varphi)$ for any $\varphi \in \text{IBr}(b)$.

Theorem A. *Let G be a p -solvable group and let B be a p -block of G . Suppose N is a normal subgroup of G , b is a p -block of N covered by B , T is the inertial group of b in G and $\varphi \in \text{IBr}(b)$. If D is an inertial defect group of B with respect to b and \hat{B} is the unique p -block of $NN_G(D)$ with defect group D such that $\hat{B}^G = B$, then*

$$|\text{IBr}_N^0(B \mid \varphi)| = \left| \bigcup_{t \in T} \text{IBr}_N^0(\hat{B} \mid \varphi^t) \right|.$$

As a consequence of this result, we have the following.

Theorem B. *Let G be a p -solvable group and let B be a p -block of G with defect group D . Suppose N is a normal subgroup of G and let \hat{B} be the unique p -block of $NN_G(D)$ with defect group D such that $\hat{B}^G = B$. Then*

$$|\text{IBr}_N^0(B)| = |\text{IBr}_N^0(\hat{B})|.$$

If we take $N = 1$, then Theorem B reduces to an equivalent form of [10, Theorem 4.1], which is a modular version of the Alperin–McKay conjecture for p -solvable groups.

We finally mention that Theorem B is not true if the group is not assumed to be p -solvable. For example, let $G = GL(3, 2)$. Let B be the principal 2-block of G and take $N = 1$. Then $|\text{IBr}_N^0(B)| = 3$, as B has three irreducible Brauer characters, all of height zero. On the other hand, if D is a Sylow 2-subgroup of G , then $N_G(D) = D$ and so if \hat{B} is the principal 2-block of $N_G(D)$, we have $|\text{IBr}_N^0(\hat{B})| = 1$.

2. Proof of the results

In this section we prove Theorems A and B of the introduction. We begin with a number of preliminary results.

Let ψ be an irreducible p -Brauer character of an arbitrary finite group G . By a vertex of ψ we mean any vertex of the simple G -module (in characteristic p) that corresponds to ψ .

Lemma 2.1. *Let $N \triangleleft G$ where G is p -solvable and let B be a p -block of G with defect group D . Suppose $\psi \in \text{IBr}(B)$.*

- (i) *$\psi \in \text{IBr}_N^0(B)$ if and only if $|D|/|D \cap N| = |P|/|P \cap N|$ where P is any vertex of ψ .*
- (ii) *Suppose Q is a vertex of ψ with $Q \subseteq D$. Then $\psi \in \text{IBr}_N^0(B)$ if and only if $QN = DN$.*

Proof. (i) Let φ be an irreducible constituent of ψ_N and let P be a vertex for ψ . Then [11, Corollary 3] implies that $|P \cap N|$ is the order of any vertex of φ . Furthermore, if b is the p -block of N to which φ belongs, $|D \cap N|$ is the order of any defect group of b from [3, Proposition 4.2]. Now $\psi \in \text{IBr}_N^0(B)$ if and only if

$$\frac{|G|_p}{\psi(1)_p} \left(\frac{|N|_p}{\varphi(1)_p} \right)^{-1} = \frac{|D|}{|D \cap N|}.$$

The result is then immediate by [1, Theorem 2.1].

- (ii) Part (i) implies that $\psi \in \text{IBr}_N^0(B)$ if and only if $|QN| = |DN|$. Since $Q \subseteq D$, we have $QN \subseteq DN$ and the result immediately follows. □

Before stating our second preliminary result, we mention an easy corollary.

Corollary 2.2. *Let $M \subseteq N$ be normal subgroups of a p -solvable group G and let B be a p -block of G . Then $\text{IBr}_M^0(B) \subseteq \text{IBr}_N^0(B)$.*

Proof. Let D be a defect group of B and assume $\psi \in \text{IBr}_M^0(B)$. Choose a vertex Q for ψ such that $Q \subseteq D$. By Lemma 2.1 (ii) we have $QM = DM$. So $QN = DN$ and again by Lemma 2.1 (ii), $\psi \in \text{IBr}_N^0(B)$. □

Our next lemma involves the concept of a relative p -block introduced in [4, 5]. For the reader's convenience, we first give a brief review of this concept and other related notions needed for our purposes.

Let G be a p -solvable group. If χ is an ordinary character of G , we denote by χ^0 the restriction of χ to the set of p -regular elements of G .

In [2], Isaacs associates to every character $\chi \in \text{Irr}(G)$ a unique (up to G -conjugacy) pair (W, γ) called nucleus of χ such that W is a subgroup of G , $\gamma \in \text{Irr}(W)$ is p -factorable (i.e., γ is a product of a p -special and a p' -special characters of W), and $\chi = \gamma^G$. Let $B_{p'}(G)$ be the set of all those $\chi \in \text{Irr}(G)$ for which γ is p' -special. Then Isaacs in [2] proved that the restriction map $\chi \mapsto \chi^0$ defines a bijection of $B_{p'}(G)$ onto $\text{IBr}(G)$.

Now let $N \triangleleft G$ and $\mu \in B_{p'}(N)$. We say that two characters $\chi, \chi' \in \text{Irr}(G \mid \mu)$ (the set of irreducible characters of G lying over μ) are linked provided that there exist irreducible characters $\chi_0, \chi_1, \dots, \chi_n$ of G lying over μ and irreducible Brauer characters $\psi_0, \dots, \psi_{n-1}$ of G such that $\chi_0 = \chi, \chi_n = \chi'$ and $d_{\chi_i \psi_i} d_{\chi_{i+1} \psi_i} \neq 0$ for all

$0 \leq i \leq n-1$. This linking clearly defines an equivalence relation on $\text{Irr}(G \mid \mu)$ and the resulting equivalence classes are called relative p -blocks of G with respect to (N, μ) . Let $\text{Bl}_p(G \mid \mu)$ be the set of all these relative p -blocks. If B is any (Brauer) p -block of G such that $\text{Irr}(B) \cap \text{Irr}(G \mid \mu) \neq \emptyset$, then it is not hard to see that $\text{Irr}(B) \cap \text{Irr}(G \mid \mu)$ is in fact a union of some relative p -blocks in $\text{Bl}_p(G \mid \mu)$.

Let $\mathcal{B} \in \text{Bl}_p(G \mid \mu)$. By $\text{IBr}(\mathcal{B})$, we mean the set of all $\psi \in \text{IBr}(G)$ such that $\psi = \chi^0$ where $\chi \in \mathcal{B}$. It turns out that $\text{IBr}(\mathcal{B})$ is always nonempty.

Finally we should mention that a notion of the defect group of a relative p -block is defined in Section 4 of [5].

