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THE SECOND VARIATIONAL FORMULA OF THE *k*-ENERGY AND *k*-HARMONIC CURVES

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Abstract

In [4], J. Eells and L. Lemaire introduced k-energy and k-harmonic maps. In 1989, S.B. Wang [17] showed the first variation formula of the k-energy. In this paper, we give the second variation formula of k-energy and a notion of weakly stable and unstable. We also study k-harmonic maps into product Riemannian manifolds and k-harmonic curves into Riemannian manifolds with constant sectional curvature. Moreover, we give some non-trivial solutions of 3-harmonic curves.

Introduction

The theory of harmonic maps has been applied into various fields in differential geometry. The harmonic maps between two Riemannian manifolds are critical maps of the energy functional $E(\phi) = (1/2)\int_M \|d\phi\|^2 v_g$, for smooth maps $\phi\colon M \to N$.

On the other hand, in 1983, J. Eells and L. Lemaire [4] proposed the problem to consider the *k-harmonic maps*: they are critical maps of the functional

$$E_k(\phi) = \int_M e_k(\phi) v_g, \quad (k = 1, 2, \dots),$$

where $e_k(\phi) = (1/2)\|(d + d^*)^k \phi\|^2$ for smooth maps $\phi\colon M \to N$. G.Y. Jiang [6] studied the first and second variation formulas of the bi-energy E_2, and critical maps of E_2 are called *biharmonic maps*. There have been extensive studies on biharmonic maps.

In 1989, S.B. Wang [17] studied the first variation formula of the k-energy E_k, whose critical maps are called k-harmonic maps. Harmonic maps are always k-harmonic maps by definition. In this paper, we study k-harmonic maps and show the second variational formula of E_k.

In §1, we introduce notation and fundamental formulas of the tension field.

In §2, we recall k-harmonic maps.

In §3, we calculate the second variation of the k-energy $E_k(\phi)$.

In §4, we show the reduction theorem of k-harmonic maps into product spaces.

Finally, in §5, we study k-harmonic curves into Riemannian manifolds with constant sectional curvature, and get non-trivial solutions. Furthermore, we determine the ODE of the 3-harmonic curve equation into a sphere.

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1. Preliminaries

Let (M, g) be an m dimensional Riemannian manifold, (N, h), an n dimensional one, and $\phi\colon M \to N$, a smooth map. We use the following notation. The second fundamental form $B(\phi)$ of ϕ is a covariant differentiation $\tilde{\nabla}\, d\phi$ of 1-form $d\phi$, which is a section of $\bigodot^2 T^*M \otimes \phi^{-1}TN$. For every $X, Y \in \Gamma(TM)$, let

$$(1)\qquad \begin{aligned} B(X, Y) &= (\tilde{\nabla}\, d\phi)(X, Y) = (\tilde{\nabla}_X\, d\phi)(Y) \\ &= \bar{\nabla}_X\, d\phi(Y) - d\phi(\nabla_X Y) = \nabla^N_{d\phi(X)}\, d\phi(Y) - d\phi(\nabla_X Y). \end{aligned}$$

Here, ∇, ∇^N, $\bar{\nabla}$ and $\tilde{\nabla}$ are the induced connections on the bundles TM, TN, $\phi^{-1}TN$ and $T^*M \otimes \phi^{-1}TN$ respectively.

If M is compact, we consider critical points of the energy functional

$$(2)\qquad E(\phi) = \int_M e(\phi) v_g,$$

where $e(\phi) = (1/2)\|d\phi\|^2 = \sum_{i=1}^m (1/2)\langle d\phi(e_i), d\phi(e_i)\rangle$ which is called the *energy density* of ϕ, and the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ is a Riemannian metric h, where $\{e_i\}_{i=1}^m$ is an orthonormal frame field on M. The *tension field* $\tau(\phi)$ of ϕ is defined by

$$(3)\qquad \tau(\phi) = \sum_{i=1}^m (\tilde{\nabla}\, d\phi)(e_i, e_i) = \sum_{i=1}^m (\tilde{\nabla}_{e_i}\, d\phi)(e_i).$$

Then, ϕ is a *harmonic map* if $\tau(\phi) = 0$.

The curvature tensor field $R^N(\,\cdot\,,\,\cdot\,)$ of the Riemannian metric on the bundle TN is defined as follows:

$$(4)\qquad R^N(X, Y)Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z - \nabla^N_{[X,Y]} Z, \quad (X, Y, Z \in \Gamma(TN)).$$

Moreover, $\bar{\Delta} = \bar{\nabla}^* \bar{\nabla} = -\sum_{k=1}^m (\bar{\nabla}_{e_k} \bar{\nabla}_{e_k} - \bar{\nabla}_{\nabla_{e_k} e_k})$ is the *rough Laplacian*, $\{e_i\}_{i=1}^m$ is an orthonormal frame field on M in this paper.

2. *k*-harmonic maps

J. Eells and L. Lemaire [4] proposed the notation of k-harmonic maps. The Euler–Lagrange equations for the k-harmonic maps were shown by S.B. Wang [17]. In this section, we recall the definition of k-harmonic maps.

DEFINITION 2.1 ([4]). For $k = 1, 2, \dots$ the *k-energy functional* is defined by

$$E_k(\phi) = \frac{1}{2} \int_M \|(d + d^*)^k \phi\|^2 v_g, \quad \phi \in C^\infty(M, N),$$

where d is a exterior differentiation and d^* is a codifferentiation. Then, ϕ is *k-harmonic* if it is a critical point of E_k, i.e., for all smooth variations $\{\phi_t\}$ of ϕ with $\phi_0 = \phi$,

$$\left.\frac{d}{dt}\right|_{t=0} E_k(\phi_t) = 0.$$

We say for a k-harmonic map to be *proper* if it is not harmonic.

G.Y. Jiang studied the case $k = 2$, and showed that $\phi\colon (M, g) \to (N, h)$ is a 2-harmonic if and only if

$$-\bar{\Delta}\tau(\phi) + R^N(\tau(\phi), d\phi(e_i))\, d\phi(e_i) = 0.$$

We consider a smooth variation $\{\phi_t\}_{t\in I_\varepsilon}$ $(I_\varepsilon = (-\varepsilon, \varepsilon))$ of ϕ with parameters t, i.e., we consider the smooth map F given by

$$F\colon I_\varepsilon \times M \to N,\ F(t, p) = \phi_t(p),$$

where $F(0, p) = \phi_0(p) = \phi(p)$, for all $p \in M$.

The corresponding variational vector field V is given by

$$V(p) = \left.\frac{d}{dt}\right|_{t=0} \phi_{t,0} \in T_{\phi(p)}N,$$

V is a section of $\phi^{-1}TN$, i.e. $V \in \Gamma(\phi^{-1}TN)$.

We also denote by $\nabla, \bar{\nabla}$ and $\tilde{\nabla}$, the induced Riemannian connection on $T(I_\varepsilon \times M)$, $F^{-1}TN$ and $T^*(I_\varepsilon \times M) \otimes F^{-1}TN$ respectively.

Lemma 2.2 ([17]).

$$\begin{aligned}
&\bar{\nabla}_{\partial/\partial t}\bar{\Delta}^{s-1}\tau(F)|_{t=0}\\
&= -\bar{\Delta}^s V + \sum_{j=1}^m \bar{\Delta}^{s-1} R^N(V, d\phi(e_j))\, d\phi(e_j)\\
&\quad + \sum_{j=1}^m \sum_{l=1}^{s-1} \bar{\Delta}^{l-1}\{-\bar{\nabla}_{e_j} R^N(V, d\phi(e_j))\bar{\Delta}^{s-l-1}\tau(\phi)\\
&\qquad - R^N(V, d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1}\tau(\phi) + R^N(V, d\phi(\nabla_{e_j}e_j))\bar{\Delta}^{s-l-1}\tau(\phi)\}.
\end{aligned}$$

Proof. For all $\omega \in \Gamma(\phi^{-1}TN)$,

$$\begin{aligned}
\bar{\nabla}_{\partial/\partial t}\bar{\Delta}\omega &= -\sum_{j=1}^m \{\bar{\nabla}_{\partial/\partial t}(\bar{\nabla}_{e_j}\bar{\nabla}_{e_j} - \bar{\nabla}_{\nabla_{e_j}e_j})\omega\}\\
&= -\sum_{j=1}^m \left\{\bar{\nabla}_{e_j}\bar{\nabla}_{\partial/\partial t}(\bar{\nabla}_{e_j}\omega) + R^N\left(dF\left(\frac{\partial}{\partial t}\right), dF(e_j)\right)\bar{\nabla}_{e_j}\omega\right.\\
&\qquad \left. - \bar{\nabla}_{\nabla_{e_j}e_j}\bar{\nabla}_{\partial/\partial t}\omega - R^N\left(dF\left(\frac{\partial}{\partial t}\right), dF(\nabla_{e_j}e_j)\right)\omega\right\}
\end{aligned}$$

$$= -\sum_{j=1}^{m}\Big\{\bar\nabla_{e_j}\Big(\bar\nabla_{e_j}\bar\nabla_{\partial/\partial t}\omega + R^N\Big(dF\Big(\frac{\partial}{\partial t}\Big), dF(e_j)\Big)\omega\Big) + R^N\Big(dF\Big(\frac{\partial}{\partial t}\Big), dF(e_j)\Big)\bar\nabla_{e_j}\omega - \bar\nabla_{\nabla_{e_j}e_j}\bar\nabla_{\partial/\partial t}\omega - R^N\Big(dF\Big(\frac{\partial}{\partial t}\Big), dF(\nabla_{e_j}e_j)\Big)\omega\Big\}.$$

