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ASYMPTOTIC PROFILE OF QUENCHING IN A SYSTEM OF HEAT EQUATIONS COUPLED AT THE BOUNDARY

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Abstract

We study finite time quenching for the radial solutions of a system of heat equations coupled at the boundary condition. This system exhibits simultaneous and non-simultaneous quenching. In particular, three kinds of simultaneous quenching profiles are obtained for different nonlinear exponent regions and appropriate initial data.

1. Introduction

In this paper we study quenching phenomena for heat equations

$$u_t = \Delta u, \quad v_t = \Delta v, \quad x \in \Omega,\ t \in (0, T), \tag{1.1}$$

with coupled boundary conditions

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = -v^{-p}, \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = -u^{-q}, \quad x \in \partial\Omega,\ t \in (0, T), \tag{1.2}$$

and positive initial data

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \overline{\Omega},$$

where $p, q > 0$ and $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain. Throughout this paper we assume that

$$u_0(x), v_0(x) \in C^2(\overline{\Omega}), \quad \Delta u_0(x), \Delta v_0(x) \leq 0, \quad x \in \overline{\Omega}.$$

In the radial symmetric case with $\Omega = B_1 = \{x \mid |x| < 1\}$, let $r = |x|$ and $u_0(x) = u_0(r)$, $v_0(x) = v_0(r)$. Then the radial solutions $u(r, t)$ and $v(r, t)$ satisfy the

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following equations,

$$
(1.3)\qquad \begin{cases} u_t = u_{rr} + \dfrac{N-1}{r} u_r,\ v_t = v_{rr} + \dfrac{N-1}{r} u_r, & (r,t) \in (0,1) \times (0,T), \\ u_r(0,t) = 0,\ u_r(1,t) = -v^{-p}(1,t), & t \in (0,T), \\ v_r(0,t) = 0,\ v_r(1,t) = -u^{-q}(1,t), & t \in (0,T), \\ u(r,0) = u_0(r),\ v(r,0) = v_0(r), & r \in [0,1]. \end{cases}
$$

Similarly, we have

$$
u_0(r),\, v_0(r) \geq 0, \quad u_t(r,0),\, v_t(r,0) \leq 0, \quad 0 \leq r \leq 1.
$$

For the convenience, we assume that

$$
u_0'(r) \leq 0, \quad v_0'(r) \leq 0, \quad 0 \leq r \leq 1.
$$

The study of quenching (in general the solution is defined up to $t = T$ but some term in the problem ceases to make sense) began with the work of Kawarada [11] appeared in 1975. In that paper he studied the semi-linear heat equation as a singular reaction at level $u = 1$. He proved that not only the reaction term, but also the time derivative blows up wherever u reaches this value, see also [1]. Quenching problems have been studied by many authors, see [2, 4, 5, 9, 10] and the references therein.

In [7], Ferreira, Quiros and Rossi studied the one-dimensional case of (1.1) and found that, due to the absorption generated by the boundary condition at $x = 0$, the solutions decrease to zero at this point. If they vanish in finite time $t = T_0$, the boundary condition $u_x(0,t) = -v^{-p}(0,t)$ and $v_x(0,t) = -u^{-q}(0,t)$ for $0 < t < T$ blows up and the solution, being classical up to $t = T$, no longer exists (as a classical solution) for greater times, thus the maximal existence time of a classical solution is $T = T_0$. They characterized in terms of the parameters involved when non-simultaneous quenching may appear. They obtained that if $p, q \geq 1$ quenching is always simultaneous, while if $p < 1$ or $q < 1$ non-simultaneous quenching indeed occurs. Moreover, if quenching is non-simultaneous they found the quenching rate, which surprisingly depends on the parameter in the flux associated to the other component. Also, the only quenching point is the origin.

In [8], Hu and Yin considered the profile near the blowup time for the solutions of the following problem:

$$
(1.4)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u & \text{for } x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial n} = u^p & \text{for } x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, n is the exterior normal vector on $\partial\Omega$, $p > 1$ and $u_0(x) \geq 0$. Under the assumptions of u_0, they obtained the blowup rate $u(x,t) \sim C(T-t)^{-1/[2(p-1)]}$, where $x \in \partial\Omega$ and $C > 0$.

In [6], Fila and Levine studied the quenching problem for the scalar case

$$(1.5)\qquad \begin{cases} u_t = u_{xx}, & (x,t) \in (0,1)\times(0,T),\\ u_x(0,t) = 0,\ u_x(1,t) = -u^{-q}(1,t), & t \in (0,T),\\ u(x,0) = u_0(x) > 0, & x \in [0,1], \end{cases}$$

and obtained that $u(1,t) \sim (T-t)^{1/[2(q+1)]}$, where $f \sim g$ means that $c_1 f \leqslant g \leqslant c_2 f$ holds for t close to T and some positive constants c_1, c_2. We will use this notation throughout this paper.

Pablo, Quiros and Rossi [13] firstly distinguished non-simultaneous quenching from simultaneous one. They considered a heat system coupled via inner absorptions:

$$(1.6)\qquad \begin{cases} u_t = u_{xx} - v^{-p},\ v_t = v_{xx} - u^{-q}, & (x,t) \in (0,1)\times(0,T),\\ u_x(0,t) = v_x(0,t) = u_x(1,t) = v_x(1,t) = 0, & t \in (0,T),\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in [0,1], \end{cases}$$

where $\min_{x\in[0,1]} u(x,t) = u(0,t)$, $\min_{x\in[0,1]} v(x,t) = v(0,t)$ under certain assumptions on the initial data $u_0, v_0 > 0$. For the coupled equations (1.6), the following quenching rates were proved in [13]:

(a) If quenching is non-simultaneous and, for instance, v is the quenching component, then $v(0,t) \sim (T-t)$ for t close to T.

