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ON KNOTS WITH ICON SURFACES

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Abstract

An ICON surface is an incompressible compact orientable nonseparating surface properly embedded in a knot exterior. We show that for any odd positive number n, there exist plenty of knots whose exteriors E contain an ICON surface F with $|\partial F| = n$. We also show that our examples satisfy the $\mathbb{Z}$-conjecture, that is, $\pi_1(E/F) \cong \mathbb{Z}$.

1. Introduction

A well known conjecture in combinatorial group theory is the so called *Kervaire conjecture*:

Conjecture 1.1. *Let G be a group, $G \neq 1$. Then $\mathbb{Z} * G$ cannot be normally generated by one element.*

F. González-Acuña and A. Ramírez proved that Kervaire conjecture is equivalent to what they called the $\mathbb{Z}$-conjecture [2]:

Conjecture 1.2. *If F is a compact orientable nonseparating surface properly embedded in a knot exterior $E(K)$, then $\pi_1(E(K)/F) \cong \mathbb{Z}$.*

We remark that by a surface we mean a connected 2-manifold.

Following González-Acuña, we define:

DEFINITION 1.3. An ICON surface is an incompressible compact orientable nonseparating surface properly embedded in a knot exterior.

An incompressible Seifert surface for a knot is then an example of an ICON surface, but as pointed out by González-Acuña and Ramírez, it is not clear whether or not there exists ICON surfaces with disconnected boundary. Here we show that there exist plenty of knots with ICON surfaces with disconnected boundary.

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Theorem 1.4. *Given any odd positive number n, there exist plenty of knots whose exteriors contain an ICON surface F with $|\partial F| = n$.*

We make a general construction that produces explicit examples of knots with ICON surfaces. This produces surfaces of genus n having n boundary components, n odd, or more generally, ICON surfaces of genus m having n boundary components, n odd, $n \leq m$. The main construction is shown in Figs. 1, 2, 3. Basically, the idea is to start with a genus one Seifert surface for the unknot, seen as a disk with two bands. Take 3 copies of the surface and join them by tubes, getting a genus 3 orientable surface with 3 boundary components, which is compressible. Then, cut the bands, link them and make them pass through the tubes and then glue them again, getting a new knot and a nonseparating surface in its exterior. Under some mild conditions the surface will be incompressible. A generalization of the construction produces, for each odd integer n, knots K whose exteriors have $(n+1)/2$ disjoint ICON surfaces, of genus $n, n-1, n-2, \ldots, n-(n-1)/2$ and with n, $n-2$, $n-4$, $\ldots$, 1 boundary components respectively.

We also make a more particular construction producing knots whose exteriors contain an ICON surface of genus 2 with 3 boundary components, shown in Fig. 8. Here the idea is to start with the unknot K and a disk bounding it. Now take 3 copies of the disk, join them with 4 tubes, getting an orientable genus 2 nonseparating surface S with 3 boundary components. Now find an unknot L in the complement of the surface, so that S is incompressible in the complement of the link $K \cup L$. By doing $1/n$-Dehn surgery on L, we get a new knot K_n and the corresponding surface S_n remain incompressible. It is easy to find knots in the complement of S so that S is incompressible in the complement of $K \cup L$; the difficult part is to find one which is trivial.

Finally, we show that all our examples satisfy the $\mathbb{Z}$-conjecture. We remark that the $\mathbb{Z}$-conjecture is known to hold for surfaces with 1 or 3 boundary components [2], and that J. Rodríguez-Viorato [4] has recently shown that several infinite families of pretzel knots satisfy it.

2. The construction of ICON surfaces

Let K be the trivial knot, and let F be a disk properly embedded in the exterior of K, $E(K)$, whose boundary is a longitude of $E(K)$. Let $N(F)$ be a regular neighborhood of F in $E(K)$, $N(F) \cong F \times I$. Take 3 parallel copies of F in $N(F)$, say $F_1 = F \times \{1\}$, $F_2 = F \times \{1/2\}$ and $F_3 = F \times \{0\}$. Let x, y be distinct points in the interior of F, and let $t_1 = x \times [1/2, 1]$, $t_2 = y \times [0, 1/2]$, i.e., t_1 is a straight arc connecting F_1 and F_2, and t_2 is a straight arc connecting F_2 and F_3.

Connect F_1, F_2 and F_3 with tubes following the arcs t_1 and t_2. That is, consider disjoint regular neighborhoods $N(t_1)$, $N(t_2)$ of t_1 and t_2, in $F \times [1/2, 1]$ and $F \times [0, 1/2]$, respectively, and let $G = (F_1 \cup \partial N(t_1) \cup F_2 \cup \partial N(t_2) \cup F_3) - \mathrm{int}((F_1 \cup F_2 \cup F_3) \cap (N(t_1) \cup N(t_2)))$. Note that G is a compact orientable nonseparating surface with 3 boundary

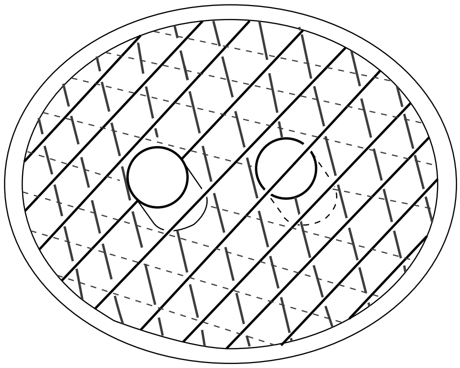

Fig. 1.

components in $E(K)$, but of course it is compressible. Such a surface G is shown in Fig. 1.