Lemma 2.3. *Let G be a p -solvable group and let B be a p -block of G with defect group D . Suppose $N \triangleleft G$ and let $\mu \in \mathbb{B}_p(N)$ be G -invariant. If Q is a vertex of the unique irreducible Brauer character ψ of DN lying over the irreducible Brauer character μ^0 , then*

- (i) $QN = DN$;
- (ii) *there exists a nucleus (W, γ) for μ such that Q is contained in the stabilizer S of (W, γ) in G and $Q \cap W$ is a Sylow p -subgroup of W ;*
- (iii) *assume $\mathcal{B} \in \text{Bl}_p(G \mid \mu)$ has defect group Q and let \hat{B} be the relative p -block in $\text{Bl}_p(N_G(QN) \mid \mu)$ with defect group Q corresponding to \mathcal{B} via Proposition 3.4 (c) in [6]. Then if \hat{B} is the p -block of $N_G(QN)$ ($= N_G(DN)$) with defect group D such that $\hat{B}^G = B$, we have $\mathcal{B} \subseteq \text{Irr}(B)$ if and only if $\hat{B} \subseteq \text{Irr}(\hat{B})$.*

Proof. (i) Since μ is G -invariant, then so is μ^0 . It follows by [9, Theorem 3.5.11 (ii)] that $\psi_N = \mu^0$, and so in particular $\psi(1)_p = \mu^0(1)_p$. Next as Q is a vertex for ψ , [11, Corollary 3] implies that $|Q \cap N|$ is the order of any vertex of μ^0 . Then by [1, Theorem 2.1], $|DN|_p/|Q| = |N|_p/|Q \cap N|$, and hence $|D|/|D \cap N| = |Q|/|Q \cap N|$. Therefore $|DN| = |QN|$. But $QN \subseteq DN$ as $Q \subseteq DN$. It follows that $QN = DN$.

(ii) Since $QN = DN$ by (i), Lemma 2.4 (b) in [7] tells us that there exists a nucleus (W, γ) for μ such that Q is contained in the stabilizer S of (W, γ) in G .

Next by Lemma 2.3 (c) of [7], there is $\eta \in \text{IBr}(S \cap (QN))$ lying over γ^0 and having vertex Q such that $\eta^{QN} = \psi$. Now as γ is an S -invariant p' -special character of W , we conclude by [6, Lemma 2.1 (b)] that $Q \cap W$ is a Sylow p -subgroup of W .

(iii) Let β (resp. $\tilde{\beta}$) be the p -block of G (resp. $N_G(QN)$) such that $\mathcal{B} \subseteq \text{Irr}(\beta)$ (resp. $\hat{B} \subseteq \text{Irr}(\tilde{\beta})$). We claim that $\tilde{\beta}^G$ is defined and $\tilde{\beta}^G = \beta$.

Since \hat{B} has Q as a defect group, Corollary 2.5 in [6] says that there exists $\nu \in \text{IBr}(\hat{B})$ having vertex Q . Now $\nu \in \text{IBr}(\tilde{\beta})$ and hence by [9, Theorem 5.1.9], there is some defect group P of $\tilde{\beta}$ such that $Q \subseteq P$. As $QC_G(Q) \subseteq N_G(QN)$, it follows by Corollary 5.3.7 and Theorem 5.3.6 of [9] that $\tilde{\beta}^G$ is defined. Next we show that $\tilde{\beta}^G = \beta$.

Choose $\alpha \in \hat{B}$. Since α lies over μ , we may write $\alpha^G = \sum c_\delta \delta$ for $\delta \in \text{Irr}(G \mid \mu)$. Let $\mathcal{B}_1 = \mathcal{B}, \dots, \mathcal{B}_m$ be all the (distinct) relative p -blocks in $\text{Bl}_p(G \mid \mu)$ that are

contained in $\text{Irr}(\beta)$. Then

$$\sum_{\delta \in \text{Irr}(\beta)} c_\delta \delta = \sum_{i=1}^m \left(\sum_{\delta \in \mathcal{B}_i} c_\delta \delta \right).$$

Now as $[\sum_{\delta \in \mathcal{B}_1} c_\delta \delta(1)]_p = [\alpha^G(1)]_p$ and $[\sum_{\delta \in \mathcal{B}_i} c_\delta \delta(1)]_p > [\alpha^G(1)]_p$ for $i \neq 1$ by [6, Proposition 3.4 (a)], we get that $[\sum_{\delta \in \text{Irr}(\beta)} c_\delta \delta(1)]_p = [\alpha^G(1)]_p$. Since $\alpha \in \hat{\mathcal{B}} \subseteq \text{Irr}(\tilde{\beta})$, it follows by [9, Corollary 5.3.2] that $\tilde{\beta}^G = \beta$, as claimed.

Now if $\hat{\mathcal{B}} \subseteq \text{Irr}(\hat{B})$, then $\mathcal{B} \subseteq \text{Irr}(B)$ by the above. Next suppose $\mathcal{B} \subseteq \text{Irr}(B)$. Then $\tilde{\beta}^G = B$ and so to finish the proof of (iii), by the uniqueness of the p -block \hat{B} , it suffices to show that $\tilde{\beta}$ has defect group D .

Recall that P is a defect group of $\tilde{\beta}$. Then in view of Lemma 5.3.3 of [9], there exists $g \in G$ for which

$$(*) \quad P \subseteq D^g.$$

So in particular $QN \subseteq PN \subseteq D^g N$. Since $QN = DN$ by (i) and $|DN| = |D^g N|$, it follows that

$$(**) \quad QN = PN = D^g N.$$

Next let b be the p -block of N such that $\mu \in \text{Irr}(b)$. Since $\alpha \in \text{Irr}(\tilde{\beta}) \cap \text{Irr}(N_G(QN) \mid \mu)$, we have that $\tilde{\beta}$ covers b . Also B covers b , as $\mathcal{B} \subseteq \text{Irr}(B)$. Proposition 4.2 in [3] now implies that $|P \cap N| = |D^g \cap N| = p^{d(b)}$, where $d(b)$ is the defect of b . In view of (**), we are then forced to have $|P| = |D^g|$. Hence $P = D^g$ from (*). Also, since $DN = QN = D^g N$, it is clear that $g \in N_G(DN)$. Then as P is a defect group of $\tilde{\beta}$ (a p -block of $N_G(DN)$), we deduce that D also is a defect group for $\tilde{\beta}$, as needed to be shown. □

Lemma 2.4. *Let $N \triangleleft G$ where G is p -solvable and let B be a p -block of G with defect group D . Assume $\mu \in \text{B}_p(N)$ is G -invariant and $\omega \in \text{IBr}_N^0(B \mid \varphi)$ where $\varphi = \mu^0$. If Q is a vertex for the unique irreducible Brauer character ψ of DN lying over φ , then*

- (i) ω has vertex Q ;
- (ii) $\omega \in \text{IBr}(\mathcal{B})$ for some relative p -block $\mathcal{B} \in \text{Bl}_p(G \mid \mu)$ having defect group Q and such that $\mathcal{B} \subseteq \text{Irr}(B)$.