(b) If quenching is simultaneous, then for t close to T:
1. $u(0,t) \sim (T-t)^{(p-1)/(pq-1)}$, $v(0,t) \sim (T-t)^{(q-1)/(pq-1)}$, if $p,q > 1$ or $p,q < 1$;
2. $u(0,t)$, $v(0,t) \sim (T-t)^{1/2}$, if $p = q = 1$;
3. $u(0,t) \sim |\log(T-t)|^{-1/(q-1)}$, $v(0,t) \sim (T-t)|\log(T-t)|^{q/(q-1)}$, if $q > p = 1$.

For the system

$$(1.7)\qquad \begin{cases} u_t = u_{xx},\ v_t = v_{xx}, & (x,t) \in (0,1)\times(0,T),\\ u_x(0,t) = 0,\ u_x(1,t) = -v^{-p}(1,t), & t \in (0,T),\\ v_x(0,t) = 0,\ v_x(1,t) = -u^{-q}(1,t), & t \in (0,T),\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in [0,1], \end{cases}$$

the finite time quenching results from the coupled singular nonlinear boundary flux by Zheng and Song [14], other than the situation in the model of (1.6) with coupled nonlinear absorption terms. The quenching in (1.7) may be either simultaneous or non-simultaneous. This is determined by particular ranges of nonlinear exponents and initial data. They showed that $\{x = 1\}$ is the only quenching point and there are three kinds of simultaneous quenching rates in time can be briefly described in the following conclusions:

1. $u(1,t) \sim (T-t)^{\alpha/2}$, $v(1,t) \sim (T-t)^{\beta/2}$ for $p,q > 1$ or $p,q < 1$;
2. $u(1,t) \sim (T-t)^{1/4}$, $v(1,t) \sim (T-t)^{1/4}$ for $p = q = 1$;
3. $u(1,t) \sim |\log(T-t)|^{-1/(q-1)}$, $v(1,t) \sim (T-t)|\log(T-t)|^{q/(q-1)}$ for $1 = p < q$,

where $\alpha = (p-1)/(pq-1)$, $\beta = (q-1)/(pq-1)$.

And $v(1,t) \sim (T-t)^{1/(p+1)}$ for non-simultaneous quenching with v quenching only.

The quenching in (1.3) may be either simultaneous or non-simultaneous. This is determined by particular ranges of nonlinear exponents and initial data, denoted as follows:

(H1) (i) $q \geq p > 1$: $v_0^{1-p}(r) \leq C_1(\mathrm{i}) u_0^{1-q}(r)$ with $C_1(\mathrm{i}) \geq (p-1)/(q-1)$,

(ii) $q > p = 1$: $-\log v_0(r) \leq C_1(\mathrm{ii}) u_0^{1-q}(r)$ with $C_1(\mathrm{ii}) \geq 1/(q-1)$,

(iii) $p = q = 1$: $c_1 u_0(r) \leq v_0(r) \leq c_2 u_0(r)$ with $c_1, c_2 > 0$,

(H2) $0 < p \leq q < 1$: $v_0^{1-p}(r) \leq C_2 u_0^{1-q}(r)$ with $0 < C_2 \leq (p-1)/(q-1)$,

(H3) $0 < p < 1 \leq q$: $v_0^{1-p}(r) \leq C_3 u_0^{1-q}(r)$ with $C_3 \geq 0$,

(H4) $u_0^{-(q+1)/2}(1)(u_0''(r) + ((N-1)/r)u_0'(r)) \geq c_0 v_0^{-(p+1)/2}(1)(v_0''(r) + ((N-1)/r)v_0'(r))$ with $\sqrt{p}/\sqrt{q} < c_0 < (p+1)/(\sqrt{C}_4(q+1))$,

for $r \in [0, 1]$, where C_4 is one of the constants in the assumptions (H1)–(H3).

In this paper, we extend the problem (1.1)–(1.4) in [7] to higher dimensional space and study the asymptotic profile of quenching for radial solutions. In comparison with the one dimensional case, some additional terms need to be taken care of while a verity of auxiliary functions are constructed by the maximum principle. We consider radial solutions in a ball and we will propose a criterion to identify simultaneous and non-simultaneous quenching for (1.3) under some assumptions, and then establish asymptotic estimates of quenching with different conditions, exactly the asymptotic profile near the quenching point. We will show that $\{r = 1\}$ is the only quenching point and that the three kinds of simultaneous quenching profiles can be briefly described in the following conclusions:

1. $u(r, T) \sim (1-r)^{(p-1)/(pq-1)}$, $v(r, T) \sim (1-r)^{(q-1)/(pq-1)}$, if $p, q > 1$ or $p, q < 1$;
2. $u(r, T), v(r, T) \sim (1-r)^{1/2}$, if $p = q = 1$;
3. $u(r, T) \sim |\log(1-r)|^{-1/(q-1)}$, $v(r, T) \sim (1-r)|\log(1-r)|^{q/(q-1)}$, if $q > p = 1$,

for r close to 1.

If quenching is non-simultaneous and, for instance, v is the quenching component, then $v(r, T) \sim (1-r)$ for r close to 1.

For simultaneous and non-simultaneous quenching cases, the quenching rates of radial solutions are very similar to those in the one dimensional case (see [14]), which will be described in Remarks 3.1 and 4.2 below.

REMARK 1.1. It's interesting that an open problem is left on whether the quenching profile and rate are unique without the assumption on the initial data so that the solution is monotonically decreasing both in t and r. Also, the quenching behavior of non-radial solutions in higher dimensional space is still open. They will be the subjects of future research.