Let D_1 and D_2 be disjoint disks properly embedded in $E(K) - \operatorname{int} N(F)$, such that ∂D_1 (resp. ∂D_2) consists of one arc in F_1, one arc in F_3, and two arcs in $\partial N(K)$, so ∂D_1 (resp. ∂D_2) bounds a disk E_1 (resp. E_2) in $\partial(E(K) - \operatorname{int} N(F))$, which we assume to be disjoint from the points x and y. We assume also that the disks E_1 and E_2 are disjoint, i.e., D_1 and D_2 are not nested. Let B_1 (resp. B_2) be the 3-ball bounded by D_1 and E_1 (resp. D_2 and E_2) contained in $E(K) - \operatorname{int} N(F)$.

Let α_1 and α_2 be two disjoint arcs properly embedded in $E(K)$, which are disjoint from G. Assume that $\alpha_i \cap B_1$ consists of one arc, having one endpoint in $\partial N(K)$ and one endpoint in D_1, and that $\alpha_i \cap B_2$ consists of one arc, having one endpoint in $\partial N(K)$ and one endpoint in D_2, for $i = 1, 2$. The intersections of the arcs α_1 and α_2 with B_1 (reps. B_2) determine a 2-tangle in B_1 (resp. B_2), with ∂D_1 (resp. ∂D_2) as a meridian; assume that this is not a rational tangle of the form $R(1/n)$, i.e., there is no a disk D embedded in B_1 (resp. B_2), with interior disjoint from the arcs of the tangle, so that ∂D consists of the union of one arc in $\partial N(K)$, one arc in D_1 (resp. D_2), and the pair of arcs of the tangle. Assume that the part of α_1 outside $B_1 \cup B_2$ is an arc that starts at ∂B_1, passes through $N(t_1)$, wraps around $N(t_2)$, i.e. it has winding number $\neq 0$ in the solid torus $F \times [0, 1/2] - N(t_2)$, then passes again through $N(t_1)$ and finishes at ∂B_2, as in Fig. 2. Assume also that the part of α_2 outside $B_1 \cup B_2$ is an arc that starts at ∂B_2, passes through $N(t_2)$, wraps around $N(t_1)$, passes again through $N(t_2)$ and finishes at ∂B_1, as in Fig. 2. More precisely, assume that $\alpha_1 \cap N(t_1)$ consists of two straight arcs in $N(t_1)$, that is, arcs which are fibers in the product structure of $N(t_1)$, and that the knot k, obtained from the arc of α_1 contained in $F \times [0, 1/2]$, after joining its endpoint with an arc lying in $N(t_1) \cap F_2$, has winding number $\neq 0$ in the solid torus $F \times [0, 1/2] - N(t_2)$. Similarly for the arc α_2. Outside the region $N(F) \cup B_1 \cup B_2$, there are no restrictions for the arcs α_1 and α_2.

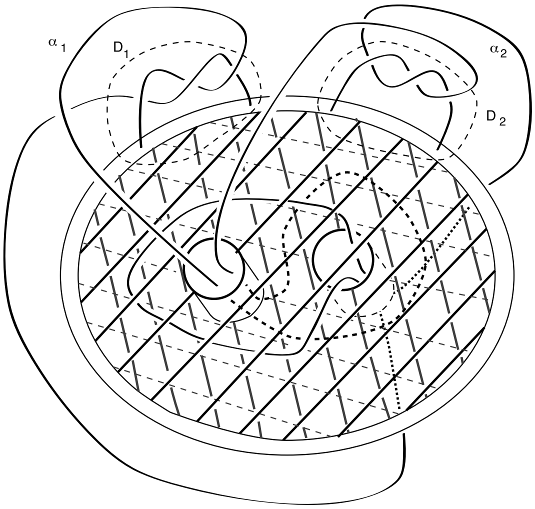

Fig. 2.

We can think of α_1 and α_2 as arcs with endpoints in K; assume then that the endpoints of α_1 and α_2 on K alternate. Following α_1 and α_2 we can add two bands to K. That is, consider embeddings $b_1\colon I \times I \to S^3$, so that $b_1(I \times I) \cap K = b_1(\{0\} \times I) \cup b_1(\{1\} \times I)$, and that $b_1(I \times \{1/2\}) = \alpha_1$, and $b_2\colon I \times I \to S^3$, so that $b_2(I \times I) \cap K = b_2(\{0\} \times I) \cup b_2(\{1\} \times I)$, and that $b_2(I \times \{1/2\}) = \alpha_2$. Of course, assume that the two embeddings are disjoint. By twisting the bands, we see that there are many possible bands given by α_1 and α_2; take any two of them, just assume that the disk F, though as a disk with boundary K, union the bands b_1 and b_2 is an orientable (singular) surface. As the endpoints of α_1 and α_2 alternate, this surface has to be a once punctured torus (with ribbon singularities), and then its boundary is a new knot K_1. Namely, K_1 is the knot $K_1 = (K - (b_1(\{0\} \times I) \cup b_1(\{1\} \times I) \cup b_2(\{0\} \times I) \cup b_2(\{1\} \times I))) \cup b_1(I \times \{0\}) \cup b_1(I \times \{1\}) \cup b_2(I \times \{0\}) \cup b_2(I \times \{1\})$.

