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CONFORMAL SYMMETRIES OF SELF-DUAL HYPERBOLIC MONOPOLE METRICS

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Abstract

We determine the group of conformal automorphisms of the self-dual metrics on $n\,\#\,\mathbb{CP}^2$ due to LeBrun for $n \geq 3$, and Poon for $n = 2$. These metrics arise from an ansatz involving a circle bundle over hyperbolic three-space $\mathcal{H}^3$ minus a finite number of points, called monopole points. We show that for $n \geq 3$, any conformal automorphism is a lift of an isometry of $\mathcal{H}^3$ which preserves the set of monopole points. Furthermore, we prove that for $n = 2$, such lifts form a subgroup of index 2 in the full automorphism group, which we show to be a semi-direct product $(\mathrm{U}(1) \times \mathrm{U}(1)) \rtimes \mathrm{D}_4$, where D_4 is the dihedral group of order 8.

Contents

1. Introduction

In [15] and [16], Yat-Sun Poon found examples of self-dual conformal classes on the connected sums $\mathbb{CP}^2\#\mathbb{CP}^2$ and $3\#\mathbb{CP}^2$ using techniques from algebraic geometry. In [13], Claude LeBrun gave a more explicit construction of U(1)-invariant self-dual conformal classes on $n\#\mathbb{CP}^2$ for any n. Briefly, the idea is to choose n distinct points $\{p_1, \ldots, p_n\}$ in hyperbolic 3-space $\mathcal{H}^3$, and consider a certain U(1)-bundle $X_0 \to M_0$, where $M_0 = \mathcal{H}^3 \setminus \{p_1, \ldots, p_n\}$. A scalar-flat Kähler metric is written explicitly on X_0 in terms of a connection 1-form, and extends to the metric completion X of X_0, which is biholomorphic to $\mathbb{C}^2$ blown up at n points along a line. This metric conformally compactifies to give a self-dual conformal class on $\hat{X} = n\#\mathbb{CP}^2$, which we denote by $[g_{\mathrm{LB}}]$. It turns out that any hyperbolic isometry which preserves the set of monopole

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points lifts to a conformal automorphism of $(n\,\#\,\mathbb{CP}^2, [g_{\mathrm{LB}}])$. The main result of this paper is that the converse is also true for $n \geq 3$, and when $n = 2$, such lifts form a subgroup of index 2 in the full conformal group.

Theorem 1.1. *Let $n \geq 3$, and $[g_{\mathrm{LB}}]$ be any LeBrun self-dual conformal class on $\hat{X} = n\,\#\,\mathbb{CP}^2$. A map $\Phi\colon \hat{X} \to \hat{X}$ is a conformal automorphism if and only if it is the lift of an isometry of $\mathcal{H}^3$ which preserves the set of monopole points.*

For $n = 2$, there is a conformal involution $\Lambda\colon \hat{X} \to \hat{X}$ with the following property. For any conformal automorphism $\Phi\colon \hat{X} \to \hat{X}$, exactly one of Φ or $\Phi \circ \Lambda$ is the lift of an isometry of $\mathcal{H}^3$ which preserves the set of the two monopole points.

REMARK 1.2. The involution Λ arises as follows. For $n = 2$, there are exactly *two* semi-free conformal S^1-actions, which yield a double fibration of an open subset of $\hat{X}$ over $\mathcal{H}^3 \setminus \{\text{two points}\}$. The map Λ interchanges the fibers of these two fibrations. Moreover, since Λ does not commute with either semi-free S^1-action, one obtains an S^1-family of involutions with the same properties as Λ by conjugating with either of the semi-free S^1-actions. These facts will be proved in Section 6. To visualize this map, we recall that $\mathbb{CP}^2\,\#\,\mathbb{CP}^2$ can be viewed as a boundary connect sum of two Eguchi–Hanson ALE spaces (glued along the boundary $\mathbb{RP}^3$-s). The involution Λ interchanges the Eguchi–Hanson spaces, and has an invariant $\mathbb{RP}^3$ (with fixed point set an S^2). The existence of such an automorphism is not difficult from the topological perspective, but finding one that is *conformal* is highly nontrivial.

We will let $\mathrm{Aut}(g)$ denote the conformal automorphism group, and $\mathrm{Aut}_0(g)$ denote the identity component. Theorem 1.1 implies the following.

Theorem 1.3. *Let $[g_{\mathrm{LB}}]$ be any LeBrun self-dual conformal class on $\hat{X} = n\#\mathbb{CP}^2$ and $n \geq 2$. All conformal automorphisms are orientation preserving.*

If the monopole points do not lie on any common geodesic, then

$$\mathrm{Aut}(g_{\mathrm{LB}}) = \mathrm{U}(1) \rtimes \Gamma, \quad \mathrm{Aut}_0(g_{\mathrm{LB}}) = \mathrm{U}(1), \tag{1.1}$$

where Γ is a finite subgroup of $\mathrm{O}(3)$.

If the monopole points all lie on a common hyperbolic geodesic, then

$$\mathrm{Aut}_0(g_{\mathrm{LB}}) = \mathrm{U}(1) \times \mathrm{U}(1). \tag{1.2}$$

In this case, for $n \geq 3$ the full conformal group is

$$\mathrm{Aut}(g_{\mathrm{LB}}) = \mathrm{Aut}_0(g_{\mathrm{LB}}) \rtimes \mathbb{Z}_2, \tag{1.3}$$

unless the points are configured symmetrically about a midpoint, in which case

$$\mathrm{Aut}(g_{\mathrm{LB}}) = \mathrm{Aut}_0(g_{\mathrm{LB}}) \rtimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2). \tag{1.4}$$

In the case $n = 2$, (1.2) *necessarily holds, and the full conformal group is*

$$\mathrm{Aut}(g_{\mathrm{LB}}) = \mathrm{Aut}_0(g_{\mathrm{LB}}) \rtimes \mathrm{D}_4, \tag{1.5}$$

where D_4 *is the dihedral group of order* 8.

The symmetry condition in the case of (1.4) is, more precisely, that there exists an extra hyperbolic reflection preserving the set of monopoles which fixes only a midpoint on the common geodesic. We will give explicit generators for each of the finite subgroups appearing in the semi-direct products (1.3)–(1.5), see Theorem 3.11.

We next give a brief outline of the paper. We review the construction of LeBrun metrics in Section 2, and we will detail the procedure for lifting hyperbolic isometries to conformal automorphisms of the LeBrun metrics. In Section 3, we present an explicit form of the LeBrun metrics in the toric case when $n = 2$, and discuss the extra involution. In Subsection 3.2, we give a summary of the results, and give a short discussion of the fixed point set of involutions and invariant sets, and the action on cohomology.

The remainder of the paper will use twistor methods to prove that there are no other conformal automorphisms. Section 4 will cover the case of $n \geq 3$, while Section 5 will cover the case when $n = 2$. The case of $n \geq 3$ is relatively easy, since in this case a (rational) quotient map for the $\mathbb{C}^*$-action on the twistor space corresponding to the semi-free U(1)-action is induced by a *complete* linear system, which implies that any automorphism descends to the quotient space. For $n = 2$, this is not true, and for this reason we instead use Poon's model of the twistor space, which is a small resolution of the intersection of two quadrics in $\mathbb{CP}^5$, see Section 5. In Subsection 5.1, we show that the holomorphic automorphisms of the intersection of the two quadrics which commute with the real structure consist of 16 tori. In Subsection 5.2, we determine explicitly which small resolutions actually give the twistor space. Then in Subsection 5.3, we show that the conformal automorphism group of Poon's metric consists of 8 tori, by explicitly determining which automorphisms among the 16 tori lift to the small resolutions obtained in Subsection 5.2. Finally, we interpret these automorphisms geometrically in Section 6, focusing on the involution Λ when $n = 2$.

We could have alternatively started the paper with the sections on twistor theory—this completely determines the automorphism group using only algebraic methods. However, one would like to understand the automorphisms geometrically, so we begin with the metric definition. From this perspective, it is easier to visualize the automorphisms for $n \geq 3$, as they are lifts of hyperbolic isometries. However, the existence of the extra conformal involution for $n = 2$ is not at all obvious from the metric perspective (in fact we first discovered it from the twistor viewpoint).

After acceptance of this paper, Fujiki [4] has determined the full conformal automorphism group for arbitrary Joyce metrics on $n\,\#\,\mathbb{CP}^2$ for any n.

2. Hyperbolic monopole metrics

We briefly recall the construction of LeBrun's self-dual hyperbolic monopole metrics from [13]. Consider the upper half-space model of hyperbolic space

$$\mathcal{H}^3 = \{(x, y, z) \in \mathbb{R}^3,\ z > 0\}, \tag{2.1}$$

with the hyperbolic metric $g_{\mathcal{H}^3} = z^{-2}(dx^2 + dy^2 + dz^2)$. Choose n distinct points $p_1, \dots, p_n$ in $\mathcal{H}^3$, and let $P = p_1 \cup \cdots \cup p_n$. Let Γ_{p_j} denote the fundamental solution for the hyperbolic Laplacian based at p_j with normalization $\Delta\Gamma_{p_j} = -2\pi\delta_{p_j}$, and let $V = 1 + \sum_{i=1}^n \Gamma_{p_i}$. Then $*dV$ is a closed 2-form on $\mathcal{H}^3 \setminus P$, and $(1/2\pi)[*dV]$ is an integral class in $H^2(\mathcal{H}^3 \setminus P, \mathbb{Z})$. Let $\pi : X_0 \to \mathcal{H}^3 \setminus P$ be the unique principal U(1)-bundle determined by the the above integral class. By Chern–Weil theory, there is a connection form $\omega \in H^1(X_0, i\mathbb{R})$ with curvature form $i(*dV)$. LeBrun's metric is defined by

$$g_{\mathrm{LB}} = z^2(V \cdot g_{\mathcal{H}^3} - V^{-1}\omega \odot \omega). \tag{2.2}$$

Note the minus sign appears, since by convention our connection form is imaginary valued. We define a larger manifold X by attaching points $\tilde{p}_j$ over each p_j, and by attaching an $\mathbb{R}^2$ at $z = 0$. The space X is non-compact, and has the topology of an ALE space. Adding the point at infinity will result in a compact manifold $\hat{X}$.

REMARK 2.1. Choosing a different connection form will result in the same metric, up to diffeomorphism, see the proof of Proposition 2.6 below.

We summarize the main properties of (X, g_{LB}) in the following proposition.

Proposition 2.2 (LeBrun [13]). *The metric g_{LB} extends to X as a smooth Riemannian metric. The space (X, g_{LB}) is asymptotically flat Kähler scalar-flat with a single end, and is biholomorphic to $\mathbb{C}^2$ blown up n points on a line. By adding one point, this metric conformally compactifies to a self-dual conformal class on the compactification $(\hat{X}, [g_{\mathrm{LB}}])$, which is diffeomorphic to $n\,\#\,\mathbb{CP}^2$.*

We next review some facts from bundle theory, which will then be applied to LeBrun's metrics.

2.1. Bundle methods. In this section $\mathrm{U}(1) \to X_0 \xrightarrow{\pi} M$ will be a principal U(1)-bundle over a connected oriented base manifold M. The group U(1) acts on X_0 from the right, we will denote this action by R_g for $g \in \mathrm{U}(1)$. Recall that a connection $\omega \in \Lambda^1(X_0; i\mathbb{R})$ is a 1-form on X_0 with values in the Lie algebra of U(1). The connection satisfies

(i) ω restricted to the fiber $\pi^{-1}(z)$ is $i \cdot d\theta$, the Maurer–Cartan form on U(1), and (ii) $R_g^* \omega = \omega$. Since the group is abelian, the *curvature* 2*-form* of the connection is given by $\Omega_\omega = d\omega \in H^2(X_0, i\mathbb{R})$, and this forms descends to M.

DEFINITION 2.3. The connections ω and ω' are said to be *gauge equivalent* if there exists a function $f: M \to \mathbb{R}$ such that $\omega = \omega' + i \cdot df$.

REMARK 2.4. If $\Omega_\omega = \Omega_{\omega'}$ then $d(\omega - \omega') = 0$. If $H^1(M; \mathbb{R}) = 0$, then $\omega - \omega' = i \cdot df$, so ω and ω' are gauge equivalent.

DEFINITION 2.5. The connections ω and ω' are said to be *bundle equivalent* if there exists a fiber-preserving map $B: X_0 \to X_0$ covering the identity map of M, that is $\pi \circ B = \pi$, and which commutes with the right action of U(1), satisfying $B^* \omega' = \omega$.

Proposition 2.6. *If the connections ω and ω' are gauge equivalent then they are bundle equivalent. The converse holds if $H^1(M, \mathbb{R}) = 0$.*

Proof. If the connections are gauge equivalent, then $\omega = \omega' + i \cdot df$. Define a bundle map $B: X_0 \to X_0$ by $Bv = v \cdot e^{if}$ (right action). Letting ω_1' denote a local connection form on the base, we have

$$B^* \omega' = B^*(\omega_1' + i \cdot d\theta) = \omega_1' + i B^* d\theta = \omega_1' + i(d\theta + df) = \omega' + i \cdot df = \omega. \tag{2.3}$$

Conversely, if $B^* \omega' = \omega$, then $\Omega_\omega = d\omega = dB^* \omega' = B^* \Omega_{\omega'}$. These are forms on the base, and B covers the identity map, so $\Omega_\omega = \Omega_{\omega'}$, which implies that ω and ω' are gauge equivalent by Remark 2.4. □

Since X_0 is a U(1)-bundle, it has a first Chern class $c_1(X_0) \in H^2(M; \mathbb{Z})$. From the exponential sheaf sequence, $H^1(M, \mathcal{E}^*) = H^2(M; \mathbb{Z})$, so X_0 is determined up to smooth bundle equivalence by $c_1(X_0)$. By Chern–Weil theory, the image of $c_1(X_0)$ in $H^2(M; i\mathbb{R})$ is cohomologous to Ω_ω, for any connection ω on X_0.

Proposition 2.7. *Assume that $H^1(M; \mathbb{Z}) = 0$, and that $H^2(M; \mathbb{Z})$ has no torsion. Let ω be a connection on X_0, and $\phi: M \to M$ an orientation preserving diffeomorphism satisfying $\phi^* \Omega_\omega = \Omega_\omega$. Then there exists an equivariant lift of ϕ to $\Phi: X_0 \to X_0$ satisfying $\Phi^* \omega = \omega$. If $\phi: M \to M$ is an orientation reversing diffeomorphism satisfying $\phi^* \Omega_\omega = -\Omega_\omega$, then there exists such a lift satisfying $\Phi^* \omega = -\omega$. These lifts are unique up to right action by a constant in* U(1)*. In both cases, Φ is orientation preserving.*

Proof. First assume that ϕ is orientation preserving. Consider the pull-back bundle $\phi^* X_0$. By naturality,

$$c_1(\phi^* X_0) = \phi^* c_1(X_0) = \phi^* [\Omega_\omega] = [\Omega_\omega] = c_1(X_0). \tag{2.4}$$

Consequently, there exists a bundle equivalence $A\colon \phi^* X_0 \to X_0$, which is an equivariant map covering the identity map on M. Denote by π_2 the natural map $\pi_2\colon \phi^* X_0 \to X_0$. This is summarized in the following diagram.

$$\begin{array}{ccccc} X_0 & \xrightarrow{A^{-1}} & \phi^* X_0 & \xrightarrow{\pi_2} & X_0 \\ \downarrow & & \downarrow & & \downarrow \pi \\ M & \xrightarrow{Id} & M & \xrightarrow{\phi} & M. \end{array} \tag{2.5}$$

The pull-back $\omega' = (A^{-1})^* \pi_2^* \omega$ is a connection on X_0. Since $\pi_2 \circ A^{-1}$ covers ϕ, we have

$$\Omega_{\omega'} = d\omega' = d((\pi_2 \circ A^{-1})^* \omega) = (\pi_2 \circ A^{-1})^* \Omega_\omega = \phi^* \Omega_\omega = \Omega_\omega. \tag{2.6}$$

From Remark 2.4, it follows that ω' and ω are gauge equivalent. By Proposition 2.6, ω' and ω are bundle equivalent, so there exists a bundle map $B\colon X_0 \to X_0$ satisfying $B^* \omega' = \omega$. The desired map is given $\pi_2 \circ A^{-1} \circ B$. In the construction of the map B in the proof of Proposition 2.6 above, there is a freedom to replace the function f by $f + c$ for any constant c, and the uniqueness statement follows.

If ϕ is orientation reversing, then the pull-back bundle $\phi^* X_0$ will satisfy $c_1(\phi^* X_0) = -c_1(X_0)$. In this case we need to add an additional map identifying the bundle with its conjugate bundle using complex conjugation, which corresponds geometrically to making a reflection in each fiber (such a choice is not canonical). Clearly, this makes the lift orientation preserving. □

REMARK 2.8. These lifts can be computed explicitly once the transition functions of the bundle are known (with respect to some open cover). Assume that the bundle is trivialized over a simply connected open set U, and that U is a ϕ-invariant set. Tracing through the above proof, to find the lift, we must first find a function $f\colon U \to \mathbb{R}$ such that

$$\phi^* \omega - \omega = i \cdot df, \tag{2.7}$$

and the lift is then right multiplication by e^{if} in each fiber (if ϕ is orientation-reversing, then we add a reflection in each fiber). The action in other coordinate systems is then found using the transition functions.

Proposition 2.9. *Let p be a fixed point of ϕ. If ϕ is orientation reversing, then any lift Φ of ϕ fixes exactly 2 points over p.*

Proof. From the above proof, any lift is a reflection in the fiber over a fixed point. A reflection always has exactly 2 fixed points. □

2.2. Lifts of hyperbolic isometries. We begin with a brief summary of the group of hyperbolic isometries. This is the group of time-oriented Lorentz transformations $\mathrm{SO}_+(3,1)$, which is clear from the hyperboloid model of hyperbolic space. The identity component is isomorphic to $\mathrm{PSL}(2,\mathbb{C})$; an isomorphism can be seen explicitly as follows. Using the quaternions, write hyperbolic upper half space as

$$\mathcal{H}^3 = \{x + yi + zj \mid (x, y, z) \in \mathbb{R}^3,\ z > 0\}. \tag{2.8}$$

Any orientation preserving hyperbolic isometry $\phi\colon \mathcal{H}^3 \to \mathcal{H}^3$ may be written as a quaternionic Möbius transformation

$$\phi(w) = (aw + b)(cw + d)^{-1}, \tag{2.9}$$

with $(a,b,c,d) \in \mathbb{C}^4$, and $ad - bc = 1$. For the non-identity component, any orientation reversing hyperbolic isometry $\phi\colon \mathcal{H}^3 \to \mathcal{H}^3$ may be written

$$\phi(w) = (a(-\bar{w}) + b)(c(-\bar{w}) + d)^{-1}, \tag{2.10}$$

with $(a,b,c,d) \in \mathbb{C}^4$, and $ad - bc = 1$. For more details on this isomorphism, see [19, Chapter 4].

Proposition 2.10. *Let $\{p_1, \ldots, p_n\} \subset \mathcal{H}^3$, and $n \geq 2$. Let G denote the group of all hyperbolic isometries preserving this set of points. If all points lie on a single hyperbolic geodesic γ, then $G = \mathrm{O}(2)$ acting as rotations and reflections about γ, unless the points are configured symmetrically about a midpoint, in which case $G = \mathrm{O}(2) \times \mathrm{O}(2) = \mathrm{U}(1) \rtimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ (more precisely, this symmetry condition is that there is another reflection preserving the set of points, and G is generated by $\mathrm{O}(2)$ and this reflection). Finally, if the points do not lie on any common geodesic, then G is conjugate to finite subgroup of $\mathrm{O}(3)$.*

Proof. This can be proved by a direct computation using the presentations (2.9) and (2.10). The proof is finished by noting that *any* finite subgroup of $\mathrm{SO}_+(3,1)$ is conjugate to a subgroup of $\mathrm{O}(3)$, see [19, Theorem 5.5.2]. □

The following proposition shows the lifts obtained in Proposition 2.7 yield conformal automorphisms of LeBrun's metrics.