The paper is organized as follows. We begin with a theorem on finite time quenching and quenching sets in Section 2 together with two basic lemmas as preliminaries of the paper, and then, in Section 3, we propose the criterion to identify the simultaneous and non-simultaneous quenching. As the main results of the paper, the three kinds of simultaneous quenching behaviors will be proved in Section 4.

Throughout this paper, C and c denote different positive constants.

2. Finite time quenching and preliminaries

Let (u, v) be a solution of (1.3) with $0 < u_0 \le M$, $0 < v_0 \le K$ on $[0, 1]$. Then $0 < u \le M$, $0 < v \le K$ for all t in the existence interval and $r \in [0, 1]$.

At first, we consider the following quenching theorem.

Theorem 2.1. *Assume $p, q > 0$. Then every solution (u, v) of* (1.3) *quenches in finite time with the only quenching point $r = 1$.*

Proof. Since $u_0', v_0' \le 0$, we know that $u_r, v_r \le 0$ by the maximum principle. Thus,

$$\min_{r\in[0,1]} u(r, t) = u(1, t), \quad \min_{r\in[0,1]} v(r, t) = v(1, t), \quad t \in [0, T).$$

For $F(t) = \int_0^1 r^{N-1}u(r, t)\,dr$ and $G(t) = \int_0^1 r^{N-1}v(r, t)\,dr$, we have

$$\begin{aligned} F'(t) &= \int_0^1 r^{N-1}u_t(r, t)\,dr = \int_0^1 (r^{N-1}u_{rr} + (N-1)r^{N-2}u_r)\,dr \\ &= u_r(1, t) = -v^{-p}(1, t) \le -K^{-p}, \\ G'(t) &= \int_0^1 r^{N-1}v_t(r, t)\,dr = \int_0^1 (r^{N-1}v_{rr} + (N-1)r^{N-2}v_r)\,dr \\ &= v_r(1, t) = -u^{-q}(1, t) \le -M^{-q}, \end{aligned}$$

and so

$$F(t) \le F(0) - tK^{-p} \le \frac{M}{N} - tK^{-p}, \quad G(t) \le G(0) - tM^{-q} \le \frac{K}{N} - tM^{-q}.$$

On the other hand,

$$\begin{aligned} F(t) &= \int_0^1 r^{N-1}u(r, t)\,dr \ge u(1, t)\int_0^1 r^{N-1}\,dr = \frac{u(1, t)}{N}, \\ G(t) &= \int_0^1 r^{N-1}v(r, t)\,dr \ge v(1, t)\int_0^1 r^{N-1}\,dr = \frac{v(1, t)}{N}. \end{aligned}$$

Then we have

$$u(1, t) \le M - NtK^{-p}, \quad v(1, t) \le K - NtM^{-q},$$

which means that there exists $T > 0$ such that $\lim_{t\to T^-} \min\{u(1, t), v(1, t)\} = 0$.

To show that $r = 1$ is the unique quenching point, it suffices to prove that the quenching cannot occur at any inner point $r_0 \in (1/2, 1)$. Define

$$h(r,t) = u_r(r,t) + \frac{\varepsilon}{2K^p}\left(r - \frac{1}{4}\right)^2,$$

where $\varepsilon > 0$. Since $u_r(r, T/2) < 0$ for $r \in (0, 1]$, there exists $\varepsilon_0 > 0$ such that $u_r(r, T/2) \le -\varepsilon_0 < 0$ for $r \in [1/4, 1]$. If we take $\varepsilon \le 32K^p\varepsilon_0/9$, then $h(r, T/2) \le 0$, $r \in [1/4, 1]$. We have

$$\begin{aligned}
&h_t - h_{rr} - \frac{N-1}{r}h_r + \frac{N-1}{r^2}h \\
&= -\frac{\varepsilon}{K^p} - \frac{\varepsilon(N-1)}{rK^p}\left(r - \frac{1}{4}\right) + \frac{\varepsilon(N-1)}{2r^2K^p}\left(r - \frac{1}{4}\right)^2 \\
&= -\frac{\varepsilon}{K^p}\left(1 + \frac{N-1}{r}\left(r - \frac{1}{4}\right) - \frac{N-1}{2r^2}\left(r - \frac{1}{4}\right)^2\right) \\
&= -\frac{\varepsilon}{2r^2K^p}\left((N+1)r^2 - \frac{N-1}{16}\right) \\
&\le 0
\end{aligned}$$

for $(r,t) \in (1/4, 1) \times (T/2, T)$. And

$$h\left(\frac{1}{4}, t\right) = u_r\left(\frac{1}{4}, t\right) \le 0, \quad h(1,t) = -v^{-p}(1,t) + \frac{9\varepsilon}{32K^p} \le 0$$

for $t \in (T/2, T)$. By the maximum principle, $h \le 0$ in $(1/4, 1) \times (T/2, T)$, which means that

$$u_r(r,t) + \frac{\varepsilon}{2K^p}\left(r - \frac{1}{4}\right)^2 \le 0, \quad (r,t) \in \left(\frac{1}{4}, 1\right) \times \left(\frac{T}{2}, T\right).$$

Integrating with respect to r, we obtain

$$\begin{aligned}
u(r,t) &\ge u(1,t) + \frac{\varepsilon(1-r)}{6K^p}\left(\frac{9}{16} + \frac{3}{4}\left(r - \frac{1}{4}\right) + \left(r - \frac{1}{4}\right)^2\right) \\
&\ge u(1,t) + \frac{3\varepsilon(1-r)}{32K^p}, \quad (r,t) \in \left(\frac{1}{4}, 1\right) \times \left(\frac{T}{2}, T\right).
\end{aligned}$$

Hence, for any $r_0 \in (1/2, 1)$,

$$\liminf_{t \to T^-} u(r_0, t) \ge \frac{3\varepsilon(1-r_0)}{32K^p} > 0.$$

Similarly, we have also $\liminf_{t \to T^-} v(r_0, t) > 0$.

