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EULER CHARACTERISTICS ON A CLASS OF FINITELY GENERATED NILPOTENT GROUPS

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Abstract

A finitely generated torsion free nilpotent group is called an $\mathscr{F}$-group. To each $\mathscr{F}$-group Γ there is associated a connected, simply connected nilpotent Lie group G_Γ. Let TUF be the class of all $\mathscr{F}$-group Γ such that G_Γ is totally unimodular. A group in TUF is called TUF-group. In this paper, we are interested in finding non-zero Euler characteristic on the class TUF and therefore, on TUFF, the class of groups K having a subgroup Γ of finite index in TUF. An immediate consequence we obtain that any two isomorphic finite index subgroups of a TUFF-group have the same index. As applications, we give two results, the first is a generalization of Belegradek's result, in which we prove that every TUFF-group is co-hopfian. The second is a known result due to G.C. Smith, asserting that every TUFF-group is not compressible.

1. Introduction and main results

We follow [5, p. 222] in defining an Euler characteristic on a class of groups as follows (see also [2, p. 1]).

DEFINITION 1.1 (Euler characteristic). Let $\mathfrak{X}$ be a class of groups closed under taking subgroups of finite index. By an Euler characteristic on $\mathfrak{X}$ it meant a function $\chi\colon \mathfrak{X} \to \mathbb{R}$ satisfying
(Ec1) If K and H are in $\mathfrak{X}$, and K is isomorphic to H, then $\chi(K) = \chi(H)$.
(Ec2) If K is in $\mathfrak{X}$, and H is a subgroup of K of finite index, then $\chi(H) = [K : H]\chi(K)$, where $[K : H]$ denotes the index of H in K.

In this paper, we are interested in finding non zero Euler characteristics defined on a class of finitely generated nilpotent groups.

Let G be a connected Lie group and $\mathrm{Aut}(G)$ its group of continuous automorphisms. Let μ be a Haar measure on G. For every $\alpha \in \mathrm{Aut}(G)$ we have

$$\alpha_*^{-1}\mu = \Delta(\alpha)\mu,$$

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where $\alpha_*^{-1}\mu$ is the push forward of μ under α^{-1}, and $\Delta\colon \mathrm{Aut}(G) \to \mathbb{R}_+^*$ is a homomorphism of $\mathrm{Aut}(G)$ into the multiplicative group of the positive reals. If G is a connected, simply connected nilpotent Lie group, then

$$\Delta(\alpha) = |\det(\alpha)|.$$

DEFINITION 1.2 ([10, p. 627]). A connected, simply connected nilpotent Lie group G is called totally unimodular if the image of Δ is $\{1\}$.

Let TULG be the class of connected, simply connected totally unimodular nilpotent Lie groups.

A real Lie algebra is called *characteristically nilpotent* if all its derivations are nilpotent ([4, p. 157], [6, p. 623]). We note that a characteristically nilpotent Lie algebra is nilpotent. Let CNLG be the class of connected, simply connected nilpotent Lie groups $G = \exp \mathfrak{g}$ such that $\mathfrak{g}$ is a characteristically nilpotent Lie algebra.

Proposition 1.3 ([10, (1.1)]). *We have*

$$\mathit{CNLG} \subset \mathit{TULG}.$$

A finitely generated torsion free nilpotent group is called an $\mathscr{F}$-group. Any $\mathscr{F}$-group Γ is isomorphic to a discrete uniform subgroup of a connected, simply connected nilpotent Lie group G_Γ whose Lie algebra $\mathfrak{g}_\Gamma$ has rational structure constants ([8, Theorem 6]). Let TUF be the class of all $\mathscr{F}$-groups Γ such that $G_\Gamma \in$ TULG. We call a group Γ a TUF-group if $\Gamma \in$ TUF. For every integer $n \geq 7$ there exists a n-dimensional characteristically nilpotent Lie algebra with rational structure ([14, Theorem 5]). By the Mal'cev rationality criterion (Theorem 2.1) we derive the following.

Proposition 1.4. *For every integer $n \geq 7$ there exists a TUF-group with Hirsch length n.*

The main result of this paper is the following.

Theorem 1.5. *The class TUF admits Euler characteristics.*

In Section 3, we give an explicit Euler characteristic on TUF.

By [5, p. 222] (see also [15], [2]) every Euler characteristic χ on TUF can be extended to TUFF, the class of groups K having a subgroup Γ of finite index in TUF, by setting

$$\chi(K) = \frac{1}{[K:\Gamma]}\chi(\Gamma).$$

As an immediate consequence we have the following.

Proposition 1.6. *Any two isomorphic finite index subgroups of a* TUFF*-group have the same index.*

DEFINITION 1.7 (Co-hopfian group). A group is called co-hopfian if it satisfies the following equivalent conditions:
(1) It is not isomorphic to any proper subgroup.
(2) Every injective endomorphism of the group is an automorphism.

As an easy consequence of Proposition 1.6, we obtain a generalization for I. Belegradek's result ([1, Corollary 2.4]).

