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EQUIVARIANT MAPS BETWEEN COMPLEX STIEFEL MANIFOLDS

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Abstract

The canonical circle action on complex Stiefel manifolds is considered. The corresponding mod p index for all primes p is computed and some Borsuk–Ulam type theorems are proved.

1. Introduction

Let $V_k(\mathbb{C}^n)$ be the complex Stiefel manifold of orthonormal k-frames in $\mathbb{C}^n$. We consider the free S^1-action on $V_k(\mathbb{C}^n)$ given by $z(v_1, v_2, \dots, v_k) = (zv_1, zv_2, \dots, zv_k)$. The orbit space $V_k(\mathbb{C}^n)/S^1$ is complex projective Stiefel manifold. We want to investigate whether there exists an S^1-equivariant map from $V_k(\mathbb{C}^n)$ to $V_l(\mathbb{C}^m)$ for positive integers k, n, l, m such that $k \leq n$ and $l \leq m$. The analogous question for the real Stiefel manifolds $V_k(\mathbb{R}^n)$ (with $\mathbb{Z}_2$-action) was treated in [7].

In [4] and [5], the authors considered the actions of the unitary group $U(k)$ and its subgroup $(\mathbb{Z}_p)^k$ on $V_k(\mathbb{C}^n)$. They studied the degree of equivariant maps and, among other things, they proved that there is no such map from $V_k(\mathbb{C}^n)$ to $V_k(\mathbb{C}^m)$ when $n > m$. In this paper, we prove that this statement is also true for S^1-equivariant maps. Moreover, we obtain additional theorems of this kind. In particular, we show that a sufficient and necessary condition for the existence of an S^1-equivariant map from $U(n)$ to $U(m)$ is that n divides m.

Our method is the cohomological ideal-valued index theory introduced by Fadell and Husseini ([3]) and independently by Jaworowski ([6]). The mod p cohomology of $V_k(\mathbb{C}^n)/S^1$ for all primes p was calculated in [1]. Using this result, in Section 2 we determine $\operatorname{Ind}^{S^1}(V_k(\mathbb{C}^n); \mathbb{Z}_p)$ for all primes p and we also prove some facts concerning $\operatorname{Ind}^{S^1}(V_k(\mathbb{C}^n); \mathbb{Z})$. These results are applied in Section 3 and the above mentioned Borsuk–Ulam type theorems are obtained.

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2. Index of complex Stiefel manifolds

Let G be a group acting on a space X (X is a G-space). For a coefficient ring R, the (cohomological) index of X, denoted by $\operatorname{Ind}^G(X;R)$, is defined as the kernel of the map $c^*\colon H^*(BG;R)\to H^*(EG\times_G X;R)$ induced by $c\colon X\to *$, where $*$ is any one-point space. If X is a free G-space, then the orbit spaces $EG\times_G X$ and X/G have the same homotopy type.

2.1. Mod p index of $V_k(\mathbb{C}^n)$. Let us recall the structure of the mod p cohomology algebra of complex projective Stiefel manifolds. In [1], Astey, Gitler, Micha and Pastor consider the homotopy commutative diagram of fibrations

$$\begin{tikzcd} V_k(\mathbb{C}^n) \arrow[r, equal] \arrow[d] & V_k(\mathbb{C}^n) \arrow[d] \\ V_k(\mathbb{C}^n)/S^1 \arrow[r, "\tilde{f}"] \arrow[d, "g"'] & BU(n-k) \arrow[d, "i"] \\ \mathbb{C}\mathrm{P}^\infty \arrow[r, "f"] & BU(n) \end{tikzcd} \tag{2.1}$$