We have shown that quenching cannot occur in the interior of $(0, 1)$. The proof is complete. □

Next, we introduce two basic lemmas as preliminaries.

Lemma 2.1. *Let* (u, v) *be a solution to* (1.3) *with assumptions* (H1)–(H3).
(i) *If* $q \geq p > 0$, $q, p \neq 1$, *then there exists a positive constant* C *such that*

$$v^{1-p}(r, t) \leq Cu^{1-q}(r, t), \quad (r, t) \in [0, 1] \times [0, T), \tag{2.1}$$

where C *can be one of the constants in the assumptions* (H1)-(i)–(H3).
(ii) *If* $q > p = 1$, *then there exists a positive constant* $C = C_1$(ii) *such that*

$$-\log v(r, t) \leq Cu^{-q+1}(r, t), \quad (r, t) \in [0, 1] \times [0, T). \tag{2.2}$$

(iii) *If* $q = p = 1$, *then* $u \sim v$ *for* t *close to* T.

Proof. (i) For $q \geq p > 0$ with $p,q \neq 1$, set $\Phi(r,t) = v^{1-p} - Cu^{1-q}(r,t)$. We know

$$\begin{aligned}&\Phi_t - \Phi_{rr} - \frac{N-1}{r}\Phi_r - (pv^{-1}v_r + qu^{-1}u_r)\Phi_r + q(1-p)u^{-1}v^{-1}u_r v_r \Phi\\&= C(p-q)v^{-1}u^{-q}u_r v_r \leq 0\end{aligned}$$

in $(0, 1) \times (0, T)$, and moreover

$$\Phi_r(1, t) = ((p-1) - C(q-1))u^{-q}(1, t)v^{-p}(1, t) \leq 0, \quad t \in (0, T)$$

for each of (H1)–(H3), $q, p \neq 1$. The facts $\Phi_r(0, t) = 0$ for $t \in (0, T)$ and $\Phi(r, 0) = v_0^{1-p}(r) - Cu_0^{1-q}(r) \leq 0$ for $r \in [0, 1]$ are obviously true under the assumptions of the lemma. By the maximum principle, $\Phi(r, t) \leq 0$, i.e., $v^{1-p}(r, t) \leq Cu^{1-q}(r, t)$ for $(r, t) \in [0, 1] \times [0, T)$.

(ii) For the case of $q > p = 1$, let $\Psi(r, t) = -(\log v + Cu^{-q+1})(r, t)$. By taking C large enough, we get

$$\begin{aligned}&\Psi_t - \Psi_{rr} - \frac{N-1}{r}\Psi_r - (v^{-1}v_r + qu^{-1}u_r)\Psi_r\\&= (C(1-q)u^{-q+1} + q)u^{-1}v^{-1}u_r v_r \leq 0\end{aligned}$$

for $(0, 1) \times (0, T)$, and

$$\begin{aligned}&\Psi_r(0, t) = 0, \quad \Psi_r(1, t) = (C(1-q) + 1)u^{-q}(1, t)v^{-1}(1, t) \leq 0, \quad t \in (0, T),\\&\Psi(r, 0) = -(\log v_0(r) + Cu_0^{-q+1}(r)) \leq 0, \quad r \in [0, 1].\end{aligned}$$

It follows by the maximum principle that

$$\Psi(r,t) = -(\log v + Cu^{-q+1})(r,t) \le 0, \quad (r,t) \in [0,1] \times [0,T).$$

(iii) For $q = p = 1$, let $w = v - cu$. Then $w_t - w_{rr} - (N-1)w_r/r = 0$ in $(0,1) \times (0,T)$. $w_r(0,t) = 0$ and $w_r(1,t) + w(1,t)/(vu) = 0$. Therefore, we can show by the maximum principle that $u \sim v$.

The lemma is proved. $\square$

Lemma 2.2. *If* (H4) *and one of* (H1)–(H3) *hold, then*

$$u^{-(q+1)/2}(1,t)u_t(r,t) \ge c_0 v^{-(p+1)/2}(1,t)v_t(r,t) \tag{2.3}$$

for $(r,t) \in [0,1] \times (0,T)$ *and the positive constant* c_0 *in* (H4).

Proof. Set

$$J(r,t) = u^{-(q+1)/2}(1,t)u_t(r,t) - c_0 v^{-(p+1)/2}(1,t)v_t(r,t).$$

Since $u, v \ge 0$, $u_t, v_t \le 0$, we have on the parabolic boundary that

$$\begin{aligned}
J(r,0) &= u_0^{-(q+1)/2}(1)\left(u_0''(r) + \frac{N-1}{r}u_0'(r)\right) \\
&\quad - c_0 v_0^{-(p+1)/2}(1)\left(v_0''(r) + \frac{N-1}{r}v_0'(r)\right) \ge 0, \quad r \in [0,1], \\
J_r(0,t) &= u^{-(q+1)/2}(1,t)(u_r(0,t))_t - c_0 v^{-(p+1)/2}(1,t)(v_r(0,t))_t = 0, \quad t \in (0,T), \\
&J_r(1,t) + c_0 q v^{-(p+1)/2}(1,t)u^{-(q+1)/2}(1,t)J(1,t) \\
&= (p - c_0^2 q)u^{-(q+1)/2}v^{-p-1}(1,t)J(1,t)v_t(1,t) \ge 0, \quad t \in (0,T)
\end{aligned}$$