Proof. Let Q be a vertex of the unique irreducible Brauer character ψ of DN lying over φ . Then $QN = DN$ by Lemma 2.3 (i).

- (i) Choose a vertex R for ω such that $R \subseteq D$. Then by Lemma 2.1 (ii), we have $RN = DN$. Now as φ is G -invariant, [7, Lemma 2.5] tells us that R is a vertex of ψ . So Q is DN -conjugate to R , and hence is a vertex for ω .

(ii) Let χ be the unique character in $B_{p'}(G)$ such that $\chi^0 = \omega$. As ω lies over φ and all the irreducible constituents of χ_N are in $B_{p'}(N)$ by [2, Corollary 7.5], χ must lie over μ . Let \mathcal{B} be the relative p -block in $\text{Bl}_p(G \mid \mu)$ in which χ lies. Then $\omega \in \text{IBr}(\mathcal{B})$ and since $\chi \in \text{Irr}(B)$, we have $\mathcal{B} \subseteq \text{Irr}(B)$. Next we show that \mathcal{B} has Q as a defect group.

By Corollary 2.5 in [6], \mathcal{B} has a defect group P such that $Q \subseteq P$ (as $\omega \in \text{IBr}(\mathcal{B})$), and there is $\xi \in \text{IBr}(\mathcal{B})$ with P as a vertex. Now since $\xi \in \text{IBr}(B)$, we have $P \subseteq D^g$ for some $g \in G$, and so in particular $PN \subseteq (DN)^g$. But $QN \subseteq PN$ and $QN = DN$. We deduce then that $QN = PN$. Also, as ω and ξ both lie over φ , [11, Corollary 3] implies that $|Q \cap N| = |P \cap N|$. It follows that $|Q| = |P|$ and therefore $Q = P$ as $Q \subseteq P$. We have thus shown that Q is a defect group for \mathcal{B} , as needed. \square

The following is an essential step toward the proof of Theorem A.

Proposition 2.5. *Let $N \triangleleft G$ where G is p -solvable and let B and b be p -blocks of G and N respectively such that B covers b . Suppose $\varphi \in \text{IBr}(b)$ is G -invariant. Let D be a defect group of B and write \hat{B} for the unique p -block of $N_G(DN)$ with defect group D such that $\hat{B}^G = B$, then*

$$|\text{IBr}_N^0(B \mid \varphi)| = |\text{IBr}_N^0(\hat{B} \mid \varphi)|.$$

Proof. First let μ be the character in $B_{p'}(N)$ for which $\mu^0 = \varphi$. Since μ is uniquely determined by φ , we note that μ is G -invariant. Next fix a vertex Q for the unique irreducible Brauer character ψ of DN that lies over φ . Then, in view of Lemma 2.3 (i), we have $QN = DN$.

Next by [8, Lemma 3.2 (ii)], $\text{IBr}_N^0(B \mid \varphi) \neq \emptyset$. Let $\omega \in \text{IBr}_N^0(B \mid \varphi)$. Then Lemma 2.4 tells us that ω has vertex Q and that $\omega \in \text{IBr}(\mathcal{B})$ for some relative p -block $\mathcal{B} \in \text{Bl}_p(G \mid \mu)$ having Q as a defect group and such that $\mathcal{B} \subseteq \text{Irr}(B)$.

Let now $\mathcal{B}_1, \dots, \mathcal{B}_m$ be all the (distinct) relative p -blocks in $\text{Bl}_p(G \mid \mu)$ having defect group Q and such that $\mathcal{B}_i \subseteq \text{Irr}(B)$ for all $i \in \{1, \dots, m\}$. Next write A for the set of all those $\eta \in \bigcup_{i=1}^m \text{IBr}(\mathcal{B}_i)$ having Q as a vertex. Then $\text{IBr}_N^0(B \mid \varphi) \subseteq A$ by the preceding paragraph. Since $QN = DN$, Lemma 2.1 implies that $A \subseteq \text{IBr}_N^0(B \mid \varphi)$. Consequently, we have $\text{IBr}_N^0(B \mid \varphi) = A$.

Next by Lemma 2.3 (ii), there exists a nucleus (W, γ) of μ such that Q is contained in the stabilizer of (W, γ) in G and $Q \cap W$ is a Sylow p -subgroup of W . Then for each i , let $\hat{\mathcal{B}}_i$ be the relative p -block in $\text{Bl}_p(N_G(QN) \mid \mu)$ with defect group Q corresponding to \mathcal{B}_i through Proposition 3.4 (c) of [6]. By Lemma 2.3 (iii), we have $\hat{\mathcal{B}}_i \subseteq \text{Irr}(\hat{B})$.

Suppose $\hat{\beta}$ is a relative p -block in $\text{Bl}_p(N_G(QN) \mid \mu)$ with defect group Q and such that $\hat{\beta} \subseteq \text{Irr}(\hat{B})$. If β is the relative p -block in $\text{Bl}_p(G \mid \mu)$ that corresponds to $\hat{\beta}$ via [6, Proposition 3.4 (c)], then Q is a defect group for β , and by Lemma 2.3 (iii)

we have $\beta \subseteq \text{Irr}(B)$. Therefore β is one of $\mathcal{B}_1, \dots, \mathcal{B}_m$ and it follows that $\hat{\beta}$ is one of $\hat{\mathcal{B}}_1, \dots, \hat{\mathcal{B}}_m$. This tells us that $\hat{\mathcal{B}}_1, \dots, \hat{\mathcal{B}}_m$ are all the (distinct) relative p -blocks in $\text{Bl}_p(N_G(QN) \mid \mu)$ having Q as a defect group and which are contained in $\text{Irr}(\hat{B})$.

If $\rho \in \text{Irr}(B)$, then the irreducible constituents of ρ_N all belong to b , as b is G -invariant. It follows from the remark following Lemma 3.1 in [6] that the p -block \hat{B} covers b . Now let C be the set of all $\tau \in \bigcup_{i=1}^m \text{IBr}(\hat{\mathcal{B}}_i)$ having vertex Q . Then by repeating the argument used above to show that $\text{IBr}_N^0(B \mid \varphi) = A$, we get that $\text{IBr}_N^0(\hat{B} \mid \varphi) = C$.