Repeating this and using

$$\bar\nabla_{\partial/\partial t}\tau(F)|_{t=0} = -\bar\Delta V + \sum_{j=1}^{m} R^N(V, d\phi(e_j))\, d\phi(e_j),$$

we have the lemma. □

Lemma 2.3 ([17]). *For any* e_i $(i = 1, \dots, m)$,

$$\begin{aligned}
&\bar\nabla_{\partial/\partial t}\bar\nabla_{e_i}\bar\Delta^{s-1}\tau(F)|_{t=0}\\
&= -\bar\nabla_{e_i}\bar\Delta^{s}V + \sum_{j=1}^{m}\bar\nabla_{e_i}\bar\Delta^{s-1}R^N(V, d\phi(e_j))\, d\phi(e_j)\\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{s-1}\bar\nabla_{e_i}\bar\Delta^{l-1}\{-\bar\nabla_{e_j}R^N(V, d\phi(e_j))\bar\Delta^{s-l-1}\tau(\phi)\\
&\qquad - R^N(V, d\phi(e_j))\bar\nabla_{e_j}\bar\Delta^{s-l-1}\tau(\phi) + R^N(V, d\phi(\nabla_{e_j}e_j))\bar\Delta^{s-l-1}\tau(\phi)\}\\
&\quad + R^N(V, d\phi(e_i))\bar\Delta^{s-1}\tau(\phi).
\end{aligned}$$

Proof.

$$\bar\nabla_{\partial/\partial t}\bar\nabla_{e_i}\bar\Delta^{s-1}\tau(F) = \bar\nabla_{e_i}\bar\nabla_{\partial/\partial t}\bar\Delta^{s-1}\tau(F) + R^N\Big(dF\Big(\frac{\partial}{\partial t}\Big), dF(e_i)\Big)\bar\Delta^{s-1}\tau(F).$$

By using Lemma 2.2, we have the lemma. □

Lemma 2.4 ([17]). *For any* e_j $(j = 1, \dots, m)$,

$$\begin{aligned}
&\int_M \langle \bar\nabla_{e_j}R^N(V, d\phi(e_j))V_1 - R^N(V, d\phi(\nabla_{e_j}e_j))V_1, V_2\rangle v_g\\
&= -\int_M \langle R^N(V, d\phi(e_j))V_1, \bar\nabla_{e_j}V_2\rangle v_g,
\end{aligned}$$

where $V_1, V_2 \in \Gamma(\phi^{-1}TN)$.

Proof.

$$\begin{aligned}
&\mathrm{div}(\langle R^N(V, d\phi(e_i))V_1, V_2\rangle e_i)\\
&= \sum_{j=1}^{m}\langle \nabla_{e_j}\langle R^N(V, d\phi(e_i))V_1, V_2\rangle e_i, e_j\rangle
\end{aligned}$$

$$= \sum_{j=1}^{m} \langle \langle \bar{\nabla}_{e_j} R^N(V, d\phi(e_i))V_1, V_2\rangle e_i + \langle R^N(V, d\phi(e_i))V_1, \bar{\nabla}_{e_j} V_2\rangle e_i + \langle R^N(V, d\phi(e_i))V_1, V_2\rangle \nabla_{e_j} e_i, e_j\rangle.$$

By Green's theorem, we have

$$\begin{aligned} 0 &= \int_M \operatorname{div}\langle R^N(V, d\phi(e_i))V_1, V_2\rangle e_i v_g \\ &= \sum_{j=1}^{m} \int_M \langle \bar{\nabla}_{e_j} R^N(V, d\phi(e_i))V_1, V_2)\rangle \delta_{ij} \\ &\qquad + \langle R^N(V, d\phi(e_i))V_1, \bar{\nabla}_{e_j} V_2)\delta_{ij} + \langle R^N(V, d\phi(e_i))V_1, V_2\rangle \langle \nabla_{e_j} e_i, e_j\rangle v_g. \end{aligned}$$

Here,

$$\begin{aligned} \sum_{j=1}^{m} \langle R^N(V, d\phi(e_i))V_1, V_2\rangle \langle \nabla_{e_j} e_i, e_j\rangle &= \sum_{j=1}^{m} \langle R^N(V, d\phi(\langle \nabla_{e_j} e_i, e_j\rangle e_i))V_1, V_2\rangle \\ &= -\langle R^N(V, d\phi(\nabla_{e_i} e_i))V_1, V_2\rangle. \end{aligned}$$

Therefore, we have the lemma. □

Theorem 2.5 ([17]). *Let* $k = 2s$ $(s = 1, 2, \ldots)$, *then*

$$\frac{d}{dt}\bigg|_{t=0} E_{2s}(\phi_t) = -\int_M \langle \tau_{2s}(\phi), V\rangle,$$

where

$$\begin{aligned} \tau_{2s}(\phi) = \bar{\Delta}^{2s-1}\tau(\phi) &- \sum_{j=1}^{m} R^N(\bar{\Delta}^{2s-2}\tau(\phi), d\phi(e_j))\, d\phi(e_j) \\ &- \sum_{j=1}^{m}\sum_{l=1}^{s-1} \{R^N(\bar{\nabla}_{e_j} \bar{\Delta}^{s+l-2}\tau(\phi), \bar{\Delta}^{s-l-1}\tau(\phi))\, d\phi(e_j) \\ &\qquad - R^N(\bar{\Delta}^{s+l-2}\tau(\phi), \bar{\nabla}_{e_j} \bar{\Delta}^{s-l-1}\tau(\phi))\, d\phi(e_j)\}, \end{aligned}$$

where $\bar{\Delta}^{-1} = 0$.

Proof.

$$\begin{aligned} E_{2s}(\phi) &= \int_M \langle \underbrace{(d^*d)\cdots(d^*d)}_{s}\phi, \underbrace{(d^*d)\cdots(d^*d)}_{s}\phi\rangle v_g \\ &= \int_M \langle \bar{\Delta}^{s-1}\tau(\phi), \bar{\Delta}^{s-1}\tau(\phi)\rangle v_g. \end{aligned}$$

By using Lemma 2.2 and Lemma 2.4, we calculate $(d/dt)E_{2s}(\phi_t)$,

(5)

$$
\begin{aligned}
&\frac{d}{dt}E_{2s}(\phi_t)|_{t=0} \\
&= \int_M \langle \bar{\nabla}_{\partial/\partial t}\bar{\Delta}^{s-1}\tau(F),\, \bar{\Delta}^{s-1}\tau(F)\rangle v_g|_{t=0} \\
&= \int_M \Big\langle -\bar{\Delta}^s V + \sum_{j=1}^m \bar{\Delta}^{s-1}R^N(V, d\phi(e_j))\, d\phi(e_j) \\
&\qquad + \sum_{j=1}^m\sum_{l=1}^{s-1} \bar{\Delta}^{l-1}\{-\bar{\nabla}_{e_j}R^N(V, d\phi(e_j))\bar{\Delta}^{s-l-1}\tau(\phi) - R^N(V, d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1}\tau(\phi) \\
&\qquad\qquad + R^N(V, d\phi(\nabla_{e_j}e_j))\bar{\Delta}^{s-l-1}\tau(\phi),\, \bar{\Delta}^{s-1}\tau(\phi)\}\Big\rangle v_g \\
&= \int_M \langle V, -\bar{\Delta}^{2s-1}\tau(\phi)\rangle v_g + \sum_{j=1}^m \int_M \langle V,\, R^N(\bar{\Delta}^{2s-2}\tau(\phi), d\phi(e_j))\, d\phi(e_j)\rangle v_g \\
&\quad + \sum_{j=1}^m\sum_{l=1}^{s-1} \int_M \langle -\bar{\nabla}_{e_j}R^N(V, d\phi(e_j))\bar{\Delta}^{s-l-1}\tau(\phi) - R^N(V, d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1}\tau(\phi) \\
&\qquad\qquad + R^N(V, d\phi(\nabla_{e_j}e_j))\bar{\Delta}^{s-l-1}\tau(\phi),\, \bar{\Delta}^{s+l-2}\tau(\phi)\rangle v_g \\
&= \int_M \langle V, -\bar{\Delta}^{2s-1}\tau(\phi)\rangle v_g + \sum_{j=1}^m \int_M \langle V,\, R^N(\bar{\Delta}^{2s-2}\tau(\phi), d\phi(e_j))\, d\phi(e_j)\rangle v_g \\
&\quad + \sum_{j=1}^m\sum_{l=1}^{s-1}\Big\{\int_M \langle R^N(V, d\phi(e_j))\bar{\Delta}^{s-l-1}\tau(\phi),\, \bar{\nabla}_{e_j}\bar{\Delta}^{s+l-2}\tau(\phi)\rangle v_g \\
&\qquad\qquad + \int_M \langle -R^N(V, d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1}\tau(\phi),\, \bar{\Delta}^{s+l-2}\tau(\phi)\rangle v_g\Big\} \\
&= \int_M \langle V, -\bar{\Delta}^{2s-1}\tau(\phi)\rangle v_g + \sum_{j=1}^m \int_M \langle V,\, R^N(\bar{\Delta}^{2s-2}\tau(\phi), d\phi(e_j))\, d\phi(e_j)\rangle v_g \\
&\quad + \sum_{j=1}^m\sum_{l=1}^{s-1}\Big\{\int_M \langle R^N(\bar{\nabla}_{e_j}\bar{\Delta}^{s+l-2}\tau(\phi),\, \bar{\Delta}^{s-l-1}\tau(\phi))\, d\phi(e_j),\, V\rangle v_g \\
&\qquad\qquad - \int_M \langle R^N(\bar{\Delta}^{s+l-2}\tau(\phi),\, \bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1}\tau(\phi))\, d\phi(e_j),\, V\rangle v_g\Big\} \\
&= \int_M \Big\langle V, -\bar{\Delta}^{2s-1}\tau(\phi) + \sum_{j=1}^m R^N(\bar{\Delta}^{2s-2}\tau(\phi), d\phi(e_j))\, d\phi(e_j) \\
&\qquad + \sum_{j=1}^m\sum_{l=1}^{s-1}\{R^N(\bar{\nabla}_{e_j}\bar{\Delta}^{s+l-2}\tau(\phi),\, \bar{\Delta}^{s-l-1}\tau(\phi))\, d\phi(e_j) \\
&\qquad\qquad - R^N(\bar{\Delta}^{s+l-2}\tau(\phi),\, \bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1}\tau(\phi))\, d\phi(e_j)\}\Big\rangle v_g.
\end{aligned}
$$