since $p \le c_0^2 p$. Moreover, by Lemma 2.1 (i), we have with $c_0 < (p+1)/(\sqrt{C}_4(q+1))$ that

$$\begin{aligned}
&J_t - J_{rr} - \frac{N-1}{r}J_r + \frac{c_0(q+1)}{2}u^{(q-1)/2}(1,t)v^{-(p+1)/2}(1,t)v_t(1,t)J(r,t) \\
&+ \frac{q+1}{2}u^{-1}(1,t)u_t(r,t)J(1,t) \\
&= \frac{c_0}{2}((p+1)v^{(p-1)/2}(1,t) - c_0(q+1)u^{(q-1)/2}(1,t))v^{-p-1}(1,t)v_t(1,t)v_t(r,t) \\
&\ge 0
\end{aligned}$$

for $(r,t) \in (0,1) \times (0,T)$. By the maximum principle (see, e.g., Lemma 2.1 of [3]), $J(r,t) \ge 0$, or equivalently, $u^{-(q+1)/2}(1,t)u_t(r,t) \ge c_0 v^{-(p+1)/2}(1,t)v_t(r,t)$ for $(r,t) \in [0,1] \times (0,T)$. This completes the proof. $\square$

3. Simultaneous and non-simultaneous quenching

In this section, we will prove a criterion to identify the simultaneous and non-simultaneous quenching, which is given as the following theorem.

Theorem 3.1. *Let* (u, v) *be a solution of* (1.3) *with quenching time* T.
(i) *If* $p, q \geq 1$, *then simultaneous quenching will occur for any positive initial data satisfying* (H1) *and* (H4). *If* $0 < p < 1 \leq q$ *with* (H3) *and* (H4) *hold, then the quenching is non-simultaneous.*
(ii) *If* $0 < p, q < 1$, *then for any positive* $u_0(r)$ *there exists* $v_0(r)$ *such that* (H2) *and* (H4) *hold, and the quenching is non-simultaneous.*
(iii) *In the case of non-simultaneous quenching, for instance, if* v *is the quenching component, then* $v(r, T) \sim (1 - r)$ *for* r *close to* 1.

We needs three lemmas to prove the three parts of the theorem, respectively.

Lemma 3.1. *Assume that the quenching is non-simultaneous and, for instance,* v *is the quenching component with quenching time* T. *Then* $v(r, T) \sim (1 - r)$ *for* $0 < 1 - r \ll 1$.

Proof. Notice that $\lim_{t\to T^-} v(1, t) = 0$ and $0 < c < u(1, t) \leq M$ for $0 < 1 - r \ll 1$. Set

$$J(r, t) = v_r(r, t) + \varphi(r)u^{-q},$$

where $\varphi, \varphi', \varphi'' \geq 0$, $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(1) \geq 0$, $\varphi(r) \leq -u_0^q(r)v_0'(r)$. It is easy to see that $J(r, 0) \leq 0$, $J(0, t) = J(1, t) = 0$, and

$$\begin{aligned}
&J_t - J_{rr} - \frac{N-1}{r}J_r + \frac{N-1}{r^2}J \\
&= \frac{N-1}{r^2}\varphi u^{-q} - \varphi'' u^{-q} + 2q\varphi' u^{-q-1}u_r - q(q+1)\varphi u^{-q-2}u_r^2 - \frac{N-1}{r}\varphi' u^{-q} \\
&= \varphi u^{-q-2}\left(\frac{N-1}{r^2}u^2 - q(q+1)u_r^2\right) - \varphi'' u^{-q} + 2q\varphi' u^{-q-1}u_r - \frac{N-1}{r}\varphi' u^{-q} \\
&\leq 0
\end{aligned}$$

for $(r,t) \in (0,1)\times(0,T)$. By the maximum principle, $J(r,t) \leq 0$ for $(r,t) \in [0,1]\times[0,T)$. By $\varphi'(r) \geq 0$, we know that there exists $0 < 1 - r_1 \ll 1$ such that $\varphi(r) \geq c > 0$ for any $r \in [r_1, 1]$. By $0 < c < u(1, t) \leq M$, we have $-v_r \geq cu^{-q} \geq C$. Integrating the above inequality from r to 1, we obtain

$$v(r, t) \geq C(1 - r)$$

for $r \in (r_1, 1]$, $0 < T - t \ll 1$.

On the other hand, set $J(r,t) = v_r(r,t) + Cu^\beta$, where $0 < \beta < 1$ and C is large enough. Then

$$J_t - J_{rr} - \frac{N-1}{r} J_r + \frac{N-1}{r^2} J = -C\beta(\beta-1)u^{\beta-2}u_r^2 + C\frac{N-1}{r^2}u^\beta \geq 0$$

for $(r,t) \in (0,1)\times(0,T)$. And $J(0,t) = Cu^\beta(0,r) \geq 0$, $J(1,t) = -u^{-q}(1,t) + Cu^\beta(1,t) \geq 0$ where C is large enough and $0 < c < u(1,t) \leq M$. By the maximum principle, $J(r,t) \geq 0$, that is

$$-v_r \leq Cu^\beta \leq C.$$

Integrating this inequality from r to 1, we get for $0 < T - t \ll 1$, $0 < 1 - r \ll 1$,

$$v(r,t) \leq C(1-r).$$

Let $t \to T$ and the proof is complete. □

Lemma 3.2. *The quenching in* (1.3) *is simultaneous under the assumptions* (H1) *and* (H4), *and is non-simultaneous if* (H3) *and* (H4) *are satisfied.*

Lemma 3.3. *Assume that* (H2) *and* (H4) *hold. Then for any initial data* u_0, *there exists an open set* (*in the* C^2 *topology*) *of initial data* v_0 *such that* v *quenches while* u *keeps strictly positive.*

The proofs of Lemmas 3.2 and 3.3 are similar to the ones of Lemmas 3.2 and 3.3 in [14]. Here we omit them.