Proposition 2.11. *If $\phi\colon \mathcal{H}^3 \to \mathcal{H}^3$ is a hyperbolic isometry preserving the set of monopole points, then there exists a unique $\mathrm{U}(1)$-family of lifts Φ as in* Proposition 2.7 *which are orientation preserving conformal automorphisms of (X_0, g_{LB}). Furthermore, any such lift extends to a conformal automorphism of the compactification $(n\,\#\,\mathbb{CP}^2, [g_{\mathrm{LB}}])$.*

Proof. If ϕ is a hyperbolic isometry, then

$$V \circ \Phi = 1 + \sum_{j=1}^{n} \Gamma_{\Phi^{-1}(p_j)}. \tag{2.11}$$

Since ϕ fixes the set of monopole points, we have $\phi^* V = V$. We chose the connection above so that $\Omega_\omega = i(*dV)$. This implies that $\phi^*\Omega_\omega = \Omega_\omega$ if ϕ is orientation preserving, and $\phi^*\Omega_\omega = -\Omega_\omega$ if ϕ is orientation reversing. In either case, we may apply Proposition 2.7 to find a lift of ϕ satisfying $\Phi^*\omega = \pm\omega$. By assumption, $\phi^* g_{\mathcal{H}^3} = g_{\mathcal{H}^3}$, so we have

$$\begin{aligned} \Phi^* g_{\mathrm{LB}} &= (z \circ \Phi)^2((V \circ \Phi) \cdot \Phi^* g_{\mathcal{H}^3} - (V \circ \Phi)^{-1} \cdot \Phi^*(\omega \odot \omega)) \\ &= (z \circ \Phi)^2(V \cdot g_{\mathcal{H}^3} - V^{-1}\omega \odot \omega) = \left(\frac{z \circ \Phi}{z}\right)^2 g_{\mathrm{LB}}. \end{aligned} \tag{2.12}$$

For the last statement, the S^1-action of fiber rotation on X_0 clearly extends smoothly to the compactification, since $\hat{X}$ is obtained from X_0 by adding fixed points over the monopole points, and also adding the entire boundary of $\mathcal{H}^3$, which is also fixed by the S^1-action. The argument in [13] for extending the metric conformally to $\hat{X}$ generalizes to show that Φ yields a *smooth* conformal diffeomorphism of $\hat{X}$, we omit the details. □

We emphasize that Proposition 2.7 only provides a lift of a single isometry. The lifting of a group of isometries is more subtle. We define $\mathrm{Aut}(\mathcal{H}^3; p_1, \ldots, p_n)$ to be the group of isometries of $\mathcal{H}^3$ which preserve the set of monopole points, and let $\mathrm{Aut}(g_{\mathrm{LB}}; p_1, \ldots, p_n)$ denote the subgroup of conformal automorphisms which are lifts of elements in $\mathrm{Aut}(\mathcal{H}^3; p_1, \ldots, p_n)$. Clearly, we have an exact sequence

$$1 \to \mathrm{U}(1) \to \mathrm{Aut}(g_{\mathrm{LB}}; p_1, \ldots, p_n) \xrightarrow{\rho} \mathrm{Aut}(\mathcal{H}^3; p_1, \ldots, p_n) \to 1, \tag{2.13}$$

where ρ is the obvious projection. A natural question is whether this sequence *splits*, that is, does there exist a homomorphism

$$\mu : \mathrm{Aut}(\mathcal{H}^3; p_1, \ldots, p_n) \to \mathrm{Aut}(g_{\mathrm{LB}}; p_1, \ldots, p_n) \tag{2.14}$$

such that $\rho \circ \mu = Id$?

In general, this sequence does not split (the automorphism group will in general be a semi-direct product with U(1), not a direct product, see Theorem 3.11 below). However, we next give a condition for the sequence to split when restricted to a subgroup of $G \subset \mathrm{Aut}(\mathcal{H}^3; p_1, \ldots, p_n)$.

Proposition 2.12. *Let the subgroup G consist of orientation preserving elements. If the subgroup G has a fixed point $p \in \mathcal{H}^3 \setminus \{p_1, \ldots, p_n\}$, then there is a splitting homomorphism*

$$\mu : G \to \mathrm{Aut}(g_{\mathrm{LB}}; p_1, \ldots, p_n), \quad \rho \circ \mu = Id_G. \tag{2.15}$$

Furthermore,

$$\mathrm{U}(1) \times G \subset \mathrm{Aut}(g_{\mathrm{LB}}; p_1, \ldots, p_n). \tag{2.16}$$

Proof. Since p is not one of the monopole points, then any element of G has a unique lift which fixes the fiber over p. This defines the splitting map μ. To see that μ is a homomorphism: given $g_1 \in G$ and $g_2 \in G$, we compare $\mu(g_1 g_2)$ with $\mu(g_1)\mu(g_2)$. The former is, by definition, the lift of $g_1 g_2$ in the unique lift which fixes the fiber over p. The latter is also a lift of $g_1 g_2$, and fixes the fiber over p, since both $\mu(g_1)$ and $\mu(g_2)$ fix this fiber. By uniqueness, they are the same.

Next, U(1) is the identity component, which is normal. We claim that $\mu(G)$ is also normal. To see this, let $\mu(g) \in \mu(G)$, and $\Phi \in \mathrm{Aut}(g_{\mathrm{LB}}; p_1, \ldots, p_n)$. Then $\Phi\mu(g)\Phi^{-1}$ fixes $\{p_1, \ldots, p_n\}$ and fixes the fiber over p, therefore must be of the form $\mu(h)$ for some element $h \in G$.

Finally, since both subgroups are normal, by an elementary theorem in group theory, we have a direct product. □

REMARK 2.13. Consider the case when the points are not contained on a common geodesic. Then, as mentioned above in Proposition 2.10, $\mathrm{Aut}(\mathcal{H}^3; p_1, \ldots, p_n)$ is conjugate to a finite subgroup of O(3). Let us assume for simplicity that the symmetry group G is conjugate to a subgroup of SO(3). The group G either fixes a geodesic, or has a single fixed point. In the former case, there must be a non-monopole fixed point, and Proposition 2.12 can be applied. In the latter case, if the fixed point is not a monopole point, then again Proposition 2.12 can be applied. But if the fixed point is a monopole point, then the entire group might not lift. In this case, it is possible that the group G appearing in (1.1) is a strictly smaller subgroup of $\mathrm{Aut}(\mathcal{H}^3; p_1, \ldots, p_n)$, and which might not necessarily lift to a normal subgroup. However, we do not know of any such example for which this happens.

Proposition 2.14. *If all of the monopole points lie on a common geodesic, then*

$$\mathrm{U}(1) \times \mathrm{SO}(2) = \mathrm{U}(1) \times \mathrm{U}(1) \subseteq \mathrm{Aut}(g_{\mathrm{LB}}). \tag{2.17}$$

Proof. The subgroup SO(2) of rotations around a geodesic fix the entire geodesic. Let p be any non-monopole point on the geodesic, and apply Proposition 2.12. □

In the next section we present a direct method of finding such lifts, via an explicit connection form.

3. An explicit global connection

We will call a conformal class *toric* if the automorphism group contains $\mathrm{U}(1) \times \mathrm{U}(1)$. In this section we give an explicit connection for the LeBrun ansatz in the toric case. Here we consider the case of 2 monopole points. Let the monopole points lie on the z-axis, $p_1 = (0, 0, c_1)$, and $p_2 = (0, 0, c_2)$, with $c_1 < c_2$. Choose cylindrical coordinates

$$\mathcal{H}^3 = \{(x, y, z) = (r\cos\theta_3,\ r\sin\theta_3, z),\ z > 0\}. \tag{3.1}$$

Theorem 3.1. *Let $U = \mathcal{H}^3 \setminus \{z\text{-axis}\}$, and write*

$$\mathcal{H}^3 \setminus \{p_1, p_2\} = U_1 \cup U_2 \cup U_3, \tag{3.2}$$

where

$$U_1 = U \cup \{(0, 0, z),\ z < c_1\} = U \cup I_1, \tag{3.3}$$

$$U_2 = U \cup \{(0, 0, z),\ c_1 < z < c_2\} = U \cup I_2, \tag{3.4}$$

$$U_3 = U \cup \{(0, 0, z),\ z > c_2\} = U \cup I_3. \tag{3.5}$$

Let $f_c \colon \mathcal{H}^3 \setminus \{p_1, p_2\} \to \mathbb{R}$ denote the function

$$f_c(r, z) = \frac{r^2 + z^2 - c^2}{2\sqrt{(c^2 + r^2 + z^2)^2 - 4c^2z^2}} - \frac{1}{2}. \tag{3.6}$$

Then $f = f_{c_1} + f_{c_2}$ satisfies

$$d(f\, d\theta_3) = *dV, \tag{3.7}$$

in U. That is, the form $i f\, d\theta_3$ is a local connection form in U. Define

$$\omega_1(x) = i(f + 2)\, d\theta_3, \quad x \in U_1, \tag{3.8}$$

$$\omega_2(x) = i(f + 1)\, d\theta_3, \quad x \in U_2, \tag{3.9}$$

$$\omega_3(x) = i f\, d\theta_3, \quad x \in U_3. \tag{3.10}$$

These 1-forms define a global connection (with values in $\mathfrak{u}(1) = i\mathbb{R}$) on the total space $X_0 \to M$. That is, there is a global connection ω on X_0, such that over U_j, ω has the form $\omega_j + i \cdot d\theta_1$, where θ_1 is an angular coordinate on the fiber.

Proof. Recall we want the connection to have curvature form $\Omega_\omega = *dV$, where $V = 1 + \Gamma_{p_1} + \Gamma_{p_2}$. The Green's function is given by

$$\Gamma_{(0,0,c)}(x, y, z) = -\frac{1}{2} + \frac{1}{2}\left[1 - \frac{4c^2z^2}{(r^2 + z^2 + c^2)^2}\right]^{-1/2}, \tag{3.11}$$

where $r^2 = x^2 + y^2$, see [12, Section 2]. An important point is that Γ only depends upon z and r. A computation shows that in cylindrical coordinates

$$*dV = \left(\frac{r}{z}(-V_r\, dz + V_z\, dr)\right) \wedge d\theta_3, \tag{3.12}$$

since $d * dV = 0$, this implies that

$$d\left(\frac{r}{z}(-V_r\, dz + V_z\, dr)\right) = 0. \tag{3.13}$$

The first quadrant $Q_1 = \{(r, z),\ r > 0,\ z > 0\}$ is contractible, so there exists a function $f = f(r, z)$ such that

$$df = \frac{r}{z}(-V_r\, dz + V_z\, dr). \tag{3.14}$$

We let

$$\kappa = \frac{r}{z}V_z\, dr - \frac{r}{z}V_r\, dz = \kappa_1\, dr + \kappa_2\, dz. \tag{3.15}$$

An explicit potential f satisfying $df = \kappa$ is

$$f = \left(\int_0^1 \kappa_1(tr, tz)\, dt\right)r + \left(\int_0^1 \kappa_2(tr, tz)\, dt\right)z + \text{const.} \tag{3.16}$$

A computation, which we omit, shows that

$$f = f_{c_1} + f_{c_2}, \tag{3.17}$$

is a solution where f_c is given by

$$f_c(r, z) = \frac{r^2 + z^2 - c^2}{2\sqrt{(c^2 + r^2 + z^2)^2 - 4c^2z^2}} - \frac{1}{2}, \tag{3.18}$$

and any other solution differs from this by a constant, since U is connected. An important remark is that

$$(c^2 + r^2 + z^2)^2 - 4c^2z^2 \geq 0, \tag{3.19}$$

and if $(c^2 + r^2 + z^2)^2 - 4c^2z^2 = 0$, then $(z - c)^2 + r^2 = 0$, so f_c is well-defined on all of $\mathcal{H}^3 \setminus \{(0, 0, c)\}$. We then have on U,

$$d(f\, d\theta_3) = df \wedge d\theta_3 = *dV. \tag{3.20}$$

Along the z-axis, we have the following

$$f(0,z) = \begin{cases} -2 & z < c_1, \\ -1 & c_1 < z < c_2, \\ 0 & z > c_2. \end{cases} \tag{3.21}$$

Furthermore, $(\partial/\partial r) f(0,z) = 0$. Consequently, $\omega_1 = (f+2)\, d\theta_3$ is a smooth 1-form in U_1, $\omega_2 = (f+1)\, d\theta_3$ is a smooth 1-form in U_2, and $\omega_3 = f\, d\theta_3$ is a smooth 1-form in U_3. □

REMARK 3.2. For $n > 2$, simply take $f = f_{c_1} + \cdots + f_{c_n}$, and U_j to be the union of U with the corresponding interval on the z-axis, with the connection form in each chart adding the appropriate constant multiple of $d\theta_3$. This explicit connection form can be used to exhibit a direct proof that toric LeBrun metrics are Joyce metrics, for this we refer the reader to [7]. We remark that an explicit potential in the case $n = 2$ was written down in [5] in pseudospherical coordinates, but only in a single chart; our method above yields a *global* connection form.

We can use the above to write down explicit transition functions for the bundle $X_0 \to M$.

Proposition 3.3. *With respect to the covering* $\{U_1, U_2, U_3\}$, *the transition functions of the bundle are given by* $g_{21} = e^{-i\theta_3}$, *and* $g_{23} = e^{i\theta_3}$.

Proof. From above

$$\omega_2 - \omega_1 = i(f+1)\, d\theta_3 - i(f+2)\, d\theta_3 = -i \cdot d\theta_3. \tag{3.22}$$

The formula for the change of connection is given by

$$\omega_2 - \omega_1 = g_{21}^{-1}\, dg_{21}, \tag{3.23}$$

which implies that $g_{21} = e^{-i\theta_3}$. Also,

$$\omega_2 - \omega_3 = i(f+1)\, d\theta_3 - if\, d\theta_3 = i \cdot d\theta_3 = g_{23}^{-1}\, dg_{23}, \tag{3.24}$$

which implies that $g_{23} = e^{i\theta_3}$. □

For the transition functions in the case $n > 2$, we refer the reader to [7].

We name two points on the boundary of $\mathcal{H}^3$: $q_1 = (0,0,0)$, and $q_2 = (0,0,\infty)$. We denote the union of the fibers over $\overline{I_j}$ by Σ_j $(1 \le j \le 3)$, which is a 2-sphere. We also let Σ_4 denote the 2-sphere corresponding to the boundary of hyperbolic space. Using the above, we next show that the S^1-action on $\mathcal{H}^3$ given by rotation around the z-axis

has infinitely many lifts to conformal S^1-actions on $(\hat{X}, [g_{LB}])$. We recall that a *semi-free* action is a non-trivial action of a group G on a connected space M such that for every $x \in M$, the corresponding isotropy subgroup is either all of G or is trivial.

Proposition 3.4. *Consider the S^1-action on $\mathcal{H}^3$ by oriented rotations around the z-axis. Then for any integer k, there exists a lift to a conformal S^1-action on $(\hat{X}, [g_{LB}])$ such that $e^{i\theta}$ lifts with following property: the lifted action rotates the fibers over I_2 by $e^{ik\theta}$, it rotates the fibers over I_1 by $e^{i(k+1)\theta}$, and rotates the fibers over I_3 by $e^{i(k-1)\theta}$.*

Consequently, for $k = 0$, the lift fixes only $\Sigma_2 \cup \{q_1, q_2\}$, and this action is the only only lift to a semi-free S^1-action. For $k = 1$, the fixed points are $\Sigma_1 \cup \{p_2, q_2\}$, and for $k = -1$, the lift fixes only $\Sigma_3 \cup \{p_1, q_1\}$. For any other k, the fixed set consists of four points $\{p_1, p_2, q_1, q_2\}$.

Proof. Let ϕ denote an oriented rotation about the z-axis, determined by $e^{i\theta_0}$. As in the above section, we know a lift of ϕ, call it Φ, exists, and is unique up to a right multiplication by a constant. If we choose the lift Φ so that Φ fixes fiber over a point on I_2, then Φ fixes all fibers over I_2. This follows because the connection form on U_2 chosen in Theorem 3.1 is invariant under rotations around the z-axis, see Remark 2.8. From the transition functions given in Proposition 3.3, Φ rotates the fibers over I_1 by $e^{i\theta_0}$, and the fibers over I_3 are rotated by $e^{-i\theta_0}$. Finally, it is clear that we can lift to an S^1-action by specifying the action on the fibers over I_2; there is a lift for any integer k so that the fibers of I_2 are rotated by $e^{ik\theta}$. The semi-free claim is obvious, since for $k = 0$, the lift only makes a single rotation on Σ_1 and Σ_3, while for $k \neq 0$, I_1 and I_3 are rotated multiple times. Again, the argument in [13] extends to show all of the above actions yield smooth actions on the compactification $\hat{X}$, we omit the details. □

We denote the lifted action for $k = 0$ by K_3. Since the K_3-action clearly commutes with the K_1-action, this then gives an identification of the identity component of the automorphism group with $K_1 \times K_3$, where K_1 is the group of rotations in the fiber. It will be shown below in Lemma 6.1, that for $n = 2$, K_1 and K_3 are the only semi-free S^1-actions. We will also see in Section 6 that the K_3-action yields another fibration of an open subset of X over $\mathcal{H}^3 \setminus \{\text{two points}\}$.

While for simplicity of presentation we restricted the above discussion to the case of 2 monopole points, it is clear that for the case of n monopole points all lying on a common geodesic, the SO(2)-action of rotations in $\mathcal{H}^3$ around the geodesic will have a lift to an S^1-action for any integer k. Since these actions commute with the fiber rotation, there is a torus action as identity component. However, in contrast to $n = 2$, for $n \geq 3$, none of these lifted S^1-actions are semi-free, see Lemma 6.1.

3.1. Extra involution for n = 2. Recall we have the boundary sphere Σ_4 fixed by K_1, and the sphere Σ_2, fixed by K_3. We next find a conformal transformation which interchanges these spheres, and also has the property that p_1 maps to $q_1 = (0, 0, 0)$,

and p_2 maps to $q_2 = (0, 0, \infty)$. This map will interchange the orbits of the K_1 and K_3 actions.

Let r, z and c_1, c_2 have the same meaning as in the beginning of Section 3. We first define an automorphism $\varphi : Q_1 \to Q_1$, where $Q_1 = \{(r, z) \mid r > 0,\ z > 0\}$ is the first quadrant.

DEFINITION 3.5. Let $w = r + iz$, and define

$$\varphi(w) = ic_2\sqrt{\frac{\bar{w}^2 + c_1^2}{\bar{w}^2 + c_2^2}} = \varphi_1(r, z) + i\varphi_2(r, z). \tag{3.25}$$

We recall that the intervals I_j, $j = 1, 2, 3$, were defined above as subsets of the hyperbolic upper half-space $\mathcal{H}^3$

$$I_1 = \{(0, 0, z), z < c_1\}, \tag{3.26}$$

$$I_2 = \{(0, 0, z), c_1 < z < c_2\}, \tag{3.27}$$

$$I_3 = \{(0, 0, z), z > c_2\}. \tag{3.28}$$

We also define

$$I_4 = \{(r, 0, 0),\ r > 0\}. \tag{3.29}$$

In the following, we will view $Q_1 \subset \mathcal{H}^3$ by setting $\theta_3 = 0$, that is

$$Q_1 = \{(r, 0, z),\ r > 0,\ z > 0\}, \tag{3.30}$$

and view $I_j \subset \partial Q_1$ for $j = 1, 2, 3, 4$. The map φ extends to the closure of Q_1, with the following properties:

Proposition 3.6. *The map φ interchanges I_2 and I_4, interchanges p_1 and $q_1 = (0,0,0)$, and interchanges p_2 and $q_2 = (0,0,\infty)$. Under the identification of Q_1 with upper half 2-space under the complex square $w \mapsto w^2$, the map is a hyperbolic isometry.*

Proof. We identify Q_1 with $\mathcal{H}^2$ using the complex square,

$$\zeta = x_1 + ix_2 = (r + iz)^2 = s(w). \tag{3.31}$$

Under this map, the monopole points p_j map to $(-c_j^2, 0)$. Consider the Möbius transformation defined by

$$L(\zeta) = -(c_2)^2\frac{\bar{\zeta} + (c_1)^2}{\bar{\zeta} + (c_2)^2}, \tag{3.32}$$

which is an orientation-reversing hyperbolic isometry of $\mathcal{H}^2$. It has the property that

$$L((-c_1^2, 0)) = (0, 0),\ L(0, 0) = (-c_1^2, 0), \quad \text{and} \quad L(-c_2^2, 0)) = (\infty, 0). \tag{3.33}$$

Clearly, $\varphi(w) = s^{-1} \circ L \circ s(w)$, which is (3.25). The first statement follows easily. □

REMARK 3.7. The map L is the unique orientation-reversing hyperbolic involution satisfying (3.33).

The original coordinates on $U \times S^1$ are ordered $(r, \theta_3, z, \theta_1)$, but in the following we will rearrange coordinates so that this domain is $Q_1 \times S^1 \times S^1$.