Now, in the exterior of K_1, consider the union of the surface G appropriately pasted with 3 copies of each of the bands b_1 and b_2, as in Fig. 3, and denote this surface by S. Then S is a compact connected orientable nonseparating surface properly embedded in $E(K_1)$, with $|\partial S| = 3$, and we show next that S is incompressible, that is, S is an ICON surface with 3 boundary components. Also note that genus$(S) = 3$. In Fig. 4 we show an example of such a knot K_1 without the surface S.

Theorem 2.1. *Let K_1 and S be as above. Then S is incompressible.*

Proof. Let D_1, D_2 be the disks defined above, and let D_3, D_4 be defined as $D_3 = N(t_1) \cap F_1$ and $D_4 = N(t_2) \cap F_3$. In $E(K_1) - \operatorname{int} N(S)$ the disk D_1 gives rise to a twice punctured disk plus four disks, as shown in Fig. 5; similarly for D_2. The disk D_3 also

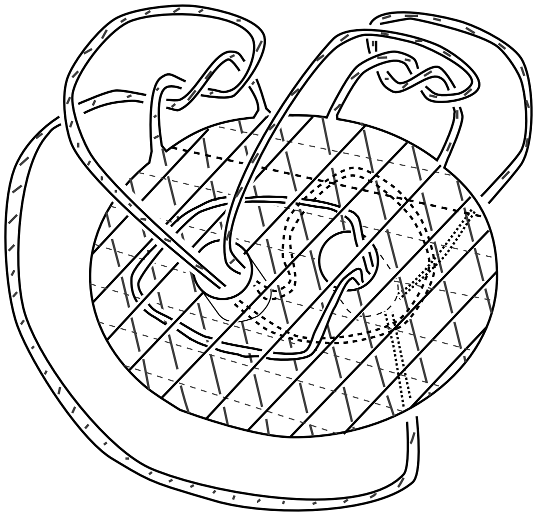

Fig. 3.

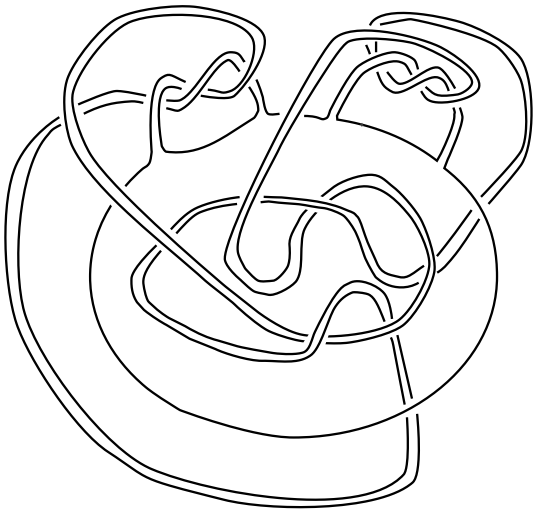

Fig. 4.

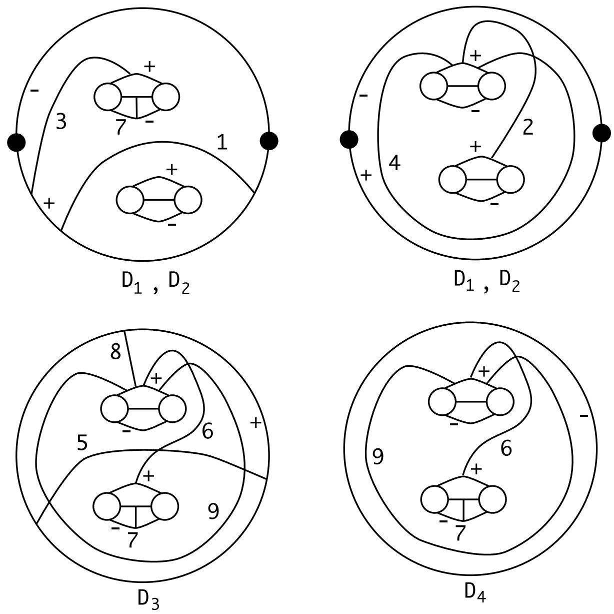

Fig. 5.

give rise to a twice punctured disk plus four disks, as shown in Fig. 5; similarly for D_4. In this figure we have indicated with signs "$+$, $-$" the side of the surface S in which a neighborhood of the boundaries of the punctured disks lie, assuming that the side of F_1 pointing out of $N(F)$ is the "$+$" side. Note that the disks D_1, D_2, D_3, D_4 cut off the "$+$" side of the surface S into two annuli, a once punctured annulus and several disks. The "$-$" side of the surface S is cut off by the disks D_i's in a similar manner.