Theorem 3.1 follows from Lemmas 3.1–3.3 directly.

REMARK 3.1. We note that, in the case of non-simultaneous case, we could also get the quenching rate. For instance, if v is the quenching component, then $v(1,t) \sim (T-t)^{1/(p+1)}$. The conclusion is similar to the one dimensional case that has been proved in [14].

4. Simultaneous quenching profiles

Now we deal with the more interesting simultaneous quenching profiles. Consider the case of $q \geq p > 1$ at first.

Theorem 4.1. *Let* (H4) *with either* (H1) *or* (H2) *hold*, (u, v) *be the solution of* (1.3) *with quenching time* T. *Then*

$$u(r,T) \sim (1-r)^{(1-p)/(1-pq)}, \quad v(r,T) \sim (1-r)^{(1-q)/(1-pq)}, \quad 0 < 1-r \ll 1.$$

We will prove the upper and lower bounds for the quenching profiles for u and v by a chain of lemmas.

Lemma 4.1. *Let* (H1) *and* (H4) *hold,* (u, v) *be the solution of* (1.3) *with quenching time* T. *Then there exists a positive constant* C *such that*

$$v(r,t) \geq C(1-r)^{(1-q)/(1-pq)}$$

for $0 < T-t \ll 1$, $0 < 1-r \ll 1$.

Proof. Set

$$J(r,t) = v_r(r,t) + \varphi(r)u^{-q},$$

where φ, φ', $\varphi'' \geq 0$, $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(1) \geq 0$, $\varphi(r) \leq -v_0^p(r)u_0'(r)$. By the Lemma 3.1 and Lemma 2.1 (i), we have

$$-v_r \geq cu^{-q} \geq cv^{-q(1-p)/(1-q)},$$

or equivalently

$$-v^{q(1-p)/(1-q)}v_r \geq c.$$

Integrating the above equality from r to 1, we can get

$$v(r,t) \geq c(1-r)^{(1-q)/(1-pq)}, \quad 0 < 1-r \ll 1,\ 0 < T-t \ll 1. \qquad \square$$

Lemma 4.2. *Let* (H1) *and* (H4) *hold,* (u, v) *be the solution of* (1.3) *with quenching time* T. *Then there exists a positive constant* C *such that*

$$u(r,t) \leq C(1-r)^{(1-p)/(1-pq)}$$

for $0 < T-t \ll 1$, $0 < 1-r \ll 1$.

Proof. Set

$$J(r,t) = (1-r)^{\alpha}u_r(r,t) + Cu^{\beta},$$

where $0 < \alpha < 1$, $0 < \beta < 1$ and $(1-\alpha)/(1-\beta) = (1-p)/(1-pq)$. We have

$$\begin{aligned}
&J_t - J_{rr} - \frac{N-1}{r}J_r + \frac{N-1}{r^2}J - 2\alpha(1-r)^{-1}J_r \\
&= -\alpha(\alpha-1)(1-r)^{\alpha-2}u_r - C\beta(\beta-1)u^{\beta-2}u_r^2 + C\frac{N-1}{r^2}u^{\beta} + \alpha\frac{N-1}{r}(1-r)^{\alpha-1}u_r \\
&\quad + 2\alpha^2(1-r)^{\alpha-2}u_r - 2C\alpha\beta u^{\beta-1}(1-r)^{-1}u_r
\end{aligned}$$

$$
\begin{aligned}
&= \alpha(\alpha+1)(1-r)^{\alpha-2}u_r + \alpha\frac{N-1}{r}(1-r)^{\alpha-1}u_r - C\beta(\beta-1)u^{\beta-2}u_r^2 \\
&\quad - 2C\alpha\beta u^{\beta-1}(1-r)^{-1}u_r + C\frac{N-1}{r^2}u^\beta \\
&= \alpha(1-r)^{\alpha-2}u_r\left((\alpha+1) + \frac{N-1}{r}(1-r) - 2C\beta u^{\beta-1}(1-r)^{1-\alpha}\right) \\
&\quad - C_2\beta(\beta-1)u^{\beta-2}u_r^2 + C\frac{N-1}{r^2}u^\beta \\
&\geq 0
\end{aligned}
$$

for $r \in (0, 1)$, $0 < T - t \ll 1$ and C large enough. It easy to see that $J(0, t) = Cu^\beta(0, t) \geq 0$, $J(1, t) = Cu^\beta(1, t) \geq 0$. By the maximum principle, $J(r, t) \geq 0$ for $0 < 1 - r \ll 1$, $0 < T - t \ll 1$. That is

$$(1-r)^\alpha u_r(r, t) + Cu^\beta \geq 0,$$

equivalently

$$-u^{-\beta}u_r \leq C(1-r)^{-\alpha}.$$

Integrating the above equality from r to 1, we get

$$u(r, t) \leq C(1-r)^{(1-\alpha)/(1-\beta)} = C(1-r)^{(1-p)/(1-pq)}$$

for $0 < T - t \ll 1$, $0 < 1 - r \ll 1$. □

Lemma 4.3. *Let* (H1) *and* (H4) *hold*, (u, v) *be the solution of* (1.3) *with quenching time* T. *Then there exists a positive constant* C *such that*

$$v(r, t) \leq C(1-r)^{(1-q)/(1-pq)}$$

for $0 < T - t \ll 1$, $0 < 1 - r \ll 1$.