where g is homotopy equivalent to the map $ES^1\times_{S^1}V_k(\mathbb{C}^n)\to BS^1=\mathbb{C}\mathrm{P}^\infty$ obtained from the map $V_k(\mathbb{C}^n)\to *$, f is the classifying map for the sum of n copies of the Hopf bundle over $\mathbb{C}\mathrm{P}^\infty$ and $\tilde{f}$ is the classifying map for the $(n-k)$-bundle over $V_k(\mathbb{C}^n)/S^1$ obtained as the pullback of the orthogonal complement of the canonical bundle over Grassmann manifold $G_k(\mathbb{C}^n)$ via the quotient map $V_k(\mathbb{C}^n)/S^1\to G_k(\mathbb{C}^n)$. Using this diagram and Leray–Serre spectral sequence, they prove the following theorems. (In both theorems, $x\in H^2(V_k(\mathbb{C}^n)/S^1;\mathbb{Z}_p)$ is the mod p reduction of the Euler class of the complex line bundle associated with the principal bundle $S^1\hookrightarrow V_k(\mathbb{C}^n)\to V_k(\mathbb{C}^n)/S^1$.)

Theorem 2.1 ([1]). *Let p be an odd prime. There are cohomology classes $y_j\in H^{2j-1}(V_k(\mathbb{C}^n)/S^1;\mathbb{Z}_p)$, $n-k+1\le j\le n$, such that*

$$H^*(V_k(\mathbb{C}^n)/S^1;\mathbb{Z}_p)=\mathbb{Z}_p[x]/(x^N)\otimes\Lambda(y_{n-k+1},\dots,y_{N-1},y_{N+1},\dots,y_n)$$

as an algebra, where $N=N_p(n,k):=\min\bigl\{j \bigm| n-k+1\le j\le n,\ p\nmid\binom{n}{j}\bigr\}$.

Theorem 2.2 ([1]). *There are $y_j\in H^{2j-1}(V_k(\mathbb{C}^n)/S^1;\mathbb{Z}_2)$, $n-k+1\le j\le n$, such that*:

(a) *if $k<n$ or else $k=n$ and* $n\equiv 0,1,3 \pmod 4$, *then*

$$H^*(V_k(\mathbb{C}^n)/S^1;\mathbb{Z}_2)=\mathbb{Z}_2[x]/(x^N)\otimes\Lambda(y_{n-k+1},\dots,y_{N-1},y_{N+1},\dots,y_n)$$

as an algebra, where $N=N_2(n,k)$ as in Theorem 2.1;

(b) *if* $k = n$ *and* $n \equiv 2 \pmod 4$, *then*

$$H^*(V_k(\mathbb{C}^n)/S^1; \mathbb{Z}_2) = \mathbb{Z}_2[y_1]/(y_1^4) \otimes \Lambda(y_3, \dots, y_n)$$

as an algebra and $x = y_1^2$.

Recall that the height of a cohomology class σ is defined as $\mathrm{ht}(\sigma) := \max\{m \mid \sigma^m \neq 0\}$. Let us also recall the Lucas formula which states that if p is a prime, $a = \sum_{i=0}^m a_i p^i$ and $b = \sum_{i=0}^m b_i p^i$, $0 \le a_i,\ b_i \le p-1$, then $\binom{a}{b} \equiv \prod_{i=0}^m \binom{a_i}{b_i} \pmod p$.

Returning to the above theorems, note that if $k = n$ and $n \equiv 2 \pmod 4$, then $N_2(n,k) = N_2(n,n) = \min\left\{j \;\middle|\; 1 \le j \le n,\ \binom{n}{j} \equiv 1 \pmod 2\right\} = 2$ by Lucas formula. Note also that $x \neq 0$ and $x^2 = y_1^4 = 0$ in this case. So, we can summarize and say that $\mathrm{ht}(x) = N_p(n,k) - 1$ for any prime p and any $n \ge k \ge 1$.

In [1], the authors also show that $g^*(w) = x$, where g is the map from diagram (2.1) and w is the generator in $H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}_p) = \mathbb{Z}_p[w]$. Since $\mathrm{Ind}^{S^1}(V_k(\mathbb{C}^n); \mathbb{Z}_p) = \ker g^*$, from the preceding observations we immediately get the following theorem.