So we have the theorem. □

Theorem 2.6 ([17]). *Let* $k = 2s + 1$ $(s = 0, 1, 2, \ldots)$, *then,*

$$\frac{d}{dt}\bigg|_{t=0} E_{2s+1}(\phi_t) = -\int_M \langle \tau_{2s+1}(\phi), V\rangle,$$

where

$$\begin{aligned}\tau_{2s+1}(\phi) = {} & \bar{\Delta}^{2s}\tau(\phi) - \sum_{j=1}^{m} R^N(\bar{\Delta}^{2s-1}\tau(\phi), d\phi(e_j))\, d\phi(e_j)\\ & - \sum_{j=1}^{m}\sum_{l=1}^{s-1}\{R^N(\bar{\nabla}_{e_j}\bar{\Delta}^{s+l-1}\tau(\phi), \bar{\Delta}^{s-l-1}\tau(\phi))\, d\phi(e_j)\\ & \qquad - R^N(\bar{\Delta}^{s+l-1}\tau(\phi), \bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1}\tau(\phi))\, d\phi(e_j)\}\\ & - \sum_{j=1}^{m} R^N(\bar{\nabla}_{e_j}\bar{\Delta}^{s-1}\tau(\phi), \bar{\Delta}^{s-1}\tau(\phi))\, d\phi(e_j),\end{aligned}$$

where $\bar{\Delta}^{-1} = 0$.

Proof. When $s = 0$, it is the first variation of harmonic maps. So we consider the case of $s = 1, 2, \ldots$.

$$\begin{aligned}E_{2s+1}(\phi) &= \int_M \langle d\, \underbrace{(d^*d)\cdots(d^*d)}_{s}\, \phi,\ d\, \underbrace{(d^*d)\cdots(d^*d)}_{s}\, \phi\rangle v_g\\ &= \sum_{i=1}^{m}\int_M \langle \bar{\nabla}_{e_i}\bar{\Delta}^{s-1}\tau(\phi), \bar{\nabla}_{e_i}\bar{\Delta}^{s-1}\tau(\phi)\rangle v_g.\end{aligned}$$

By using Lemma 2.3 and Lemma 2.4, we calculate $(d/dt)E_{2s+1}(\phi_t)$,

$$\begin{aligned}\frac{d}{dt}E_{2s+1}(\phi_t)\bigg|_{t=0} &= \sum_{i=1}^{m}\int_M \langle \bar{\nabla}_{\partial/\partial t}\bar{\nabla}_{e_i}\bar{\Delta}^{s-1}\tau(F), \bar{\nabla}_{e_i}\bar{\Delta}^{s-1}\tau(F)\rangle v_g|_{t=0}\\ = \sum_{i=1}^{m}\int_M \bigg\langle & -\bar{\nabla}_{e_i}\bar{\Delta}^{s}V + \sum_{j=1}^{m}\bar{\nabla}_{e_i}\bar{\Delta}^{s-1}R^N(V, d\phi(e_j))d\phi(e_j)\\ & + \sum_{j=1}^{m}\sum_{l=1}^{s-1}\bar{\nabla}_{e_i}\bar{\Delta}^{l-1}\{-\bar{\nabla}_{e_j}R^N(V, d\phi(e_j))\bar{\Delta}^{s-l-1}\tau(\phi)\\ & \qquad -R^N(V, d\phi(e_j))\bar{\nabla}_{e_i}\bar{\Delta}^{s-l-1}\tau(\phi) + R^N(V, d\phi(\nabla_{e_j}e_j))\bar{\Delta}^{s-l-1}\tau(\phi)\\ & \qquad + R^N(V, d\phi(e_i))\bar{\Delta}^{s-1}\tau(\phi), \bar{\nabla}_{e_i}\bar{\Delta}^{s-1}\tau(\phi)\}\bigg\rangle v_g.\end{aligned}$$

Here, using

$$\sum_{i=1}^{m}\int_M \langle \bar{\nabla}_{e_i}\omega_1, \bar{\nabla}_{e_i}\omega_2\rangle v_g = \int_M \langle \bar{\Delta}\omega_1, \omega_2\rangle v_g,$$

where $\omega_1, \omega_2 \in \Gamma(\phi^{-1}TN)$, we have

$$\begin{aligned}
&\frac{d}{dt}E_{2s+1}(\phi_t)|_{t=0}\\
&= \int_M \langle V, -\bar{\Delta}^{2s}\tau(\phi)\rangle v_g\\
&\quad + \sum_{j=1}^{m}\int_M \langle R^N(V, d\phi(e_j))\, d\phi(e_j),\ \bar{\Delta}^{2s-1}\tau(\phi)\rangle v_g\\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{s-1}\int_M \langle -\bar{\nabla}_{e_j} R^N(V, d\phi(e_j))\bar{\Delta}^{s-l-1}\tau(\phi)\\
&\qquad\qquad - R^N(V, d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1}\tau(\phi)\\
&\qquad\qquad + R^N(V, d\phi(\nabla_{e_j}e_j))\bar{\Delta}^{s-l-1}\tau(\phi),\ \bar{\Delta}^{s+l-1}\tau(\phi)\rangle v_g\\
&\quad + \sum_{j=1}^{m}\int_M \langle R^N(V, d\phi(e_j))\bar{\Delta}^{s-1}\tau(\phi),\ \bar{\nabla}_{e_j}\bar{\Delta}^{s-1}\tau(\phi)\rangle v_g\\
&= \int_M \Big\langle V, -\bar{\Delta}^{2s}\tau(\phi) + \sum_{j=1}^{m} R^N(\bar{\Delta}^{2s-1}\tau(\phi), d\phi(e_j))\, d\phi(e_j)\\
&\qquad + \sum_{j=1}^{m}\sum_{l=1}^{s-1}\{R^N(\bar{\nabla}_{e_j}\bar{\Delta}^{s+l-1}\tau(\phi), \bar{\Delta}^{s-l-1}\tau(\phi))\, d\phi(e_j)\\
&\qquad\qquad - R^N(\bar{\Delta}^{s+l-1}\tau(\phi), \bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1}\tau(\phi))\, d\phi(e_j)\}\\
&\qquad + \sum_{j=1}^{m} R^N(\bar{\nabla}_{e_j}\bar{\Delta}^{s-1}\tau(\phi), \bar{\Delta}^{s-1}\tau(\phi))\, d\phi(e_j)\Big\rangle v_g.
\end{aligned} \tag{6}$$

So we have the theorem. □

By Theorem 2.5 and 2.6, we have the following [17].

Corollary 2.7. *A harmonic map is always k-harmonic* ($k = 1, 2, \ldots$).

For $\bar{\Delta}^l$ ($k = 1, 2, \ldots$), we have Theorem 2.10. We show the following two lemmas.

Lemma 2.8. *Let $l = 1, 2, \ldots$. If for any e_i $(i = 1, \ldots, m)$, $\bar{\nabla}_{e_i} \bar{\Delta}^{(l-1)} \tau(\phi) = 0$, then*

$$\bar{\Delta}^l \tau(\phi) = 0.$$

Proof. Indeed, we can define a global vector field $X_\phi \in \Gamma(TM)$ defined by

$$X_\phi = \sum_{j=1}^{m} \langle -\bar{\nabla}_{e_j} \bar{\Delta}^{(l-1)} \tau(\phi), \bar{\Delta}^l \tau(\phi) \rangle e_j. \tag{7}$$

Then, the divergence of X_ϕ is given as

$$\begin{aligned} \operatorname{div}(X_\phi) &= \langle \bar{\Delta}^l \tau(\phi), \bar{\Delta}^l \tau(\phi) \rangle + \sum_{j=1}^{m} \langle -\bar{\nabla}_{e_j} \bar{\Delta}^{(l-1)} \tau(\phi), \bar{\nabla}_{e_j} \bar{\Delta}^l \tau(\phi) \rangle \\ &= \langle \bar{\Delta}^l \tau(\phi), \bar{\Delta}^l \tau(\phi) \rangle, \end{aligned}$$

by the assumption. Integrating this over M, we have

$$0 = \int_M \operatorname{div}(X_\phi) v_g = \int_M \langle \bar{\Delta}^l \tau(\phi), \bar{\Delta}^l \tau(\phi) \rangle v_g,$$

which implies $\bar{\Delta}^l \tau(\phi) = 0$. □

Lemma 2.9. *Let $l = 1, 2, \ldots$. If $\bar{\Delta}^l \tau(\phi) = 0$, then*

$$\bar{\nabla}_{e_i} \bar{\Delta}^{(l-1)} \tau(\phi) = 0, \quad (i = 1, \ldots, m).$$