Next let $i \in \{1, \dots, m\}$ and assume $\zeta \in \text{IBr}(\hat{\mathcal{B}}_i)$. We claim that Q is a vertex of ζ . Since ζ lies over φ and ψ is the unique irreducible Brauer character of QN lying over φ , we have that ζ lies over ψ . Then as Q is a vertex for ψ , there exists a vertex P of ζ with $Q \subseteq P$ by [9, Lemma 4.3.4]. On the other hand, Corollary 2.5 of [6] implies that some $N_G(QN)$ -conjugate of P is contained in the defect group Q of $\hat{\mathcal{B}}_i$. It follows that $P = Q$, which proves our claim.

We now have $\text{IBr}_N^0(\hat{B} \mid \varphi) = C = \bigcup_{i=1}^m \text{IBr}(\hat{\mathcal{B}}_i)$. By [6, Proposition 3.4 (b)], $|\text{IBr}(\hat{\mathcal{B}}_i)|$ is equal to the number of all those Brauer characters in $\text{IBr}(\mathcal{B}_i)$ that have Q as a vertex. It follows that $|\text{IBr}_N^0(B \mid \varphi)| = |A| = \sum_{i=1}^m |\text{IBr}(\hat{\mathcal{B}}_i)| = |\text{IBr}_N^0(\hat{B} \mid \varphi)|$, and the proof of the proposition is complete. \square

The stable case of our next proposition is the key for proving Theorem A. This result is analogous to Theorem A, where the subgroup $NN_G(D)$ is replaced by the possibly larger subgroup $N_G(DN)$ (note that $NN_G(D) \subseteq N_G(DN)$, always). We should mention that in the proof of this proposition, we will adapt some of the arguments used in the proof of the main theorem of [6].

Proposition 2.6. *Let $N \triangleleft G$ where G is p -solvable and let B and b be p -blocks of G and N respectively such that B covers b . Let T be the inertial group of b in G and suppose $\varphi \in \text{IBr}(b)$. If D is an inertial defect group of B with respect to b and \hat{B} is the unique p -block of $N_G(DN)$ with defect group D such that $\hat{B}^G = B$, then*

$$|\text{IBr}_N^0(B \mid \varphi)| = \left| \bigcup_{t \in T} \text{IBr}_N^0(\hat{B} \mid \varphi^t) \right|.$$

Proof. First let I be the inertial group of φ in G .

CASE 1. Assume $\text{IBr}_N^0(B \mid \varphi) = \emptyset$. We just need to show that $\text{IBr}_N^0(\hat{B} \mid \varphi^t) = \emptyset$ for all $t \in T$. Suppose, on the contrary, that there is $t_0 \in T$ for which $\text{IBr}_N^0(\hat{B} \mid \varphi^{t_0}) \neq \emptyset$. Then [8, Theorem 3.3] implies that an $N_G(DN)$ -conjugate of D , hence DN , is contained in the inertial group of φ^{t_0} in G . So, in particular, $D^{t_0^{-1}} \subseteq I$. Now as D is an inertial defect group of B with respect to b and $t_0^{-1} \in T$, we have that $D^{t_0^{-1}}$ also is an inertial defect group of B with respect to b . Then in view of Theorem 3.3 in [8], it follows that $\text{IBr}_N^0(B \mid \varphi) \neq \emptyset$, which contradicts our assumption.

CASE 2. Suppose that $\text{IBr}_N^0(B \mid \varphi) \neq \emptyset$. Let $\omega \in \text{IBr}_N^0(B \mid \varphi)$. Next denote by $\omega_{(I)}$ the unique irreducible Brauer character of I lying over φ such that $(\omega_{(I)})^G = \omega$ and let $\omega_{(T)}$ be the irreducible Brauer character $(\omega_{(I)})^T$. Then as $\varphi \in \text{IBr}(b)$ and $\omega_{(T)}$ lies over φ , [9, Theorem 5.5.10] implies that $\omega_{(T)} \in \text{IBr}(B')$, where B' is the Fong–Reynolds correspondent of B in T . Now let Q be a vertex of $\omega_{(I)}$. Then $\omega_{(T)}$ has vertex Q and so by [9, Theorem 5.1.9], $Q \subseteq D^t$ for some $t \in T$. Also, since Q is a vertex for ω and $\omega \in \text{IBr}_N^0(B)$, we have $\omega_{(T)} \in \text{IBr}_N^0(B')$ by Lemma 2.1 (i). Then part (ii) of Lemma 2.1 tells us that $QN = D^tN$.

Choose now a minimal subset U of T such that for each $\omega \in \text{IBr}_N^0(B \mid \varphi)$, there exists a unique element $u_\omega \in U$ such that $D^{u_\omega}N = QN$ for some vertex Q of $\omega_{(I)}$.

STEP 1. Our objective in this step is to show that

$$(1) \quad \bigcup_{t \in T} \text{IBr}_N^0(\hat{B} \mid \varphi^t) = \bigcup_{u \in U} \text{IBr}_N^0(\hat{B} \mid \varphi^{u^{-1}}).$$

So let $t \in T$ and suppose $\theta \in \text{IBr}_N^0(\hat{B} \mid \varphi^t)$. Then as a vertex of $\theta^{t^{-1}}$ is conjugate to a vertex of θ and a defect group of $\hat{B}^{t^{-1}}$ is conjugate to a defect group of \hat{B} , we have $\theta^{t^{-1}} \in \text{IBr}_N^0(\hat{B}^{t^{-1}} \mid \varphi)$ by Lemma 2.1 (i). Let now η be the unique element in $\text{IBr}(N_G(D^{t^{-1}}N) \cap I)$ lying over φ such that $\eta^{N_G(D^{t^{-1}}N)} = \theta^{t^{-1}}$.

Since $\hat{B}^{t^{-1}}$ has defect group $D^{t^{-1}}$, the Brauer character $\theta^{t^{-1}}$ has some vertex $P \subseteq D^{t^{-1}}$ by [9, Theorem 5.1.9]. Then Lemma 2.1 (ii) says that $PN = D^{t^{-1}}N$. Next there is $h \in N_G(D^{t^{-1}}N)$ for which $R = P^h$ is a vertex for η . Now $RN = (PN)^h = D^{t^{-1}}N$. So in particular, as $RN \subseteq I$, we get $D^{t^{-1}}N \subseteq I$. Therefore $N_G(D^{t^{-1}}N) \cap I = N_I(D^{t^{-1}}N)$.

Now if β is the p -block of $N_I(D^{t^{-1}}N)$ for which $\eta \in \text{IBr}(\beta)$, we have that $\beta^{N_G(D^{t^{-1}}N)}$ is defined and $\beta^{N_G(D^{t^{-1}}N)} = \hat{B}^{t^{-1}}$ by Lemma 3.1 in [8].