DEFINITION 3.8. For any angle ϑ, define the map $\tilde{\Lambda}(\vartheta)\colon X \to X$ by

$$\tilde{\Lambda}(\vartheta)\colon ((r, z), \theta_3, \theta_1) \mapsto (\varphi(r, z), \theta_1 - \vartheta, \theta_3 + \vartheta). \tag{3.34}$$

On first observation, it might appear that the map $\tilde{\Lambda}(\vartheta)$ is not well-defined at points on the z-axis corresponding the the intervals I_1 and I_3, where the coordinate θ_3 is not defined. However, the map is in fact well-defined everywhere:

Proposition 3.9. *For any angle ϑ, the map $\tilde{\Lambda}(\vartheta)$ extends to a diffeomorphic involution of $\hat{X} = 2\#\mathbb{CP}^2$. The extension interchanges Σ_2 and Σ_4, and interchanges the points p_j and q_j for $j = 1, 2$.*

Proof. We need only consider the case that $\vartheta = 0$, since $\tilde{\Lambda}(\vartheta) = (e^{-i\vartheta}, e^{i\vartheta}) \cdot \tilde{\Lambda}(0)$ (viewing this as the $K_1 \times K_3$-action). We note that initially $\tilde{\Lambda}(0)$ is defined with respect to a trivialization of the bundle on the open set U_2. To confirm that it well-defined everywhere, we must use the transition functions from Proposition 3.3. For example, in U_2, the angles change by $(\theta_3, \theta_1) \mapsto (\theta_1, \theta_3)$. Taking into account the transition function $g_{21} = e^{i\theta_3}$, in U_1 the action is $(\theta_3, \theta_1) \mapsto (\theta_1 - \theta_3, \theta_1)$. In the U_1 chart, the map $\tilde{\Lambda}(0)$ therefore takes the form

$$(r, z, \theta_3, \theta_1) \mapsto (\varphi_1(r, z), \varphi_2(r, z), \theta_1 - \theta_3, \theta_1). \tag{3.35}$$

Rewriting the map in the coordinates (x, y, z, θ_1),

$$(x, y, z, \theta_1) \mapsto (\varphi_1(r, z) \sin(\theta_1 - \theta_3), \varphi_1(r, z) \cos(\theta_1 - \theta_3), \varphi_2(r, z), \theta_1). \tag{3.36}$$

For points with $r = 0$, the map φ is given by

$$\varphi(0, z) = \left(0, c_2 \sqrt{\frac{c_1^2 - z^2}{c_2^2 - z^2}}\right), \tag{3.37}$$

which *is* well-defined on I_1. Therefore, for $(x, y) = (0, 0)$, (3.36) becomes

$$(0, 0, z, \theta_1) \mapsto (0, 0, \varphi(0, z), \theta_1), \tag{3.38}$$

which is indeed well-defined. A similar argument confirms that $\tilde{\Lambda}(0)$ is well-defined (and smooth) everywhere on $2\,\#\,\mathbb{CP}^2$.

It is easy to see that $\tilde{\Lambda}(0)$ interchanges Σ_2 and Σ_4, and interchanges the points p_j and q_j for $j = 1, 2$. Finally, it is clear that $\tilde{\Lambda}(\vartheta)$ is an involution. □

Theorem 3.10. *For any angle ϑ, the map $\tilde{\Lambda}(\vartheta)$ is a conformal involution of $[g_{\mathrm{LB}}]$.*

Proof. It would be a formidable calculation to show directly that this map is indeed conformal. In this paper, for space considerations, we therefore prefer to argue indirectly using twistor theory, see Theorem 6.13 below. □

3.2. Summary. In this section, we summarize what we have obtained so far, and we also make some remarks about the fixed point sets of various lifts.

Theorem 3.11. *Consider $(n\,\#\,\mathbb{CP}^2, [g_{\mathrm{LB}}])$ and $n \geq 2$. If the monopole points do not lie on any common geodesic (so that $n \geq 3$), then*

$$\mathrm{U}(1) \rtimes \Gamma \subseteq \mathrm{Aut}(g_{\mathrm{LB}}), \tag{3.39}$$

where Γ is a finite subgroup of O(3).

Next, assume that the monopole points all lie on a common geodesic. Let Aut_0 *denote the identity component of* $\mathrm{Aut}(g_{\mathrm{LB}})$. *Then we have*

$$\mathrm{U}(1) \times \mathrm{U}(1) = \mathrm{Aut}_0(g_{\mathrm{LB}}). \tag{3.40}$$

Let ϕ_3 be any reflection about a hemisphere on which all the monopole points belong. Then there exists a lift Φ_3 of ϕ_3 which is also an involution. Let $\mathbb{Z}_2 = \{Id, \Phi_3\}$ denote the subgroup generated by Φ_3. Then the semi-direct product

$$(\mathrm{U}(1) \times \mathrm{U}(1)) \rtimes \mathbb{Z}_2 \subseteq \mathrm{Aut}(g_{\mathrm{LB}}). \tag{3.41}$$

In the case there is an additional reflection symmetry ϕ_2 (which is always the case for $n = 2$), consider also the composition $\phi_1 = \phi_2 \circ \phi_3$. Then, in addition to Φ_3, there exist lifts Φ_j of ϕ_j, for $j = 1, 2$ such that $\{Id, \Phi_1, \Phi_2, \Phi_3\}$ is a subgroup of Aut *which is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and*

$$(\mathrm{U}(1) \times \mathrm{U}(1)) \rtimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \subseteq \mathrm{Aut}(g_{\mathrm{LB}}). \tag{3.42}$$

For $n = 2$ consider also the extra involution $\tilde{\Lambda}(0)$. Then

$$\{Id, \Phi_1, \Phi_2, \Phi_3, \tilde{\Lambda}(0), \tilde{\Lambda}(0)\Phi_1, \tilde{\Lambda}(0)\Phi_2, \tilde{\Lambda}(0)\Phi_3\}$$

is a subgroup of Aut *isomorphic to* D_4, *the dihedral group with* 8 *elements, and*

$$(U(1) \times U(1)) \rtimes D_4 \subseteq \mathrm{Aut}(g_{LB}). \tag{3.43}$$

Proof. The inclusion (3.39) was discussed above in Remark 2.13. The equality (3.40) follows from Proposition 2.14, and the fact that the identity component is a manifold, and cannot be strictly greater than dimension 2 in this case [17].

For (3.41), we let $\tilde{\Phi}_3$ be *any* lifting of ϕ_3 from Proposition 2.11. Note that $\tilde{\Phi}_3^2$ is orientation preserving and covers the identity map of $\mathcal{H}^3$. Therefore, by the uniqueness in Proposition 2.11, we must have that $\tilde{\Phi}_3^2 = R(g)$ is right multiplication by $g \in U(1)$. To find an involution, we then define Φ_3 to be $\tilde{\Phi}_3 \circ R(\sqrt{g^{-1}})$. This is an involution since any lift is equivariant. Therefore $\{Id, \Phi_3\}$ is indeed a subgroup of $\mathrm{Aut}(g_{LB})$ isomorphic to $\mathbb{Z}_2$. Since the identity component is necessarily normal, the group generated by the identity component and this $\mathbb{Z}_2$-subgroup is a semi-direct product.

For (3.42), we let Φ_3 be as in the previous paragraph. Next, the map $\phi_1 = \phi_2 \circ \phi_3$ is an orientation preserving hyperbolic isometry which fixes a geodesic. Thus we may apply Proposition 2.14, and let $\Phi_1 = \mu(\phi_1)$. Since ϕ_1 is an involution, from the definition of μ, it follows that Φ_1 is also an involution. Then we *define* $\Phi_2 = \Phi_1 \circ \Phi_3$, which is necessarily a lift of ϕ_2. Clearly, $\{Id, \Phi_1, \Phi_2, \Phi_3\}$ is a subgroup isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and for the same reason as in the previous paragraph, the generated subgroup is the semi-direct product.

Finally, the inclusion (3.43) will be proved in Section 6, see Proposition 6.9. □

REMARK 3.12. The finite subgroups of O(3) are given by the cyclic, dihedral, tetrahedral, octahedral, and icosahedral groups. For example, 3 monopole points could be arranged in a planar triangle, 4 points in a tetrahedral configuration, 8 points in a cubic configuration, etc. For a complete description of these groups, see [19, Chapter 7]. We do not go into further detail here since we are primarily concerned with the toric case in this paper.

It is the purpose of Sections 4 and 5 below to show that the inclusions (3.41)–(3.43) are in fact equalities. We end this section with a short discussion on fixed point sets of involutions, and the action on cohomology.

Theorem 3.13. *For* $(n\#\mathbb{CP}^2, [g_{LB}])$ *and* $n \geq 2$, *assume that the monopole points all lie on a common geodesic. If* ϕ_3 *is a reflection about a hemisphere containing all the monopole points, then the lift* Φ_3 *of* ϕ_3 *given in* Theorem 3.11 *has fixed point locus* $\Upsilon_3 = n\#\mathbb{RP}^2$, *which is contained in an invariant* $n\#\mathbb{RP}^3$. *Furthermore,* Φ_3 *induces minus the identity map on cohomology.*

In the case there is an additional reflection symmetry ϕ_2 (*which is always the case for* $n = 2$), *consider also the composition* $\phi_1 = \phi_2 \circ \phi_3$. *Let* Υ_j *denote the fixed locus of* Φ_j, *where* Φ_j *are the lifts of* ϕ_j *given in* Theorem 3.11. *For n even,* Υ_1 *and* Υ_2 *are*

both two-dimensional spheres, and $\Upsilon_1 \cap \Upsilon_2 = S^1$. *For n odd,* $\Upsilon_1 = S^2$ *and* $\Upsilon_2 = \mathbb{RP}^2$, *with* $\Upsilon_1 \cap \Upsilon_2 = S^1$. *The maps* ϕ_1 *and* ϕ_2 *induce the following map on cohomology*

$$(k_1, k_2, \ldots, k_n) \mapsto (k_n, k_{n-1}, \ldots, k_1), \tag{3.44}$$

with respect to an orthonormal basis of $H^2(n\#\mathbb{CP}^2;\mathbb{Z})$. *Further, for n even,* Φ_2, *leaves invariant an* S^3. *For n odd,* Φ_2 *leaves invariant an* $\mathbb{RP}^3$.

For $n = 2$, *the fixed point set of the extra involution* $\tilde{\Lambda}(0)$ *is an* S^2, *which is contained in an invariant* $\mathbb{RP}^3$. *Also,* $\tilde{\Lambda}(0)$ *induces the following map on cohomology*:

$$(k_1, k_2) \mapsto (-k_2, k_1), \tag{3.45}$$

with respect to an orthonormal basis of $H^2(2\#\mathbb{CP}^2;\mathbb{Z})$.

Proof. We let ϕ_3 be a reflection in a hemisphere containing the monopole points. Since ϕ_3 is orientation reversing, by Proposition 2.9, the lift Φ_3 will fix exactly 2 points in each fiber over this hemisphere. Let Υ_3 denote the fixed locus. Topologically, Υ_3 is a double covering of a 2-disc branched over the boundary circle and over n points. We compute

$$\chi(\Upsilon_3) = 2\chi(D^2) - \chi(S^1) - n = 2 - n. \tag{3.46}$$

It turns out that Υ_3 is non-orientable, so by the surface classification, $\Upsilon_3 = n\#\mathbb{RP}^2$ (to see non-orientability, we note that odd dimensions is clear since the Euler characteristic is odd, and the even-dimensional case be viewed as a limiting case of the next higher odd dimension). The invariant set is a circle bundle over this hemisphere, branched over n points and the boundary circle, so is $n\#\mathbb{RP}^3$.

When the points are in symmetric configuration, we let ϕ_2 denote the extra symmetry of inversion in a hemisphere. If n is even, there is no monopole point on this hemisphere. Since ϕ_2 is orientation reversing, Proposition 2.9 implies that the fixed point set of the lift Φ_2 is a double cover of D^2 branched only over the boundary circle, so $\Upsilon_2 = S^2$. The invariant set is a circle bundle over the disc branched over the boundary, so is an S^3. Next, define $\phi_1 = \phi_2 \circ \phi_3$. The fixed point set of ϕ_1 is a geodesic γ. From the proof of Theorem 3.11, our choice of the lifting Φ_1 fixes a fiber over a point of γ, thus fixes every fiber over γ. Therefore, Υ_1 is a circle bundle over γ, completed by adding two points on the boundary of $\mathcal{H}^3$, so $\Upsilon_1 = S^2$. The intersection of Υ_1 and Υ_2 gives 2 points in each fiber over γ. Adding the 2 boundary points gives that $\Upsilon_1 \cap \Upsilon_2 = S^1$.

If n is odd, then there is a monopole point on this hemisphere. From Proposition 2.9, the fixed point set of the lift Φ_2 is a double cover of D^2 branched over the boundary circle, and a single point. We have

$$\chi(\Upsilon_2) = 2\chi(D^2) - \chi(S^1) - 1 = 1, \tag{3.47}$$

which implies that Υ_2 is $\mathbb{RP}^2$. The invariant set is a circle bundle over D^2 branched over the boundary circle, and a single point, thus is an $\mathbb{RP}^3$.

As in the even case, define $\phi_1 = \phi_2 \circ \phi_3$. Again, the fixed point set of ϕ_1 is a geodesic γ. Therefore, Υ_1 is contained in the restriction of the bundle to this geodesic (including the 2 boundary points of the geodesic). Since there is a single monopole point on this geodesic, the restriction of the bundle is topologically the wedge $S^2 \vee S^2$. From the proof of Theorem 3.11, the lift Φ_1 was chosen to fix a fiber over some point on this geodesic. Since the fixed point set must be a smooth 2-dimensional manifold, Υ_1 must be one of these S^2-s, depending upon the particular choice of the lift Φ_1. The intersection of Υ_1 and Υ_2 then is 2 points in each fiber over one half of γ, together with the monopole point and a single boundary point, which implies that $\Upsilon_1 \cap \Upsilon_2 = S^1$.

In the case $n = 2$, recall the hyperbolic isometry L defined in (3.32). It is easy to verify that the fixed point set of L is given by

$$(x_1 + c_2^2)^2 + x_2^2 = c_2^2(c_2^2 - c_1^2), \tag{3.48}$$

and is therefore a semicircle centered at $(-c_2^2, 0)$ of radius $c_2\sqrt{c_2^2 - c_1^2}$. Since $z \mapsto z^2$ is a conformal transformation, the fixed point set of φ is a semi-circle centered on the z-axis at $(0, c_2)$, intersecting the positive z-axis at two points, one of them on the interval I_1, and the other on I_3. The fixed point set of $(\theta_3, \theta_1) \mapsto (\theta_1, \theta_3)$ is obviously points of the form (θ_3, θ_3). Thus the fixed point set of $\tilde{\Lambda}(0)$ is a circle bundle over the semicircle branched over the two endpoints, therefore is an S^2. The invariant set consists of all the torus fibers over the semicircle, which is easily seen to be an $\mathbb{RP}^3$ (it is the S^1 bundle restricted to a sphere containing both monopole points).

These involutions can be visualized as follows. In the case $n = 2$, it is well-known that $\mathbb{CP}^2 \# \mathbb{CP}^2$ can also be viewed as a boundary connect sum of 2 Eguchi–Hanson ALE space (glued along the boundary $\mathbb{RP}^3$-s). The involution Φ_1 reverses the two factors of the usual connect sum, and has an invariant S^3 (it flips Σ_1 and Σ_3), while the involution $\tilde{\Lambda}(0)$ interchanges the Eguchi–Hanson spaces, and has an invariant $\mathbb{RP}^3$ (it flips Σ_2 and Σ_4). For n even, then involution Φ_1 reflects the connect sum through the central neck of the connect sum, and has an invariant S^3. For n odd, then involution Φ_1 reflects the connect sum through a central $\mathbb{CP}^2$ summand, and has an invariant $\mathbb{RP}^3$. The action on cohomology follows easily from these descriptions. □

REMARK 3.14. In the case of a single monopole point, the LeBrun conformal class compactifies to the conformal class of the Fubini–Study metric on $\mathbb{CP}^2$, which is Einstein. By Obata's theorem, any conformal automorphism is an isometry, thus the conformal automorphism group for $n = 1$ is SU(3). For $n = 0$, the LeBrun conformal class compactifies to the conformal class of the round metric on S^4, thus the conformal group is $\mathrm{SO}_+(5, 1)$, the time-oriented Lorentz transformations. For $n \geq 1$, there are no orientation reversing diffeomorphisms, this follows from the Hirzebruch signature theorem since the signature is non-zero. However, S^4 does admit orientation-reversing

diffeomorphisms, which is reflected in the fact that $\mathrm{SO}_+(5,1)$ has 2 components.

4. LeBrun's twistor spaces

Let $\mathrm{Aut}(\mathcal{H}^3; p_1, \dots, p_n)$ be the group of isometries of $\mathcal{H}^3$ which preserve the set of monopole points $p_1, \dots, p_n$. In this section, we prove Theorem 1.1 in the case $n \geq 3$ by showing the following.

Proposition 4.1. *Let $[g_{\mathrm{LB}}]$ be a LeBrun self-dual conformal class on $n\,\#\,\mathbb{CP}^2$ with monopole points $p_1, \dots, p_n \in \mathcal{H}^3$. Suppose $n \geq 3$. Then there is a homomorphism*

$$\rho\colon \mathrm{Aut}(g_{\mathrm{LB}}) \to \mathrm{Aut}(\mathcal{H}^3; p_1, \dots, p_n) \tag{4.1}$$

such that $\rho(\Phi) = \phi$, where Φ is any lift of ϕ obtained in Proposition 2.11.

Together with Proposition 2.11, Proposition 4.1 means that there exists an exact sequence

$$1 \to \mathrm{U}(1) \to \mathrm{Aut}(g_{\mathrm{LB}}) \xrightarrow{\rho} \mathrm{Aut}(\mathcal{H}^3; p_1, \dots, p_n) \to 1. \tag{4.2}$$

Namely, for $n \geq 3$, the full conformal automorphism group of g_{LB} on $n\,\#\,\mathbb{CP}^2$ is an extension of the group of hyperbolic isometries preserving the set of monopole points, by U(1) (which comes from the bundle construction).

REMARK 4.2. In the previous sections, we used the upper half space model of hyperbolic space. However, in this and the following sections, $\mathcal{H}^3$ will no longer refer to any specific model of hyperbolic 3-space.

In the following we prove Proposition 4.1 by using twistor spaces; for background on twistor theory, see [1], [2]. Let Z be the twistor space of $[g_{\mathrm{LB}}]$ in Proposition 4.1, and $\mathrm{Aut}(Z)$ the group of holomorphic transformations of Z. By the twistor correspondence, there is a canonical injective homomorphism

$$\mathrm{Aut}(g_{\mathrm{LB}}) \to \mathrm{Aut}(Z) \tag{4.3}$$

(see, for example, [18, Proposition 2.1]). Using this, we regard $\mathrm{Aut}(g_{\mathrm{LB}})$ as a subgroup of $\mathrm{Aut}(Z)$. Let F be the canonical square root of $-K_Z$ (the anticanonical line bundle). Then the action of $\mathrm{Aut}(g_{\mathrm{LB}})$ on Z naturally lifts to the line bundle F. Hence we obtain a homomorphism

$$\mathrm{Aut}(g_{\mathrm{LB}}) \to \mathrm{GL}(H^0(Z, F)). \tag{4.4}$$

In general, this map will not be injective.

4.1. Proof of Proposition 4.1. For this, we first recall the following result on the structure of LeBrun twistor spaces.

Proposition 4.3. *If $n \geq 3$, $\dim H^0(Z, F) = 4$ holds. Further, if $\Psi\colon Z \to \mathbb{CP}^3$ denotes the rational map induced by the linear system $|F|$, we have the following.*
(i) *The base locus of $|F|$ consists of two smooth rational curves C_1 and $\bar{C}_1$, which are mapped to the boundary sphere $\partial\mathcal{H}^3 \subset n\,\#\,\mathbb{CP}^2$ by the twistor fibration $Z \to n\,\#\,\mathbb{CP}^2$.*
(ii) *The image $\Psi(Z)$ is a non-singular quadratic surface $\mathbb{CP}^1 \times \mathbb{CP}^1$.*
(iii) *If $Z' \to Z$ denotes the blow-up of Z at $C_1 \cup \bar{C}_1$, the composition $Z' \to Z \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is holomorphic. Further, the discriminant locus consists of n smooth rational curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$ of bidegree* (1,1)*, which canonically correspond to the monopole points $p_1, \ldots, p_n$.*

Proof. We first take any smooth member $S \in |F|$ and consider an exact sequence

$$0 \to \mathcal{O}_Z \to F \to K_S^{-1} \to 0 \tag{4.5}$$

and use $H^1(\mathcal{O}_Z) = 0$ to conclude that $\dim H^0(F) = 1 + \dim H^0(K_S^{-1})$ and $\mathrm{Bs}|F| = \mathrm{Bs}|K_S^{-1}|$. Since S is obtained from $\mathbb{CP}^1 \times \mathbb{CP}^1$ by blowing-up n points lying on a curve of bidegree $(1, 0)$ and also n points lying on another curve of the same bidegree, we readily obtain $\dim H^0(K_S^{-1}) = 3$. We also obtain that $\mathrm{Bs}|K_S^{-1}|$ is exactly the strict transform of the last two curves, for which we write C_1 and $\bar{C}_1$. (Note that to conclude these, we have used the assumption $n \geq 3$.) As $\mathbb{C}^*$ acts on S fixing any points on $C_1 \cup \bar{C}_1$ and the twistor fibration $Z \to n\,\#\,\mathbb{CP}^2$ is U(1)-equivariant, the image of C_1 under the twistor fibration must be the unique 2-sphere fixed by the U(1)-action on $n\,\#\,\mathbb{CP}^2$. Thus we obtain (i). For (ii), there are two distinguished pencils of degree-one divisors, which form a conjugate pair. These two pencils generate a 3-dimensional system in $|F|$. As $\dim|F| = 3$, this means $|F|$ is in fact generated by the two pencils. This implies that $\Psi(Z)$ is a smooth quadric. For the first part of (iii), it suffices to notice that the union of the base locus of the above 2 pencils (of degree-one divisors) are exactly $C_1 \cup \bar{C}_1$, and they are eliminated after blowing-up $C_1 \cup \bar{C}_1$. See [13, §7], [16, §3] and [10, §3] for details. □

REMARK 4.4. The proposition is true for arbitrary $n \geq 0$ if we consider $H^0(Z, F)^{\mathrm{U}(1)}$ (the subspace consisting of all U(1)-invariant sections) instead of $H^0(Z, F)$, where U(1) is the subgroup of fiber rotations of $\mathrm{Aut}(g_{\mathrm{LB}})$ coming from the bundle construction.