Proposition 1.8. *Every* TUFF*-group is co-hopfian.*

We introduce the following definition due to G.C. Smith ([13, Definition 1]).

DEFINITION 1.9 (Compressible group). A group G is called compressible if any finite index subgroup of G contains a finite index subgroup isomorphic to G.

The following proposition which is due to G.C. Smith ([13, Proposition 4]) is an immediate consequence of Proposition 1.6.

Proposition 1.10. *Every* TUFF*-group is not compressible.*

2. Rational structures and discrete uniform subgroups

General references for the material in this section are [3] and [11] as well as the original paper of Mal'cev [8].

Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then the exponential map $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism. Let $\log\colon G \to \mathfrak{g}$ denote the inverse of exp.

2.1. Rational structures. Let G be a nilpotent, connected and simply connected real Lie group and let $\mathfrak{g}$ be its Lie algebra. We say that $\mathfrak{g}$ (or G) has a *rational structure* if there is a Lie algebra $\mathfrak{g}(\mathbb{Q})$ over $\mathbb{Q}$ such that $\mathfrak{g} \cong \mathfrak{g}(\mathbb{Q}) \otimes \mathbb{R}$. It is clear that $\mathfrak{g}$ has a rational structure if and only if $\mathfrak{g}$ has an $\mathbb{R}$-basis $(X_1, \ldots, X_n)$ with rational structure constants.

2.2. Uniform subgroups. A discrete subgroup Γ is called *uniform* in G if the quotient space G/Γ is compact. A proof of the next result can be found in Theorem 7 of [8] or in Theorem 2.12 of [11].

Theorem 2.1 (The Malcev rationality criterion)**.** *Let G be a simply connected nilpotent Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then G admits a uniform subgroup Γ if and only if $\mathfrak{g}$ admits a basis $(X_1, \ldots, X_n)$ such that*

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k, \quad (\forall 1 \leq i, j \leq n),$$

where the constants c_{ijk} are all rational.

2.3. The Malcev rigidity theorem. The following is a theorem of Mal'cev ([8, Theorem 5]); see also ([9, Theorem 4]).

Theorem 2.2 (Malcev rigidity theorem)**.** *Let G_1 and G_2 be connected simply connected nilpotent Lie groups and Γ_1, Γ_2 discrete uniform subgroups of G_1 and G_2. Any abstract group isomorphism ϕ between Γ_1 and Γ_2 extends uniquely to an isomorphism $\mathrm{M}(\phi)$ of G_1 on G_2; that is, the following diagram*

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\phi} & \Gamma_2 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ G_1 & \xrightarrow[\mathrm{M}(\phi)]{} & G_2 \end{array} \tag{2.1}$$

is commutative, where i is the inclusion mapping. The isomorphism $\mathrm{M}(\phi)$ is called the Mal'cev extension of ϕ.

3. An explicit Euler characteristic on TUF. Proof of Theorem 1.5

Let G be a connected Lie group, $\mathscr{S}(G)$ be the space of discrete uniform (i.e., cocompact) subgroups of G. Let μ be a right Haar measure of G. Let $\Gamma \in \mathscr{S}(G)$, the measure μ induces a finite measure $\bar{\mu}$ over the homogeneous space G/Γ. Let

$$V_G^{\mu} : \mathscr{S}(G) \to \mathbb{R}_+$$

defined for $\Gamma \in \mathscr{S}(G)$) by

$$V_G^{\mu}(\Gamma) = \bar{\mu}(G/\Gamma).$$

REMARK 3.1. We recall that if F is a fundamental domain for G/Γ then $\bar{\mu}(G/\Gamma) = \mu(F)$ ([7, p. 430]).

The notation $H \leq_f K$ signifies that H is a finite index subgroup of the group K. A proof of the following proposition can be found in Lemma 3.2 of [7].

Proposition 3.2. *If $H, K \in \mathscr{S}(G)$ and if $H \leq_f K$ then we have*

$$V_G^{\mu}(H) = [K : H] V_G^{\mu}(K). \tag{3.1}$$

Proposition 3.3. *Let G in* TULG *and μ a Haar measure on G. Let Γ_1, Γ_2 be two isomorphic subgroups of $\mathscr{S}(G)$. Then we have*

$$V_G^{\mu}(\Gamma_1) = V_G^{\mu}(\Gamma_2). \tag{3.2}$$

Proof. Let ϕ be an isomorphism of Γ_1 onto Γ_2. Let F be a fundamental domain of G/Γ_1 and compute

$$\begin{aligned} V_G^{\mu}(\Gamma_1) &= \mu(F) \\ &= \mu(\mathrm{M}(\phi)(F)) \\ &= V_G^{\mu}(\Gamma_2). \end{aligned}$$ □

We define an equivalence relation $\simeq$ on TULG by

$$G_1 \simeq G_2 \Longleftrightarrow G_1, G_2 \quad \text{are isomorphic.}$$

For $G \in$ TULG, let $[G]$ be the equivalence class containing G. Let T be a transversal for the equivalence relation $\simeq$.