Proof. Indeed, by computing the Laplacian of the $2l$-energy density $e_{2l}(\phi)$, we have

$$\begin{aligned} \triangle e_{2l}(\phi) &= \sum_{i=1}^{m} \langle \bar{\nabla}_{e_i} \bar{\Delta}^{(l-1)} \tau(\phi), \bar{\nabla}_{e_i} \bar{\Delta}^{(l-1)} \tau(\phi) \rangle \\ &\quad - \langle \bar{\nabla}^* \bar{\nabla} (\bar{\Delta}^{(l-1)} \tau(\phi)), \bar{\Delta}^{(l-1)} \tau(\phi) \rangle \\ &= \sum_{i=1}^{m} \langle \bar{\nabla}_{e_i} \bar{\Delta}^{(l-1)} \tau(\phi), \bar{\nabla}_{e_i} \bar{\Delta}^{(l-1)} \tau(\phi) \rangle \geq 0. \end{aligned} \tag{8}$$

By Green's theorem $\int_M \triangle e_{2l}(\phi) v_g = 0$, and (8), we have $\triangle e_{2l}(\phi) = 0$. Again, by (8), we have

$$\bar{\nabla}_{e_i} \bar{\Delta}^{(l-1)} \tau(\phi) = 0, \quad (i = 1, \ldots, m, \ l = 1, 2, \ldots).$$ □

Theorem 2.10. *Let $l = 1, 2, \ldots$. If $\bar{\Delta}^l \tau(\phi) = 0$ or if for any e_i $(i = 1, \ldots, m)$, $\bar{\nabla}_{e_i} \bar{\Delta}^{(l-1)} \tau(\phi) = 0$, then $\phi \colon M \to N$ from a compact Riemannian manifold into a Riemannian manifold is a harmonic map.*

Proof. By using Lemma 2.8 and 2.9, we have Theorem 2.10. □

3. The second variational formula of the k-energy

In this section, we calculate the second variation of the k-energy. The formula was proved for $k=2$, by G.Y. Jiang [6], and for $k=3$, S.B. Wang [18].

Now let $\phi\colon (M,g)\to(N,h)$ be a k-harmonic map ($k=1,2,\dots$). We consider a smooth variation $\{\phi_{t,r}\}_{t,r\in I_\varepsilon}$ ($I_\varepsilon=(-\varepsilon,\varepsilon)$) of ϕ with two parameters t and r, i.e., we consider the smooth map F given by

$$F\colon I_\varepsilon\times I_\varepsilon\times M\to N,\quad F(t,r,p)=\phi_{t,r}(p),$$

where $F(0,0,p)=\phi_{0,0}(p)=\phi(p)$, for all $p\in M$.

The corresponding variational vector field V and W are given by

$$V(p)=\left.\frac{d}{dt}\right|_{t=0}\phi_{t,0}\in T_{\phi(p)}N,$$
$$W(p)=\left.\frac{d}{dr}\right|_{r=0}\phi_{0,r}\in T_{\phi(p)}N.$$

V and W are section of $\phi^{-1}TN$.

We also denote by ∇, $\bar\nabla$ and $\tilde\nabla$ the induced Riemannian connection on $T(I_\varepsilon\times I_\varepsilon\times M)$, $F^{-1}TN$ and $T^*(I_\varepsilon\times I_\varepsilon\times M)\otimes F^{-1}TN$ respectively.

The *Hessian* of E_k at its critical point ϕ is defined by

$$H(E_k)_\phi(V,W)=\left.\frac{\partial^2}{\partial t\partial r}\right|_{(t,r)=(0,0)}E_k(\phi_{t,r}).$$

Theorem 3.1. *Let ϕ: $(M,g)\to(N,h)$ be a $2s$-harmonic map ($s=1,2,\dots$). Then, the Hessian of the $2s$-energy E_{2s} at ϕ is given by*

$$H(E_{2s})_\phi(V,W)=\int_M\langle V,J_{2s}(W)\rangle v_g,$$

where

$$J_{2s}(W)=-\mathrm{I}_{2s}+\mathrm{II}_{2s}+\mathrm{III}_{2s}-\mathrm{IV}_{2s}.$$

where

$$\begin{aligned}\mathrm{I}_{2s}={}&-\bar\Delta^{2s}W+\sum_{j=1}^m\bar\Delta^{2s-1}R^N(W,d\phi(e_j))d\phi(e_j)\\&+\sum_{j=1}^m\sum_{l=1}^{2s-1}\bar\Delta^{l-1}\{-\bar\nabla_{e_j}R^N(W,d\phi(e_j))\bar\Delta^{2s-l-1}\tau(\phi)\\&\qquad-R^N(W,d\phi(e_j))\bar\nabla_{e_j}\bar\Delta^{2s-l-1}\tau(\phi)+R^N(W,d\phi(\nabla_{e_j}e_j))\bar\Delta^{2s-l-1}\tau(\phi)\},\end{aligned}$$

$$
\begin{aligned}
\mathrm{II}_{2s} = &-\sum_{i=1}^{m}(\nabla^N_{\bar{\Delta}^{2s-2}\tau(\phi)}R^N)(d\phi(e_i),W)d\phi(e_i)-\sum_{i=1}^{m}(\nabla^N_{d\phi(e_i)}R^N)(W,\bar{\Delta}^{2s-2}\tau(\phi))d\phi(e_i)\\
&+\sum_{i=1}^{m}R^N\Bigg(-\bar{\Delta}^{2s-1}W+\sum_{j=1}^{m}\bar{\Delta}^{2s-2}R^N(W,d\phi(e_j))d\phi(e_j)\\
&\qquad+\sum_{j=1}^{m}\sum_{l_2=1}^{2s-2}\{\bar{\Delta}^{l_2-1}\{-\bar{\nabla}_{e_j}R^N(W,d\phi(e_j))\bar{\Delta}^{2s-l_2-2}\tau(\phi)\\
&\qquad\qquad-R^N(W,d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{2s-l_2-2}\tau(\phi)\\
&\qquad\qquad+R^N(W,d\phi(\nabla_{e_j}e_j))\bar{\Delta}^{2s-l_2-2}\tau(\phi)\}\},d\phi(e_i)\Bigg)d\phi(e_i)\\
&+\sum_{i=1}^{m}R^N(\bar{\Delta}^{2s-2}\tau(\phi),\bar{\nabla}_{e_i}W)d\phi(e_i)+\sum_{i=1}^{m}R^N(\bar{\Delta}^{2s-2}\tau(\phi),d\phi(e_i))\bar{\nabla}_{e_i}W,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{III}_{2s} = &-\sum_{l=1}^{s-1}\sum_{i=1}^{m}\big(\nabla^N_{\bar{\nabla}_{e_i}\bar{\Delta}^{s+l-2}\tau(\phi)}R^N\big)(\bar{\Delta}^{s-l-1}\tau(\phi),W)d\phi(e_i)\\
&-\sum_{l=1}^{s-1}\sum_{i=1}^{m}\big(\nabla^N_{\bar{\Delta}^{s-l-1}\tau(\phi)}R^N\big)(W,\bar{\nabla}_{e_i}\bar{\Delta}^{s+l-2}\tau(\phi))d\phi(e_i)\\
&+\sum_{l=1}^{s-1}\sum_{i=1}^{m}R^N\Bigg(-\bar{\nabla}_{e_i}\bar{\Delta}^{s+l-1}W+\sum_{j=1}^{m}\bar{\nabla}_{e_i}\bar{\Delta}^{s+l-2}R^N(W,d\phi(e_j))d\phi(e_j)\\
&\qquad+\sum_{j=1}^{m}\sum_{l_2=1}^{s+l-2}\{\bar{\nabla}_{e_i}\bar{\Delta}^{l_2-1}\{-\bar{\nabla}_{e_j}R^N(W,d\phi(e_j))\bar{\Delta}^{s+l-2-l_2}\tau(\phi)\\
&\qquad\qquad-R^N(W,d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s+l-2-l_2}\tau(\phi)\\
&\qquad\qquad+R^N(W,d\phi(\nabla_{e_j}e_j))\bar{\Delta}^{s+l-2-l_2}\tau(\phi)\}\}\\
&\qquad+R^N(W,d\phi(e_i))\bar{\Delta}^{s+l-2}\tau(\phi),\bar{\Delta}^{s-l-1}\tau(\phi)\Bigg)d\phi(e_i)\\
&+\sum_{l=1}^{s-1}\sum_{i=1}^{m}R^N\Bigg(\bar{\nabla}_{e_i}\bar{\Delta}^{s+l-2}\tau(\phi),\\
&\qquad-\bar{\Delta}^{s-l}W+\sum_{j=1}^{m}\bar{\Delta}^{s-l-1}R^N(W,d\phi(e_j))d\phi(e_j)\\
&\qquad+\sum_{j=1}^{m}\sum_{l_2=1}^{s-l-1}\{\bar{\Delta}^{l_2-1}\{-\bar{\nabla}_{e_j}R^N(W,d\phi(e_j))\bar{\Delta}^{s-l-1-l_2}\tau(\phi)\\
&\qquad\qquad-R^N(W,d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1-l_2}\tau(\phi)\\
&\qquad\qquad+R^N(W,d\phi(\nabla_{e_j}e_j))\bar{\Delta}^{s-l-1-l_2}\tau(\phi)\}\}\Bigg)d\phi(e_i)\\
&+\sum_{l=1}^{s-1}\sum_{i=1}^{m}R^N(\bar{\nabla}_{e_i}\bar{\Delta}^{s+l-2}\tau(\phi),\bar{\Delta}^{s-l-1}\tau(\phi))\bar{\nabla}_{e_i}W,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{IV}_{2s} = &-\sum_{l=1}^{s-1}\sum_{i=1}^{m}\big(\nabla^N_{\bar{\Delta}^{s+l-2}\tau(\phi)}R^N\big)(\bar{\nabla}_{e_i}\bar{\Delta}^{s-l-1}\tau(\phi),W)d\phi(e_i)\\
&-\sum_{l=1}^{s-1}\sum_{i=1}^{m}\big(\nabla^N_{\bar{\nabla}_{e_i}\bar{\Delta}^{s-l-1}\tau(\phi)}R^N\big)(W,\bar{\Delta}^{s+l-2}\tau(\phi))d\phi(e_i)\\
&+\sum_{l=1}^{s-1}\sum_{i=1}^{m}R^N\Bigg(-\bar{\Delta}^{s+l-1}W+\sum_{j=1}^{m}\bar{\Delta}^{s+l-2}R^N(W,d\phi(e_j))d\phi(e_j)\\
&\qquad+\sum_{j=1}^{m}\sum_{l_2=1}^{s+l-2}\{\bar{\Delta}^{l_2-1}\{-\bar{\nabla}_{e_j}R^N(W,d\phi(e_j))\bar{\Delta}^{s+l-2-l_2}\tau(\phi)\\
&\qquad\qquad-R^N(W,d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s+l-2-l_2}\tau(\phi)\\
&\qquad\qquad+R^N(W,d\phi(\nabla_{e_j}e_j))\bar{\Delta}^{s+l-2-l_2}\tau(\phi)\}\},\\
&\qquad\bar{\nabla}_{e_i}\bar{\Delta}^{s-l-1}\tau(\phi)\Bigg)d\phi(e_i)\\
&+\sum_{l=1}^{s-1}\sum_{i=1}^{m}R^N\Bigg(\bar{\Delta}^{s+l-2}\tau(\phi),\\
&\qquad-\bar{\nabla}_{e_i}\bar{\Delta}^{s-l}W+\sum_{j=1}^{m}\bar{\nabla}_{e_i}\bar{\Delta}^{s-l-1}R^N(W,d\phi(e_j))d\phi(e_j)\\
&\qquad+\sum_{l_2=1}^{s-l-1}\Bigg\{\bar{\nabla}_{e_i}\bar{\Delta}^{l_2-1}\sum_{j=1}^{m}\{-\bar{\nabla}_{e_j}R^N(W,d\phi(e_j))\bar{\Delta}^{s-l-1-l_2}\tau(\phi)\\
&\qquad\qquad-R^N(W,d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s-l-1-l_2}\tau(\phi)\\
&\qquad\qquad+R^N(W,d\phi(\nabla_{e_j}e_j))\bar{\Delta}^{s-l-1-l_2}\tau(\phi)\}\\
&\qquad+R^N(W,d\phi(e_i))\bar{\Delta}^{s-l-1}\tau(\phi)\Bigg\}\Bigg)d\phi(e_i)\\
&+\sum_{l=1}^{s-1}\sum_{i=1}^{m}R^N(\bar{\Delta}^{s+l-2}\tau(\phi),\bar{\nabla}_{e_i}\bar{\Delta}^{s-l-1}\tau(\phi))\bar{\nabla}_{e_i}W.
\end{aligned}
$$