Next choose a defect group L for β such that $R \subseteq L$. By [9, Lemma 5.3.3], there exists $k \in N_G(D^{t^{-1}}N)$ such that $L \subseteq (D^{t^{-1}})^k$. Now

$$RN \subseteq LN \subseteq (D^{t^{-1}}N)^k = D^{t^{-1}}N.$$

Since $RN = D^{t^{-1}}N$, we get

$$(2) \quad LN = D^{t^{-1}}N.$$

Next as η lies over φ and $\varphi \in \text{IBr}(b)$, the p -block β covers b . Then [3, Proposition 4.2] implies that $|L \cap N| = p^{d(b)}$ where $d(b)$ is the defect of b . Similarly, since $\hat{B}^{t^{-1}}$ covers b , we have $|D^{t^{-1}} \cap N| = p^{d(b)}$. Now from (2) $|L|/|L \cap N| = |D^{t^{-1}}|/|D^{t^{-1}} \cap N|$ and it follows that

$$(3) \quad |L| = |D^{t^{-1}}| = |D|.$$

Since $N_I(L) \subseteq N_I(LN) \subseteq I$ and β is a p -block of $N_I(LN)$ ($= N_I(D^{t^{-1}}N)$) with defect group L , Theorem 5.3.8 in [9] tells us that β^L is defined and has L as a defect

group. Furthermore, as β covers b , note by the remark following Lemma 3.1 of [6] that β^I covers b as well.

Now since φ is I -invariant, there exists an element $\nu \in \text{IBr}_N^0(\beta^I \mid \varphi)$ by [8, Lemma 3.2 (ii)]. Next we have $\beta^{N_G(D^{t^{-1}}N)} = \hat{B}^{t^{-1}}$ and $(\hat{B}^{t^{-1}})^G = B$. Then [9, Lemma 5.3.4] says that β^G is defined and equals B . By Lemma 5.3.4 in [9] again, we get that $(\beta^I)^G$ is defined and equals B . Now as ν is an irreducible Brauer character of β^I lying over φ , [8, Lemma 3.1] implies that ν^G is an irreducible Brauer character of B lying over φ . We claim that $\nu^G \in \text{IBr}_N^0(B \mid \varphi)$.

As $\nu \in \text{IBr}_N^0(\beta^I \mid \varphi)$ and the order of any defect group of β^I is $|D|$ by (3), we have $(\nu(1)_p | D|) / |I|_p = (\varphi(1)_p p^{d(b)}) / |N|_p$. On the other hand, $(\nu^G(1)_p | D|) / |I|_p = (\nu^G(1)_p | D|) / |G|_p$. It follows that $\nu^G \in \text{IBr}_N^0(B \mid \varphi)$, as claimed.

Next choose a vertex K of ν such that $K \subseteq L$ (recall that L is a defect group of β^I). Since $\nu \in \text{IBr}_N^0(\beta^I)$, we have $KN = LN$ by Lemma 2.1 (ii). Further, by the choice of the subset U , as $\nu^G \in \text{IBr}_N^0(B \mid \varphi)$, there exist $y \in I$ and $u_0 \in U$ such that $K^y N = D^{u_0} N$. Now in view of (2), we get that $(DN)^{t^{-1}y} = (D^{t^{-1}}N)^y = (DN)^{u_0}$. Hence $z = u_0 y^{-1} t \in N_G(DN)$. Then $\varphi^t = \varphi^{y u_0^{-1} z} = (\varphi^{u_0^{-1}})^z$, which tells us that φ^t is $N_G(DN)$ -conjugate to $\varphi^{u_0^{-1}}$. Now since $\theta \in \text{IBr}_N^0(\hat{B} \mid \varphi^t)$, we conclude that $\theta \in \text{IBr}_N^0(\hat{B} \mid \varphi^{u_0^{-1}})$. This clearly proves (1).

STEP 2. For each $u \in U$, we let E_u be the set of $\omega \in \text{IBr}_N^0(B \mid \varphi)$ such that $\omega_{(I)}$ has a vertex Q with $QN = D^u N$. By the minimality of U , it is clear that $E_u \neq \emptyset$. Our final goal in this step is to prove that

$$(4) \quad |E_u| = |\text{IBr}_N^0(\hat{B} \mid \varphi^{u^{-1}})|,$$

for each $u \in U$.

Let $u \in U$. Suppose $\omega \in E_u$. If Δ is the p -block of I such that $\omega_{(I)} \in \text{IBr}(\Delta)$, then Δ^G is defined and equals B by Lemma 3.1 of [8]. Let Q be a vertex for $\omega_{(I)}$ such that

$$(5) \quad QN = D^u N.$$

Then $Q \subseteq P$ for some defect group P of Δ . Next by [9, Lemma 5.3.3], there is $g \in G$ such that $P \subseteq (D^u)^g$. Now we have $QN \subseteq PN \subseteq (D^u N)^g$. Then from (5), it follows that

$$(6) \quad D^u N = QN = PN = (D^u N)^g.$$

Hence, by Lemma 2.1, we get that $\omega_{(I)} \in \text{IBr}_N^0(\Delta \mid \varphi)$.

It is clear from (6) that $g \in N_G(D^u N)$. Also, as $PN = D^u N$ and $|P \cap N| = p^{d(b)} = |D^u \cap N|$ (as implied by [3, Proposition 4.2]), we have $|P| = |D^u|$. Since $P \subseteq (D^u)^g$, it follows that $P = (D^u)^g$. Hence Δ has some $N_G(D^u N)$ -conjugate (namely P) of D^u as a defect group.

Let now $\Delta_{u,1}, \dots, \Delta_{u,n_u}$ be all the (distinct) p -blocks of I covering b , each having some $N_G(D^u N)$ -conjugate of D^u as a defect group and such that $(\Delta_{u,i})^G = B$ for all $i \in \{1, \dots, n_u\}$.

Let $i \in \{1, \dots, n_u\}$. Assume that $\tau \in \text{IBr}_N^0(\Delta_{u,i} \mid \varphi)$. Then $\tau^G \in \text{IBr}(B \mid \varphi)$ by [8, Lemma 3.1]. Next choose a defect group $R_{u,i}$ for $\Delta_{u,i}$ which is $N_G(D^u N)$ -conjugate to D^u . Let L be a vertex of τ with $L \subseteq R_{u,i}$. Then by Lemma 2.1 (ii) we have $LN = R_{u,i}N = D^u N$. So, in particular, $|L|/|L \cap N| = |D|/|D \cap N|$. Since τ^G has vertex L , it follows by Lemma 2.1 (i) that $\tau^G \in E_u$.

Now the correspondence $\tau \mapsto \tau^G$ defines a map from $\bigcup_{i=1}^{n_u} \text{IBr}_N^0(\Delta_{u,i} \mid \varphi)$ to E_u . This map is onto by the above discussion and 1-1 by Theorem 3.3.2 of [9]. Consequently we get $|E_u| = \sum_{i=1}^{n_u} |\text{IBr}_N^0(\Delta_{u,i} \mid \varphi)|$.