Theorem 2.3. *Let* p *be a prime. Then,*

$$\mathrm{Ind}^{S^1}(V_k(\mathbb{C}^n); \mathbb{Z}_p) = (w^N),$$

where (w^N) *is the ideal in* $H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}_p) = \mathbb{Z}_p[w]$ *generated by* w^N *and*

$$N = N_p(n,k) = \min\left\{ j \;\middle|\; n-k+1 \le j \le n,\ p \nmid \binom{n}{j} \right\}.$$

Hence, the index is completely determined by the integer $N_p(n,k)$. Let us now compute this number in some cases.

It is obvious that for all primes p, $n-k+1 \le N_p(n,k) \le n$ and

$$n = N_p(n,1) \ge N_p(n,2) \ge \cdots \ge N_p(n,n). \tag{2.2}$$

The following lemma will be very useful. The proof is straightforward from Lucas formula.

Lemma 2.4. *Let* p *be a prime and let* $a = \sum_{i=0}^m a_i p^i$ *and* $b = \sum_{i=0}^m b_i p^i$, *where* $0 \le a_i,\ b_i \le p-1$. *Then* $p \nmid \binom{a}{b}$ *if and only if* $b_i \le a_i$ *for all* $i = \overline{0,m}$.

In what follows, for positive integers a and b, we denote by $\rho(a,b)$ the remainder of the division of a by b. Also, for a prime p and positive integer a, the number of factors p in the prime factorization of a is denoted by $\nu_p(a)$. If $a = \sum_{i=0}^m a_i p^i$, $0 \le a_i \le p-1$, then $\nu_p(a) = \min\{i \mid a_i \neq 0\}$.

Proposition 2.5. *Let p be a prime, r nonnegative integer and $n \geq p^r$. Then*

$$N_p(n, n - p^r + 1) = p^{\nu_p(n-\rho(n,p^r))}.$$

Proof. We want to calculate $N_p(n, n-p^r+1) = \min\{j \mid p^r \leq j \leq n,\ p \nmid \binom{n}{j}\}$. If $n = \sum_{i=0}^{m} a_i p^i$, then from Lemma 2.4 it is obvious that $N_p(n, n-p^r+1) = p^l$, where $l = \min\{i \mid r \leq i \leq m,\ a_i \neq 0\}$. Since $n - \rho(n, p^r) = \sum_{i=0}^{m} a_i p^i - \sum_{i=0}^{r-1} a_i p^i = \sum_{i=r}^{m} a_i p^i$, we see that $l = \nu_p(n - \rho(n, p^r))$. □

Since $V_n(\mathbb{C}^n) = U(n)$, the number $N_p(n, n)$ determines $\mathrm{Ind}^{S^1}(U(n); \mathbb{Z}_p)$. For $r = 0$, the preceding proposition gives us the following corollary.

Corollary 2.6. *For every prime p, $N_p(n, n) = p^{\nu_p(n)}$.*

So, the number $N_p(n, n)$ is maximal (i.e., equal to n) if and only if n is a power of p. Now, from inequalities (2.2), we get the following claim.

Corollary 2.7. *Let p be a prime and let $n = p^r$. Then for all $k \in \{1, 2, \ldots, n\}$, $N_p(n, k) = n$.*

Proposition 2.8. *Let p be a prime, r nonnegative integer and $n \geq p^r$. Then*

$$N_p(n, p^r) = n - \rho(n, p^r).$$

Proof. We wish to find the first binomial coefficient not divisible by p in the sequence $\binom{n}{n-p^r+1}, \binom{n}{n-p^r+2}, \ldots, \binom{n}{n-1}, \binom{n}{n}$. But, these binomial coefficients are respectively equal to $\binom{n}{p^r-1}, \binom{n}{p^r-2}, \ldots, \binom{n}{1}, \binom{n}{0}$, so, if we determine the number $l := \max\{i \mid 0 \leq i < p^r,\ p \nmid \binom{n}{i}\}$, we will obtain the number $N_p(n, p^r)$, since obviously $N_p(n, p^r) = n - l$. Lemma 2.4 applies and we immediately have that $l = \rho(n, p^r)$. □