Proof. By (5), we have

(9)
$$
\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial r\partial t}E_{2s}(F)\\
&= \int_M \Big\langle \bar\nabla_{\partial/\partial r}\, dF\Big(\frac{\partial}{\partial t}\Big), -\bar\Delta^{2s-1}\tau(F) + \sum_{i=1}^m R^N(\bar\Delta^{2s-2}\tau(F), dF(e_i))\, dF(e_i)\\
&\qquad + \sum_{i=1}^m \sum_{l=1}^{s-1}\{R^N(\bar\nabla_{e_i}\bar\Delta^{s+l-2}\tau(F), \bar\Delta^{s-l-1}\tau(F))\, dF(e_i)\\
&\qquad - R^N(\bar\Delta^{s+l-2}\tau(F), \bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(F))\, dF(e_i)\}\Big\rangle v_g.\\
&\quad + \int_M \Big\langle F\Big(\frac{\partial}{\partial t}\Big), \bar\nabla_{\partial/\partial r}\Big\{-\bar\Delta^{2s-1}\tau(F) + \sum_{i=1}^m R^N(\bar\Delta^{2s-2}\tau(F), dF(e_i))\, dF(e_i)\\
&\qquad + \sum_{i=1}^m \sum_{l=1}^{s-1}\{R^N(\bar\nabla_{e_i}\bar\Delta^{s+l-2}\tau(F), \bar\Delta^{s-l-1}\tau(F))\, dF(e_i)\\
&\qquad - R^N(\bar\Delta^{s+l-2}\tau(F), \bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(F))\, dF(e_i)\}\Big\}\Big\rangle v_g.
\end{aligned}
$$

Then, putting $t = 0$, the first term of (9) vanishes. Thus, we calculate the second term of (9).

Using Lemma 2.2, we have

$$
\begin{aligned}
&\bar\nabla_{\partial/\partial r}\bar\Delta^{2s-1}\tau(F)|_{t=0} = I_{2s}.\\
&\bar\nabla_{\partial/\partial r} R^N(\bar\Delta^{2s-2}\tau(F), dF(e_i))\, dF(e_i)\\
&= (\nabla^N_{dF(\partial/\partial r)}R^N)(\bar\Delta^{2s-2}\tau(F), dF(e_i))\, dF(e_i)\\
&\quad + R^N(\bar\nabla_{\partial/\partial r}\bar\Delta^{2s-2}\tau(F), dF(e_i))\, dF(e_i)\\
&\quad + R^N(\bar\Delta^{2s-2}\tau(F), \bar\nabla_{\partial/\partial r}\, dF(e_i))\, dF(e_i)\\
&\quad + R^N(\bar\Delta^{2s-2}\tau(F), dF(e_i))\bar\nabla_{\partial/\partial r}\, dF(e_i).
\end{aligned}
$$

Using second Bianch's identity, Lemma 2.2, we have

$$
\sum_{i=1}^m \bar\nabla_{\partial/\partial r} R^N(\bar\Delta^{2s-2}\tau(F), dF(e_i))\, dF(e_i)|_{t=0} = \mathrm{II}_{2s}.
$$

$$\begin{aligned}
&\bar\nabla_{\partial/\partial r} R^N(\bar\nabla_{e_i}\bar\Delta^{s+l-2}\tau(F),\ \bar\Delta^{s-l-1}\tau(F))\,dF(e_i)\\
&= (\nabla^N_{dF(\partial/\partial r)}R^N)(\bar\nabla_{e_i}\bar\Delta^{s+l-2}\tau(F),\ \bar\Delta^{s-l-1}\tau(F))\,dF(e_i)\\
&\quad + R^N(\bar\nabla_{\partial/\partial r}\bar\nabla_{e_i}\bar\Delta^{s+l-2}\tau(F),\ \bar\Delta^{s-l-1}\tau(F))\,dF(e_i)\\
&\quad + R^N(\bar\nabla_{e_i}\bar\Delta^{s+l-2}\tau(F),\ \bar\nabla_{\partial/\partial r}\bar\Delta^{s-l-1}\tau(F))\,dF(e_i)\\
&\quad + R^N(\bar\nabla_{e_i}\bar\Delta^{s+l-2}\tau(F),\ \bar\Delta^{s-l-1}\tau(F))\bar\nabla_{\partial/\partial r}\,dF(e_i).
\end{aligned}$$

Using second Bianch's identity, Lemma 2.2 and Lemma 2.3, we have

$$\sum_{l=1}^{s-1}\sum_{i=1}^{m}\bar\nabla_{\partial/\partial r} R^N(\bar\nabla_{e_i}\bar\Delta^{s+l-2}\tau(F),\ \bar\Delta^{s-l-1}\tau(F))\,dF(e_i) = \mathrm{III}_{2s}.$$

$$\begin{aligned}
&\bar\nabla_{\partial/\partial r} R^N(\bar\Delta^{s+l-2}\tau(F),\ \bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(F))\,dF(e_i)\\
&= (\nabla^N_{dF(\partial/\partial r)}R^N)(\bar\Delta^{s+l-2}\tau(F),\ \bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(F))\,dF(e_i)\\
&\quad + R^N(\bar\nabla_{\partial/\partial r}\bar\Delta^{s+l-2}\tau(F),\ \bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(F))\,dF(e_i)\\
&\quad + R^N(\bar\Delta^{s+l-2}\tau(F),\ \bar\nabla_{\partial/\partial r}\bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(F))\,dF(e_i)\\
&\quad + R^N(\bar\Delta^{s+l-2}\tau(F),\ \bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(F))\bar\nabla_{\partial/\partial r}\,dF(e_i).
\end{aligned}$$

Using second Bianch's identity, Lemma 2.2 and Lemma 2.3, we have

$$\sum_{l=1}^{s-1}\sum_{i=1}^{m}\bar\nabla_{\partial/\partial r} R^N(\bar\Delta^{s+l-2}\tau(F),\ \bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(F))\,dF(e_i) = \mathrm{IV}_{2s}. \qquad \square$$