Lemma 4.5. *Let $\Psi\colon Z \to \mathbb{CP}^3$ and $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be as in* Proposition 4.3*. Then the following are all degree-one divisors on Z:*
(i) *the inverse images of curves on $\mathbb{CP}^1 \times \mathbb{CP}^1$ whose bidegree are* (1, 0) *or* (0, 1)*,*
(ii) *the inverse images $\Psi^{-1}(\mathcal{C}_j)$, $1 \leq j \leq n$.*

Proof. If D is a degree-one divisor, then $D+\bar{D} \in |F|$ holds by a Chern-class consideration (see [16]). Hence, since the rational map Ψ is associated to $|F|$, any degree-one divisor is an irreducible component of a reducible divisor of the form $\Psi^{-1}(H)$, where H is a hyperplane in $\mathbb{CP}^3$. If the divisor $\Psi^{-1}(H)$ is reducible, then one of the following must clearly hold: $H \cap (\mathbb{CP}^1 \times \mathbb{CP}^1)$ is reducible, or $H \cap (\mathbb{CP}^1 \times \mathbb{CP}^1)$ is irreducible but $\Psi^{-1}(H)$ is reducible. The former and latter correspond to the cases (i) and (ii) in the lemma respectively. □

Lemma 4.6. *Suppose $n \geq 3$. Then we have the following.*
(i) *Any $\Phi \in \mathrm{Aut}(g_{\mathrm{LB}})$ leaves the boundary sphere $\partial\mathcal{H}^3$ (regarded as a subset of $n\,\#\,\mathbb{CP}^2$) invariant.*
(ii) *Any $\Phi \in \mathrm{Aut}(g_{\mathrm{LB}})$ leaves the set of isolated fixed points invariant.*

Proof. As before, we regard Φ as an automorphism of Z. For (i), by Proposition 4.3 (i) it suffices to show $\{\Phi(C_1), \Phi(\bar{C}_1)\} = \{C_1, \bar{C}_1\}$. But since $C_1 \cup \bar{C}_1$ are the base locus of the system $|F|$ as in Proposition 4.3 (i), this is automatic. For (ii), let $L_1, \ldots, L_n$ be the twistor lines over the isolated fixed points of the U(1)-action. Then we have $\Psi(L_j) = \mathcal{C}_j$ and $\Psi^{-1}(\mathcal{C}_j) = D_j + \bar{D}_j$, where D_j and $\bar{D}_j$ are degree one divisors intersecting transversally along L_j ([16, Proposition 3.6], [13, §7]). Further, we have

$$\{\Phi(D_j), \Phi(\bar{D}_j) \mid 1 \leq j \leq n\} = \{D_j, \bar{D}_j \mid 1 \leq j \leq n\}, \tag{4.6}$$

since the $\mathcal{C}_j$-s are discriminant curves of the morphism $Z' \to \mathbb{CP}^1 \times \mathbb{CP}^1$ by Proposition 4.3. Since Φ commutes with the real structure, this means that $\{\Phi(L_j) \mid 1 \leq j \leq n\} = \{L_j \mid 1 \leq j \leq n\}$, which implies (ii) of the lemma. □

REMARK 4.7. The lemma says that if $n \geq 3$, any $\Phi \in \mathrm{Aut}(g_{\mathrm{LB}})$ preserves the open subset X_0 (on which U(1) acts freely). Obviously this does not hold if $n = 0$ or 1. Namely, the general automorphism of the standard metrics on S^4 or $\mathbb{CP}^2$ does not preserve the boundary sphere $\partial\mathcal{H}^3$. We will show in the next subsection that the lemma also fails to hold when $n = 2$.

By Proposition 4.3, when $n \geq 3$ we obtain a homomorphism

$$\mathrm{Aut}(g_{\mathrm{LB}}) \to \mathrm{Aut}(\mathbb{CP}^1 \times \mathbb{CP}^1). \tag{4.7}$$

Further, by LeBrun's construction [13], the image quadric $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be regarded as a quotient space of the twistor space by a $\mathbb{C}^*$-action, where the last action is the complexification of the semi-free U(1)-action on Z. More intrinsically, $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be interpreted as the minitwistor space (in the sense of Hitchin [6]) of the hyperbolic space $\mathcal{H}^3$. This in particular means that $\mathcal{H}^3$ can be canonically identified with the space of minitwistor lines in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Such lines are explicitly given as real irreducible

curves of bidegree (1, 1) which are disjoint from $(\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma$ (the real locus on $\mathbb{CP}^1 \times \mathbb{CP}^1$). Furthermore, as a consequence of

$$\Psi^{-1}((\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma) = \bigcup_{x \in \partial \mathcal{H}^3} L_x, \tag{4.8}$$

where L_x denotes the twistor line over a point $x \in n\#\mathbb{CP}^2$, there is a natural identification $(\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma \simeq \partial\mathcal{H}^3$. By Lemma 4.6, we have $\Phi(\partial\mathcal{H}^3) = \partial\mathcal{H}^3$ (on $n\#\mathbb{CP}^2$). From this it follows that the automorphism of $\mathbb{CP}^1 \times \mathbb{CP}^1$ coming from any $\Phi \in \mathrm{Aut}(g_{\mathrm{LB}})$ (via (4.7)) maps real (1, 1)-curves disjoint from $(\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma$ to real (1, 1)-curves disjoint from $(\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma$. Hence it maps minitwistor lines to minitwistor lines. This way, we obtain a homomorphism

$$\rho\colon \mathrm{Aut}(g_{\mathrm{LB}}) \to \mathrm{Aut}(\mathcal{H}^3). \tag{4.9}$$

Moreover, since the action of $\mathrm{Aut}(g_{\mathrm{LB}})$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ preserves $\mathcal{C}_1, \ldots, \mathcal{C}_n$ (as they are discriminant curves), the image of (4.9) is contained in $\mathrm{Aut}(\mathcal{H}^3; p_1, \ldots, p_n)$.

To finish the proof of Proposition 4.1, it remains to show that if $\Phi \in \mathrm{Aut}(g_{\mathrm{LB}})$ is one of the lifts (obtained in Proposition 2.11) of some $\phi \in \mathrm{Aut}(\mathcal{H}^3; p_1, \ldots, p_n)$, then $\rho(\Phi) = \phi$ holds. Take any point $p \in \mathcal{H}^3 \setminus \{p_1, \ldots, p_n\}$, and put $q = \phi(p)$. Let $\tilde{p} \in X_0$ be any point belonging to the fiber over p and let $\tilde{q} = \Phi(\tilde{p})$. Let $L_{\tilde{p}}$ and $L_{\tilde{q}}$ be the twistor lines over $\tilde{p}$ and $\tilde{q}$, respectively. Letting Φ also denote the induced automorphism on $\mathbb{CP}^1 \times \mathbb{CP}^1$, we have $\Phi(\Psi(L_{\tilde{p}})) = \Psi(L_{\tilde{q}})$ by construction. By the result of Jones–Tod [9] on the relation between Penrose correspondence (between self-dual 4-manifolds and 3-dimensional twistor spaces) and Hitchin correspondence (between Einstein–Weyl 3-manifolds and minitwistor spaces), the points on $\mathcal{H}^3$ which correspond to the minitwistor lines $\Psi(L_{\tilde{p}})$ and $\Psi(L_{\tilde{q}})$ are exactly p and q respectively. This implies $(\rho(\Phi))(p) = q$, as required. This completes the proof of Proposition 4.1.

5. Poon's projective model

In this section, we determine the group of all conformal isometries of Poon's metrics on $2\#\mathbb{CP}^2$. Although Poon's metrics can be constructed by LeBrun's hyperbolic ansatz, it turns out that, in contrast to the case $n \geq 3$, not all conformal isometries come from isometries of $\mathcal{H}^3$. More precisely, we show that such lifts form a subgroup of index 2 in the full conformal isometry group.

5.1. Automorphism group of Poon's projective models. In order to analyze the automorphism group in the case of $2\#\mathbb{CP}^2$, instead of LeBrun's projective model, it is more convenient to use Poon's projective model of the twistor spaces (these are of course equivalent, see [13, Section 7]). In this subsection we investigate the holomorphic automorphism group of the projective models. We begin with recalling the following result due to Poon [15].

Proposition 5.1 ([15]). *Let g be a self-dual metric on $2\#\mathbb{CP}^2$ of positive scalar curvature and Z the twistor space of g. Then*
(i) *the linear system $|F|$ is base point free, 5-dimensional, and its associated morphism $\Psi\colon Z \to \mathbb{CP}^5$ is bimeromorphic to its image.*
(ii) *The image $\tilde{Z} := \Psi(Z)$ is an intersection of the two hyperquadrics in $\mathbb{CP}^5$ defined by*
(5.1)

$$Q_\infty = \{w_0w_1 + z_2^2 + z_3^2 + w_4w_5 = 0\}, \quad Q_0 = \left\{2w_0w_1 + \lambda z_2^2 + \frac{3}{2}z_3^2 + w_4w_5 = 0\right\}$$

where $(w_0, w_1, z_2, z_3, w_4, w_5)$ is a homogeneous coordinate on $\mathbb{CP}^5$ and λ is a real number satisfying $3/2 < \lambda < 2$.
(iii) *The singular locus of $\tilde{Z}$ consists of 4 points*

$$\begin{aligned} &P_1 := (1, 0, 0, 0, 0, 0), \quad \bar{P}_1 := (0, 1, 0, 0, 0, 0), \\ &P_3 := (0, 0, 0, 0, 1, 0), \quad \bar{P}_3 := (0, 0, 0, 0, 0, 1), \end{aligned} \tag{5.2}$$

and all these are ordinary nodes.
(iv) *The morphism $\Psi\colon Z \to \tilde{Z}$ is a small resolution of these 4 nodes.*
(v) *The real structure on $\mathbb{CP}^5$ induced by that on Z is given by*

$$(w_0, w_1, z_2, z_3, w_4, w_5) \mapsto (\bar{w}_1, \bar{w}_0, \bar{z}_2, -\bar{z}_3, -\bar{w}_5, -\bar{w}_4). \tag{5.3}$$

The identity component of the conformal transformation group of Poon's conformal class is $\mathrm{U}(1)\times\mathrm{U}(1)$. Correspondingly, the identity component of holomorphic transformation group of Poon's twistor space is $\mathbb{C}^* \times \mathbb{C}^*$. In the above coordinates, this action is explicitly given by

$$(w_0, w_1, z_2, z_3, w_4, w_5) \mapsto (sw_0, s^{-1}w_1, z_2, z_3, tw_4, t^{-1}w_5), \quad (s, t) \in \mathbb{C}^* \times \mathbb{C}^*, \tag{5.4}$$

which preserves the quadrics Q_∞ and Q_0. The map (5.4) commutes with the real structure (5.3) if and only if $|s| = |t| = 1$.

In the following we put $K = \mathrm{U}(1) \times \mathrm{U}(1)$, and $G = \mathbb{C}^* \times \mathbb{C}^*$ for simplicity. The K-action on $2\#\mathbb{CP}^2$ has exactly 4 fixed points. Correspondingly, there are four G-invariant twistor lines in Z.

DEFINITION 5.2. Define the two real numbers $\alpha := \sqrt{4 - 2\lambda}$ and $\beta := \sqrt{2\lambda - 2}$.

We remark that since $3/2 < \lambda < 2$, we have the inequalities $0 < \alpha < \beta$. These numbers will play an important role in the following.

Lemma 5.3. (i) *Any $\Phi \in \mathrm{Aut}(g)$ leaves the set of four K-fixed points (on $2\#\mathbb{CP}^2$) invariant.*

(ii) *The image under* Ψ *of the twistor lines over these* 4 *points are conics whose equations are respectively given by*

$$\{\alpha z_2 - i z_3 = w_4 = w_5 = w_0 w_1 + (2\lambda - 3) z_2^2 = 0\}, \tag{5.5}$$

$$\{\alpha z_2 + i z_3 = w_4 = w_5 = w_0 w_1 + (2\lambda - 3) z_2^2 = 0\}, \tag{5.6}$$

$$\{\beta z_2 - i z_3 = w_0 = w_1 = (3 - 2\lambda) z_2^2 + w_4 w_5 = 0\}, \tag{5.7}$$

$$\{\beta z_2 + i z_3 = w_0 = w_1 = (3 - 2\lambda) z_2^2 + w_4 w_5 = 0\}. \tag{5.8}$$

Proof. For (i), consider the two linear projections $f_j \colon \mathbb{CP}^5 \to \mathbb{CP}^3$ $(j = 1, 3)$ defined by

$$f_1(w_0, w_1, z_2, z_3, w_4, w_5) = (z_2, z_3, w_4, w_5), \tag{5.9}$$

$$f_3(w_0, w_1, z_2, z_3, w_4, w_5) = (w_0, w_1, z_2, z_3). \tag{5.10}$$

By an elementary computation, we have

$$f_1(\tilde{Z}) = \{\alpha^2 z_2^2 + z_3^2 + 2 w_4 w_5 = 0\}, \quad f_3(\tilde{Z}) = \{2 w_0 w_1 + \beta^2 z_2^2 + z_3^2 = 0\}. \tag{5.11}$$

Intrinsically, the composition $f_j \circ \Psi \colon Z \to \mathbb{CP}^3$ is the meromorphic map associated to the linear system corresponding to the subspace $H^0(Z, F)^{G_j}$, where G_1 and G_3 are $\mathbb{C}^*$-subgroups of G defined by

$$G_1 = \{\mathrm{diag}(s, s^{-1}, 1, 1, 1, 1) \in \mathrm{PGL}(6, \mathbb{C}) \mid s \in \mathbb{C}^*\} \tag{5.12}$$

and

$$G_3 = \{\mathrm{diag}(1, 1, 1, 1, t, t^{-1}) \in \mathrm{PGL}(6, \mathbb{C}) \mid t \in \mathbb{C}^*\}. \tag{5.13}$$

Since $\alpha\beta \neq 0$ by Poon's constraint $(3/2) < \lambda < 2$, (5.11) means that the images $f_1(\tilde{Z})$ and $f_3(\tilde{Z})$ are non-singular quadrics. Hence both are isomorphic to a product $\mathbb{CP}^1 \times \mathbb{CP}^1$. (Both of these two rational maps from Z to $\mathbb{CP}^1 \times \mathbb{CP}^1$ exactly correspond to the map $\Psi \colon Z \to \mathbb{CP}^1 \times \mathbb{CP}^1$ for LeBrun twistor spaces considered above for $n \geq 3$). Then by taking the pull-back of pencils on $\mathbb{CP}^1 \times \mathbb{CP}^1$ of bidegree (1, 0) and (0, 1), we obtain 2 pencils on Z for each of $j = 1$ and $j = 3$. Hence we obtain 4 pencils on Z in total. Since $(f_j \circ \Psi)^* \mathcal{O}(1) \simeq F$ and hyperplane sections of the quadrics are bidegree $(1, 1)$, members of the 4 pencils are degree one, since the intersection number of the divisor with twistor lines is one. On the other hand, by [16, Lemma 1.9], for $2 \# \mathbb{CP}^2$ there are at most 4 degree one line bundles on Z which have a non-trivial section. Further, since $\dim |D| \leq 1$ for any degree 1 divisor D on any twistor space on $n \# \mathbb{CP}^2$ by [16, Lemma 1.10 (2)], these 4 pencils have mutually different Chern classes. This implies that there are no pencils of degree one other than the above 4 ones. Obviously, the G-action preserves each of these pencils. Furthermore, it can be

readily seen by (5.4), (5.9), (5.10), and (5.11) that G acts non-trivially on the parameter space ($\mathbb{CP}^1$) of the pencils. Hence each pencil has precisely two G-invariant members, so that we have eight G-invariant degree one divisors in total. By (5.3), it is clear that the two G-invariant divisors in the same pencil form a conjugate pair. So we may write $\{D_j, \bar{D}_j \mid 1 \leq j \leq 4\}$ for the set of G-invariant degree one divisors.

We next compute the defining equations of the images of these 8 divisors in $\mathbb{CP}^5$ (under Ψ) in the following way. First, by using (5.11), we can obtain equations of the four G-invariant curves of bidegrees (1, 0) or (0, 1). (For instance, one of them is given by $\alpha z_2 - iz_3 = w_4 = 0$.) Next, substituting the equations into (any one of) (5.1), we obtain the equations of the images $\Psi(D_j)$ and $\Psi(\bar{D}_j)$. (For the above curve, the equations become $\alpha z_2 - iz_3 = w_4 = w_0 w_1 + (2\lambda - 3)z_2^2 = 0$.) The last equations imply that $\Phi(D_j)$ is a quadratic cone in $\mathbb{CP}^3$ and that its vertex is exactly one of the four singular points of $\tilde{Z} = \Psi(Z)$. (For the above case $\bar{P}_3$ is contained as the vertex.) Recall that Ψ is the morphism which contracts the four rational curves, and that the images of the curves are exactly the singular points of $\tilde{Z}$. On the other hand, by ([16, Lemma 1.9]), any degree-one divisor is non-singular. Therefore the morphisms $D_j \to \Psi(D_j)$ and $\bar{D}_j \to \Psi(\bar{D}_j)$ factor through the minimal resolution of the quadratic cones. Then again by ([16, Lemma 1.10]), D_j and $\bar{D}_j$ are obtained from $\Sigma_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ (the minimal resolution of the cone) by blowing-up one point.

In a similar fashion, we can compute the defining equations of $\Psi(D)$ for a non-G-invariant degree-one divisor D. (For instance, one of them is given by

$$w_4 - c(\alpha z_2 - iz_3) = \alpha z_2 + iz_3 + 2cw_5 = 2w_0 w_1 + (2\lambda - 2)z_2^2 + z_3^2 = 0, \tag{5.14}$$

where $c \in \mathbb{C}^*$.) From these (and also by the constraint $3/2 < \lambda < 2$), we obtain that $\Psi(D)$ is biholomorphic to a non-singular quadric in $\mathbb{CP}^3$; namely $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then again by [16, Lemmas 1.9 and 1.10] we obtain that the divisor D is obtained from $\mathbb{CP}^1 \times \mathbb{CP}^1$ by blowing-up one point. Since the one point blow-up of Σ_2 and that of $\mathbb{CP}^1 \times \mathbb{CP}^1$ cannot be biholomorphic, we conclude that the G-invariant divisors $D_j, \bar{D}_j$ and non-G-invariant divisors D cannot be biholomorphic.

For a given $\Phi \in \mathrm{Aut}(g)$, if we use the same letter to denote the induced automorphism of Z, Φ clearly preserves the set of 4 pencils (as any $\Phi \in \mathrm{Aut}(Z)$ preserves the degree of divisors). Further, by the above distinction of complex structure between G-invariant and non-G-invariant members, the set of G-invariant members (which are explicitly given by $\{D_j, \bar{D}_j \mid 1 \leq j \leq 4\}$) are preserved under Φ. As Φ preserves the real structure, this means that Φ preserves the set $\{D_j \cap \bar{D}_j \mid 1 \leq j \leq 4\}$. Since these are exactly the set of G-invariant twistor lines, this implies the claim (i) of the lemma.