Proof. Set

$$J(r, t) = (1-r)^\lambda v_r(r, t) + Cv^\gamma,$$

where $0 < \gamma < 1$, $0 < \lambda < 1$ and $(1-\lambda)/(1-\gamma) = (1-q)/(1-pq)$. Similarly to the proof of Lemma 4.2, we can get

$$v(r, t) \leq C(1-r)^{(1-\lambda)/(1-\gamma)} = C(1-r)^{(1-q)/(1-pq)}$$

for $0 < T - t \ll 1$, $0 < 1 - r \ll 1$. □

Lemma 4.4. *Let* (H1) *and* (H4) *hold,* (u, v) *be the solution of* (1.3) *with quenching time* T*. Then there exists a positive constant* c *such that*

$$u(r, t) \geq c(1-r)^{(1-p)/(1-pq)}$$

for $0 < T - t \ll 1$, $0 < 1 - r \ll 1$.

Proof. Set

$$J(r, t) = u_r(r, t) + \varphi(r) v^{-p},$$

where φ, φ', $\varphi'' \geq 0$, $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(1) \geq 0$, $\varphi(r) \leq -v_0^p(r) u_0'(r)$. It is easy to see that $J(r, 0) \leq 0$, $J(0, t) = J(1, t) = 0$, and

$$\begin{aligned}
&J_t - J_{rr} - \frac{N-1}{r} J_r + \frac{N-1}{r^2} J \\
&= \frac{N-1}{r^2} \varphi v^{-p} - \varphi'' v^{-p} + 2p\varphi' v^{-p-1} v_r - p(p+1) \varphi v^{-p-2} v_r^2 - \frac{N-1}{r} \varphi' v^{-p} \\
&= \varphi v^{-p-2} \left(\frac{N-1}{r^2} v^2 - p(p+1) v_r^2 \right) - \varphi'' v^{-p} + 2p\varphi' v^{-p-1} v_r - \frac{N-1}{r} \varphi' v^{-p} \\
&\leq 0
\end{aligned}$$

for $(r,t) \in (0,1) \times (0,T)$. By the maximum principle, $J(r,t) \leq 0$ for $(r,t) \in [0,1] \times [0,T)$. By $\varphi'(r) \geq 0$, we know that there exists $0 < 1 - r_1 \ll 1$ such that $\varphi(r) \geq c > 0$ for any $r \in (r_1, 1]$. Then we have

$$-u_r(r, t) \geq \varphi(r) v^{-p} \geq c(1-r)^{-p(1-q)/(1-pq)},$$

equivalently where we use the conclusion of Lemma 4.3. Integrating the above equality from r to 1, we can obtain $u(r,t) \geq c(1-r)^{(1-p)/(1-pq)}$ for $r \in (r_1, 1]$, $0 < T - t \ll 1$. □

REMARK 4.1. We obtain the quenching profiles for (H1) ($1 < p \leq q$) by letting $t \to T$ and combining Lemmas 4.1–4.4. The subcase of (H2) ($0 < p \leq q < 1$) can be treated by a similar way. The main difference between the two subcases is that, by Theorem 3.1, the quenching should be assumed simultaneous for the second case, while the quenching in the first case is always simultaneous.

Theorem 4.1 is proved by Lemmas 4.1–4.4 and Remark 4.1.

Finally, we consider the simultaneous quenching profiles for the other cases.

Theorem 4.2. *Let* (u, v) *be the solution of* (1.3) *with quenching time* T*. For* $0 < 1 - r \ll 1$,

(1) *If* $p = q = 1$, *the simultaneous quenching profile is*

$$u(r, T) \sim (1-r)^{1/2}, \quad v(r, T) \sim (1-r)^{1/2}.$$

(2) *If* $p = 1 < q$, *the simultaneous quenching profile is*

$$u(r, T) \sim |\log(1-r)|^{-1/(q-1)}, \quad v(r, T) \sim (1-r)|\log(1-r)|^{q/(q-1)}.$$

Proof. (1) For $p = q = 1$, by using Lemma 4.1 with $q = 1$ and noticing $u \sim v$, we can obtain $v(r, t) \geq C(1-r)^{1/2}$, and $v(r, t) \leq C(1-r)^{1/2}$ from Lemma 4.3 where $\lambda = 5/8$, $\gamma = 1/4$. Thus, $u(r, t) \sim v(r, t) \sim (1-r)^{1/2}$ for $0 < T - t \ll 1$, $0 < 1 - r \ll 1$.

(2) Now we consider the case of $q > p = 1$. We know from the Lemma 2.1 (ii) that $-\log v(r, t) \leq cu^{-q+1}(r, t)$, that is

$$v(x, t) \geq e^{-cu^{-q+1}(r,t)}, \quad (r, t) \in [0, 1] \times [0, T).$$

To get the upper bound of the $v(r, t)$, set

$$J(r, t) = v_r(r, t) - C \log(1-r) v^{1/q}(r, t).$$

It easy to see that $J(0, t) = 0$ and C is large to make $J(1, t) \geq 0$. We have

$$\begin{aligned}
&J_t - J_{rr} - \frac{N-1}{r} J_r + \frac{N-1}{r^2} J \\
&= \frac{C}{(1-r)^2} v^{1/q} - \frac{2C}{q(1-r)} v^{1/q-1} v_r + \frac{C}{q}\left(\frac{1}{q} - 1\right) \log(1-r) v^{1/q-2} v_r^2 \\
&\quad - \frac{C(N-1)}{r(1-r)} v^{1/q} - \frac{C(N-1)}{r^2} \log(1-r) v^{1/q} \\
&= -\frac{2C}{q(1-r)} v^{1/q-1} v_r + \frac{C}{q}\left(\frac{1}{q} - 1\right) \log(1-r) v^{1/q-2} v_r^2 \\
&\quad + \frac{C}{1-r} v^{1/q} \left(\frac{1}{1-r} - \frac{N-1}{r} - \frac{N-1}{r^2}(1-r)\log(1-r)\right) \\
&\geq 0
\end{aligned}$$