Corollary 2.9. *If p is a prime, $k = p^r$ and $k \mid n$, then $N_p(n, k) = n$.*

Corollaries 2.7 and 2.9 provide some sufficient conditions for $N_p(n, k)$ to be maximal. Let us add one more proposition of this type.

Proposition 2.10. *Let $n \geq k \geq 1$ and assume that $n \nmid (k-1)!$. Then there exists a prime p such that $N_p(n, k) = n$.*

Proof. The condition $n \nmid (k-1)!$ implies that for some prime p,

$$\nu_p(n) > \nu_p((k-1)!).$$

This means that the binomial coefficient $\binom{n}{n-k+1} = \binom{n}{k-1} = n(n-1)\cdots(n-k+2)/((k-1)!)$ is divisible by p. Obviously, since $\nu_p((k-1)!) \geq \nu_p((k-2)!) \geq \cdots \geq \nu_p(2!) \geq \nu_p(1) = 0$, the same conclusion holds for $\binom{n}{n-k+2}, \ldots, \binom{n}{n-1}$ and we have that $N_p(n, k) = \min\{ j \mid n-k+1 \leq j \leq n,\ p \nmid \binom{n}{j}\} = n$. $\square$

Of course, for given integers $n \geq k \geq 1$, there is always a prime p such that $N_p(n, k)$ is minimal, i.e., $N_p(n, k) = n - k + 1$ (simply take p such that $p \nmid \binom{n}{n-k+1}$). We conclude this section with the proposition which corresponds to Corollary 2.7 and which is proven directly from Lemma 2.4.

Proposition 2.11. *Let p be a prime and let $n = p^r - 1$. Then for all $k \in \{1,2,\ldots,n\}$, $N_p(n, k) = n - k + 1$.*

2.2. Integral index of $V_k(\mathbb{C}^n)$. Referring to diagram (2.1) again, we are interested in the kernel of $g^*\colon H^*(\mathbb{C}P^\infty;\mathbb{Z}) \to H^*(V_k(\mathbb{C}^n)/S^1;\mathbb{Z})$. Let $c_j \in H^{2j}(BU(n);\mathbb{Z})$, $j = \overline{1, n}$, be the Chern classes of the canonical bundle over $BU(n)$. Since the map f classifies the Whitney sum of n copies of the Hopf bundle over $\mathbb{C}P^\infty$ and since the total Chern class of this sum is $(1 + w)^n$, we have that $f^*(c_j) = \binom{n}{j} w^j$, where $w \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ is a generator. On the other hand, for $n - k + 1 \leq j \leq n$, commutativity of the diagram gives us

$$g^*\left(\binom{n}{j} w^j\right) = g^* f^*(c_j) = \tilde{f}^* i^*(c_j) = \tilde{f}^*(0) = 0,$$

and so,

$$\left\{\binom{n}{j} w^j \;\middle|\; n - k + 1 \leq j \leq n\right\} \subseteq \ker g^* = \mathrm{Ind}^{S^1}(V_k(\mathbb{C}^n); \mathbb{Z}). \tag{2.3}$$

Theorem 2.12. *Let* $\mathrm{Ind}_q^{S^1}(V_k(\mathbb{C}^n); \mathbb{Z}) := \mathrm{Ind}^{S^1}(V_k(\mathbb{C}^n); \mathbb{Z}) \cap H^q(\mathbb{C}P^\infty; \mathbb{Z})$.