For (ii) we notice that each $D_j + \bar{D}_j$ is contracted to a reducible curve of bidegree (1, 1) under precisely one of the two rational maps $f_1 \circ \Psi$ and $f_3 \circ \Psi$. Therefore each twistor line $L_j = D_j \cap \bar{D}_j$ is mapped to a real G-fixed point on (one of) the image quadrics. On the quadric $f_1(\tilde{Z}) \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, there are exactly two real G-fixed points,

and they are explicitly given by

$$\{\alpha z_2 - iz_3 = w_4 = w_5 = 0\}, \quad \text{and} \quad \{\alpha z_2 + iz_3 = w_4 = w_5 = 0\}.$$

Similarly, on the quadric $f_3(\tilde{Z})$, real G-fixed points are explicitly given by

$$\{\beta z_2 - iz_3 = w_0 = w_1 = 0\}, \quad \text{and} \quad \{\beta z_2 + iz_3 = w_0 = w_1 = 0\}.$$

Computing the inverse images of these 4 points under f_1 and f_3 (namely substituting these into the equations (5.1)), we obtain the desired equations for the images of G-invariant twistor lines. □

The homomorphism (4.4) and the coordinates $(w_0, w_1, z_2, z_3, w_4, w_5)$ give a homomorphism

$$\text{Aut}(g) \to \text{GL}(6, \mathbb{C}). \tag{5.15}$$

We shall obtain the image of (5.15) explicitly. Take any $\Phi \in \text{Aut}(g)$ and let $U \in \text{GL}(6, \mathbb{C})$ be its image. Then as in the case of $n \geq 3$, U preserves the variety $\tilde{Z}$. Hence U preserves the singular set $\{P_1, P_3, \bar{P}_1, \bar{P}_3\}$. Taking the real structure into account, the following two possibilities can occur:
(I) $\{U(P_1), U(\bar{P}_1)\} = \{P_1, \bar{P}_1\}$ and $\{U(P_3), U(\bar{P}_3)\} = \{P_3, \bar{P}_3\}$,
(II) $\{U(P_1), U(\bar{P}_1)\} = \{P_3, \bar{P}_3\}$ and $\{U(P_3), U(\bar{P}_3)\} = \{P_1, \bar{P}_1\}$.
For case (I), using the fact that U commutes with the real structure (5.3), it is easy to deduce that U is of the form

$$\begin{pmatrix} A_{11} & A_{12} & O \\ O & A_{22} & O \\ O & A_{32} & A_{33} \end{pmatrix}, \tag{5.16}$$

where A_{12}, A_{22} and A_{32} are 2×2 matrices with $\det A_{22} \neq 0$ and

$$A_{11} = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix}, \quad A_{33} = \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix}, \tag{5.17}$$

where $a, b \in \mathbb{C}^*$. Similarly, for case (II), U is of the form

$$\begin{pmatrix} O & A_{12} & A_{13} \\ O & A_{22} & O \\ A_{31} & A_{32} & O \end{pmatrix}, \tag{5.18}$$

where A_{12}, A_{22} and A_{32} are 2×2 matrices with $\det A_{22} \neq 0$ and

$$A_{13} = \begin{pmatrix} a & 0 \\ 0 & -\bar{a} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix}, \quad A_{31} = \begin{pmatrix} b & 0 \\ 0 & -\bar{b} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \tag{5.19}$$

where $a, b \in \mathbb{C}^*$.

Using Lemma 5.3, we can deduce another restriction for the 6×6 matrix U as follows.

Lemma 5.4. (i) *In the presentations* (5.16) *and* (5.18), $A_{12} = A_{32} = O$ *holds.*
(ii) *If U belongs to the case* (I), *the matrix A_{22} must be of the form*

$$A_{22} = c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad or \quad c\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad c \in \mathbb{R}^*. \tag{5.20}$$

(iii) *If U belongs to the case* (II), *we have*

$$A_{22} = c\begin{pmatrix} 0 & 1 \\ \alpha\beta & 0 \end{pmatrix} \quad or \quad c\begin{pmatrix} 0 & -1 \\ \alpha\beta & 0 \end{pmatrix}, \quad c \in i\mathbb{R}^*. \tag{5.21}$$

Proof. First, we note that by Lemma 5.3, U has to leave the set of 4 conics (5.5)–(5.8) invariant. In the case (I), namely if $\{U(P_1), U(\bar{P}_1)\} = \{P_1, \bar{P}_1\}$, the set of the two conics {(5.5), (5.6)} must be preserved under U, since (5.5) and (5.6) contain P_1 and $\bar{P}_1$, and (5.7) and (5.8) do not. Similarly, the set {(5.7), (5.8)} must also be preserved under U.

A generic point on the conics (5.5) and (5.6) is of the form $(w_0, w_1, 1, \mp i\alpha, 0, 0)$. Since

$$\begin{pmatrix} A_{11} & A_{12} & O \\ O & A_{22} & O \\ O & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ 1 \\ \mp i\alpha \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ * \\ A_{22}\begin{pmatrix} 1 \\ \mp i\alpha \end{pmatrix} \\ A_{32}\begin{pmatrix} 1 \\ \mp i\alpha \end{pmatrix} \end{pmatrix}, \tag{5.22}$$

and these points still belong to (5.5) or (5.6), we obtain

$$A_{32}\begin{pmatrix} 1 \\ -i\alpha \end{pmatrix} = A_{32}\begin{pmatrix} 1 \\ i\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\alpha = \sqrt{4 - 2\lambda} \neq 0$, we obtain $A_{32} = 0$. Similarly, considering the analogous requirement for (5.7) and (5.8), we obtain $A_{12} = 0$.

Thus we have obtained the claim (i) for the case (I). For the case (II), namely if $\{U(P_1), U(\bar{P}_1)\} = \{P_3, \bar{P}_3\}$, the sets of the two conics {(5.5), (5.6)} and {(5.7), (5.8)} must be interchanged under U. From this we can again deduce $A_{12} = A_{32} = O$ by similar computations. Hence we obtain (i).

Next we show (ii). Suppose U belongs to the case (I). Then since the right hand side of (5.22) belongs to the conics (5.5) or (5.6), as points on $\mathbb{CP}^1$ we have either

$$A_{22}\begin{pmatrix} 1 \\ i\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ -i\alpha \end{pmatrix} \quad \text{and} \quad A_{22}\begin{pmatrix} 1 \\ -i\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ i\alpha \end{pmatrix} \tag{5.23}$$

or

$$A_{22}\begin{pmatrix} 1 \\ i\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ i\alpha \end{pmatrix} \quad \text{and} \quad A_{22}\begin{pmatrix} 1 \\ -i\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ -i\alpha \end{pmatrix}, \tag{5.24}$$

according to whether U interchanges (5.5) and (5.6) or not. Similarly, by using the computations to deduce $A_{12} = 0$, either

$$A_{22}\begin{pmatrix} 1 \\ i\beta \end{pmatrix} = \begin{pmatrix} 1 \\ -i\beta \end{pmatrix} \quad \text{and} \quad A_{22}\begin{pmatrix} 1 \\ -i\beta \end{pmatrix} = \begin{pmatrix} 1 \\ i\beta \end{pmatrix} \tag{5.25}$$

or

$$A_{22}\begin{pmatrix} 1 \\ i\beta \end{pmatrix} = \begin{pmatrix} 1 \\ i\beta \end{pmatrix} \quad \text{and} \quad A_{22}\begin{pmatrix} 1 \\ -i\beta \end{pmatrix} = \begin{pmatrix} 1 \\ -i\beta \end{pmatrix}, \tag{5.26}$$

according to whether U interchanges (5.7) and (5.8) or not. We note that as points on $\mathbb{CP}^1$

$$(1, i\alpha), (1, -i\alpha), (1, i\beta), (1, -i\beta) \tag{5.27}$$

are four distinct points. Thus (5.23)–(5.26) mean that in any case the projective transformation determined by the matrix A_{22} leaves the set of the 4 points (5.27) invariant. If (5.24) and (5.26) happen, then A_{22} fixes all 4 points. This means $A_{22} = cI_2$ for some $c \in \mathbb{C}^*$. If (5.23) and (5.25) happen, then A_{22} interchanges $(1, i\alpha)$ and $(1, -i\alpha)$ and also $(1, i\beta)$ and $(1, -i\beta)$. This means $A_{22} = c\,\mathrm{diag}(1, -1)$. A simple computation also shows that there exists no projective transformation realizing the remaining two cases. Moreover, since U commutes with the real structure (5.3), we readily obtain $c \in \mathbb{R}$. Thus we obtain the claim (ii) in case (I).

If U belongs to the case (II), by similar computation as above, we deduce that, as a projective transformation, A_{22} maps $(1, i\alpha)$ to either $(1, i\beta)$ or $(1, -i\beta)$ (so that $(1, -i\alpha)$ is mapped to $(1, -i\beta)$ or $(1, i\beta)$ respectively). Further, A_{22} maps $(1, i\beta)$ to either $(1, i\alpha)$ or $(1, -i\alpha)$ (so that $(1, -i\beta)$ is mapped to $(1, -i\alpha)$ or $(1, i\alpha)$ respectively). Among these $2 \cdot 2 = 4$ possibilities, only the two cases

$$A_{22}\colon (1, i\alpha) \mapsto (1, i\beta) \quad \text{and} \quad (1, i\beta) \mapsto (1, i\alpha),$$

and

$$A_{22}\colon (1, i\alpha) \mapsto (1, -i\beta) \quad \text{and} \quad (1, i\beta) \mapsto (1, -i\alpha)$$

can actually occur, and for each these cases A_{22} is represented by the matrices

$$A_{22} = c\begin{pmatrix} 0 & 1 \\ \alpha\beta & 0 \end{pmatrix} \quad \text{and} \quad c\begin{pmatrix} 0 & -1 \\ \alpha\beta & 0 \end{pmatrix}, \tag{5.28}$$

respectively for some $c \in \mathbb{C}^*$. Finally, again by commutativity with (5.3) we obtain $c \in i\mathbb{R}$. This completes the proof of claim (iii). □

The next lemma determines all automorphisms of the projective model $\tilde{Z}$ which commute with the real structure.

Lemma 5.5. (i) *Let U be a 6×6 matrix in* Case (I) *of the form* (5.16), *where A_{11} and A_{33} are as in* (5.17) *and $A_{12} = A_{32} = O$ and A_{22} is as in* (5.20) *by* Lemma 5.4. *Then after normalizing by a scalar multiplication to make $c = 1$, U preserves the projective model $\tilde{Z}$ if and only if the entries in* (5.17) *satisfy $|a| = |b| = 1$.*
(ii) *Let U be the 6×6 matrix in* Case (II) *of the form* (5.18), *where A_{13} and A_{31} are as in* (5.19) *and $A_{12} = A_{32} = O$ and A_{22} is as in* (5.21) *by* Lemma 5.4. *Then after normalizing by a scalar multiplication to make $c = 1$, U preserves the projective model $\tilde{Z}$ if and only if the entries in* (5.19) *satisfy*

$$|a| = \beta,\ |b| = \alpha \quad (\text{and } c = i). \tag{5.29}$$

Proof. We only show (ii) since (i) can be proved by a similar (and simpler) computation. We recall that $\tilde{Z}$ is defined by the following 2 quadratic polynomials:

$$h_0 = 2w_0w_1 + \lambda z_2^2 + \frac{3}{2}z_3^2 + w_4w_5, \tag{5.30}$$

$$h_\infty = w_0w_1 + z_2^2 + z_3^2 + w_4w_5. \tag{5.31}$$

We also recall $\alpha^2 = 4-2\lambda$, $\beta^2 = 2\lambda - 2$. Let the constants (a,b,c) be arbitrary satisfying $c \in i\mathbb{R}$. Then by substitution, we obtain

$$Uh_0 = -2|a|^2w_4w_5 + c^2\lambda z_3^2 + \frac{3}{2}c^2\alpha^2\beta^2z_2^2 - |b|^2w_0w_1, \tag{5.32}$$

$$Uh_\infty = -|a|^2w_4w_5 + c^2z_3^2 + c^2\alpha^2\beta^2z_2^2 - |b|^2w_0w_1. \tag{5.33}$$

By multiplying a real constant to U, we may suppose $|b| = 1$. So we have constants (a, b, c) with $|b| = 1$ determining U in Case (II). This gives,

$$Uh_0 = -2|a|^2w_4w_5 + c^2\lambda z_3^2 + \frac{3}{2}c^2\alpha^2\beta^2z_2^2 - w_0w_1, \tag{5.34}$$

$$Uh_\infty = -|a|^2w_4w_5 + c^2z_3^2 + c^2\alpha^2\beta^2z_2^2 - w_0w_1. \tag{5.35}$$

If U preserves $\tilde{Z}$, then preserves the quadratic ideal (h_0, h_∞), so there exist constants c_1 and c_2 so that

$$c_1 U h_0 - c_2 U h_\infty = h_0 - h_\infty = w_0 w_1 + (\lambda - 1) z_2^2 + \frac{1}{2} z_3^2. \tag{5.36}$$

A computation gives

$$\begin{aligned} c_1 U h_0 - c_2 U h_\infty = & -(c_1 - c_2) w_0 w_1 + |a|^2 (-2c_1 + c_2) w_4 w_5 \\ & + c^2 (c_1 \lambda - c_2) z_3^2 + \left(\frac{3}{2} c_1 - c_2 \right) c^2 \alpha^2 \beta^2 z_2^2. \end{aligned} \tag{5.37}$$

Comparing with (5.36), we see that $c_2 - c_1 = 1$ and $|a|^2(-2c_1 + c_2) = 0$. Since $|a| \neq 0$, we obtain

$$c_1 = 1, \quad c_2 = 2. \tag{5.38}$$

Then we have

$$U h_0 - 2 U h_\infty = w_0 w_1 + c^2 (\lambda - 2) z_3^2 - \frac{1}{2} c^2 \alpha^2 \beta^2 z_2^2. \tag{5.39}$$

Comparing coefficients with (5.36), we have

$$c^2 (\lambda - 2) = \frac{1}{2} \tag{5.40}$$

$$-\frac{1}{2} c^2 \alpha^2 \beta^2 = (\lambda - 1). \tag{5.41}$$

By (5.40) we obtain $c^2 = -\alpha^{-2}$. Further, if this is satisfied, (5.41) automatically holds. So we find that $h_0 - h_\infty \in (U h_0, U h_\infty)$ holds if and only if after a rescaling the entries of U satisfy $c = i$ and $|b| = \alpha$.

Next, by rescaling, we assume $|a| = 1$. We compute that

$$U h_0 = -2 w_4 w_5 + c^2 \lambda z_3^2 + \frac{3}{2} c^2 \alpha^2 \beta^2 z_2^2 - |b|^2 w_0 w_1 \tag{5.42}$$

$$U h_\infty = -w_4 w_5 + c^2 z_3^2 + c^2 \alpha^2 \beta^2 z_2^2 - |b|^2 w_0 w_1. \tag{5.43}$$

Consider the element

$$h_0 - 2 h_\infty = (\lambda - 2) z_2^2 - \frac{1}{2} z_3^2 - w_4 w_5. \tag{5.44}$$

We next find c_1 and c_2 so that

$$c_1 U h_0 - c_2 U h_\infty = (\lambda - 2) z_2^2 - \frac{1}{2} z_3^2 - w_4 w_5. \tag{5.45}$$

We compute

$$
\begin{aligned}
c_1 U h_0 - c_2 U h_\infty = {} & (-2c_1 + c_2) w_4 w_5 + c^2 (c_1 \lambda - c_2) z_3^2 \\
& + \left(\frac{3}{2} c_1 - c_2 \right) c^2 \alpha^2 \beta^2 z_2^2 - |b|^2 (c_1 - c_2) w_0 w_1 .
\end{aligned} \tag{5.46}
$$

We find that

$$
|b|^2 (c_1 - c_2) = 0, \quad -2c_1 + c_2 = -1, \tag{5.47}
$$

which implies that $c_1 = c_2 = 1$. We then have

$$
\frac{1}{2} c^2 \alpha^2 \beta^2 = \lambda - 2, \quad c^2 (\lambda - 1) = -\frac{1}{2}. \tag{5.48}
$$

The latter equation implies $c^2 = -\beta^{-2}$, which implies the former equation. So we find that $h_0 - 2h_\infty \in (Uh_0, Uh_\infty)$ holds if and only if after a rescaling the entries of U satisfy $c = 1$ and $|a| = \beta$.

On the other hand, as $(h_0, h_\infty) = (h_0 - h_\infty, h_0 - 2h_\infty)$, and $(h_0, h_\infty) = (Uh_0, Uh_\infty)$ holds if and only if $h_0 \in (Uh_0, Uh_\infty)$ and $h_\infty \in (Uh_0, Uh_\infty)$. Hence by a combination of the above two, we conclude that U preserves $\tilde{Z}$ if and only if U can be rescaled to satisfy $c = 1$, $|a| = \beta$ and $|b| = \alpha$. □

According to Lemma 5.5, in Case (I), each of A_{11}, A_{22} and A_{33} has 2 types of choices, with $|a| = |b| = 1$. Hence the automorphisms in (i) of Lemma 5.5 constitute $2^3 = 8$ tori. Similarly by Lemma 5.5 (ii), the same is true for Case (II), so that we again obtain 8 tori. Thus we obtain 16 tori in the holomorphic automorphism group of $\tilde{Z}$. All automorphisms in these 16 tori commute with the real structure.

5.2. Determination of small resolutions. As in Proposition 5.1, the projective model $\tilde{Z}$ of Poon's twistor spaces on $2\#\mathbb{CP}^2$ has precisely 4 ordinary nodes $P_1, \bar{P}_1, P_3$ and $\bar{P}_3$. The actual twistor space Z is obtained from $\tilde{Z}$ by taking small resolutions for each node. Of course, there are exactly 2 ways of small resolutions for each node. (We refer the reader to [11, Section 12] for a discussion of the small resolutions of ordinary nodes of threefolds.) Since the resolution must preserve the real structure, the small resolutions of P_1 and P_3 uniquely determine those of $\bar{P}_1$ and $\bar{P}_3$ respectively, so there are exactly 4 ways to obtain small resolutions of the variety $\tilde{Z}$ which preserve the real structure. In this subsection we explicitly determine which small resolutions yield the twistor space. This gives a completely explicit construction of the twistor spaces of Poon's metrics on $2\#\mathbb{CP}^2$, starting from his projective models in $\mathbb{CP}^5$.

For the purpose of specifying the small resolutions of ordinary nodes of $\tilde{Z}$, we first investigate local structure of $\tilde{Z}$ in neighborhoods of the singularities. First we take

$P_1 = (1, 0, 0, 0, 0, 0)$ and $\bar{P}_1 = (0, 1, 0, 0, 0, 0)$. If we define two hyperplanes in $\mathbb{CP}^5$ by $H_\alpha = \{\alpha z_2 - iz_3 = 0\}$ and $H_{-\alpha} = \{\alpha z_2 + iz_3 = 0\}$, then by (5.5) and (5.6) the two irreducible components of the two reducible hyperplane sections $\tilde{Z} \cap H_\alpha$ and $\tilde{Z} \cap H_{-\alpha}$ contain P_1 and $\bar{P}_1$ as smooth points. Namely, the 4 surfaces

$$D'_1 := \{\alpha z_2 - iz_3 = w_4 = w_0 w_1 + (2\lambda - 3)z_2^2 = 0\}, \tag{5.49}$$

$$\bar{D}'_1 := \{\alpha z_2 - iz_3 = w_5 = w_0 w_1 + (2\lambda - 3)z_2^2 = 0\}, \tag{5.50}$$

$$D'_2 := \{\alpha z_2 + iz_3 = w_4 = w_0 w_1 + (2\lambda - 3)z_2^2 = 0\}, \tag{5.51}$$

$$\bar{D}'_2 := \{\alpha z_2 + iz_3 = w_5 = w_0 w_1 + (2\lambda - 3)z_2^2 = 0\}, \tag{5.52}$$

all of which are cones over a smooth conic, share P_1 and $\bar{P}_1$ as smooth points. Note that $\bar{D}'_1 = \sigma(D'_1)$ and $\bar{D}'_2 = \sigma(D'_2)$ hold, and that all of these 4 surfaces are G-invariant. The configuration of these 4 surfaces and the ordinary nodes is illustrated in the diagram on the left in Fig. 1. In a neighborhood of P_1, by setting $w_0 = 1$ in the defining equations in (5.1) and eliminating w_1, we can think of $\tilde{Z}$ as defined in $\mathbb{C}^4 = \{(z_2, z_3, w_4, w_5)\}$ by the equation

$$\alpha^2 z_2^2 + z_3^2 + 2w_4 w_5 = 0, \tag{5.53}$$

from which one can see that P_1 is an ordinary node of $\tilde{Z}$. Similarly, by neglecting the last common hyperquadric in (5.49)–(5.52), these 4 surfaces can be considered to be defined in the same $\mathbb{C}^4$ (at least in a neighborhood of P_1).

By the equations (5.53) and (5.49)–(5.52) (with the last common quadratic equation neglected), a small resolution of $\tilde{Z}$ at P_1 is clearly specified by which pair among $\{D'_1, \bar{D}'_2\}$ or $\{\bar{D}'_1, D'_2\}$ is blown-up at P_1. By exchanging the role of w_0 and w_1 in the above argument, we see that a small resolution at the conjugate point $\bar{P}_1$ can also be specified by which pair of $\{D'_1, \bar{D}'_2\}$ or $\{\bar{D}'_1, D'_2\}$ is blown-up at $\bar{P}_1$.