For each i , we have seen above that $\Delta_{u,i}$ has a defect group $R_{u,i}$ such that $R_{u,i}N = D^u N$. Let $\hat{\Delta}_{u,i}$ be the unique p -block of $N_I(D^u N)$ having $R_{u,i}$ as a defect group and such that $(\hat{\Delta}_{u,i})^I = \Delta_{u,i}$. By Proposition 2.5, we have $|\text{IBr}_N^0(\Delta_{u,i} \mid \varphi)| = |\text{IBr}_N^0(\hat{\Delta}_{u,i} \mid \varphi)|$. It follows that

$$(7) \quad |E_u| = \sum_{i=1}^{n_u} |\text{IBr}_N^0(\hat{\Delta}_{u,i} \mid \varphi)|.$$

Now let $i \in \{1, \dots, n_u\}$. As φ is I -invariant, Lemma 3.2 (ii) in [8] tells us that $\text{IBr}_N^0(\Delta_{u,i} \mid \varphi) \neq \emptyset$. So $\text{IBr}_N^0(\hat{\Delta}_{u,i} \mid \varphi) \neq \emptyset$. Suppose $\alpha \in \text{IBr}_N^0(\hat{\Delta}_{u,i} \mid \varphi)$. Then $\alpha^{N_G(D^u N)}$ is an irreducible Brauer character of $N_G(D^u N)$ lying over φ . Now by [8, Lemma 3.1], $(\hat{\Delta}_{u,i})^{N_G(D^u N)}$ is defined and $\alpha^{N_G(D^u N)} \in \text{IBr}((\hat{\Delta}_{u,i})^{N_G(D^u N)})$. Next we have $(\hat{\Delta}_{u,i})^I = \Delta_{u,i}$ and $(\Delta_{u,i})^G = B$. So $(\hat{\Delta}_{u,i})^G$ is defined and equals B by [9, Lemma 5.3.4]. Applying Lemma 5.3.4 of [9] once again, we get that $((\hat{\Delta}_{u,i})^{N_G(D^u N)})^G$ is defined and equals B .

Next we know that $\hat{\Delta}_{u,i}$ has $R_{u,i}$ as a defect group. Then by Lemma 5.3.3 in [9], $(\hat{\Delta}_{u,i})^{N_G(D^u N)}$ has a defect group M for which $R_{u,i} \subseteq M \subseteq D^f$, where f is some element of G . But as $R_{u,i}$ is G -conjugate to D , we conclude that $M = R_{u,i}$. Further, since $R_{u,i}$ is $N_G(D^u N)$ -conjugate to D^u , it follows that $(\hat{\Delta}_{u,i})^{N_G(D^u N)}$ has D^u as a defect group. However, \hat{B}^u is the only p -block of $N_G(D^u N)$ that has D^u as a defect group and such that $(\hat{B}^u)^G = B$. We must then have that $(\hat{\Delta}_{u,i})^{N_G(D^u N)} = \hat{B}^u$. Then $\alpha^{N_G(D^u N)} \in \text{IBr}(\hat{B}^u)$.

Let X be a vertex of α . As $\alpha \in \text{IBr}_N^0(\hat{\Delta}_{u,i})$, we have $|R_{u,i}|/|R_{u,i} \cap N| = |X|/|X \cap N|$ by Lemma 2.1. Now since X is also a vertex for $\alpha^{N_G(D^u N)}$ and $R_{u,i}$ is a defect group of \hat{B}^u , it follows by the same lemma that $\alpha^{N_G(D^u N)} \in \text{IBr}_N^0(\hat{B}^u \mid \varphi)$.

We may now define a map F from $\bigcup_{i=1}^{n_u} \text{IBr}_N^0(\hat{\Delta}_{u,i} \mid \varphi)$ to $\text{IBr}_N^0(\hat{B}^u \mid \varphi)$ by $F(\alpha) = \alpha^{N_G(D^u N)}$. We claim that F is a bijection. First F is 1-1 by [9, Theorem 3.3.2]. Next we show that F is onto.

Let δ be an element of $\text{IBr}_N^0(\hat{B}^u \mid \varphi)$. Next call γ , the element in $\text{IBr}(N_I(D^u N))$ that lies over φ and such that $\gamma^{N_G(D^u N)} = \delta$. Now if Λ is the p -block of $N_I(D^u N)$

to which γ belongs, we have that $\Lambda^{N_G(D^u N)}$ is defined and equals \hat{B}^u by Lemma 3.1 of [8].

Let Y be a defect group for Λ . Then by [9, Lemma 5.3.3], $Y \subseteq (D^u)^e$ for some $e \in N_G(D^u N)$. So, in particular, $YN \subseteq D^u N$. Now choose a vertex V for γ such that $V \subseteq Y$. Since $\delta \in \text{IBr}_N^0(\hat{B}^u)$, V is a vertex of δ and $(D^u)^e$ is a defect group of \hat{B}^u , Lemma 2.1 (ii) tells us that $VN = (D^u)^e N = D^u N$. It follows that

$$(8) \quad VN = YN = D^u N.$$

Then, by Lemma 2.1 (ii) again, we have $\gamma \in \text{IBr}_N^0(\Lambda \mid \varphi)$. Also by (8) we have $|Y|/|Y \cap N| = |D^u|/|D^u \cap N|$. As each of Λ and \hat{B}^u covers the p -block b , Proposition 4.2 of [3] implies that $|Y \cap N| = |D^u \cap N|$. Hence $|Y| = |D^u|$ and so $Y = (D^u)^e$.

By [9, Theorem 5.3.8], we have that Λ^I is defined and has Y as a defect group. Next as $\Lambda^{N_G(D^u N)} = \hat{B}^u$ and $(\hat{B}^u)^G = B$, Lemma 5.3.4 in [9] tells us that Λ^G is defined and $\Lambda^G = B$. Another application of [9, Lemma 5.3.4] now gives us that $(\Lambda^I)^G$ is defined and $(\Lambda^I)^G = B$.

Since Λ covers b , then so does Λ^I . Furthermore, as Λ^I has defect group Y , which is $N_G(D^u N)$ -conjugate to D^u , we deduce that $\Lambda^I = \Delta_{u,i_0}$ for some $i_0 \in \{1, \dots, n_u\}$.

Recall that R_{u,i_0} is a defect group for Δ_{u,i_0} and that $R_{u,i_0}N = D^u N$. Since Y is also a defect group of Δ_{u,i_0} , we have $Y = (R_{u,i_0})^c$ with $c \in I$. Now as $YN = D^u N$ from (8), we get $R_{u,i_0}N = (R_{u,i_0}N)^c$. This tells us that $c \in N_I(R_{u,i_0}N) = N_I(D^u N)$. It follows that $R_{u,i_0} (= Y^{c^{-1}})$ is a defect group of Λ . Then we must have $\Lambda = \hat{\Delta}_{u,i_0}$.