for C large enough and $(r, t) \in (r_1, 1) \times (0, T)$, where $0 < 1 - r_1 \ll 1$. By the maximum principle, $J(r, t) \geq 0$ for $(r, t) \in (r_1, 1] \times [0, T)$. Then we have

$$-v^{-1/q} v_r \leq -C \log(1-r).$$

Integrating the inequality from r to 1, we get

$$v^{1-1/q}(r, t) \leq -C(1-r)\log(1-r),$$

or equivalently

$$v(r, t) \leq C(1-r)^{q/(q-1)} |\log(1-r)|^{q/(q-1)} \leq C(1-r)|\log(1-r)|^{q/(q-1)}$$

for $0 < T - t \ll 1$, $0 < 1 - r \ll 1$.

By $-\log v(r,t) \le cu^{-q+1}(r,t)$, we have $u^{1-q}(r,t) \ge -\log v(r,t)$. Using the above upper bound of $v(r,t)$, we obtain

$$\begin{aligned} u^{1-q}(r,t) &\ge -\log v(r,t) \ge -C\log((1-r)|\log(1-r)|^{q/(q-1)}) \\ &= -C\log(1-r) - \frac{Cq}{q-1}\log(|\log(1-r)|) \\ &\ge -C\log(1-r) \end{aligned}$$

for $(r,t) \in (0,1) \times (0,T)$. Then

$$u(r,t) \le C|\log(1-r)|^{-1/(q-1)}$$

for $0 < T - t \ll 1$, $0 < 1 - r \ll 1$.

To get the lower bound of the $v(r,t)$, we set $J(r,t) = v_r(r,t) + ru^{-q}$. Obviously, $J(0,t) = J(1,t) = 0$. We have

$$J_t - J_{rr} - \frac{N-1}{r}J_r + \frac{N-1}{r^2}J = 2qu^{-q-1}u_r - rq(q+1)u^{-q-2}u_r^2 \le 0$$

for $(r,t) \in (0,1) \times (0,T)$. By the maximum principle, $J(r,t) \le 0$ for $(r,t) \in [0,1] \times [0,T)$. Then we have

$$-v_r(r,t) \ge ru^{-q} \ge C|\log(1-r)|^{q/(q-1)}.$$

Integrating the inequality from r to 1 where $0 < 1 - r \ll 1$,

$$v(r,t) \ge C\int_r^1 |\log(1-s)|^{q/(q-1)}\,ds.$$

Setting $\log(1-s) = -w$, we get

$$v(r,t) \ge C\int_{-\log(1-r)}^{\infty} w^{q/(q-1)}e^{-w}\,dw.$$

It is known that the incomplete Gamma function $\Gamma(a,z) = \int_z^\infty w^{a-1}e^{-w}\,dw$ satisfies $\Gamma(a,z) \sim z^{a-1}e^{-z}$ for $z \to \infty$. For the incomplete Gamma function $\Gamma(a, -\log(1-r))$ with $a - 1 = q/(q-1)$, we obtain

$$v(r,t) \ge C(1-r)|\log(1-r)|^{q/(q-1)}$$

ford $0 < T - t \ll 1$, $0 < 1 - r \ll 1$.

As for the lower bound of the $v(r,t)$, we set

$$J(r,t) = u_r(r,t) + rv^{-1}.$$

It is easy to see that $J(0,t) = J(1,t) = 0$. And

$$J_t - J_{rr} - \frac{N-1}{r} J_r + \frac{N-1}{r^2} J = 2v^{-2} v_r - 2r v^{-3} v_r^2 \le 0$$

for $(r,t) \in (0,1)\times(0,T)$. By the maximum principle, $J(r,t) \le 0$ for $(r,t) \in [0,1]\times[0,T)$. Then we have

$$-u_r \ge r v^{-1} \ge C(1-r)^{-1} |\log(1-r)|^{-q/(q-1)}$$

for $0 < T - t \ll 1$, $0 < 1 - r \ll 1$. Integrating the above inequality from r to 1, we can get

$$u(r,t) \ge C|\log(1-r)|^{-1/(q-1)}$$

for $0 < T - t \ll 1$, $0 < 1 - r \ll 1$. Let $t \to T$ and the proof is complete. □

REMARK 4.2. For the simultaneous quenching case, we note that $\{r = 1\}$ is the only quenching point. Moreover, by virtue of Lemma 2.1, we also could get three kinds of simultaneous quenching rates described briefly as the following conclusions, which are very similar to those in the one dimensional case (see [14]),

$$u(1,t) \sim (T-t)^{\alpha/2}, \quad v(1,t) \sim (T-t)^{\beta/2} \quad \text{for} \quad p,q > 1 \quad \text{or} \quad p,q < 1;$$
$$u(1,t) \sim (T-t)^{1/4}, \quad v(1,t) \sim (T-t)^{1/4} \quad \text{for} \quad p = q = 1;$$
$$u(1,t) \sim |\log(T-t)|^{-1/(q-1)}, \quad v(1,t) \sim (T-t)|\log(T-t)|^{q/(q-1)} \quad \text{for} \quad 1 = p < q,$$

where $\alpha = (p-1)/(pq-1)$, $\beta = (q-1)/(pq-1)$.

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