Suppose that S is compressible, and let E be a compression disk. Consider the intersections between E and the collection of disks D_i's, which consist of simple closed curves and arcs. Let γ be an innermost simple closed curve of intersection in E, which bounds a disk E'. The curve γ is contained in one of the disks D_i; suppose first that it lies in D_1 or D_2, say in D_1. Let D' be the disk bounded by γ in D_1. If the disk D' is not disjoint from K_1, then it must contain one point of intersection between D_1 and one of the arcs α_1 or α_2, but then the sphere $E' \cup D'$ would intersect the simple closed curve formed by α_1 or α_2 plus one arc of K in one point, which is not possible. Suppose now that γ is contained in D_3 or D_4, say in D_3. Again, let D' be the disk bounded by γ in D_3. If the disk D' is not disjoint from K_1, then it must contain one or two points of intersection between D_3 and the arc α_1. If it contains just one point, then the sphere $E' \cup D'$ would intersect the simple closed curve formed by α_1 plus one arc of K in one point, which is not possible. Suppose then that D' contains two points of intersection with α_1. If the disk E' is contained in $E(K_1) - N(F)$, then the arc α_1 could not join

$N(K)$ and D', so E' must be contained in $N(t_1) \cup (F \times [0, 1/2] - N(t_2))$. But then the winding number of α_1 in $F \times [0, 1/2] - N(t_2)$ would be 0, which is a contradiction. Then in both cases the disk D' has interior disjoint from K_1 and from the surface. So by doing an isotopy of the disk E, the curve γ of intersection can be removed.

So assume that the intersection between E and the disks D_i's consists of arcs. A neighborhood of the boundary of E lies on a side of S, so assume that it lies in the "+" side of S. The proof in the other case is similar. Let γ be an outermost arc of intersection in E, which cuts off a disk E'' from E, where $\partial E'' = \gamma \cup \beta$, with β being an arc in S, and the interior of E'' is disjoint from the disks D_i's. If the arc γ is trivial in the corresponding disk D_i, i.e., there is a disk $D' \subset D_i$, so that $\partial D' = \gamma \cup \delta$, where $\delta \subset S$, and the interior of D' is disjoint from K_1 and the surface S, then by cutting E with an outermost such disk contained in D', we would get another compression disk E with fewer intersections with the D_i's. So suppose that the arc γ is non-trivial in the corresponding disk D_i.

It is not difficult to check that the arc γ must be as one of the types of arcs shown in Fig. 5, numbered 1 to 9. Suppose we have Case 1. In this case the arc γ cuts off a disk D'' from D_1 or D_2, whose interior intersects $N(K_1)$ in two disks and S in three arcs, and such that $\partial D'' = \gamma \cup \delta$, where δ is an arc in the "+" side of S. The curve $\beta \cup \delta$ lies in the "+" side of S and after possibly isotoping it, we can assume it bounds a disk C contained in the disk F_1. Let $C' = E'' \cup D'' \cup C$; this is a sphere which intersects $\alpha_1 \cup \alpha_2$ in 1 or 3 points, depending if the disk C contains or not the disk D_3. In any case we can find a simple closed curve which intersects the sphere C' is one point, which is not possible.

Suppose we have Case 2 or 3. Note that in those cases the disk E'' must be contained inside the 3-ball B_1 or B_2. In Case 2 the arc β consists of an arc on the disk F_1 and then an arc along one of the bands. In Case 3, the arc β consists of an arc on one of the bands, then an arc on F_1 and then another arc on the other band. In both cases it would follow that the tangle inside B_1 or B_2 is of the form $R(1/n)$, which is not possible by hypothesis.

Suppose we have Case 4. In this case the disk E'' must be contained in B_1 or B_2. The arc γ determines a disk D'' in D_1 or D_2, which intersects both arcs α_1 and α_2. The arc β consists of an arc on one of the bands, then an arc on F_1 and then another arc on the same band. This configuration is not possible, for it implies that the arc α_1 or α_2 intersects E''.

Suppose we have Case 5. There are two cases, depending of the position of the disk E''. The first case is that E'' lies in the exterior of $N(F)$, and then the arc β lies in F_1. Then necessarily one of the arcs α_1 or α_2 would intersect the disk E'', which is not possible. The other possibility is that E'' lies in $N(t_1) \cup (F \times [0, 1/2] - N(t_2))$. Note that the region of $S - E$ in which β lies is a once punctured annulus, and β has its endpoints in the same component of the boundary of this region. This would imply that the arc α_1 would intersect E''.

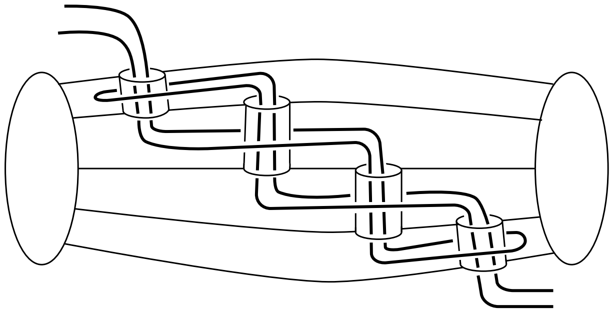

Fig. 6.

Suppose we have Case 6, and assume that γ lies in D_3. In this case the disk E'' must be contained in $N(t_1) \cup (F \times [0, 1/2] - N(t_2))$, for otherwise it would intersect D_1 and D_2. Note that the part of the arc α_1 contained in $N(t_1) \cup (F \times [0, 1/2] - N(t_2))$ can be made to coincide with the arc β, and then can be pulled out of $N(F \times [0, 1/2] - N(t_2))$, by using E''. So this implies that the winding number of α_1 in $F \times [0, 1/2] - N(t_2)$ is 0, which is a contradiction.