Similarly, by (5.7) and (5.8), the other two reducible hyperplane sections $\tilde{Z} \cap H_\beta$ and $\tilde{Z} \cap H_{-\beta}$ contain P_3 and $\bar{P}_3$ as smooth points. They consist of the four G-invariant surfaces

$$D'_3 := \{\beta z_2 - iz_3 = w_0 = (3 - 2\lambda)z_2^2 + w_4 w_5 = 0\}, \tag{5.54}$$

$$\bar{D}'_3 := \{\beta z_2 - iz_3 = w_1 = (3 - 2\lambda)z_2^2 + w_4 w_5 = 0\}, \tag{5.55}$$

$$D'_4 := \{\beta z_2 + iz_3 = w_0 = (3 - 2\lambda)z_2^2 + w_4 w_5 = 0\}, \tag{5.56}$$

$$\bar{D}'_4 := \{\beta z_2 + iz_3 = w_1 = (3 - 2\lambda)z_2^2 + w_4 w_5 = 0\}. \tag{5.57}$$

These are illustrated in the diagram on the right in Figure 1. By the same reasons as for P_1 and $\bar{P}_1$, the small resolutions of $\tilde{Z}$ at P_3 and $\bar{P}_3$ are specified by which pair among $\{D'_3, \bar{D}'_4\}$ or $\{\bar{D}'_3, D'_4\}$ is blown-up at P_3 and $\bar{P}_3$ respectively.

Hence any small resolution of $\tilde{Z}$ preserving the real structure falls into exactly one of the following:

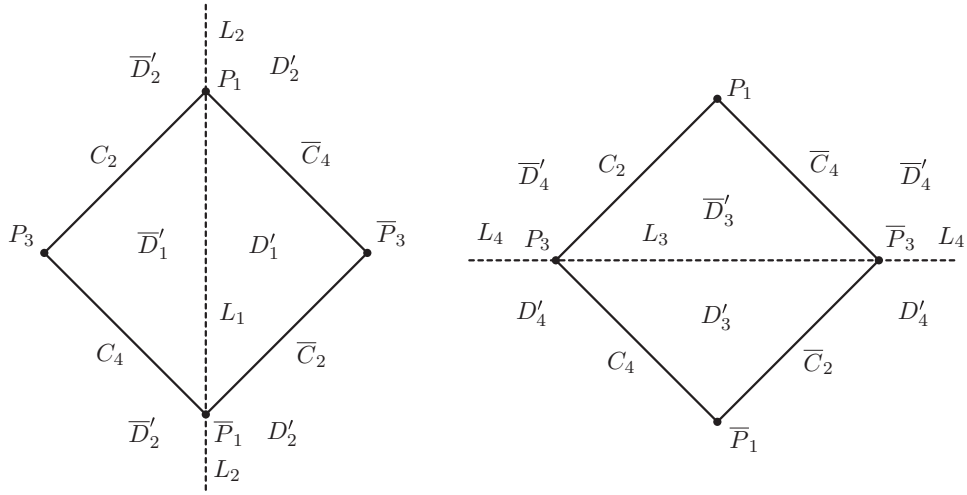

Fig. 1. The 8 cones meeting at singularities of $\tilde{Z}$. The broken lines are the images of the four G-invariant twistor lines, which separate D_j and $\bar{D}_j$ for $1 \leq j \leq 4$. The rational curves C_j, $\bar{C}_j$, $j = 2, 4$ and L_j, $1 \leq j \leq 4$, are the intersection of the corresponding divisors.

$(*)$ $\{D'_1, \bar{D}'_2\}$ and $\{D'_3, \bar{D}'_4\}$ are blown-up pairs near P_1 and P_3, respectively, or $\{\bar{D}'_1, D'_2\}$ and $\{\bar{D}'_3, D'_4\}$ (the complementary pairs) are blown-up pairs near P_1 and P_3 respectively.
$(*)'$ $\{D'_1, \bar{D}'_2\}$ and $\{\bar{D}'_3, D'_4\}$ are blown-up pairs near P_1 and P_3, respectively, or $\{\bar{D}'_1, D'_2\}$ and $\{D'_3, \bar{D}'_4\}$ (the complementary pairs) are blown-up pairs near P_1 and P_3 respectively.

Here, we are specifying blown-up pairs only for P_1 and P_3 since blown-up pairs at $\bar{P}_1$ and $\bar{P}_3$ are automatically determined from those for P_1 and P_3 respectively, by the real structure. For example, for the first case in $(*)$, the blown-up pair near P_1 is $\{D'_1, \bar{D}'_2\}$, and by the real structure this is mapped to the pair $\{\bar{D}'_1, D'_2\}$, and we choose this as a blown-up pair at the point $\bar{P}_1 = \sigma(P_1)$.

Obviously, each of these cases contain two ways of resolutions. Consequently, for each case, we obtain two (non-singular) 3-folds. Next we see that these two spaces in each case are biholomorphic. For this, we define a new matrix U_0 by

$$U_0 := \mathrm{diag}(1, 1, 1, -1, 1, 1). \tag{5.58}$$

It is immediate to see (from (5.1)) that $U_0(\tilde{Z}) = \tilde{Z}$ holds. We denote this involution on $\tilde{Z}$ by the same letter U_0. Note that U_0 commutes with the real structure.

Proposition 5.6. *Let $\nu_1 \colon Z_1 \to \tilde{Z}$ and $\nu_2 \colon Z_2 \to \tilde{Z}$ be the two resolutions of $\tilde{Z}$ in the case $(*)$, and $\nu'_1 \colon Z'_1 \to \tilde{Z}$ and $\nu'_2 \colon Z'_2 \to \tilde{Z}$ be the two resolutions of $\tilde{Z}$ in the case*

$(*)'$. *Then the involution U_0 on $\tilde{Z}$ lifts as a biholomorphic map $Z_1 \to Z_2$ and $Z_1' \to Z_2'$. Furthermore, the last two biholomorphic maps commute with the real structure.*

Proof. We first note that the real structure on the projective model $\tilde{Z}$ naturally lifts to any of the four small resolutions Z_1, Z_2, Z_1' and Z_2' since we are choosing the blowup pairs in such a way that the real structure maps blowup pairs to blowup pairs. In order to prove the proposition, it suffices to verify that U_0 maps the blow-up pairs to the (complementary) blow-up pairs. By elementary computations, we have $U_0(D_1') = D_2'$, $U_0(D_3') = D_4'$. This immediately implies the former claim of the proposition. The latter claim follows from the commutativity of U_0 with the real structure. □

By Proposition 5.6, we can identify Z_1 and Z_2, and also Z_1' and Z_2'. Next we show that the latter two spaces are *not* twistor spaces:

Proposition 5.7. *Let Z_1' and Z_2' be as above and σ_1' and σ_2' the real structure induced by that on $\tilde{Z}$. Then (Z_1', σ_1') and (Z_2', σ_2') are not twistor spaces.*

Proof. By Proposition 5.6, it suffices to show the claim for (Z_1', σ_1'). In $\mathbb{CP}^5$ we define

$$\begin{aligned} &C_2 := \{z_2 = z_3 = w_1 = w_5 = 0\}, \quad \bar{C}_2 := \{z_2 = z_3 = w_0 = w_4 = 0\}, \\ &C_4 := \{z_2 = z_3 = w_0 = w_5 = 0\}, \quad \bar{C}_4 := \{z_2 = z_3 = w_1 = w_4 = 0\}. \end{aligned} \tag{5.59}$$

It is immediate to see that these are G-invariant non-singular rational curves in $\tilde{Z}$. Moreover, each of these 4 curves goes through exactly two singular points of $\tilde{Z}$ (see Fig. 1). We further define

$$L_j := D_j' \cap \bar{D}_j', \quad 1 \le j \le 4, \tag{5.60}$$

recalling from above that these are precisely the images of the G-invariant twistor lines. Suppose that Z_1' is a twistor space. Then by Lemma 5.3, these are the images of the four G-invariant twistor lines (under Ψ). We use the same letters to mean the strict transforms into Z_1' of these curves. Further, let $C_1, \bar{C}_1, C_3, \bar{C}_3$ be the exceptional curves of the small resolution $Z_1' \to Z$. Then in the small resolution Z_1', the 8 curves $C_1, C_2, C_3, C_4, \bar{C}_1, \bar{C}_2, \bar{C}_3$ and $\bar{C}_4$ form an 'octagon'. (This is true for any small resolution of $\tilde{Z}$.) Further, under the present choice of the small resolution, the curves L_j in $\tilde{Z}_1'$ can be seen to be configured as in the left diagram in Fig. 2.

We make a short remark on how Fig. 2 is obtained. For example, consider the first small resolution in $(*)'$. Then the blow-up pair at P_1 is $\{D_1', \bar{D}_2'\}$. This means that by the effect of the resolution, L_1 and $\bar{C}_4$ are separated by the exceptional curve C_1 since D_1' is blown-up at P_1 $(= L_1 \cap \bar{C}_4)$. At the same time, C_2 and L_2 are separated by C_1 since $\bar{D}_2'$ is blown-up at P_1 $(= C_2 \cap L_2)$. As a result, near C_1 the situation

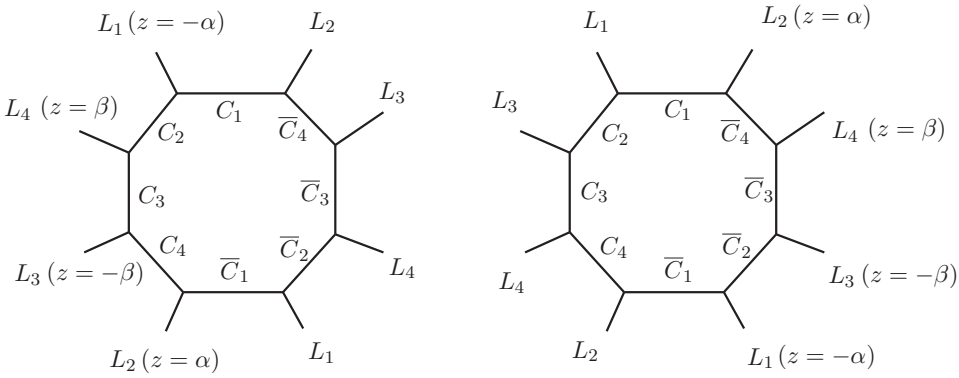

Fig. 2. Octagons formed by the 8 torus-invariant rational curves and the configuration of torus-invariant twistor lines. The left figure is for one of the two incorrect small resolutions, and the right figure is for one of the two correct small resolutions.

becomes as in the left diagram in Fig. 2. Similar reasoning applies to all other edges of the octagon.

Next, we let $z := z_3/z_2$ (where z_2, z_3 are part of the homogeneous coordinates of $\mathbb{CP}^5$) and consider it as a non-homogeneous coordinate on $\mathbb{CP}^1 = \{(z_2, z_3)\}$. Then by (5.3) the real structure on the last $\mathbb{CP}^1$ is given by $z \mapsto -\bar{z}$, so that the real locus is given by $\{z \in \mathbb{C} \mid z \in i\mathbb{R}\}$. Moreover by the definition of L_j, we have

$$\begin{aligned} &z = -i\alpha \quad \text{on} \quad L_1, \quad z = i\alpha \quad \text{on} \quad L_2, \\ &z = -i\beta \quad \text{on} \quad L_3, \quad z = i\beta \quad \text{on} \quad L_4. \end{aligned} \tag{5.61}$$

These mean that under the (meromorphic) G-quotient map $Z' \to \mathbb{CP}^1$ which is induced by the projection $(w_0, w_1, z_2, z_3, w_4, w_5) \mapsto (z_2, z_3)$, each L_j is mapped to the point

$$\begin{aligned} &z = -i\alpha \quad \text{for} \quad j = 1, \ z = i\alpha \quad \text{for} \quad j = 2, \\ &z = -i\beta \quad \text{for} \quad j = 3, \ z = i\beta \quad \text{for} \quad j = 4. \end{aligned} \tag{5.62}$$

As Poon's metric is a special form of a Joyce metric, we will next apply the theorem of Fujiki [3, Theorem 9.1, 1)], which identifies the $(n+2)$ real parameters involved in the construction of Joyce metrics on $n\#\mathbb{CP}^2$ and the twistorial invariant that specifies the positions of the reducible members in the pencil $|F|^K$ (which in our case are $D_j + \bar{D}_j$, $1 \le j \le 4$). Consequently, the four points in (5.62) can be canonically regarded as points on the boundary $\partial\mathcal{H}^2$ (where the Joyce metric is constructed on $K \times \mathcal{H}^2$).

Furthermore, since the twistor fibration map $Z \to 2\,\#\,\mathbb{CP}^2$ is K-equivariant, we have the diagram

$$\begin{array}{ccc} Z & \longleftarrow & \left(\bigcup_{i=1}^4 C_i\right) \cup \left(\bigcup_{i=1}^4 \bar{C}_i\right) \\ \downarrow & & \downarrow /\langle\sigma_1'\rangle \\ 2\,\#\,\mathbb{CP}^2 & \longleftarrow & 4K\text{-invariant 2-spheres} \\ \downarrow /K & & \downarrow /K \\ \mathcal{H}^2 \cup \partial\mathcal{H}^2 & \longleftarrow & \partial\mathcal{H}^2 \end{array} \tag{5.63}$$

where all horizontal arrows mean the obvious inclusions as subsets. In particular, we have an isomorphism

$$\left(\left(\bigcup_{i=1}^4 C_i\right) \cup \left(\bigcup_{i=1}^4 \bar{C}_i\right)\right)/\langle K, \sigma_1'\rangle \simeq \partial\mathcal{H}^2, \tag{5.64}$$

where $\langle K, \sigma_1'\rangle$ means the automorphism group of Z' generated by K and σ_1'. Therefore, looking at the left diagram of Fig. 2, we see that the image of the four K-fixed points of the K-action on $2\mathbb{CP}^2$ under the quotient map

$$2\,\#\,\mathbb{CP}^2 \to (2\,\#\,\mathbb{CP}^2)/K \simeq \mathcal{H}^2 \cup \partial\mathcal{H}^2$$

are configured along $\partial\mathcal{H}^2$ in the order

$$-\alpha,\ \beta,\ -\beta,\ \alpha. \tag{5.65}$$

But as $\alpha > 0$ and $\beta > 0$, the 4 numbers cannot be configured in this order, even up to cyclic permutation and reversing the orientation. Therefore, the L_j-s cannot be configured as in the left diagram in Fig. 2. This means that the small resolutions in $(*)'$ are not the twistor space, as claimed. □

Thus we have obtained the small resolutions of the projective variety $\tilde{Z}$ which give the twistor space in completely explicit form. Namely, such small resolutions are exactly the two ones in $(*)$. We remark that for the former among the two correct small resolutions, the torus-invariant twistor lines are configured as in the right diagram in Fig. 2; the latter case becomes the mirror image of this.

5.3. Determination of the conformal isometry group (for $2\,\#\,\mathbb{CP}^2$). In this subsection we show that, among the automorphisms in Lemma 5.5 (parameterized by 16 tori), only 8 tori lift to the twistor space. (Note that in general automorphisms of the base do not necessarily lift to a small resolution.) We begin with Case (I).

Proposition 5.8. *Let U be the 6×6 matrix of the form*

$$U = \begin{pmatrix} A_{11} & O & O \\ O & A_{22} & O \\ O & O & A_{33} \end{pmatrix}, \tag{5.66}$$

where

$$A_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.67}$$

and

(5.68)

$$A_{11} = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix}, \quad A_{33} = \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix}, \quad |a| = |b| = 1.$$

(*These are necessary conditions obtained in* Lemmas 5.4 *and* 5.5.) *Then U lifts to the twistor space if and only if A_{22}, A_{11} and A_{33} take the following combinations*:

- $A_{22} = I_2$ *and* A_{11}, A_{33} *are diagonal,*
- $A_{22} = I_2$ *and* A_{11}, A_{33} *are off-diagonal,*
- $A_{22} = \mathrm{diag}(1, -1)$, A_{11} *is diagonal and* A_{33} *is off-diagonal,*
- $A_{22} = \mathrm{diag}(1, -1)$, A_{11} *is off-diagonal and* A_{33} *is diagonal.*

REMARK 5.9. This proposition means that the natural injective homomorphism

$$\mathrm{Aut}^{\sigma}\ Z \to \mathrm{Aut}^{\sigma}\ \tilde{Z} \tag{5.69}$$

is not surjective. Namely, even if we restrict to the real resolutions, the projective models can have strictly larger symmetries than that of the twistor space.

Proof of Proposition 5.8. We determine whether the projective transformation U lifts to a small resolution, by using the obvious fact that an automorphism U of $\tilde{Z}$ lifts to a small resolution Z if and only if U maps blow-up pairs at any ordinary nodes of $\tilde{Z}$ (in the sense of Section 5.2; see $(*)$) to a blow-up pair. More concretely:

1) If U fixes P_j ($j = 1$ or 3), then U can be lifted to a small resolution of $\tilde{Z}$ at P_j if and only if U preserves *each* pair of divisors. (If $j = 1$, this means $\{U(D_1'), U(\bar{D}_2')\} = \{D_1', \bar{D}_2'\}$; if $\{U(D_1'), U(\bar{D}_2')\} = \{\bar{D}_1', D_2'\}$, U does not lift on any small resolutions. If $j = 3$, this means $\{U(D_3'), U(\bar{D}_4')\} = \{D_3', \bar{D}_4'\}$; if $\{U(D_3'), U(\bar{D}_4')\} = \{\bar{D}_3', D_4'\}$, U does not lift on any small resolutions.) In these cases, U can also be lifted to any small resolution (of P_j) automatically.

2) If $U(P_1) = \bar{P}_1$, then U can be lifted to small resolutions of $\tilde{Z}$ at P_1 and $\bar{P}_1$ which preserve the real structure if and only if $\{U(D_1'), U(\bar{D}_2')\} = \{\bar{D}_1', D_2'\}$. Similarly, if

$U(P_3) = \bar{P}_3$, U can be lifted to small resolutions at P_3 and $\bar{P}_3$ which preserves the real structure if and only if $\{U(D'_3), U(\bar{D}'_4)\} = \{\bar{D}'_3, D'_4\}$.

First we examine U of (5.66) in the case where $A_{22} = I_2$ and A_{11}, A_{33} are diagonal. These U fix all four singularities of $\tilde{Z}$ and leave any D'_j and $\bar{D}'_j$ $(1 \leq j \leq 4)$ invariant. Hence by 1) above, we conclude that such U lift to any small resolution of $\tilde{Z}$. In particular, U lifts to an automorphism of the twistor space Z. Since these U include the identity matrix, they form the identity component of the automorphism group.

Next, if $A_{22} = \mathrm{diag}(1, -1)$, and A_{11}, A_{33} are diagonal, then $U(P_1) = P_1$ and $U(D'_1) = D'_2$ hold. Hence by 1), these U do not lift to any small resolution. If $A_{22} = I_2$ and A_{11} is diagonal, and A_{33} is off-diagonal, then $U(P_1) = P_1$ and $U(D'_1) = \bar{D}'_1$ hold. Hence by 1), these U do not lift to any small resolution. If $A_{22} = \mathrm{diag}(1, -1)$, and A_{11} is diagonal and A_{33} is off-diagonal, then $U(P_1) = P_1$ and $U(D'_1) = \bar{D}'_2$. Hence by 1), these U lift to any small resolution at P_1 and $\bar{P}_1$. Further since $U(P_3) = \bar{P}_3$ and $U(D'_3) = D'_4$, by 2) this time, we conclude that these U lift to any small resolution at P_3 and $\bar{P}_3$ as long as they preserve the real structure. Hence these U lift to an automorphism of the twistor space Z. If $A_{22} = I_2$, A_{11} is off-diagonal and A_{33} is diagonal, then we have $U(P_3) = P_3$ and $U(D'_3) = \bar{D}'_3$. Hence by 1), these U do not lift to Z. If $A_{22} = \mathrm{diag}(1, -1)$, A_{11} is off-diagonal and A_{33} is diagonal, then we have $U(P_1) = \bar{P}_1$, $U(D'_1) = D'_2$, $U(P_3) = P_3$ and $U(D'_3) = \bar{D}'_4$. Hence by 2) and 1), these U do lift to the twistor space Z. If $A_{22} = I_2$ and A_{11} and A_{33} are off-diagonal, then we have $U(P_1) = \bar{P}_1$, $U(D'_1) = \bar{D}'_1$, $U(P_3) = \bar{P}_3$, and $U(D'_3) = \bar{D}'_3$. Hence by 2), these U do lift to the twistor space Z. Finally, if $A_{22} = \mathrm{diag}(1, -1)$ and A_{11} and A_{33} are off-diagonal, then we have $U(P_1) = \bar{P}_1$, $U(D'_1) = \bar{D}'_2$. Hence by 2), these U do not lift to Z. This completes the proof of Proposition 5.8. □

Next we consider Case (II). In order to simplify notation, we put

$$A_{22}^{+} = i\begin{pmatrix} 0 & 1 \\ \alpha\beta & 0 \end{pmatrix}, \quad A_{22}^{-} = i\begin{pmatrix} 0 & -1 \\ \alpha\beta & 0 \end{pmatrix}. \tag{5.70}$$

Proposition 5.10. *Let U be a 6×6 matrix of the form*

$$U = \begin{pmatrix} O & O & A_{13} \\ O & A_{22} & O \\ A_{31} & O & O \end{pmatrix}, \tag{5.71}$$

where $A_{22} = A_{22}^{+}$ or $A_{22} = A_{22}^{-}$ and

(5.72)

$$A_{13} = \begin{pmatrix} a & 0 \\ 0 & -\bar{a} \end{pmatrix} \text{ or } \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix}, \quad A_{31} = \begin{pmatrix} b & 0 \\ 0 & -\bar{b} \end{pmatrix} \text{ or } \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}, \quad |a| = \beta, \quad |b| = \alpha.$$

(*These are necessary conditions obtained in* Lemmas 5.4 *and* 5.5.) *Then U lifts to the twistor space if and only if A_{22}, A_{13} and A_{31} take the following combinations*:

- $A_{22} = A_{22}^{-}$ *and* A_{13}, A_{31} *are diagonal,*
- $A_{22} = A_{22}^{-}$ *and* A_{13}, A_{31} *are off-diagonal,*
- $A_{22} = A_{22}^{+}$, A_{13} *is diagonal and* A_{31} *is off-diagonal,*
- $A_{22} = A_{22}^{+}$, A_{13} *is off-diagonal and* A_{31} *is diagonal.*

Proof. We define a matrix of the form (5.71) (satisfying (5.72)) by

$$\Lambda := \begin{pmatrix} 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & i\alpha\beta & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.73}$$

We note $\Lambda^2 = \alpha\beta I$, so that Λ defines an involution of $\tilde{Z}$. Moreover, we have

$$\Lambda(D_1') = D_3', \quad \Lambda(\bar{D}_2') = \bar{D}_4', \quad \Lambda(D_3') = D_1', \quad \Lambda(\bar{D}_4') = \bar{D}_2'. \tag{5.74}$$

In particular we have $\{\Lambda(D_1'), \Lambda(\bar{D}_2')\} = \{D_3', \bar{D}_4'\}$ and $\{\Lambda(D_3'), \Lambda(\bar{D}_4')\} = \{D_1', \bar{D}_2'\}$ Noting that $\Lambda(P_1) = P_3$, $\Lambda(P_3) = P_1$, this means that Λ maps any blow-up pairs to blow-up pairs for the small resolutions in the case $(*)$. Therefore Λ lifts to Z if (and only if) the above condition $(*)$ is satisfied. Hence Λ lifts to the twistor space Z.