Theorem 3.2. *Let $\phi\colon (M,g)\to(N,h)$ be a $(2s+1)$-harmonic map $(s=0,1,\dots)$. Then, the* Hessian *of the $(2s+1)$-energy E_{2s+1} at ϕ is given by*

$$H(E_{2s+1})_\phi(V,\ W) = \int_M \langle V,\ J_{2s+1}(W)\rangle v_g,$$

where

$$J_{2s+1}(W) = -\mathrm{I}_{2s+1} + \mathrm{II}_{2s+1} + \mathrm{III}_{2s+1} - \mathrm{IV}_{2s+1} + \mathrm{V}_{2s+1},$$

where

$$\begin{aligned}
\mathrm{I}_{2s+1} = -\bar\Delta^{2s+1}W &+ \sum_{j=1}^{m}\bar\Delta^{2s}R^N(W,\ d\phi(e_j))\,d\phi(e_j)\\
&+ \sum_{j=1}^{m}\sum_{l=1}^{2s}\bar\Delta^{l-1}\{-\bar\nabla_{e_j}R^N(W,\ d\phi(e_j))\bar\Delta^{2s-l}\tau(\phi)\\
&\qquad - R^N(W,\ d\phi(e_j))\bar\nabla_{e_j}\bar\Delta^{2s-l}\tau(\phi)\\
&\qquad + R^N(W,\ d\phi(\nabla_{e_j}e_j))\bar\Delta^{2s-l}\tau(\phi)\},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{II}_{2s+1} = & -\sum_{i=1}^{m} (\nabla^N_{\bar{\Delta}^{2s-1}\tau(\phi)} R^N)(d\phi(e_i), W) d\phi(e_i) \\
& -\sum_{i=1}^{m} (\nabla^N_{d\phi(e_i)} R^N)(W, \bar{\Delta}^{2s-1}\tau(\phi)) d\phi(e_i) \\
& +\sum_{i=1}^{m} R^N \Bigg(-\bar{\Delta}^{2s} W + \sum_{j=1}^{m} \bar{\Delta}^{2s-1} R^N(W, d\phi(e_j)) d\phi(e_j) \\
& \qquad + \sum_{j=1}^{m} \sum_{l_2=1}^{2s-1} \{ \bar{\Delta}^{l_2-1} \{ -\bar{\nabla}_{e_j} R^N(W, d\phi(e_j)) \bar{\Delta}^{2s-l_2-1}\tau(\phi) \\
& \qquad\qquad - R^N(W, d\phi(e_j)) \bar{\nabla}_{e_j} \bar{\Delta}^{2s-l_2-1}\tau(\phi) \\
& \qquad\qquad + R^N(W, d\phi(\nabla_{e_j} e_j)) \bar{\Delta}^{2s-l_2-1}\tau(\phi) \} \}, d\phi(e_i) \Bigg) d\phi(e_i) \\
& \qquad + \sum_{i=1}^{m} R^N(\bar{\Delta}^{2s-1}\tau(\phi), \bar{\nabla}_{e_i} W) d\phi(e_i) \\
& \qquad + \sum_{i=1}^{m} R^N(\bar{\Delta}^{2s-1}\tau(\phi), d\phi(e_i)) \bar{\nabla}_{e_i} W,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{III}_{2s+1} = & -\sum_{l=1}^{s-1} \sum_{i=1}^{m} \big(\nabla^N_{\bar{\nabla}_{e_i} \bar{\Delta}^{s+l-1}\tau(\phi)} R^N \big) (\bar{\Delta}^{s-l-1}\tau(\phi), W) d\phi(e_i) \\
& -\sum_{l=1}^{s-1} \sum_{i=1}^{m} \big(\nabla^N_{\bar{\Delta}^{s-l-1}\tau(\phi)} R^N \big) (W, \bar{\nabla}_{e_i} \bar{\Delta}^{s+l-1}\tau(\phi)) d\phi(e_i) \\
& +\sum_{l=1}^{s-1} \sum_{i=1}^{m} R^N \Bigg(-\bar{\nabla}_{e_i} \bar{\Delta}^{s+l} W + \sum_{j=1}^{m} \bar{\nabla}_{e_i} \bar{\Delta}^{s+l-1} R^N(W, d\phi(e_j)) d\phi(e_j) \\
& \qquad + \sum_{j=1}^{m} \sum_{l_2=1}^{s+l-1} \{ \bar{\nabla}_{e_i} \bar{\Delta}^{l_2-1} \{ -\bar{\nabla}_{e_j} R^N(W, d\phi(e_j)) \bar{\Delta}^{s+l-1-l_2}\tau(\phi) \\
& \qquad\qquad - R^N(W, d\phi(e_j)) \bar{\nabla}_{e_j} \bar{\Delta}^{s+l-1-l_2}\tau(\phi) \\
& \qquad\qquad + R^N(W, d\phi(\nabla_{e_j} e_j)) \bar{\Delta}^{s+l-1-l_2}\tau(\phi) \} \} \\
& \qquad + R^N(W, d\phi(e_i)) \bar{\Delta}^{s+l-1}\tau(\phi), \bar{\Delta}^{s-l-1}\tau(\phi) \Bigg) d\phi(e_i)
\end{aligned}
$$

$$+\sum_{l=1}^{s-1}\sum_{i=1}^{m} R^N\Bigg(\bar\nabla_{e_i}\bar\Delta^{s+l-1}\tau(\phi),$$

$$-\bar\Delta^{s-l}W+\sum_{j=1}^{m}\bar\Delta^{s-l-1}R^N(W,d\phi(e_j))d\phi(e_j)$$

$$+\sum_{j=1}^{m}\sum_{l_2=1}^{s-l-1}\{\bar\Delta^{l_2-1}\{-\bar\nabla_{e_j}R^N(W,d\phi(e_j))\bar\Delta^{s-l-1-l_2}\tau(\phi)$$

$$-R^N(W,d\phi(e_j))\bar\nabla_{e_j}\bar\Delta^{s-l-1-l_2}\tau(\phi)$$

$$+R^N(W,d\phi(\nabla_{e_j}e_j))\bar\Delta^{s-l-1-l_2}\tau(\phi)\}\}\Bigg)d\phi(e_i)$$

$$+\sum_{l=1}^{s-1}\sum_{i=1}^{m} R^N(\bar\nabla_{e_i}\bar\Delta^{s+l-1}\tau(\phi),\bar\Delta^{s-l-1}\tau(\phi))\bar\nabla_{e_i}W,$$

$$\mathrm{IV}_{2s}=-\sum_{l=1}^{s-1}\sum_{i=1}^{m}\big(\nabla^N_{\bar\Delta^{s+l-1}\tau(\phi)}R^N\big)(\bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(\phi),\,W)\,d\phi(e_i)$$

$$-\sum_{l=1}^{s-1}\sum_{i=1}^{m}\big(\nabla^N_{\bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(\phi)}R^N\big)(W,\,\bar\Delta^{s+l-1}\tau(\phi))\,d\phi(e_i)$$

$$+\sum_{l=1}^{s-1}\sum_{i=1}^{m} R^N\Bigg(-\bar\Delta^{s+l}W+\sum_{j=1}^{m}\bar\Delta^{s+l-1}R^N(W,\,d\phi(e_j))\,d\phi(e_j)$$

$$+\sum_{j=1}^{m}\sum_{l_2=1}^{s+l-1}\{\bar\Delta^{l_2-1}\{-\bar\nabla_{e_j}R^N(W,\,d\phi(e_j))\bar\Delta^{s+l-1-l_2}\tau(\phi)$$

$$-\,R^N(W,\,d\phi(e_j))\bar\nabla_{e_j}\bar\Delta^{s+l-1-l_2}\tau(\phi)$$

$$+\,R^N(W,\,d\phi(\nabla_{e_j}e_j))\bar\Delta^{s+l-1-l_2}\tau(\phi)\}\},$$

$$\bar\nabla_{e_i}\bar\Delta^{s-l-1}\tau(\phi)\Bigg)\,d\phi(e_i)$$

$$+\sum_{l=1}^{s-1}\sum_{i=1}^{m} R^N\Bigg(\bar\Delta^{s+l-1}\tau(\phi),$$

$$-\,\bar\nabla_{e_i}\bar\Delta^{s-l}W+\sum_{j=1}^{m}\bar\nabla_{e_i}\bar\Delta^{s-l-1}R^N(W,\,d\phi(e_j))\,d\phi(e_j)$$

$$+\sum_{j=1}^{m}\sum_{l_2=1}^{s-l-1}\{\bar\nabla_{e_i}\bar\Delta^{l_2-1}\{-\bar\nabla_{e_j}R^N(W,\,d\phi(e_j))\bar\Delta^{s-l-1-l_2}\tau(\phi)$$

$$-\,R^N(W,\,d\phi(e_j))\bar\nabla_{e_j}\bar\Delta^{s-l-1-l_2}\tau(\phi)$$

$$+ R^N(W, d\phi(\nabla_{e_j} e_j))\bar{\Delta}^{s-l-1-l_2}\tau(\phi)\}\}$$

$$+ R^N(W, d\phi(e_i))\bar{\Delta}^{s-l-1}\tau(\phi)\Bigg) d\phi(e_i)$$

$$+ \sum_{l=1}^{s-1}\sum_{i=1}^{m} R^N(\bar{\Delta}^{s+l-1}\tau(\phi), \bar{\nabla}_{e_i}\bar{\Delta}^{s-l-1}\tau(\phi))\bar{\nabla}_{e_i}W,$$

$$\mathrm{V}_{2s+1} = -\sum_{l=1}^{s-1}\sum_{i=1}^{m}\big(\nabla^N_{\bar{\nabla}_{e_i}\bar{\Delta}^{s-l}\tau(\phi)}R^N\big)(\bar{\Delta}^{s-1}\tau(\phi), W)\, d\phi(e_i)$$

$$-\sum_{l=1}^{s-1}\sum_{i=1}^{m}\big(\nabla^N_{\bar{\Delta}^{s-1}\tau(\phi)}R^N\big)(W, \bar{\nabla}_{e_i}\bar{\Delta}^{s-1}\tau(\phi))\, d\phi(e_i)$$

$$+\sum_{l=1}^{s-1}\sum_{i=1}^{m} R^N\Bigg(-\bar{\nabla}_{e_i}\bar{\Delta}^{s}W + \sum_{j=1}^{m}\bar{\nabla}_{e_i}\bar{\Delta}^{s-1}R^N(W, d\phi(e_j))\, d\phi(e_j)$$