Suppose we have Case 7. In this case the arc γ must be contained in D_1 or D_2, for if it is contained in D_3, then the disk E'' would also intersect D_1 or D_2 an then it would not be outermost. The only possibility is that the disk E'' is contained in a region consisting of the product of one of the bands union $F \times [0, 1/2] - N(t_2)$, and then the arc β goes once through $\partial N(t_2)$. But this implies again that the arc α_1 has winding number 0 in $F \times [0, 1/2] - N(t_2)$.

Finally note that cases 8 and 9 are not possible simply because there cannot be an arc β with the given endpoints, and with interior disjoint from D_1, D_2 and D_3.

The only possibility left is that the disk E is disjoint from the disks D_i's. As we say before, the "$+$" side and the "$-$" side of S are cut off by the disks D_i's into annuli, once punctured annuli and disks. Now, it is not difficult to see that there are no compression disks for these subsurfaces. So the surface S must be incompressible. □

To get knots with an ICON surface having n boundary components, n odd, proceed in a similar manner. Take the trivial knot K and a disk F in its exterior as before. Take now n copies of F, denoted by $F_1, \ldots F_n$. Connect the disks with $n - 1$ tubes $T_1 \cdots T_{n-1}$, so that the tube T_i connects the disks F_i and F_{i+1}. We get a surface G. Consider now two arcs in $E(K)$ disjoint from G, such that in $E(K) - \operatorname{int} N(G)$ the arcs behave exactly as before, and so that α_1 passes through the odd numbered tubes and wrap around the even numbered tubes, and α_2 passes through the even numbered tubes and wraps around the odd numbered tubes, as shown schematically in Fig. 6 for the case $n = 5$. Suppose that the winding number of these arcs in the corresponding solid tori is $\neq 0$. More precisely, let N_i be the solid torus determined by the region between the disks F_i and F_{i+1} when we remove a solid tube given by T_i. The arc

α_1 intersects the solid torus N_i, i odd, in one or two arcs. Join the endpoints of α_1 lying in F_i with an arc lying in the intersection between the disk F_i and the solid tube T_{i-1}, and if $i+1 \leq n-2$, join the endpoints of α_1 lying in F_{i+1}, with an arc lying in the intersection between the disk F_{i+1} and the solid tube T_{i+1}. By doing this we get a knot k_i; now assume that the winding number of k_i in N_i is $\neq 0$. Do a similar assumption for the arc α_2. Take now bands following the arcs α_1 and α_2 to get a knot K_1. By taking the union of the surface G and n copies of each of the bands we get an ICON surface S for K_1 with n boundary components. Note that genus$(S) = n$. The proof that S is incompressible is just the same as the proof of Theorem 2.1.

For the following construction assume further that the winding number of the arcs α_1 and α_2 in the solid tori N_i formed in $N(F)$ is $\neq 0, \pm 1$. Let s_i be the i-th component of ∂S, that is, the component coming from ∂F_i. Note that if each s_i has the orientation induced by that of S, then s_i and s_{i+1} are oppositely oriented, for $i = 1, \ldots, n-1$. Let A_i be the annulus in $\partial N(K_1)$ cobounded by s_i and s_{i+1} whose interior is disjoint from ∂S, for $i = 1, \ldots, n-1$. Let S_i' be the surface obtained by taking the union $S \cup A_i$, and then pushing its interior into the interior of $E(K_1)$. Note that S_i' is an orientable surface of genus $n+1$ and has $n-2$ boundary components. The surface S_i' is compressible, to see this just note that two tubes were formed in a neighborhood of the bands. By compressing these tubes, i.e., by compressing S_i' twice, we get a surface S_i of genus $n-1$ and with $n-2$ boundary components. Equivalently, S_i is obtained by joining the disks F_i and F_{i+1} with an annulus before the bands are attached, and then only $n-2$ copies of the bands are attached to G. Note also that S and S_i can be made disjoint. Starting with S_1 and then repeating the operation with the annulus A_3, and then with A_5, etc., we get a collection of ICON surfaces as stated in the next theorem.

Theorem 2.2. *Given any odd integer n, there are knots K whose exteriors contain $(n+1)/2$ disjoint ICON surfaces, of genus $n, n-1, n-2, \ldots, n-(n-1)/2$ and with $n, n-2, n-4, \ldots, 1$ boundary components respectively.*

Proof. The knots K_1 just constructed satisfy the required properties. Note that there is a twice punctured torus T embedded in the exterior of K_1, so that one boundary component of T lies in $\partial E(K_1)$ and is parallel to s_1, and the other boundary component lies in S_1, it is just a core of the annulus A_1. In fact, T is the union of an annulus cobounded by a core of the annulus A_1 and a curve on $\partial N(K)$, with a copy of each of the bands. To see that the surface S_1 is incompressible do an innermost disk-outermost arc argument as in Theorem 2.1, but also using the torus T. We have assumed that the winding number of the arcs α_1, α_2 in the solid tori N_i is $\neq 0, \pm 1$, just to avoid outermost arcs of intersection in a compression disk which are of Type 5 as in Fig. 5. Those arcs can be ruled out when proving that the surface S is incompressible, but cannot be ruled out when proving the incompressibility of S_1. The remaining surfaces are shown to be incompressible by a similar argument. □