(a) *For $q < 2(n - k + 1)$,* $\mathrm{Ind}_q^{S^1}(V_k(\mathbb{C}^n); \mathbb{Z}) = 0$.

(b) $\mathrm{Ind}_{2(n-k+1)}^{S^1}(V_k(\mathbb{C}^n); \mathbb{Z}) = \mathbb{Z}\langle \binom{n}{n-k+1} w^{n-k+1}\rangle$.

(c) *For $q \geq 2n$,* $\mathrm{Ind}_q^{S^1}(V_k(\mathbb{C}^n); \mathbb{Z}) = H^q(\mathbb{C}P^\infty; \mathbb{Z})$.

Proof. Part (c) is a direct consequence of (2.3). For (a) and (b) we consider the Leray–Serre spectral sequence of the fibration $V_k(\mathbb{C}^n) \hookrightarrow V_k(\mathbb{C}^n)/S^1 \xrightarrow{g} \mathbb{C}P^\infty$. The map g^* is equal to the composition $H^*(\mathbb{C}P^\infty; \mathbb{Z}) = E_2^{*,0} \xrightarrow{\pi} E_\infty^{*,0} \subseteq H^*(V_k(\mathbb{C}^n)/S^1; \mathbb{Z})$, where π is the natural projection in this spectral sequence. This means that a class $\sigma \in H^q(\mathbb{C}P^\infty; \mathbb{Z})$ is in the kernel of g^* if and only if it is in the image of some differential $d_i\colon E_i^{q-i,i-1} \to E_i^{q,0}$.

It is known that the fibre $V_k(\mathbb{C}^n)$ is $2(n-k)$-connected and so all differentials $d_i\colon E_i^{q-i,i-1} \to E_i^{q,0}$ for $q \le 2(n-k)+1$ are trivial. This proves (a).

Moreover, it is known that $H^*(V_k(\mathbb{C}^n);\mathbb{Z}) = \Lambda(z_{n-k+1},\dots,z_n)$, the exterior algebra on generators $z_j \in H^{2j-1}(V_k(\mathbb{C}^n);\mathbb{Z})$, and that, in the fibration $V_k(\mathbb{C}^n) \hookrightarrow BU(n-k) \to BU(n)$, the generator z_{n-k+1} transgresses to the Chern class c_{n-k+1} ([2]). If τ is the transgression in the fibration $V_k(\mathbb{C}^n) \hookrightarrow V_k(\mathbb{C}^n)/S^1 \xrightarrow{g} \mathbb{C}P^\infty$, by diagram (2.1) and naturality of the transgression, we conclude that

$$d_{2(n-k+1)}(z_{n-k+1}) = \tau(z_{n-k+1}) = f^*(c_{n-k+1}) = \binom{n}{n-k+1} w^{n-k+1}.$$

Since $d_{2(n-k+1)}$ is the only nontrivial differential d_i with target $E_i^{2(n-k+1),0}$, we have obtained (b). $\square$

3. Equivariant maps

A map $f\colon X \to Y$ between G-spaces X and Y is said to be a G-equivariant map (or just a G-map) if for all $x \in X$ and all $g \in G$, $f(gx) = gf(x)$. The following theorem is the crucial tool in this section.

Theorem 3.1 ([3], [6]). *If there exists a G-map $f\colon X \to Y$, then for any coefficient ring R,*

$$\operatorname{Ind}^G(Y;R) \subset \operatorname{Ind}^G(X;R).$$

Henceforth, k, n, l, m are understood to be positive integers such that $k \le n$ and $l \le m$.

Proposition 3.2. *If there exists an S^1-map $V_k(\mathbb{C}^n) \to V_l(\mathbb{C}^m)$, then*

$$n-k \le m-l.$$

Moreover, if $n-k = m-l$, then $\binom{n}{n-k+1}$ divides $\binom{m}{m-l+1}$.