Let H, K be two groups (resp. Lie groups), the set of all isomorphisms (resp. Lie groups isomorphisms) of H onto K is denoted by $\mathscr{R}(H, K)$.

Lemma 3.4. *Let $G_0 \in T$ and $G \in [G_0]$. For every $\phi, \psi \in \mathscr{R}(G_0, G)$ we have*

$$\phi_* \mu_0 = \psi_* \mu_0,$$

where $\phi_* \mu_0$ (resp. $\psi_* \mu_0$) is the push forward of μ_0 under ϕ (resp. ψ).

Proof. Let F be a measurable set and compute

$$\begin{aligned} \phi_* \mu_0(F) &= \mu_0(\phi^{-1}(F)) \\ &= \mu_0(\psi^{-1}\phi(\phi^{-1}(F))) \qquad (\psi^{-1}\phi \in \mathrm{Aut}(G_0)) \\ &= \mu_0(\psi^{-1}(F)) \\ &= \psi_* \mu_0(F). \end{aligned}$$ □

Let $G_0 \in T$ and μ_0 a Haar measure on G_0. Let $G \in [G_0]$ and $\Gamma \in \mathscr{S}(G)$. The function

$$\mathscr{R}(G_0, G) \to \mathbb{R}, \quad \phi \to V_G^{\phi_* \mu_0}(\Gamma)$$

is constant. In the sequel, we note

$$V[G_0, \mu_0, G](\Gamma) = V_G^{\phi_*\mu_0}(\Gamma) \quad (\forall \phi \in \mathscr{R}(G_0, G)).$$

Let $G_1, G_2 \in [G_0]$. For every $\psi \in \mathscr{R}(G_0, G_2)$ and $\phi \in \mathscr{R}(G_1, G_2)$, we note

$$V[G_0, \mu_0, G_1] * \phi = V[G_0, \mu_0, G_2] \circ \phi^*,$$
$$\psi * V[G_0, \mu_0, G_1] = V[\psi(G_0), \psi_*\mu_0, G_1],$$

where $\phi^* \colon \mathscr{S}(G_1) \to \mathscr{S}(G_2),\ \Gamma \mapsto \phi(\Gamma)$.

Proposition 3.5. *With the same notation as above we have*:

$$V[G_0, \mu_0, G_1] * \phi = V[G_0, \mu_0, G_1], \tag{3.3a}$$
$$\psi * V[G_0, \mu_0, G_1] = V[G_0, \mu_0, G_1]. \tag{3.3b}$$

Proof. Let $\Gamma \in \mathscr{S}(G_1)$. Let F be a fundamental domain for G_1/Γ and compute

$$\begin{aligned} V[G_0, \mu_0, G_1] * \phi(\Gamma) &= V[G_0, \mu_0, G_2](\phi(\Gamma)) \\ &= V_{G_2}^{\varphi_*\mu_0}(\phi(\Gamma)) \qquad (\varphi \in \mathscr{R}(G_0, G_2)) \\ &= \varphi_*\mu_0(\phi(F)) \\ &= (\phi^{-1}\varphi)_*\mu_0(F) \\ &= V_{G_1}^{(\phi^{-1}\varphi)_*\mu_0}(F) \\ &= V[G_0, \mu_0, G_1](\Gamma). \end{aligned}$$

Similarly, we prove (3.3b). □

We come now to the principal theorem of this paper, in which we give an explicit Euler characteristic on TUF.

Theorem 3.6. *The mapping*

$$\chi \colon \mathit{TUF} \to \mathbb{R}, \quad \Gamma \mapsto V[G_0, \mu_0, G_\Gamma](\Gamma),$$

where $\{G_0\} = [G_\Gamma] \cap T$, *is an Euler characteristic on TUF, which does not depend of the choice of transversal.*

Proof. Let Γ be a TUF-group and $\Gamma_0 \leq_f \Gamma$. Then Γ_0 and Γ have the same Hirsch length, it follows that $\mathfrak{g}_{\Gamma_0} = \mathfrak{g}_\Gamma$ and hence the class TUF is closed under taking subgroups of finite index. Let Γ_1 and Γ_2 be two isomorphic TUF-groups. By Theorem 2.2, the Lie groups G_{Γ_1} and G_{Γ_2} are isomorphic and hence $[G_{\Gamma_1}] = [G_{\Gamma_2}]$. Let

$$\{G_0\} = [G_{\Gamma_1}] \cap T.$$

Let $\phi \in \mathscr{R}(\Gamma_1, \Gamma_2)$ and compute

$$\begin{aligned}\chi(\Gamma_2) &= V[G_0, \mu_0, G_{\Gamma_2}](\Gamma_2)\\ &= V[G_0, \mu_0, G_{\Gamma_1}] * \mathrm{M}(\phi)(\Gamma_1)\\ &= V[G_0, \mu_0, G_{\Gamma_1}](\Gamma_1) \qquad \text{(by (3.3a))}\\ &= \chi(\Gamma_1).\end{aligned}$$

This completes the proof of (Ec1). (Ec2) follows from (3.1). Finally, the formula (3.3b) implies that the mapping χ is independent of the choice of the transversal T. □

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