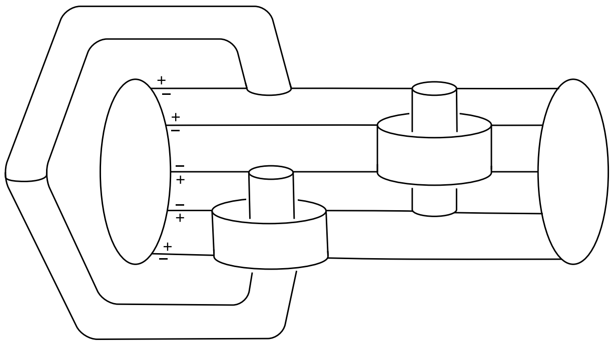

Fig. 7.

By complicating the construction, it is not difficult to construct examples of ICON surfaces of genus qn and n boundary components, q being any positive integer. To do that, just start with a knot K having an incompressible Seifert surface H of genus $q-1$, and use this surface instead of the disk F, i.e., take n copies of H, join them by $n-1$ tubes and then add two bands to the surface which go through the tubes.

Another way of complicating the construction is to start as before with n copies of a disk F, but now join the disks with many tubes, say consider a collection of arcs between the disks F_i and F_{i+1}, possibly knotted and tangled, and then add bands which go through the tubes and regions between the disks in a complicated manner. This will give ICON surfaces of genus m with n boundary components, where $m \geq n$.

In all the surfaces just constructed, if $s_1, s_2, \ldots, s_n$ denote the boundary components of an ICON surface, then s_i and s_{i+1} are oppositely oriented, for all $i = 1, \ldots, n-1$. This is just a consequence of the construction. It is possible to construct examples where this does not happen, for example in Fig. 7 such a surface is shown schematically; the surface is formed by 5 disks and 4 tubes arranged appropriately, and then to ensure incompressibility we have to add two bands which go through each of the tubes and regions.

As we said before, given positive integers n and m, with n odd and $n \leq m$, there is a knot whose exterior contains an ICON surface of genus m with n boundary components. On the other hand, it is no clear whether there exists or not knots with ICON surfaces of genus m with n boundary components, but where $m < n$.

Now we construct a genus 2 ICON surface with 3 boundary components. Let K be the trivial knot, and let F be a disk properly embedded in the exterior of K, $E(K)$, whose boundary is a longitude of $E(K)$. Let $N(F)$ be a regular neighborhood of F in $E(K)$, $N(F) \cong F \times I$. Take 3 parallel copies of F in $N(F)$, say $F_1 = F \times \{1\}$, $F_2 = F \times \{1/2\}$ and $F_3 = F \times \{0\}$. Take 4 disjoint tubes T_1, T_2, T_3 and T_4, so that T_1 and T_2 join F_1 with F_2, and T_3 and T_4 join F_2 with F_3, exactly as shown in Fig. 8, getting a surface G. Now take a knot L in the complement of G, just as shown in Fig. 8. Note that L is the trivial knot.

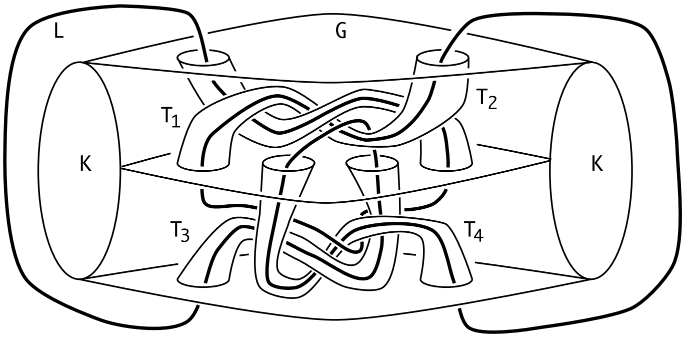

Fig. 8.

Lemma 2.3. *The surface G is incompressible in the exterior of $K \cup L$.*

Proof. Let D_i be a disk that compress the tube T_i in S^3, so that L intersects D_i in one point, for $i = 1, 2, 3, 4$. Suppose that E is a compression disk for G disjoint from L. The intersection between E and the disks D_i consist of a collection of simple closed curves and arcs. Simple closed curves are removed as usual. Intersection arcs can also be removed, for these are trivial in the punctured disks D_i. So, if G is compressible, there must be a compression disk disjoint from the disks D_i. It is easy to see that such a disk cannot be outside $N(F)$, so it has to be, say, in the region between F_1 and F_2. So, ∂E lies in the surface Σ obtained from F_1 and F_2 after adding the tubes T_1 and T_2, which is a twice punctured torus intersecting the knot L in two points. Cap off the boundary components of Σ with two disks embedded in S^3, lying in the outside of $N(F)$, getting a torus Σ'. Let τ be an arc contained in $N(F) \cap \partial N(K)$ connecting the two attached disks. Note that Σ' is knotted as a trefoil knot, and that the disk E lies in the side of Σ' not bounding a solid torus. So ∂E must in fact bound a disk E' contained in Σ'. One possibility is that E' contains the two points of intersection of L with Σ, and $E \cup E'$ cobound a 3-ball containing the arc of L lying between F_1 and F_2. Note that such an arc is an unknotting tunnel for the trefoil knot, so it cannot lie inside a 3-ball. The other possibility is that E' contains the two points of intersection of the arc τ with Σ'; but this is not possible for the arc τ is also an unknotting tunnel for the trefoil knot. Then the disk E must be parallel to a disk in G, and so it is not a compression disk. A similar argument shows that there is no compression disk in the region between F_2 and F_3. □