Having done this, for any matrix U of the form (5.71) (subject to (5.72)) we consider the product ΛU. If A_{13} and A_{31} (in the matrix U) are diagonal and $A_{22} = A_{22}^{-}$, up to a non-zero constant, the product ΛU becomes of the first form in Proposition 5.8. Hence by the proposition ΛU lifts to Z. Therefore, as Λ lifts to Z for the small resolutions in $(*)$ as above, we obtain that these U lift to Z for the small resolutions in the case $(*)$. Similarly, if A_{13} and A_{31} (in the matrix U) are off-diagonal and $A_{22} = A_{22}^{-}$, then up to a non-zero constant, the product ΛU becomes of the second form, so that U lifts to Z for the small resolutions in $(*)$. If A_{13} and A_{31} are diagonal and off-diagonal respectively and $A_{22} = A_{22}^{+}$, then up to a non-zero constant, ΛU becomes of the fourth form, so that U lifts to Z for the small resolutions in $(*)$. If A_{13} and A_{31} are off-diagonal and diagonal respectively and $A_{22} = A_{22}^{+}$, then up to a non-zero constant, ΛU becomes of the third form, so that U lifts to Z for the small resolutions in $(*)$. Further, it can be readily checked that if U is not of these 4 forms, then ΛU does not coincide with any of the 4 forms and therefore U does not lift to Z for the small resolutions in $(*)$ by Proposition 5.8. Thus we have proved the claim of the proposition. □

By Propositions 5.8, 5.10 and 5.7, we have obtained explicit representations of all conformal isometries of Poon's metrics on $2\#\mathbb{CP}^2$ by 6×6 matrices. Namely, we have obtained the image of the (injective) homomorphism (5.15) explicitly.

6. Geometric interpretation

In this subsection, we investigate the geometry of the conformal automorphisms obtained in the previous sections. We begin with the following

Lemma 6.1. *Let $n \geq 2$ and $[g_{\mathrm{LB}}]$ be a LeBrun metric on $n\,\#\,\mathbb{CP}^2$. Then*
(i) *if $n \geq 3$, there exists a unique* U(1)*-subgroup of* $\mathrm{Aut}(g_{\mathrm{LB}})$ *which acts semi-freely on $n\,\#\,\mathbb{CP}^2$,*
(ii) *if $n = 2$, the number of such* U(1)*-subgroups is two.*

Proof. Let $p_1, \ldots, p_n \in \mathcal{H}^3$ be the monopole points of $[g_{\mathrm{LB}}]$. Then the structure group U(1) of the principal bundle over $\mathcal{H}^3 \setminus \{p_1, \ldots, p_n\}$ acts semi-freely on $n\mathbb{CP}^2$, and it coincides with the identity component of $\mathrm{Aut}(g_{\mathrm{LB}})$ if and only if the n points do not lie on a common geodesic. Therefore to prove (i) it suffices to consider the case that $p_1, \ldots, p_n$ are contained on a common geodesic. If the last condition is satisfied, the identity component of $\mathrm{Aut}(g_{\mathrm{LB}})$ becomes the torus K. Note that for $n = 2$, this condition is automatically satisfied.

The K-action on $n\#\mathbb{CP}^2$ is obtained as follows. First consider a standard K-action on $\mathbb{C}^2$, which is given by $(z, w) \mapsto (sz, tw)$ for $(s,t) \in \mathrm{U}(1)\times\mathrm{U}(1)$. We blow-up $\mathbb{C}^2$ at n points in such a way that the blown-up points are always on the unique K-fixed point of the strict transform of the z-axis. Let $\tilde{\mathbb{C}}^2$ be the resulting complex (toric) surface. Next, we add a point at infinity to $\tilde{\mathbb{C}}^2$. Then by reversing the standard orientation, we obtain $n\#\mathbb{CP}^2$ with a K-action. (Over the open subset $\tilde{\mathbb{C}}^2 \subset n\#\mathbb{CP}^2$, $[g_{\mathrm{LB}}]$ contains a Kähler scalar-flat metric with a K-action.) As this K-action contains a U(1)-subgroup acting semi-freely (which is explicitly given by $\{(s, t) \mid s = 1\}$), it can be identified with the identity component of $\mathrm{Aut}(g_{\mathrm{LB}})$ (in the present situation). Hence to prove the lemma it is enough to classify all U(1)-subgroups of K which act semi-freely on $\tilde{\mathbb{C}}^2$.

If $K_1 \subset K$ is such a U(1)-subgroup, K_1 has non-isolated fixed points [14, Proposition 1]. Hence, since the K-action on $\tilde{\mathbb{C}}^2$ is free on the preimage of $\mathbb{C}^2 \setminus \{zw = 0\}$, the subgroup K_1 has to fix the strict transform of the z-axis or the w-axis, or some exceptional curve of the blow-up $\tilde{\mathbb{C}}^2 \to \mathbb{C}^2$. On these K-invariant subsets, the K-action is explicitly given by multiplication by

$$t,\ s^{-1},\ ts^{-1},\ ts^{-2},\ \ldots,\ ts^{-n}, \tag{6.1}$$

respectively. Namely, all subgroups having non-isolated fixed locus are explicitly given by

$$\{t = 1\},\ \{s = 1\},\quad \text{and}\quad \{t = s^k\}\quad (1 \leq k \leq n). \tag{6.2}$$

Since $n \geq 2$ the first one acts non-semi-freely, whereas the second one acts semi-freely.

For the remaining subgroups $\{t = s^k\}$ $(1 \leq k \leq n)$, the action on the $(n+2)$ K-invariant subsets (in the last paragraph)

$$s^k, s^{k-1}, s^{k-2}, \ldots, s^{k-n}, s^{-1}. \tag{6.3}$$

Hence the action becomes semi-free if and only if $n = 2$ and $k = 1$. This means that if $n \geq 3$ the subgroup $\{s = 1\}$ is the unique U(1)-subgroup acting semi-freely, and if $n = 2$, the subgroups $\{s = 1\}$ and $\{t = s\}$ are all such subgroups. Thus we have obtained the claim of the lemma. □

We return to the case of $2\#\mathbb{CP}^2$. Recall that in the proof of Lemma 5.3 we have defined two $\mathbb{C}^*$-subgroups G_1 and G_3 (explicitly defined as (5.12) and (5.13)).

Lemma 6.2. *Viewing the group $G = \mathbb{C}^* \times \mathbb{C}^*$ (acting on Poon's twistor space) as the complexification of $K = \mathrm{U}(1) \times \mathrm{U}(1)$ (acting on Poon's metric), the subgroups G_1 and G_3 of G are exactly the complexification of the two* U(1)*-subgroups acting semi-freely on $2\#\mathbb{CP}^2$.*

Proof. We freely use notations in the previous section. It suffices to show that G_1 and G_3 act semi-freely on the twistor space Z. By their explicit forms (5.12) and (5.13), G_1 and G_3 clearly act semi-freely on $\mathbb{CP}^5$. Therefore they act semi-freely on the projective model $\tilde{Z}$. Hence it is enough to show that they act semi-freely on the exceptional curves $C_1, C_3, \bar{C}_1$ and $\bar{C}_3$ of the small resolutions $Z \to \tilde{Z}$. The weights for the G_1 and G_3-actions on these curves can readily computed by using the G-invariant divisors D_i' and $\bar{D}_i'$ $(1 \leq i \leq 4)$, and they become either 0 or 1. Thus we conclude that G_1 and G_3 act semi-freely on Z. □

Let K_1 and K_3 be the U(1)-subgroups of K whose complexifications are G_1 and G_3, respectively. We know that these are all of the U(1)-subgroups acting semi-freely. For these subgroups, we set

$$X_0 = \{p \in 2\mathbb{CP}^2 \mid \text{the isotropy subgroup of } K_1 \text{ at } p \text{ is } \{Id\}\}, \tag{6.4}$$

and

$$Y_0 = \{p \in 2\mathbb{CP}^2 \mid \text{the isotropy subgroup of } K_3 \text{ at } p \text{ is } \{Id\}\}. \tag{6.5}$$

From the proof of Lemma 6.1 we know $X_0 \neq Y_0$. Let p_1 and p_2 be the image of the two isolated fixed points of the K_1-action under the quotient map $2\#\mathbb{CP}^2 \to 2\#\mathbb{CP}^2/K_1$. Similarly, let q_1 and q_2 be the image of the two isolated fixed points of the K_3-action under the quotient map $2\#\mathbb{CP}^2 \to 2\#\mathbb{CP}^2/K_3$. Then since $[g_{\mathrm{LB}}]$ is K_1-invariant, by the result of LeBrun [14], the quotient space $\mathcal{H}_1^3 := (X_0/K_1) \cup \{p_1, p_2\}$ becomes a 3-manifold equipped with a hyperbolic metric and g_{LB} is obtained by the

hyperbolic ansatz with monopole points p_1 and p_2. Similarly, $\mathcal{H}^3_3 := (Y_0/K_3) \cup \{q_1, q_2\}$ becomes a 3-manifold equipped with a hyperbolic metric and g_{LB} is obtained by the hyperbolic ansatz whose monopole points are q_1 and q_2. Thus any Poon metric on $2\,\#\,\mathbb{CP}^2$ has the following double fibration:

(6.6)
$$\begin{array}{ccc} & 2\,\#\,\mathbb{CP}^2 & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_3} \\ \mathcal{H}^3_1 \cup \partial\mathcal{H}^3_1 & & \mathcal{H}^3_3 \cup \partial\mathcal{H}^3_3. \end{array}$$

Here, π_1 and π_3 are the quotient maps by the K_1-action and K_3-action, respectively, and $\partial\mathcal{H}^3_1 (\simeq S^2)$ and $\partial\mathcal{H}^3_3 (\simeq S^2)$ are the images of the non-isolated fixed locus of the K_1-action and K_3-action, respectively. Note that if $n \geq 3$, an analogous double fibration does not exist by Lemma 6.1.

By Propositions 5.8 and 5.10, when $n = 2$ the group $\mathrm{Aut}(g_{\mathrm{LB}})$ consists of 8 tori.

DEFINITION 6.3. We define H to be a subgroup of the full conformal isometry group $\mathrm{Aut}(g_{\mathrm{LB}})$ consisting of the 4 tori in Proposition 5.8; namely H consists of automorphisms which are lifts of automorphisms of the projective model $\tilde{Z}$ represented by matrices of 'diagonal type'.

Proposition 6.4. *The image of the subgroup H under the homomorphism*

$$\mathrm{Aut}(g_{\mathrm{LB}}) \to \mathrm{GL}(H^0(Z, F))$$

in (4.4) *preserves the two subspaces $H^0(Z, F)^{G_1}$ and $H^0(Z, F)^{G_3}$.*

Proof. Take any $\Phi \in H$ and let $U \in \mathrm{GL}(6,\mathbb{C})$ be the image of H under the homomorphism, where we are using $\{w_0, w_1, z_2, z_3, w_4, w_5\}$ as a basis of $H^0(Z, F) \simeq \mathbb{C}^6$ as before. By the definition of the subgroup H, U must be of the form

(6.7)
$$\begin{pmatrix} A_{11} & O & O \\ O & A_{22} & O \\ O & O & A_{33} \end{pmatrix}, \quad A_{11}, A_{22}, A_{33} \in \mathrm{GL}(2, \mathbb{C}).$$

On the other hand, by (5.4), the two subspaces are explicitly given by

(6.8)
$$H^0(Z, F)^{G_1} = \langle z_2, z_3, w_4, w_5 \rangle, \quad \text{and} \quad H^0(Z, F)^{G_3} = \langle w_0, w_1, z_2, z_3 \rangle.$$

These directly imply the claim of the proposition. □

Proposition 6.5. *Let $n = 2$ and $H \subset \mathrm{Aut}(g_{\mathrm{LB}})$ be as in* Definition 6.3. *Then there are homomorphisms*

$$\rho_1 \colon H \to \mathrm{Aut}(\mathcal{H}^3_1; p_1, p_2) \tag{6.9}$$

and

$$\rho_3 \colon H \to \mathrm{Aut}(\mathcal{H}^3_3; q_1, q_2) \tag{6.10}$$

such that $\rho_j(\Phi) = \phi$, where Φ is any lift of ϕ obtained in Proposition 2.11.

Proof. We recall that we have defined the linear projections $f_j \colon \mathbb{CP}^5 \to \mathbb{CP}^3$ for $j = 1, 3$ which are explicitly given by (5.9)-(5.10). By the definition and (5.4), the composition $f_j \circ \Psi \colon Z \to \mathbb{CP}^3$ is exactly the rational map associated to the vector space $H^0(Z, F)^{G_j}$. The image $f_j \circ \Psi(Z) = f_j(\tilde{Z})$ (explicitly given as (5.11)) is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, on which K_j acts trivially. Moreover, by Proposition 6.4, H automatically preserves the quadric $f_j(\tilde{Z})$ for $j = 1, 3$. (This is also clear from Proposition 5.8 and (5.11).) Hence for $j = 1, 3$ we obtain two homomorphisms

$$H \to \mathrm{Aut}(\mathbb{CP}^1 \times \mathbb{CP}^1). \tag{6.11}$$

Furthermore, as we have considered those matrices U which commute with the real structure, the image of these homomorphisms commutes with the natural real structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Moreover, if U is a matrix representing an element of H, we have $\{U(P_1), U(\tilde{P}_1)\} = \{P_1, \tilde{P}_1\}$ and $\{U(P_3), U(\tilde{P}_3)\} = \{P_3, \tilde{P}_3\}$. If C_1 and C_3 respectively denote the exceptional curves (for Ψ) over the singular points P_1 and P_3 of $\tilde{Z}$ as before, by the twistor fibration C_1 and C_3 are mapped to the 2-spheres which are fixed by K_1-action and K_3-action, respectively. Hence any $\Phi \in H$ leaves the boundary sphere $\partial\mathcal{H}^3_j \subset 2\,\#\,\mathbb{CP}^2$ invariant for $j = 1$ and 3. Therefore, viewing $\mathbb{CP}^1 \times \mathbb{CP}^1$ as the minitwistor space of the hyperbolic space $\mathcal{H}^3_j$ as in the case $n \geq 3$, we obtain a homomorphism

$$\rho_j \colon H \to \mathrm{Aut}(\mathcal{H}^3_j) \quad (j = 1, 3). \tag{6.12}$$

Moreover, the image of (6.3) preserves the set of discriminant curves $\{\mathcal{C}_1, \mathcal{C}_2\}$ of the map $f_j \circ \Psi$ by the same reason for the case $n \geq 3$ given in Proposition 4.3 (iii). Hence the image of (6.4) is contained in $\mathrm{Aut}(\mathcal{H}^3_1; p_1, p_2)$ for $j = 1$ and $\mathrm{Aut}(\mathcal{H}^3_3; q_1, q_2)$ for $j = 3$. Furthermore, the homomorphism ρ_j is an inverse of the lift in Proposition 2.11 by the same reason for the case $n \geq 3$ given in the final part of the proof of Proposition 4.1. This finishes the proof. □

This means that the action of the subgroup H preserves each of the two fibrations in (6.6) respectively. On the other hand, for automorphisms not belonging to H, we have the following

Proposition 6.6. *If $\Phi \in \mathrm{Aut}(g_{\mathrm{LB}})$ satisfies $\Phi \notin H$, Φ maps any fiber of π_1 to a fiber of π_3, and any fiber of π_3 to a fiber of π_1, where π_1 and π_3 are the quotient maps by the K_1-action and the K_3-action, respectively, as before.*

Proof. Since the lift of the K_j-actions ($j = 1, 3$) on $2\#\mathbb{CP}^2$ to the twistor space is given by the restriction of the G_j-action to the real forms by Lemma 6.2, it suffices to show that by any $\Phi \notin H$, G_1-orbits are mapped to G_3-orbits, and G_3-orbits are mapped to G_1-orbits. Let U be a 6×6 matrix corresponding to $\Phi \notin H$. Then U is as in Proposition 5.10. As U contains 2 parameters a and b (satisfying $|a| = \beta$ and $|b| = \alpha$), we write $U = U(a,b)$ (to simplify notation). On the other hand, the subgroups G_1 and G_3 are explicitly given in (5.12) and (5.13). Let $B(s) := \mathrm{diag}(s, s^{-1}, 1, 1, 1, 1)$ and $C(t) := \mathrm{diag}(1,1,1,1,t,t^{-1})$. Then as 6×6 matrices, we have the following relations

$$\begin{aligned} &B(s)U(a,b) = U(sa,b), \quad U(a,b)B(s) = U(a,s^{-1}b),\\ &C(t)U(a,b) = U(a,tb), \quad U(a,b)C(t) = U(t^{-1}a,b). \end{aligned} \tag{6.13}$$

These imply that $U(a, b)$ interchanges G_1-orbits and G_3-orbits, as required. □

As an immediate consequence of the above discussion, we obtain the following

Corollary 6.7. *Let d_1 and d_3 be the hyperbolic distance between p_1 and $p_2 \in \mathcal{H}^3_1$, and q_1 and $q_2 \in \mathcal{H}^3_3$, respectively. Then $d_1 = d_3$ holds.*

6.1. Generators of the automorphism group. Finally, we give generators of the full automorphism group $\mathrm{Aut}(g_{\mathrm{LB}})$ in the case $n = 2$. (For $n \geq 3$ generators of $\mathrm{Aut}(g_{\mathrm{LB}})$ were already given in Theorem 3.11).

Proposition 6.8. *Suppose $n = 2$ and let $H \subset \mathrm{Aut}(g_{\mathrm{LB}})$ be as in* Definition 6.3, *and let $\mathrm{Aut}_0(g_{\mathrm{LB}})$ ($\simeq K$) be the identity component of $\mathrm{Aut}(g_{\mathrm{LB}})$. Then we have*:
(i) *The subgroup H is generated by $\mathrm{Aut}_0(g_{\mathrm{LB}})$ and two involutions.*
(ii) *$\mathrm{Aut}(g_{\mathrm{LB}})$ is generated by H and an involution Λ not belonging to H.*

Proof. This is easy since we have explicit representation of $\mathrm{Aut}(g_{\mathrm{LB}})$ as 6×6 matrices. For (i), as the two involutions in H we choose the ones represented by the following matrices

$$\Lambda_1 := \begin{pmatrix} -1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} \quad \text{and} \quad \Lambda_2 := \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}, \tag{6.14}$$

where a blank entry means 0. It is readily seen that $\Lambda_1^2 = \Lambda_2^2 = I$, Λ_1 and Λ_2 belong to mutually different non-identity connected components of H, and that the product $\Lambda_1\Lambda_2$ belongs to the remaining connected component of H. This means that the identity component and Λ_1 and Λ_2 generate the subgroup H. Hence we obtain (i). Note that these correspond to the transformations described in Theorem 3.11.