Proof. According to Theorem 2.12 (a), $\operatorname{Ind}_q^{S^1}(V_k(\mathbb{C}^n);\mathbb{Z}) = 0$ for $q < 2(n-k+1)$ and by part (b) of the same theorem $\operatorname{Ind}_{2(m-l+1)}^{S^1}(V_l(\mathbb{C}^m);\mathbb{Z}) \ne 0$. From Theorem 3.1, we immediately get that $2(m-l+1)$ must be $\ge 2(n-k+1)$, i.e., $n-k \le m-l$.

If $n-k = m-l$, then for $t := n-k+1 = m-l+1$, by Theorem 3.1 we have that $\operatorname{Ind}_{2t}^{S^1}(V_l(\mathbb{C}^m);\mathbb{Z}) \subset \operatorname{Ind}_{2t}^{S^1}(V_k(\mathbb{C}^n);\mathbb{Z})$ and by Theorem 2.12 (b), these indices are equal to $\mathbb{Z}\langle\binom{m}{t}w^t\rangle$ and $\mathbb{Z}\langle\binom{n}{t}w^t\rangle$ respectively. We conclude that $\binom{n}{t} \mid \binom{m}{t}$. $\square$

Theorem 3.3. (a) *There exists an S^1-map $V_k(\mathbb{C}^n) \to V_k(\mathbb{C}^m)$ if and only if $n \leq m$.*
(b) *There exists an S^1-map $V_k(\mathbb{C}^n) \to V_l(\mathbb{C}^n)$ if and only if $k \geq l$.*
(c) *If there exists an S^1-map $V_{n-s}(\mathbb{C}^n) \to V_{m-s}(\mathbb{C}^m)$, then $n \leq m$.*

Proof. (a) If there exists an S^1-map from $V_k(\mathbb{C}^n)$ to $V_k(\mathbb{C}^m)$, then by Proposition 3.2, $n-k \leq m-k$, i.e., $n \leq m$. Conversely, if $n \leq m$, then the natural embedding $\mathbb{C}^n \hookrightarrow \mathbb{C}^m$ induces an S^1-map $V_k(\mathbb{C}^n) \to V_k(\mathbb{C}^m)$. A k-frame $(v_1, \ldots, v_k)$ in $\mathbb{C}^n$ maps to the k-frame $(\tilde{v}_1, \ldots, \tilde{v}_k)$ in $\mathbb{C}^m$, where the vector $\tilde{v}_i$ is obtained from v_i by adding $m-n$ zeros at the end.

(b) According to Proposition 3.2 again, from the existence of S^1-map $V_k(\mathbb{C}^n) \to V_l(\mathbb{C}^n)$, we directly get the inequality $k \geq l$. Likewise, if $k \geq l$, there is an obvious S^1-equivariant map from $V_k(\mathbb{C}^n)$ to $V_l(\mathbb{C}^n)$ which forgets the last $k-l$ vectors in a frame.

(c) The second part of the preceding proposition gives us that $\binom{m}{s+1}$ must be divisible by $\binom{n}{s+1}$. Since $\binom{a}{b}$, considered as a function of a, is increasing, we have that $n \leq m$. □

If k is any positive integer, we can see S^1 as the subgroup of $U(k)$ consisting of all scalar matrices. Furthermore, the action of $U(k)$ on $V_k(\mathbb{C}^n)$ which was treated in [4] (matrix multiplication), restricts on this subgroup to the action which is being investigated in this paper. This means that, as a consequence of the part (a) of Theorem 3.3, we have obtained the following result of Hara [4, p. 120].

Corollary 3.4. *If there exists a $U(k)$-map $V_k(\mathbb{C}^n) \to V_k(\mathbb{C}^m)$, then $n \leq m$.*

Since $U(m) = V_m(\mathbb{C}^m)$, by Proposition 3.2 we see that the only complex Stiefel manifolds which could be S^1-equivariantly mapped to $U(m)$ are unitary groups $U(n)$. Moreover, we are going to prove the following theorem.