Theorem 2.4. *Let K_n be the knot obtained after performing $1/n$-Dehn surgery on L, $n \neq 0$, and let G_n be the surface properly embbeded in $E(K_n)$ obtained from G after the surgery. Then G_n is an ICON surface in $E(K_n)$, of genus 2 and having 3 boundary components*

Proof. The proof is essentially the same as that of Theorem 4 of [3]. Let D_i be a disk that compress the tube T_i, so that L intersects D_i in one point, for $i = 1, 2, 3, 4$, and let $A_i = D_i - \text{int}\, N(L)$. Suppose that E is a compression disk for G_n after performing $1/n$-Dehn surgery on L. Assume that the core of the Dehn surgery torus intersects E transversely, and let $P = E - \text{int}\, N(L)$; this is a planar surface having one boundary component in G_n, which we call the outer boundary component, and, say, p boundary components in $\partial N(L)$, called the inner boundary components, each of slope $1/n$ in $\partial N(L)$. Look at the intersection between P and the annuli A_i's. If there is a simple closed curve of intersection which is trivial in some A_i, or there is a trivial arc of intersection in some A_i, then the intersection between P and the A_i's is not minimal, or the intersection between E and the core of the surgered torus is not minimal. So assume that the intersection between P and the A_i's consist of spanning arcs in the annuli A_i's. Look now at the intersection pattern in P. It must consist of arcs, all going from the inner boundary components to the outer boundary. Note that each inner boundary component of P intersects each A_i in n points, so it intersects the collection of the A_i's in $4n$ points. So there are $4n$ arcs of intersection incident to each inner boundary component, which connect this boundary component to the outer boundary component of P. The arcs incident to an inner boundary component divide E into $4n$ regions, which may contain some other inner boundary components of P. By taking one outermost of such regions, taken over all regions determined by the intersections arcs between P and the D_i's, we see that there must be a disk $Q \subset P$, so that $\partial Q = \delta_1 \cup \delta_2 \cup \delta_3 \cup \delta_4$, where δ_1 is in one of the inner boundary components of P, δ_2 is in the intersection between P and A_i, for some i, δ_3 is in the outer boundary component of P, and δ_4 is in the intersection between P and $A_{i\pm 1}$. It is not difficult to see that such a disk cannot exist. So, G_n is incompressible. □

QUESTION 2.5. Is there a knot K having an ICON surface of genus 1 with more than one boundary component? Is there a lower bound for the genus of an ICON surface having n boundary components?

If a knot K has an ICON surface of genus n, then by a result of Gabai [1], genus$(K) \leq n$. In particular, for the knots constructed in Theorem 2.1, and their generalization to n boundary components, it is not difficult to see that each of these knots bounds a genus $(n+1)/2$ Seifert surface, as expressed in Theorem 2.2. Also, note that the knots K_n of Theorem 2.4 are genus one knots; to see that take a copy of the disk F_1 and add one tube following one arc of the knot L.

3. The surfaces satisfy the $\mathbb{Z}$-conjecture

Here we show that the surfaces constructed in the previous section satisfy the $\mathbb{Z}$-conjecture. The proof follows the same ideas as in [4], consisting in pushing an arc contained in $\partial N(K)$ with endpoints in ∂S into the surface S.

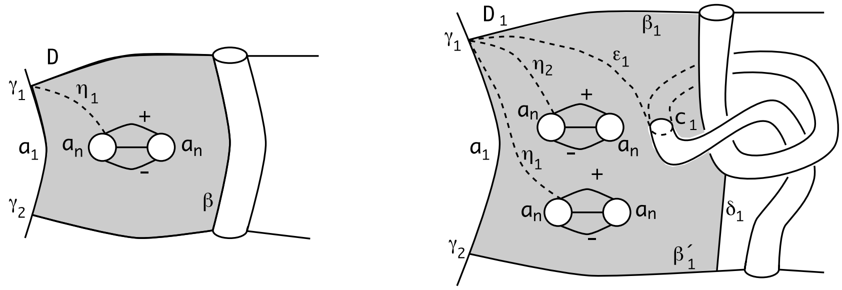

Fig. 9.