$$+\sum_{j=1}^{m}\sum_{l_2=1}^{s-1}\{\bar{\nabla}_{e_i}\bar{\Delta}^{l_2-1}\{-\bar{\nabla}_{e_j}R^N(W, d\phi(e_j))\bar{\Delta}^{s-l_2-1}\tau(\phi)$$

$$- R^N(W, d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s-l_2-1}\tau(\phi)$$

$$+ R^N(W, d\phi(\nabla_{e_j} e_j))\bar{\Delta}^{s-l_2-1}\tau(\phi)\}\}$$

$$+ R^N(W, d\phi(e_i))\bar{\Delta}^{s-1}\tau(\phi), \bar{\Delta}^{s-1}\tau(\phi)\Bigg) d\phi(e_i)$$

$$+\sum_{l=1}^{s-1}\sum_{i=1}^{m} R^N\Bigg(\bar{\nabla}_{e_i}\bar{\Delta}^{s-1}\tau(\phi),$$

$$-\bar{\Delta}^{s}W + \sum_{j=1}^{m}\bar{\Delta}^{s-1}R^N(W, d\phi(e_j))\, d\phi(e_j)$$

$$+\sum_{j=1}^{m}\sum_{l_2=1}^{s-1}\{\bar{\Delta}^{l_2-1}\{-\bar{\nabla}_{e_j}R^N(W, d\phi(e_j))\bar{\Delta}^{s-l_2-1}\tau(\phi)$$

$$- R^N(W, d\phi(e_j))\bar{\nabla}_{e_j}\bar{\Delta}^{s-l_2-1}\tau(\phi)$$

$$+ R^N(W, d\phi(\nabla_{e_j} e_j))\bar{\Delta}^{s-l_2-1}\tau(\phi)\}\}\Bigg) d\phi(e_i)$$

$$+\sum_{l=1}^{s-1}\sum_{i=1}^{m} R^N(\bar{\nabla}_{e_i}\bar{\Delta}^{s-1}\tau(\phi), \bar{\Delta}^{s-1}\tau(\phi))\bar{\nabla}_{e_i}W.$$

Proof. By (6), we have

(10)

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial r\,\partial t}E_{2s+1}(F)\\
&=\int_M\Big\langle \bar\nabla_{\partial/\partial r}\,dF\Big(\frac{\partial}{\partial t}\Big),\,-\bar\Delta^{2s}\tau(F)+\sum_{j=1}^m R^N(\bar\Delta^{2s-1}\tau(F),\,dF(e_j))\,dF(e_j)\\
&\qquad+\sum_{j=1}^m\sum_{l=1}^{s-1}\{R^N(\bar\nabla_{e_j}\bar\Delta^{s+l-1}\tau(F),\,\bar\Delta^{s-l-1}\tau(F))\,dF(e_j)\\
&\qquad\qquad-R^N(\bar\Delta^{s+l-1}\tau(F),\,\bar\nabla_{e_j}\bar\Delta^{s-l-1}\tau(F))\,dF(e_j)\}\\
&\qquad+\sum_{i=1}^m R^N(\bar\nabla_{e_i}\bar\Delta^{s-1}\tau(F),\,\bar\Delta^{s-1}\tau(F))\,dF(e_i)\Big\rangle v_g\\
&\quad+\int_M\Big\langle F\Big(\frac{\partial}{\partial t}\Big),\,\bar\nabla_{\partial/\partial r}\Big\{-\bar\Delta^{2s}\tau(F)+\sum_{j=1}^m R^N(\bar\Delta^{2s-1}\tau(F),\,dF(e_j))\,dF(e_j)\\
&\qquad+\sum_{j=1}^m\sum_{l=1}^{s-1}\{R^N(\bar\nabla_{e_j}\bar\Delta^{s+l-1}\tau(F),\,\bar\Delta^{s-l-1}\tau(F))\,dF(e_j)\\
&\qquad\qquad-R^N(\bar\Delta^{s+l-1}\tau(F),\,\bar\nabla_{e_j}\bar\Delta^{s-l-1}\tau(F))\,dF(e_j)\}\\
&\qquad+\sum_{i=1}^m R^N(\bar\nabla_{e_i}\bar\Delta^{s-1}\tau(F),\,\bar\Delta^{s-1}\tau(F))\,dF(e_i)\Big\}\Big\rangle v_g.
\end{aligned}$$

Then, putting $t=0$, the first term of (10) vanishes. Thus, we calculate the second term of (10).

Using Lemma 2.2, we have

$$\begin{aligned}
&\bar\nabla_{\partial/\partial r}\bar\Delta^{2s}\tau(F)|_{t=0}=\mathrm{I}_{2s+1}.\\
&\bar\nabla_{\partial/\partial r}R^N(\bar\Delta^{2s-1}\tau(F),\,dF(e_j))\,dF(e_j)\\
&=(\nabla^N_{dF(\partial/\partial r)}R^N)(\bar\Delta^{2s-1}\tau(F),\,dF(e_j))\,dF(e_j)\\
&\quad+R^N(\bar\nabla_{\partial/\partial r}\bar\Delta^{2s-1}\tau(F),\,dF(e_j))\,dF(e_j)\\
&\quad+R^N(\bar\Delta^{2s-1}\tau(F),\,\bar\nabla_{\partial/\partial r}\,dF(e_j))\,dF(e_j)\\
&\quad+R^N(\bar\Delta^{2s-1}\tau(F),\,dF(e_j))\bar\nabla_{\partial/\partial r}\,dF(e_j).
\end{aligned}$$

Using second Bianch's identity, Lemma 2.2, we have

$$\sum_{j=1}^m\bar\nabla_{\partial/\partial r}R^N(\bar\Delta^{2s-1}\tau(F),\,dF(e_j))\,dF(e_j)|_{t=0}=\mathrm{II}_{2s+1}.$$

$$\begin{aligned}
&\bar{\nabla}_{\partial/\partial r} R^N(\bar{\nabla}_{e_j} \bar{\Delta}^{s+l-1}\tau(F), \bar{\Delta}^{s-l-1}\tau(F))\, dF(e_j) \\
&= (\nabla^N_{dF(\partial/\partial r)} R^N)(\bar{\nabla}_{e_j} \bar{\Delta}^{s+l-1}\tau(F), \bar{\Delta}^{s-l-1}\tau(F))\, dF(e_j) \\
&\quad + R^N(\bar{\nabla}_{\partial/\partial r} \bar{\nabla}_{e_j} \bar{\Delta}^{s+l-1}\tau(F), \bar{\Delta}^{s-l-1}\tau(F))\, dF(e_j) \\
&\quad + R^N(\bar{\nabla}_{e_j} \bar{\Delta}^{s+l-1}\tau(F), \bar{\nabla}_{\partial/\partial r} \bar{\Delta}^{s-l-1}\tau(F))\, dF(e_j) \\
&\quad + R^N(\bar{\nabla}_{e_j} \bar{\Delta}^{s+l-1}\tau(F), \bar{\Delta}^{s-l-1}\tau(F)) \bar{\nabla}_{\partial/\partial r}\, dF(e_j).
\end{aligned}$$

Using second Bianch's identity, Lemmas 2.2 and 2.3, we have

$$\sum_{l=1}^{s-1} \sum_{j=1}^{m} \bar{\nabla}_{\partial/\partial r} R^N(\bar{\nabla}_{e_j} \bar{\Delta}^{s+l-1}\tau(F), \bar{\Delta}^{s-l-1}\tau(F))\, dF(e_j) = \mathrm{III}_{2s+1}.$$

$$\begin{aligned}
&\bar{\nabla}_{\partial/\partial r} R^N(\bar{\Delta}^{s+l-1}\tau(F), \bar{\nabla}_{e_j} \bar{\Delta}^{s-l-1}\tau(F))\, dF(e_j) \\
&= (\nabla^N_{dF(\partial/\partial r)} R^N)(\bar{\Delta}^{s+l-1}\tau(F), \bar{\nabla}_{e_j} \bar{\Delta}^{s-l-1}\tau(F))\, dF(e_j) \\
&\quad + R^N(\bar{\nabla}_{\partial/\partial r} \bar{\Delta}^{s+l-1}\tau(F), \bar{\nabla}_{e_j} \bar{\Delta}^{s-l-1}\tau(F))\, dF(e_j) \\
&\quad + R^N(\bar{\Delta}^{s+l-1}\tau(F), \bar{\nabla}_{\partial/\partial r} \bar{\nabla}_{e_j} \bar{\Delta}^{s-l-1}\tau(F))\, dF(e_j) \\
&\quad + R^N(\bar{\Delta}^{s+l-1}\tau(F), \bar{\nabla}_{e_j} \bar{\Delta}^{s-l-1}\tau(F)) \bar{\nabla}_{\partial/\partial r}\, dF(e_j).
\end{aligned}$$

Using second Bianch's identity, Lemmas 2.2 and 2.3, we have

$$\sum_{l=1}^{s-1} \sum_{j=1}^{m} \bar{\nabla}_{\partial/\partial r} R^N(\bar{\Delta}^{s+l-1}\tau(F), \bar{\nabla}_{e_j} \bar{\Delta}^{s-l-1}\tau(F))\, dF(e_j) = \mathrm{IV}_{2s+1}.$$

$$\begin{aligned}
&\bar{\nabla}_{\partial/\partial r} R^N(\bar{\nabla}_{e_j} \bar{\Delta}^{s-1}\tau(F), \bar{\Delta}^{s-1}\tau(F))\, dF(e_j) \\
&= (\nabla^N_{dF(\partial/\partial r)} R^N)(\bar{\nabla}_{e_j} \bar{\Delta}^{s-1}\tau(F), \bar{\Delta}^{s-1}\tau(F))\, dF(e_j) \\
&\quad + R^N(\bar{\nabla}_{\partial/\partial r} \bar{\nabla}_{e_j} \bar{\Delta}^{s-1}\tau(F), \bar{\Delta}^{s-1}\tau(F))\, dF(e_j) \\
&\quad + R^N(\bar{\nabla}_{e_j} \bar{\Delta}^{s-1}\tau(F), \bar{\nabla}_{\partial/\partial r} \bar{\Delta}^{s-1}\tau(F))\, dF(e_j) \\
&\quad + R^N(\bar{\nabla}_{e_j} \bar{\Delta}^{s-1}\tau(F), \bar{\Delta}^{s-1}\tau(F)) \bar{\nabla}_{\partial/\partial r}\, dF(e_j).
\end{aligned}$$