Theorem 3.5. *There exists an S^1-map $U(n) \to U(m)$ if and only if $n \mid m$.*

Proof. If there is an S^1-map from $U(n)$ to $U(m)$, then from the second part of Proposition 3.2, we conclude that m must be divisible by n.

Assume now that $n \mid m$. Since the action of S^1 on $U(n)$ is just the scalar multiplication, the following map $f\colon U(n) \to U(m)$ is S^1-equivariant. If $m = r \cdot n$, each matrix in $U(m)$ can be divided in r^2 $(n \times n)$-blocks. For a matrix $A \in U(n)$, we define $f(A)$ as the matrix with an A in each block on the diagonal and zero matrix in all other blocks. This is clearly an element of $U(m)$ and the proof is completed. □

REMARK 3.6. One can identify $V_{n-1}(\mathbb{C}^n)$ with the special unitary group $SU(n)$, e.g., the $n-1$ vectors could be written in the first $n-1$ rows of the matrix and the last row is filled with the coordinates of the (unique) vector such that the obtained matrix belongs to $SU(n)$. Then the map f constructed in the preceding proof (as a map from

$SU(n)$ to $SU(m)$) is not S^1-equivariant. This is because the action of S^1 on $SU(n)$ is not the scalar multiplication. Actually, the element $z \in S^1$ acts on the matrix $A \in SU(n)$ by multiplying the first $n-1$ rows and the last row is multiplied by $\overline{z}^{n-1}$. Furthermore, if $n \mid m$, then S^1-map from $SU(n)$ to $SU(m)$ may not exist at all. For example, S^1-map from $SU(5)$ to $SU(10)$ does not exist. Namely, by Proposition 3.2, the existence of S^1-map $SU(n) \to SU(m)$ implies that $\binom{n}{2} \mid \binom{m}{2}$, but $\binom{5}{2} = 10$ does not divide $\binom{10}{2} = 45$.

Since $S^{2n-1} = V_1(\mathbb{C}^n)$, the following statements are easy consequences of Proposition 3.2 and Theorem 3.3 (a).

Corollary 3.7. (a) *If there exists an S^1-map $S^{2n-1} \to V_l(\mathbb{C}^m)$, then $n \le m - l + 1$.*
(b) *If there exists an S^1-map $V_k(\mathbb{C}^n) \to S^{2m-1}$, then $m \ge n - k + 1$.*
(c) *There exists an S^1-map $S^{2n-1} \to S^{2m-1}$ if and only if $n \le m$.*

The following proposition is a direct consequence of Theorem 2.3 and Theorem 3.1.

Proposition 3.8. *If there exists an S^1-map $V_k(\mathbb{C}^n) \to V_l(\mathbb{C}^m)$, then for all primes p,*

$$N_p(n, k) \le N_p(m, l).$$

EXAMPLE 3.9. There is no S^1-map $V_4(\mathbb{C}^6) \to V_6(\mathbb{C}^9)$. Namely, $\binom{6}{3} = 20$, $\binom{6}{4} = 15$, $\binom{6}{5} = 6$, so we have that $N_5(6, 4) = 5$. On the other hand, $\binom{9}{4} = 126$ implies that $N_5(9, 6) = 4$ and the conclusion follows from Proposition 3.8.

It is clear that if $n \le m$ and $k \ge l$, then there is an S^1-map from $V_k(\mathbb{C}^n)$ to $V_l(\mathbb{C}^m)$ (one can compose S^1-maps $V_k(\mathbb{C}^n) \to V_k(\mathbb{C}^m) \to V_l(\mathbb{C}^m)$ constructed in the proof of Theorem 3.3). Furthermore, it is obvious from parts (a) and (b) of Theorem 3.3 that if one of these two inequalities is turned into equality, then the other one is equivalent to the existence of an S^1-map.