Theorem 3.1. *Let K and S be any of the knots and ICON surfaces constructed in* Theorems 2.1, 2.2, 2.4. *Then S satisfy the $\mathbb{Z}$-conjecture.*

Proof. Suppose we have a knot K and an ICON surface S as constructed in Theorems 2.1 and 2.2. Let $\gamma_1, \gamma_2, \dots, \gamma_n$ be the boundary components of S. It follows from the construction of the surfaces that γ_i and γ_{i+1} are oppositely oriented, for $i = 1, 2, \dots, n-1$, and that γ_1 and γ_n have the same orientation. Let A_i be the annulus in $\partial N(K)$ lying between γ_i and γ_{i+1}, $i = 1, 2, \dots, n-1, n$, mod n; let a_i be a spanning arc of A_i, $i = 1, 2, \dots, n$, oriented from γ_i to γ_{i+1}. Let $[a_i]$ be the class of a_i in $\pi_1(E(K)/S)$. Note that in the simplest construction, that of Figs. 3, 6, there is a disk D embedded in the region between the disks F_1 and F_2, such that $\partial D = a_1 \cup \beta$, where β is an arc on S, and $\operatorname{int} D$ intersects $N(K)$ in two meridian disks and S in n arcs joining these meridian disks, as shown in the left side of Fig. 9. Note the the points of intersection of K with $intD$ are oppositely oriented. The arc a_1 can be homotoped, keeping its endpoints fixed, to an arc of the form $\eta_1 \cdot a_n^{-1} \cdot \eta_1' \cdot a_n \cdot \eta_1'' \cdot \eta_1^{-1} \cdot \beta$, where η_1 is an arc in D that goes from one endpoint of a_1 to an endpoint of a_n, and η_1', η_1'' are the arcs of intersection between D and S that have endpoints in γ_n. From this follows that in $\pi_1(E(K)/S)$, $[a_1] = [\eta_1][a_n]^{-1}[\eta_1'][a_n][\eta_1''][\eta_1]^{-1}[\beta]$, so $[a_1] = [\eta_1][a_n]^{-1}[a_n][\eta_1]^{-1}$, for $[\eta_1']$, $[\eta_1'']$ and $[\beta]$ are trivial in $\pi_1(E(K)/S)$. This implies that $[a_1] = 1$.

In a more general case, where there are many tubes which may be knotted and entangled with the bands, by sliding a parallel copy of the arc a_1 along S and then along one of the tubes that connect F_1 and F_2, we see that there is a collection of disks $D_1, D_2, \dots, D_r$, embedded in the region between F_1 and F_2 so that $\partial D_1 = a_1 \cup \beta_1 \cup \delta_1 \cup \beta_1'$, where β_1 and β_1' lie in S and δ_1 is disjoint from S and $N(K)$, $\partial D_2 = \delta_1 \cup \beta_2 \cup \delta_2 \cup \beta_2'$, where β_2 and β_2' lie in S and δ_2 is disjoint from S and $N(K)$, $\partial D_i = \delta_{i-1} \cup \beta_i \cup \delta_i \cup \beta_i'$, where β_i lies in S and δ_i is disjoint from S and $N(K)$, until $\partial D_r = \delta_{r-1} \cup \beta_r$, where β_r lies in S. Also, the interior of each D_i intersects $N(K)$ in pairs of meridian disks oppositely oriented, and intersects S in collection of arcs joining those pairs of disks and possibly in simple closed curves. Then by homotoping a_1, we have that in $\pi_1(E(K)/S)$, $[a_1] = [\eta_1][a_n]^{-1}[a_n][\eta_1]^{-1}[\eta_2][a_n]^{-1}[a_n][\eta_2]^{-1} \cdots [\eta_k][a_n]^{-1}[a_n][\eta_k]^{-1}[\epsilon_1][c_1][\epsilon_1]^{-1} \cdots [\epsilon_t][c_t][\epsilon_t]^{-1}[\beta_1][\delta_1][\beta_1']$, where the η_i's are arcs in

D joining one endpoint of a_1 with one of the endpoints of a_n, the c_i's are simple closed curves lying in $D \cap S$, and the ϵ_i's are arcs in D joining one endpoint of a_1 with the c_i's. See Fig. 9. From this follows that $[a_1] = [\delta_1]$ in $\pi_1(E(K)/S)$. Similarly, $[\delta_1] = [\delta_2] = \cdots [\delta_{r-1}] = [\beta_r] = 1$, so $[a_1] = 1$ in $\pi_1(E(K)/S)$.

Let S_1 be the surface obtained by taking $S \cup A_1$ and then pushing it into the interior $E(K)$. As in the proof of Theorem 14 of [2], the fact that $[a_1] = 1$ in $\pi_1(E(K)/S)$ implies that $\pi_1(E(K)/S) = \pi_1(E(K)/S_1)$. Repeating the argument but now with the arc a_3, if follows that $\pi_1(E(K)/S_1) = \pi_1(E(K)/S_{1,3})$, where $S_{1,3}$ is the surface obtained from S_1 by attaching the annulus A_3 and pushing it into the interior of $E(K)$. So by induction, after attaching the odd numbered annuli A_i, we get that $\pi_1(E(K)/S) = \pi_1(E(K)/S_{1,3,\dots,n-2})$, where $S_{1,3,\dots,n-2}$ is a Seifert surface for K. Now, it follows from Proposition 11 of [2] that $\pi_1(E(K)/S) = \pi_1(E(K)/S_{1,3,\dots,n-1}) \cong \mathbb{Z}$.

For the knots and surfaces constructed in Theorem 2.4, a similar argument show that the surfaces satisfy the $\mathbb{Z}$-conjecture. □

In this proof we use the fact that consecutive curves of ∂S are oppositely oriented, but as mentioned after the proof of Theorem 2.2, this is not always the case. In explicit cases, as this shown in Fig. 7, the same argument shows that the surface constructed satisfies the $\mathbb{Z}$ conjecture, but it is not clear that the same proof works for all the possible examples.

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