Now $\gamma \in \text{IBr}_N^0(\hat{\Delta}_{u,i_0} \mid \varphi)$ and $\gamma^{N_G(D^u N)} = \delta$. This proves that the map F is onto. Hence F is a bijection as claimed.

Now $|\text{IBr}_N^0(\hat{B} \mid \varphi^{u^{-1}})| = |\text{IBr}_N^0(\hat{B}^u \mid \varphi)| = |\bigcup_{i=1}^{n_u} \text{IBr}_N^0(\hat{\Delta}_{u,i} \mid \varphi)|$ (the second equality follows from the fact that F is a bijection). Since $\hat{\Delta}_{u,1}, \dots, \hat{\Delta}_{u,n_u}$ are distinct p -blocks of $N_I(D^u N)$ (because $\Delta_{u,1}, \dots, \Delta_{u,n_u}$ are distinct p -blocks) and $|E_u| = \sum_{i=1}^{n_u} |\text{IBr}_N^0(\hat{\Delta}_{u,i} \mid \varphi)|$ by (7), equality (4) follows.

STEP 3. By our choice of the set U , since $\{E_u : u \in U\}$ is a partition of $\text{IBr}_N^0(B \mid \varphi)$, then

$$\begin{aligned} |\text{IBr}_N^0(B \mid \varphi)| &= \sum_{u \in U} |E_u| \\ &= \sum_{u \in U} |\text{IBr}_N^0(\hat{B} \mid \varphi^{u^{-1}})| \quad (\text{by (4)}). \end{aligned}$$

Now to complete the proof of the proposition, in view of (1), it suffices to show that the sets $\text{IBr}_N^0(\hat{B} \mid \varphi^{u^{-1}})$ are mutually disjoint.

So let $u, u' \in U$ and suppose $\text{IBr}_N^0(\hat{B} \mid \varphi^{u^{-1}}) \cap \text{IBr}_N^0(\hat{B} \mid \varphi^{u'^{-1}}) \neq \emptyset$. Then there is $x \in N_G(DN)$ for which $\varphi^{u^{-1}} = (\varphi^{u'^{-1}})^x$. So $u = x^{-1}u'w$ for some $w \in I$. Then $D^u N = (D^{u'}N)^w$ and by the choice of U , it follows that $u = u'$. The proof of the proposition is now complete. \square

We need two more lemmas to be able to prove the main theorems.

Lemma 2.7. *Let $N \triangleleft G$ where G is p -solvable and let B and b be p -blocks of G and N respectively such that B covers b . Let T be the inertial group of b in G and let B' be the Fong–Reynolds correspondent of B in T . Suppose $\varphi \in \text{IBr}(b)$. Then the map $\psi \mapsto \psi^G$ defines bijections from $\text{IBr}_N^0(B')$ onto $\text{IBr}_N^0(B)$ and from $\text{IBr}_N^0(B' \mid \varphi)$ onto $\text{IBr}_N^0(B \mid \varphi)$.*

Proof. In view of [9, Theorem 5.5.10 (ii)], the map $\psi \mapsto \psi^G$ defines a bijection of $\text{IBr}(B')$ onto $\text{IBr}(B)$. We shall see that this bijection restricts to a bijection from $\text{IBr}_N^0(B')$ onto $\text{IBr}_N^0(B)$. For that, it suffices to show that $\psi \in \text{IBr}_N^0(B')$ if and only if $\psi^G \in \text{IBr}_N^0(B)$. So let $\psi \in \text{IBr}(B')$ and choose $\omega \in \text{IBr}(b)$ under ψ . Then ψ^G lies over ω . Since $\text{ht}(\psi) = \text{ht}(\psi^G)$, we have $\text{ht}(\psi) - \text{ht}(\omega) = \text{ht}(\psi^G) - \text{ht}(\omega)$. The result then immediately follows.

By the above, the map $\psi \mapsto \psi^G$ defines an injection from $\text{IBr}_N^0(B' \mid \varphi)$ to $\text{IBr}_N^0(B \mid \varphi)$. Next suppose $\tau \in \text{IBr}_N^0(B \mid \varphi)$. Since T contains the inertial group of φ (in G), then by Clifford's theorem ([9, Exercise 3.3.4]), there exists an irreducible Brauer character ν of T over φ with $\nu^G = \tau$. Let B_0 be the p -block of T to which ν belongs. Then B_0 covers b and by [9, Theorem 5.5.10 (ii)], we have $\tau \in \text{IBr}(B_0^G)$. Hence $B_0^G = B$ and it follows (by Theorem 5.5.10 (i) of [9]) that $B_0 = B'$. So ν is an irreducible Brauer character of B' lying over φ . Since $\tau \in \text{IBr}_N^0(B)$, we have $\nu \in \text{IBr}_N^0(B')$ by the discussion in the first paragraph. This shows that our injection is, in fact, a bijection and the proof of the lemma is complete. \square

Lemma 2.8. *Let $N \triangleleft G$ where G is p -solvable and let B and b be p -blocks of G and N respectively such that B covers b . Let T be the inertial group of b in G and let B' be the Fong–Reynolds correspondent of B in T . Suppose D is a defect group of B' . Then B has defect group D . Moreover, if \hat{B} (resp. \hat{B}') is the unique p -block of $NN_G(D)$ (resp. $NN_T(D)$) with defect group D such that $\hat{B}^G = B$ (resp. $\hat{B}'^T = B'$), then \hat{B}' covers b , $\hat{B}'^{NN_G(D)}$ is defined and $\hat{B}'^{NN_G(D)} = \hat{B}$.*

Proof. The fact that B has defect group D is immediate from [9, Theorem 5.5.10 (iv)]. Next, since b is T -stable and B' covers b , then by the remark following Lemma 3.1 in [6], it is easy to see that \hat{B}' covers b . Also, as $NN_T(D)$ is the inertial group of b in $NN_G(D)$, we have that $\hat{B}'^{NN_G(D)}$ is defined and has defect group D by [9, Theorem 5.5.10]. Next $\hat{B}'^T = B'$ and $(B')^G = B$, and so in view of [9, Lemma 5.3.4], \hat{B}'^G is defined and $\hat{B}'^G = B$. Using Lemma 5.3.4 of [9] once more, we get that $(\hat{B}'^{NN_G(D)})^G$ is defined and equals B . Now by the uniqueness of \hat{B} , we are forced to have $\hat{B}'^{NN_G(D)} = \hat{B}$. This ends the proof of the lemma. \square

We are now ready to present a proof for Theorem A.