If we drop one of the conditions $n \le m$ and/or $k \ge l$, can there exist an S^1-map from $V_k(\mathbb{C}^n)$ to $V_l(\mathbb{C}^m)$? If $k < l$, the positive answer to this question is provided by Theorem 3.5. We are not aware of any example of S^1-map $V_k(\mathbb{C}^n) \to V_l(\mathbb{C}^m)$ when $n > m$. On the other hand, in the following theorem, we outline some necessary conditions for existence of such maps.

Theorem 3.10. *Let $n > m$. If there exists an S^1-map $V_k(\mathbb{C}^n) \to V_l(\mathbb{C}^m)$, then all of the following conditions must be satisfied*:
(i) *n is not a power of a prime*;
(ii) $n \mid (k-1)!$;
(iii) $n - k < m - l$;
(iv) *if k is a power of a prime, then $k \nmid n$.*

Proof. The condition (iii) is a consequence of Proposition 3.2 and Theorem 3.3 (c). The idea of the proof of (i), (ii) and (iv) is the following: if some of these conditions fails, we find a prime p such that $N_p(n, k) = n$ and then we apply Proposition 3.8 obtaining a contradiction. If (i) fails, we refer to Corollary 2.7. If (ii) is false, we use Proposition 2.10. Finally, if condition (iv) is not satisfied (i.e., if k is a power of a prime and $k \mid n$), then we have a contradiction by Corollary 2.9. $\square$

REMARK 3.11. Obviously, if $n > s > m$ and if there exists an S^1-map $V_k(\mathbb{C}^n) \to V_l(\mathbb{C}^m)$, then there is also an S^1-map $V_k(\mathbb{C}^s) \to V_l(\mathbb{C}^m)$ obtained by composing the former one with S^1-map $V_k(\mathbb{C}^s) \to V_k(\mathbb{C}^n)$ (Theorem 3.3 (a)). This means that we can strengthen the previous theorem by indicating that the conditions (i), (ii) and (iv) must hold not just for n, but for all integers $m+1, m+2, \dots, n$.

At the end, we present a few examples of the usage of Theorem 3.10.

EXAMPLE 3.12. For $k = 1$ or $k = 2$, (ii) implies that $n = 1$. Thus, there is no S^1-map $V_k(\mathbb{C}^n) \to V_l(\mathbb{C}^m)$ when $k = 1$ or $k = 2$ and $n > m \geq 1$. The same conclusion holds for $k = 3$ since there is no positive integer n such that $n \geq 3$ and $n \mid 2!$.

EXAMPLE 3.13. If there is an S^1-map $V_4(\mathbb{C}^n) \to V_l(\mathbb{C}^m)$ for some $n > m \geq l \geq 1$, by (ii) we see that n must be equal to 6. From the condition (iii), we conclude that there is no S^1-map $V_4(\mathbb{C}^6) \to V_l(\mathbb{C}^5)$ for $l \geq 3$. For the same reason, there is no S^1-map $V_4(\mathbb{C}^6) \to V_l(\mathbb{C}^4)$ for $l \geq 2$. But, there is no S^1-map $V_4(\mathbb{C}^6) \to V_1(\mathbb{C}^4) = S^7$ either, since otherwise we would have an S^1-map $V_4(\mathbb{C}^5) \to V_4(\mathbb{C}^6) \to V_1(\mathbb{C}^4)$ and this contradicts the condition (i) of the previous theorem since 5 is a (power of a) prime number.

EXAMPLE 3.14. There is no S^1-equivariant map $V_8(\mathbb{C}^{24}) \to V_1(\mathbb{C}^{23}) = S^{45}$. Namely, although the conditions (i), (ii) and (iii) from Theorem 3.10 hold, (iv) fails since $8 = 2^3 \mid 24$. Moreover, there is no S^1-map $V_8(\mathbb{C}^{24}) \to V_l(\mathbb{C}^m)$ for any $l \leq m < 24$.

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