For (ii) we choose the involution Λ given in (5.73). As in the proof of Proposition 5.8, Λ defines an involution on the twistor space Z. Since Λ is of off-diagonal type, we have $\Lambda \notin H$. Furthermore, by using Propositions 5.8 and 5.10, it is elementary to show that for any one of the other 3 components of $\mathrm{Aut}(g_{\mathrm{LB}}) \setminus H$, we can find an element $U \in H$ for which the product $U \cdot \Lambda$ belongs to that component. This means that H and Λ generate $\mathrm{Aut}(g_{\mathrm{LB}})$. □

The following proposition completes the proof of Theorem 3.11 above.

Proposition 6.9. *As before, let* Aut_0 *be the identity component of* $\mathrm{Aut}(g_{\mathrm{LB}})$, *which is obviously a normal subgroup of* H. *Then the quotient group* H/Aut_0 *is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$. *Moreover, the quotient* $\mathrm{Aut}\,/\mathrm{Aut}_0$ *is isomorphic to* D_4 (*the dihedral group of order* 8).

Proof. The former claim readily follows from the explicit form of the matrices U in Proposition 5.8 (the two matrices Λ_1 and Λ_2 generate the group $\mathbb{Z}_2 \times \mathbb{Z}_2$). For the second claim, we first note that the group is non-Abelian, by the explicit form of the matrices U in Proposition 5.10. Therefore, it is isomorphic to either the quaternion group (the subgroup generated by i and j in the quaternions), or the dihedral group D_4. But the former group cannot contain a subgroup which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore $\mathrm{Aut}\,/\mathrm{Aut}_0$ is isomorphic to D_4. (Alternatively, one can simply show directly that the three matrices Λ_1, Λ_2 and Λ generate the group D_4, without using any classification.) □

6.2. Einstein–Weyl spaces. We end this section by reconciling the automorphisms found using twistor theory with the automorphisms given in Theorem 3.11, and also proving that $\tilde{\Lambda}(\vartheta)$ defined in Section 3 is indeed a conformal map. To do this, we need to study more closely the associated Einstein–Weyl spaces of the G_1 and G_3 actions on the twistor space. Recall that in the proof of Lemma 5.3, we defined two linear projections $f_j\colon \mathbb{CP}^5 \to \mathbb{CP}^3$ $(j = 1, 3)$ whose restriction to $\tilde{Z}$ can be viewed as the quotient map with respect to the G_i-action. Also recall that the images $f_j(\tilde{Z})$ are non-singular quadrics whose equations are given by

$$f_1(\tilde{Z}) = \{\alpha^2 z_2^2 + z_3^2 + 2w_4 w_5 = 0\}, \quad f_3(\tilde{Z}) = \{2w_0 w_1 + \beta^2 z_2^2 + z_3^2 = 0\}. \tag{6.15}$$

For fibers of f_1 and hyperplane sections of the image $f_1(\tilde{Z})$, we have the following

Lemma 6.10. (i) *The closures of general fibers of* f_1 *are smooth conics.*
(ii) *If* h *is a* G_3*-invariant plane in* $\mathbb{CP}^3$*, the inverse image* $f_1^{-1}(h)$ *is reducible if and only if* $h = \{z_2 = \pm i z_3/\alpha\}$ *or* $\{z_2 = \pm i z_3/\beta\}$.

Since everything is explicit, we omit a proof of the lemma. Of course, an analogous result holds for the other quotient map f_3. We also note that the three involutions on $\mathbb{CP}^5$ determined by the matrices Λ_1, Λ_2 (defined in (6.14)), and $\Lambda_3 := \Lambda_2\Lambda_1$ naturally descend to the target space for both of the quotient maps. We note that under the the quotient map $\mathrm{Aut}(g_{\mathrm{LB}}) \to \mathrm{Aut}(g_{\mathrm{LB}})/\mathrm{Aut}_0 \simeq \mathrm{D}_4$, the third element Λ_3 corresponds to the non-trivial center of D_4, which is $\mathbb{Z}_2$.

By [13, Section 7], the minitwistor lines of these minitwistor spaces are precisely the hyperplane sections $h \cap f_j(\tilde{Z})$, where the plane h satisfies
(A) h is real with respect to the naturally induced real structure on $\mathbb{CP}^3$ (so that the real locus on h is necessarily $\mathbb{RP}^2$).
(B) $h \cap f_j(\tilde{Z})$ does not contain a real point.
In other words, the 3-dimensional Einstein–Weyl space appears as the parameter space of these planes. In particular, since the involutions Λ_1, Λ_2 and Λ_3 naturally induce those on $\mathbb{CP}^3$ as above, these also induce involutions on $\mathcal{H}^3$, which we denote by ϕ_1, ϕ_2, ϕ_3, respectively. For the purpose of writing these down in explicit form, next we determine all the planes h satisfying (A) and (B):

Lemma 6.11. (i) *Any plane in* $\mathbb{CP}^3$ *having* (z_2, z_3, w_4, w_5) *as homogeneous coordinates as in* (5.9) *satisfying the above conditions* (A) *and* (B)*, is of the form*

$$z_2 = ibz_3 + cw_4 - \bar{c}w_5, \tag{6.16}$$

where $b \in \mathbb{R}$, $c \in \mathbb{C}$ *satisfy the following inequality*:

$$b^2 + 2|c|^2 < \frac{1}{\alpha^2}. \tag{6.17}$$

(ii) *Alternatively any plane in* $\mathbb{CP}^3$ *having* (w_0, w_1, z_2, z_3) *as homogeneous coordinates as in* (5.10) *satisfying the conditions* (A) *and* (B)*, is either of the form*

$$z_2 = ib'z_3 + c'w_0 + \bar{c}'w_1, \tag{6.18}$$

where $b' \in \mathbb{R}$, $c' \in \mathbb{C}$ *satisfy the inequality*

$$(b')^2 - 2|c'|^2 > \frac{1}{\beta^2}, \tag{6.19}$$

or otherwise of the form

$$z_3 = cw_0 - \bar{c}w_1, \tag{6.20}$$

where $c \in \mathbb{C}$ *satisfies* $|c|^2 < 1/2$.

Proof. Since the real structure on $\mathbb{CP}^3$ is given by

$$(z_2, z_3, w_4, w_5) \mapsto (\bar{z}_2, -\bar{z}_3, -\bar{w}_5, -\bar{w}_4), \tag{6.21}$$

a plane $h = \{az_2 + bz_3 + cw_4 + dw_5 = 0\}$ is real if and only if $a \in \mathbb{R}, b \in i\mathbb{R}, d = -\bar{c}$. It can be verified by simple computations that if $a = 0$, $h \cap f_1(\tilde{Z})$ always contains real points. Hence we may suppose

$$h = \{z_2 = ibz_3 + cw_4 - \bar{c}w_5\}, \quad b \in \mathbb{R},\ c \in \mathbb{C}. \tag{6.22}$$

Substituting into (6.15), putting $w_5 = -\bar{w}_4$ and replacing z_3 by iz_3 using the reality requirement, the condition (B) is equivalent to the condition that the equation

$$\alpha^2(-bz_3 + cw_4 + \bar{c}w_4)^2 - z_3^2 - 2|w_4|^2 = 0 \tag{6.23}$$

has no solution in $(z_3, w_4) \in \mathbb{R} \times \mathbb{C}$. If we write $c = c_1 + ic_2$ and $w_4 = x + iy$, the left hand side can be seen to be equal to

$$\begin{aligned} &(\alpha^2 b^2 - 1)\left(z_3 - \frac{2\alpha^2 b}{\alpha^2 b^2 - 1}(c_1 x - c_2 y)\right)^2 \\ &-2\frac{2\alpha^2 c_1^2 + \alpha^2 b^2 - 1}{\alpha^2 b^2 - 1}\left(x - \frac{2\alpha^2 c_1 c_2}{2\alpha^2 c_1^2 + \alpha^2 b^2 - 1} y\right)^2 \\ &-2\frac{2\alpha^2 c_1^2 + 2\alpha^2 c_2^2 + \alpha^2 b^2 - 1}{2\alpha^2 c_1^2 + \alpha^2 b^2 - 1} y^2. \end{aligned} \tag{6.24}$$

The condition is equivalent to the definiteness of (6.24), viewed as a real quadratic form of (z_3, x, y). If this is positive definite, we have $\alpha^2 b^2 - 1 > 0$ from the first term. But then the coefficient of y^2 necessarily becomes negative, contradicting the definiteness. Hence (6.24) must be negative definite. Hence we have $\alpha^2 b^2 - 1 < 0$. Then looking the coefficient of the second square, we obtain $2\alpha^2 c_1^2 + \alpha^2 b^2 - 1 < 0$. Then by negativity of the coefficient of y^2, we obtain $2\alpha^2 c_1^2 + 2\alpha^2 c_2^2 + \alpha^2 b^2 - 1 < 0$. Conversely, if this last equality holds, all of the three coefficients are easily seen to be negative. Thus the quadratic form (6.24) is definite if and only $2\alpha^2 c_1^2 + 2\alpha^2 c_2^2 + \alpha^2 b^2 - 1 < 0$. This is equivalent to (6.17), and we obtain (i).

The claim (ii) can be argued in a similar way, as long as we notice that the real structure on $\mathbb{CP}^3$ with the coordinates (w_0, w_1, z_2, z_3) is given by $(w_0, w_1, z_2, z_3) \mapsto (\bar{w}_1, \bar{w}_0, \bar{z}_2, -\bar{z}_3)$, which is in a slightly different form than (6.21). We omit the details of the computations, as they are similar to the above. □

The region defined by (6.17) is an ellipsoid, which we will denote by $\mathcal{B}(\alpha)$. Although the region defined by (6.19) is disconnected, it becomes connected by adding the

last disc $\{|c|^2 < 1/2\}$, and we will denote this connected region by $\tilde{\mathcal{B}}(\beta)$. Lemma 6.11 says that the planes satisfying (A) and (B) are parameterized by the ellipsoid $\mathcal{B}(\alpha)$, for $f_1(\tilde{Z})$, and by the region $\tilde{\mathcal{B}}(\beta)$ for $f_3(\tilde{Z})$. If we think of the Einstein–Weyl space as the space of real hyperplane sections of the minitwistor space, these regions naturally appear for the two semi-free U(1)-actions, rather than the upper-half space model, as long as we adopt the present coordinates. By [14, Theorem 2], an Einstein–Weyl structure is naturally induced on these regions and it is precisely the hyperbolic structure. Using this, it is now easy to explicitly write down the three involutions ϕ_1, ϕ_2 and ϕ_3 on the Einstein–Weyl space $\mathcal{B}(\alpha)$ (with respect to G_1):

Lemma 6.12. *For $(b, c) \in \mathcal{B}(\alpha)$, we have*

$$\phi_1(b, c) = (-b, -\bar{c}), \quad \phi_2(b, c) = (-b, c), \quad \phi_3(b, c) = (b, -\bar{c}). \tag{6.25}$$

Furthermore, the image of the two isolated fixed points of the K_1-action on $2\,\#\,\mathbb{CP}^2$ (the monopole points) under the quotient map to $\mathcal{B}(\alpha)$ are given by $(b, c) = (\pm 1/\beta, 0)$. The images of the two isolated fixed points of the K_3-action are given by $(b', c') = (\pm 1/\alpha, 0)$.

Proof. The formulas for ϕ_j immediately follow from (6.16) and the explicit forms of Λ_1, Λ_2, and Λ_3 on $\mathbb{CP}^5$. The second statement follows from Lemma 6.10 (ii). □

It follows from Lemma 6.12 that among the 4 connected components of the subgroup H, the component which is mapped to (under the the quotient map $\mathrm{Aut}(g_{\mathrm{LB}}) \to \mathrm{Aut}(g_{\mathrm{LB}})/\mathrm{Aut}_0 \simeq \mathrm{D}_4$) the nontrivial center of D_4 can be characterized by the property that the induced automorphisms on $\mathcal{H}^3_1$ and $\mathcal{H}^3_3$ (by the homomorphisms ρ_1 and ρ_3 in Proposition 6.5) are *both* orientation reversing.

Since the K_3-action acts by isometries on $\mathcal{B}(\alpha)$, the fixed locus of K_3 must be a hyperbolic geodesic in $\mathcal{B}(\alpha)$. By Lemma 6.12, this geodesic contains the monopole points. The formulas (6.25) then clearly imply that the involutions ϕ_j induced by Λ_j correspond exactly with those in Theorem 3.11.

In conclusion, we show that the maps $\tilde{\Lambda}(\vartheta)$ defined in Subsection 3.1 above are conformal automorphisms. We first define

$$\Lambda(\vartheta) = B(e^{i\vartheta})\,\Lambda\,B(e^{-i\vartheta}), \tag{6.26}$$

recalling the diagonal matrices $B(s)$ defined in the proof of Proposition 6.6.

Theorem 6.13. *For any angle ϑ, $\Lambda(\vartheta)$ is an involution of the twistor space, which induces a conformal involution of $[g_{\mathrm{LB}}]$. The induced involution is $\tilde{\Lambda}(\vartheta + \pi/2)$, thus the map $\tilde{\Lambda}(\vartheta + \pi/2)$ is a conformal automorphism of $(2\,\#\,\mathbb{CP}^2, [g_{\mathrm{LB}}])$.*

Proof. It is easy to see that $\Lambda(\vartheta)$ is also an involution. For the moment, let us consider only Λ. We first note that the involution Λ induces a diffeomorphism from $\mathcal{H}^2$ to itself. To see this, we argue as follows: in the 6×6 matrix representation, the involution is off-diagonal type. The middle coordinates (z_2, z_3) in Section 4 can be regarded as a (homogeneous) coordinate on the quotient space $Z/(\mathbb{C}^* \times \mathbb{C}^*) \simeq \mathbb{CP}^1$, while $\mathcal{H}^2$ is the space of maximal orbits in the quotient space $2\,\#\,\mathbb{CP}^2/K$. By the explicit form of the matrix Λ and the $\mathbb{C}^* \times \mathbb{C}^*$-action (given in (5.4)) these involutions map $\mathbb{C}^* \times \mathbb{C}^*$-orbits to $\mathbb{C}^* \times \mathbb{C}^*$-orbits, which means that the involution is indeed a lift of some diffeomorphism of $\mathcal{H}^2$.

By [3, Theorem 9.1], the induced involution on $\mathcal{H}^2$ must be a hyperbolic isometry. To see this, we first note that as the coordinate z in the equation (53) on [3, p. 276] is a non-homogeneous coordinate on the parameter space of the pencil $|F|^K$ (consisting of torus-invariant members of the system $|F|$), and since the same is true for the coordinate z_3/z_2 of ours, it follows that z in Fujiki's paper is related to z_3/z_2 by a fractional transformation. (It is possible to write the precise relation between these two coordinates; but we do not need the explicit form). On the other hand [3, Theorem 9.1] states that the coordinate z can be used as a conformal coordinate on $\mathcal{H}^2$. This means that any conformal automorphism of Poon's metric on $2\,\#\,\mathbb{CP}^2$ (which is of course a special form of Joyce metrics) induces a conformal map on $\mathcal{H}^2$ as long as the automorphism descends to a map on $\mathcal{H}^2$. Since the conformal group of $\mathcal{H}^2$ is equal to the isometry group, this implies the involution must be a hyperbolic isometry.

We next discuss the angular transformation induced by Λ. The K-action on $\mathbb{CP}^5$ in (5.4) naturally induces $K_3 \simeq K/K_1$-action on $\mathbb{CP}^3 = \{(z_2, z_3, w_4, w_5)\}$, which is explicitly written as

$$(z_2, z_3, w_4, w_5) \mapsto (z_2, z_3, t\,w_4, t^{-1}w_5),\ t \in K_3. \tag{6.27}$$

This K_3-action naturally induces the (dual) action on the dual space $(\mathbb{CP}^3)^*$. If (a,b,c,d) means the dual coordinates as before, the action is concretely given by $(a, b, c, d) \mapsto (a, b, tc, t^{-1}d)$. By putting $a = 1$ and using (b, c, d) as non-homogeneous coordinates, the action can be written as

$$(b, c, d) \mapsto (b, tc, t^{-1}d). \tag{6.28}$$

Then recalling $b \in \mathbb{R}$ and $d = -\bar{c}$ on the real locus, we obtain that the K_3-action on $\mathcal{B}(\alpha)$ is given by

$$(b, c) \mapsto (b, tc). \tag{6.29}$$

Then since this must be an isometric U(1)-action on the hyperbolic space, and since any non-trivial isometric U(1)-action must be rotations around a geodesic, (6.29) means that $\mathrm{Arg}(c)$ can be used as a coordinate on the hyperbolic space $\mathcal{B}(\alpha) \simeq \mathcal{H}^3_1$. Then $\mathrm{Arg}(t)$ can be naturally identified with the coordinate θ_3, where θ_3 is the coordinate on $\mathrm{U}(1) \simeq K_3$ we have used throughout Section 3.

Similarly, replacing the role of K_1 and K_3 in the above argument, we first obtain that $K_1(\simeq K/K_3)$ naturally acts on $\mathbb{CP}^3 = \{(w_0, w_1, z_2, z_3)\}$ by $(w_0, w_1, z_2, z_3) \mapsto (sw_0, s^{-1}w_1, z_2, z_3)$. Taking the dual, we obtain the K_1-action on $(\mathbb{CP}^3)^*$ equipped with dual coordinates (c', d', a', b') given by $(c', d', a', b') \mapsto (sc', s^{-1}d', a', b')$. On the locus $a' \neq 0$ if we use (b', c', d') as non-homogeneous coordinates by putting $a' = 1$, the action is written as $(b', c', d') \mapsto (b', sc', s^{-1}d')$. Therefore $\mathrm{Arg}(s)$ can be naturally identified with the coordinate θ_1, where θ_1 is the coordinate on $\mathrm{U}(1) \simeq K_1$ we used in Sections 2 and 3.

The involution $\Lambda : \mathbb{CP}^5 \to \mathbb{CP}^5$ induces an isomorphism from $\mathbb{CP}^3$ with coordinates (z_2, z_3, w_4, w_5) to $\mathbb{CP}^3$ with coordinates (w_0, w_1, z_2, z_3), which is given by

$$(z_2, z_3, w_4, w_5) \mapsto (w_0, w_1, z_2, z_3) = (\beta w_4, -\beta w_5, -iz_3, i\alpha\beta z_2). \tag{6.30}$$

This induces an isomorphism between the dual spaces which is given by

$$(c', d', a', b') \mapsto (a, b, c, d) = (i\alpha\beta b', -ia', \beta c', -\beta d'). \tag{6.31}$$

In the above non-homogeneous coordinates on these two $(\mathbb{CP}^3)^*$-s, this can be written as

$$(b', c', d') \mapsto (b, c, d) = \left(-\frac{1}{\alpha\beta b'}, -\frac{ic'}{\alpha b'}, \frac{id'}{\alpha b'}\right). \tag{6.32}$$

Restricting to the real locus, we obtain

$$\mathbb{R} \times \mathbb{C} \ni (b', c') \mapsto (b, c) = \left(-\frac{1}{\alpha\beta b'}, -\frac{ic'}{\alpha b'}\right) \in \mathbb{R} \times \mathbb{C}. \tag{6.33}$$

In particular, $\Lambda^* c' = -ic'/(\alpha b')$. Because θ_1 (resp. θ_3) corresponds to the argument of c' (resp. c), this means that under Λ^*, the transformation of the two angular coordinates θ_3 and θ_1 is given by $\theta_1 \mapsto \theta_3 = \theta_1 - (\pi/2)$. Hence the angular action induced by Λ is given by

$$(\theta_3, \theta_1) \mapsto \left(\theta_1 - \frac{\pi}{2}, \theta_3 + \frac{\pi}{2}\right). \tag{6.34}$$

Since the angular map induced by Λ is orientation-reversing, the induced hyperbolic isometry must also be orientation-reversing. Since the map $L(\zeta)$ defined above in (3.32) is the unique orientation-reversing isometry with the correct properties (see Remark 3.7), Λ must therefore induce the map $\tilde{\Lambda}(\pi/2)$ (recalling Definition 3.8 for the definition of

$\tilde{\Lambda}(\vartheta)$). We next compute (with a slight abuse of notation)

$$\begin{aligned}\Lambda(\vartheta)(\theta_3, \theta_1) &= B(e^{i\vartheta})\Lambda B(e^{-i\vartheta})(\theta_3, \theta_1)\\ &= B(e^{i\vartheta})\Lambda(\theta_3, \theta_1 - \vartheta)\\ &= B(e^{i\vartheta})\Big(\theta_1 - \vartheta - \frac{\pi}{2}, \theta_3 + \frac{\pi}{2}\Big)\\ &= \Big(\theta_1 - \vartheta - \frac{\pi}{2}, \theta_3 + \vartheta + \frac{\pi}{2}\Big).\end{aligned}$$

This clearly implies that $\Lambda(\vartheta)$ induces the map $\tilde{\Lambda}(\vartheta + \pi/2)$, and the proof is complete. □

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