Proof of Theorem A. Let B' be the Fong–Reynolds correspondent of B in T . Then Lemma 2.7 implies that

$$(*) \quad |\mathrm{IBr}_N^0(B' \mid \varphi)| = |\mathrm{IBr}_N^0(B \mid \varphi)|.$$

Next we have that D is a defect group for B' and we write \hat{B}' for the unique p -block of $NN_T(D)$ with defect group D such that $\hat{B}'^T = B'$. Then Lemma 2.8 says that \hat{B}' covers b and that, as $NN_T(D)$ is the inertial group of b in $NN_G(D)$, \hat{B}' is the Fong–Reynolds correspondent of \hat{B} in $NN_T(D)$.

Note that $\varphi^t \in \mathrm{IBr}(b)$ for every $t \in T$. Then by Lemma 2.7, the map $\psi \mapsto \psi^G$ defines a bijection from $\bigcup_{t \in T} \mathrm{IBr}_N^0(\hat{B}' \mid \varphi^t)$ onto $\bigcup_{t \in T} \mathrm{IBr}_N^0(\hat{B} \mid \varphi^t)$. It follows, in particular, that

$$(**) \quad \left| \bigcup_{t \in T} \mathrm{IBr}_N^0(\hat{B}' \mid \varphi^t) \right| = \left| \bigcup_{t \in T} \mathrm{IBr}_N^0(\hat{B} \mid \varphi^t) \right|.$$

Next we claim that $NN_T(D) = N_T(DN)$.

First we easily see that $NN_T(D) \subseteq N_T(DN)$. Next, by [9, Corollary 5.5.6], there exists a unique p -block b_0 of DN covering b . Since b is $N_T(DN)$ -stable, it follows that b_0 is $N_T(DN)$ -stable. Also, as \hat{B}' covers b , then \hat{B}' must cover b_0 as well. Now in view of [3, Proposition 4.2], since \hat{B}' has defect group D , there is $g \in N$ for which D^g is a defect group for b_0 . Hence b_0 has D as a defect group. Let now x be any element of $N_T(DN)$. Then, as $b_0^x = b_0$, the subgroup D^x is a defect group of b_0 . It follows that D^x is DN -conjugate to D . Hence $D^x = D^y$ for some $y \in N$. Therefore $x \in NN_T(D)$. We have thus shown that $N_T(DN) \subseteq NN_T(D)$, and our claim is valid.

By Proposition 2.6, we have $|\mathrm{IBr}_N^0(B' \mid \varphi)| = |\bigcup_{t \in T} \mathrm{IBr}_N^0(\hat{B}' \mid \varphi^t)|$. Then, taking into account (*) and (**), we get $|\mathrm{IBr}_N^0(B \mid \varphi)| = |\bigcup_{t \in T} \mathrm{IBr}_N^0(\hat{B} \mid \varphi^t)|$, as wanted. \square

Next, we take care of Theorem B.

Proof of Theorem B. Let b_0 be a p -block of N which is covered by B . Call T_0 the inertial group of b_0 in G and let B'_0 be the Fong–Reynolds correspondent of B in T_0 . Then by [9, Theorem 5.5.10 (iv)], there is $g \in G$ for which D^g is a defect group for B'_0 .

Let $b = (b_0)^{g^{-1}}$. Then B covers b , the inertial group of b in G is $T = (T_0)^{g^{-1}}$ and $B' = (B'_0)^{g^{-1}}$ is the Fong–Reynolds correspondent of B (with respect to b) in T . Note also that B' has D as a defect group. Let \hat{B}' be the unique p -block of $NN_T(D)$ with defect group D such that $\hat{B}'^T = B'$. Then, by Lemma 2.8, we have that the p -block \hat{B}' covers b and that it is the Fong–Reynolds correspondent of \hat{B} in the inertial group $NN_T(D)$ of b in $NN_G(D)$.

Next T acts by conjugation on $\text{IBr}(b)$. Let $\mathcal{O}_1, \dots, \mathcal{O}_r$ be the resulting orbits and for each $i \in \{1, \dots, r\}$, choose $\varphi_i \in \mathcal{O}_i$. By Theorem A, we have

$$|\text{IBr}_N^0(B' \mid \varphi_i)| = \left| \bigcup_{\theta \in \mathcal{O}_i} \text{IBr}_N^0(\hat{B}' \mid \theta) \right|.$$

It follows that

$$\begin{aligned} |\text{IBr}_N^0(B')| &= \sum_{i=1}^r |\text{IBr}_N^0(B' \mid \varphi_i)| \\ &= \sum_{i=1}^r \left| \bigcup_{\theta \in \mathcal{O}_i} \text{IBr}_N^0(\hat{B}' \mid \theta) \right| \\ &= \left| \bigcup_{\theta \in \text{IBr}(b)} \text{IBr}_N^0(\hat{B}' \mid \theta) \right| \\ &= |\text{IBr}_N^0(\hat{B}')|. \end{aligned}$$

Finally since $|\text{IBr}_N^0(B)| = |\text{IBr}_N^0(B')|$ and $|\text{IBr}_N^0(\hat{B})| = |\text{IBr}_N^0(\hat{B}')|$, as implied by Lemma 2.7, we get $|\text{IBr}_N^0(B)| = |\text{IBr}_N^0(\hat{B})|$, as required. \square

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References

- [1] W. Hamernik and G. Michler: *On vertices of simple modules in p -solvable groups*, Mitt. Math. Sem. Giessen Heft **121** (1976), 147–162.
- [2] I.M. Isaacs: *Characters of π -separable groups*, J. Algebra **86** (1984), 98–128.
- [3] R. Knörr: *Blocks, vertices and normal subgroups*, Math. Z. **148** (1976), 53–60.
- [4] A. Laradji: *Relative π -blocks of π -separable groups*, J. Algebra **220** (1999), 449–465.
- [5] A. Laradji: *Relative π -blocks of π -separable groups II*, J. Algebra **237** (2001), 521–532.
- [6] A. Laradji: *On normal subgroups and height zero Brauer characters in a p -solvable group*, J. Algebra **295** (2006), 543–561.
- [7] A. Laradji: *On normal subgroups and simple modules with a given vertex in a p -solvable group*, J. Algebra **308** (2007), 484–492.
- [8] M. Murai: *Normal subgroups and heights of characters*, J. Math. Kyoto Univ. **36** (1996), 31–43.
- [9] H. Nagao and Y. Tsushima: *Representations of Finite Groups*, Academic Press, Boston, MA, 1989.
- [10] T. Okuyama: *Module correspondence in finite groups*, Hokkaido Math. J. **10** (1981), 299–318.
- [11] A. Watanabe: *Normal subgroups and multiplicities of indecomposable modules*, Osaka J. Math. **33** (1996), 629–635.

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