Using second Bianch's identity, Lemmas 2.2 and 2.3, we have

$$\sum_{j=1}^{m} \bar{\nabla}_{\partial/\partial r} R^N(\bar{\nabla}_{e_j} \bar{\Delta}^{s-1}\tau(F), \bar{\Delta}^{s-1}\tau(F))\, dF(e_j) = \mathrm{V}_{2s+1}.$$ □

DEFINITION 3.3. Assume that $\phi\colon (M, g) \to (N, h)$ is a k-harmonic map. Then, ϕ is *weakly stable* if $H(E_k)_\phi(V, V) \geq 0$, for all $V \in \Gamma(\phi^{-1}TN)$. ϕ is *unstable* if it is not *weakly stable*.

Proposition 3.4. *Any harmonic map is a weakly stable k-harmonic map.*

Proof. CASE 1. $k = 2s$, $(s = 1, 2, \dots)$.

By assumption we have

(11)

$$
\begin{aligned}
&H(E_{2s})_\phi(V, V)\\
&= \int_M \Big\langle V, -\Big(-\bar{\Delta}^{2s} V + \sum_{j=1}^m \bar{\Delta}^{2s-1} R^N(V, d\phi(e_j))\, d\phi(e_j)\Big)\\
&\qquad + \sum_{i=1}^m R^N\Big(-\bar{\Delta}^{2s-1} V + \sum_{j=1}^m \bar{\Delta}^{2s-2} R^N(V, d\phi(e_j))\, d\phi(e_j), d\phi(e_i)\Big)\, d\phi(e_i)\Big\rangle v_g\\
&= \int_M \Big\langle -V, -\bar{\Delta}^{2s} V + \sum_{j=1}^m \bar{\Delta}^{2s-1} R^N(V, d\phi(e_j))\, d\phi(e_j)\Big\rangle v_g\\
&\quad + \int_M \Big\langle \sum_{i=1}^m R^N(V, d\phi(e_i))\, d\phi(e_i), -\bar{\Delta}^{2s-1} V + \sum_{j=1}^m \bar{\Delta}^{2s-2} R^N(V, d\phi(e_j))\, d\phi(e_j)\Big\rangle v_g\\
&= \int_M \Big\| -\bar{\Delta}^{s} V + \sum_{j=1}^m \bar{\Delta}^{s-1} R^N(V, d\phi(e_j))\, d\phi(e_j)\Big\|^2 v_g \geq 0.
\end{aligned}
$$

CASE 2. $k = 2s+1$, $(s = 0, 1, 2, \dots)$.

By assumption we have

(12)

$$
\begin{aligned}
&H(E_{2s+1})_\phi(V, V)\\
&= \int_M \Big\langle V, -\Big(-\bar{\Delta}^{2s+1} V + \sum_{j=1}^m \bar{\Delta}^{2s} R^N(V, d\phi(e_j))\, d\phi(e_j)\Big)\\
&\qquad + \sum_{i=1}^m R^N\Big(-\bar{\Delta}^{2s} V + \sum_{j=1}^m \bar{\Delta}^{2s-1} R^N(V, d\phi(e_j))\, d\phi(e_j), d\phi(e_i)\Big)\, d\phi(e_i)\Big\rangle v_g\\
&= \int_M \Big\langle -V, -\bar{\Delta}^{2s+1} V + \sum_{j=1}^m \bar{\Delta}^{2s} R^N(V, d\phi(e_j))\, d\phi(e_j)\Big\rangle v_g\\
&\quad + \int_M \Big\langle \sum_{i=1}^m R^N(V, d\phi(e_i))\, d\phi(e_i), -\bar{\Delta}^{2s} V + \sum_{j=1}^m \bar{\Delta}^{2s-1} R^N(V, d\phi(e_j))\, d\phi(e_j)\Big\rangle v_g\\
&= \int_M \Big\| \bar{\nabla}\Big(-\bar{\Delta}^{s} V + \sum_{j=1}^m \bar{\Delta}^{s-1} R^N(V, d\phi(e_j))\, d\phi(e_j)\Big)\Big\|^2 v_g \geq 0. \qquad \square
\end{aligned}
$$

Corollary 3.5. *Assume that* $\phi\colon (M, g) \to (N, h)$ *is a harmonic map. Then,*

$$J_k(V) = J(\bar{\Delta}^{k-2} J(V)).$$

for all $V \in \Gamma(\phi^{-1}TN)$.

Proof. If ϕ is harmonic map, then, $\tau(\phi) = 0$. Thus we have

$$J_k(V) = J(\bar{\Delta}^{k-2} J(V)),$$

for all $V \in \Gamma(\phi^{-1}TN)$. Therefore, we have the corollary. □

4. The k-harmonic maps into product spaces

In this section, we describe the necessary and sufficient condition of k-harmonic maps into product spaces. Let us recall the result of Y.-L. Ou [10].

Theorem 4.1 ([10]). *Let* $\varphi\colon (M, g) \to (N_1, h_1)$ *and* $\psi\colon (M, g) \to (N_2, h_2)$ *be two maps. Then, the map* $\phi\colon (M, g) \to (N_1 \times N_2, h_1 \times h_2)$ *with* $\phi(x) = (\varphi(x), \psi(x))$ *is* 2*-harmonic if and only if both map* φ *or* ψ *are* 2*-harmonic. Furthermore, if one of* φ *or* ψ *is* 2*-harmonic and the other is a proper* 2*-harmonic map, then* ϕ *is a proper* 2*-harmonic map.*

We generalize Theorem 4.1 for k-harmonic maps. We have the following theorem which is useful to construct examples the k-harmonic maps.

Theorem 4.2. *Let* $\varphi\colon (M, g) \to (N_1, h_1)$ *and* $\psi\colon (M, g) \to (N_2, h_2)$ *be two maps. Then, the map* $\phi\colon (M, g) \to (N_1 \times N_2, h_1 \times h_2)$ *with* $\phi(x) = (\varphi(x), \psi(x))$ *is* k*-harmonic if and only if both map* φ *or* ψ *are* k*-harmonic. Furthermore, if one of* φ *or* ψ *is harmonic and the other is a proper* k*-harmonic map, then* ϕ *is a proper* k*-harmonic map.*

Proof. It is easily seen that

$$d\phi(X) = d\varphi(X) + d\phi(X), \quad \forall X \in \Gamma(TM). \tag{13}$$

It follows that

$$\nabla^{\phi}_X \, d\phi(Y) = \nabla^{\phi}_X \, d\varphi(Y) + \nabla^{\phi}_X \, d\psi(Y), \quad X, Y \in \Gamma(TM). \tag{14}$$

where ∇^{ϕ} is given by $\nabla^{\phi}_X = \nabla^N_{d\phi(X)}$, $\forall X \in \Gamma(TM)$.

Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame on (M, g) and $Y = Y^i e_i$, then $d\varphi(Y) = Y^i \varphi_i^{\alpha}(E_{\alpha}\varphi)$, for some function φ^{α} defined locally on M. A straight forward computation yields $\nabla^{\phi}_X \, d\varphi(Y) = \nabla^{\varphi}_X d\varphi(Y)$, $\tau(\phi) = \tau(\varphi) + \tau(\psi)$, $\bar{\Delta}_{\phi}\tau(\phi) = \bar{\Delta}_{\varphi}\tau(\varphi) + \bar{\Delta}_{\psi}\tau(\psi)$.

We notice that $\bar{\Delta}_{\varphi}\tau(\varphi)$ is tangent to N_1, $\bar{\Delta}_{\psi}\tau(\psi)$ is tangent to N_2. So we have

$$\begin{aligned}\bar{\Delta}_{\phi}\big(\bar{\Delta}_{\phi}\tau(\phi)\big) &= \bar{\Delta}_{\phi}\big(\bar{\Delta}_{\varphi}\tau(\varphi) + \bar{\Delta}_{\psi}\tau(\psi)\big)\\ &= \bar{\Delta}_{\varphi}\big(\bar{\Delta}_{\varphi}\tau(\varphi)\big) + \bar{\Delta}_{\psi}\big(\bar{\Delta}_{\psi}\tau(\psi)\big).\end{aligned} \tag{15}$$

Similarly,

$$\bar{\Delta}^{t}_{\phi}\tau(\phi) = \bar{\Delta}^{t}_{\varphi}\tau(\varphi) + \bar{\Delta}^{t}_{\psi}\tau(\psi).$$

for all $t = 0, 1, 2, \dots$.

We use the property of the curvature of the product manifold to have

$$\begin{aligned}&R^{N_1\times N_2}\big(\bar{\Delta}^{t}_{\phi}\tau(\phi), d\phi(e_i)\big)\, d\phi(e_i)\\ &= R^{N_1}\big(\bar{\Delta}^{t}_{\varphi}\tau(\varphi), d\varphi(e_i)\big)\, d\varphi(e_i) + R^{N_2}\big(\bar{\Delta}^{t}_{\psi}\tau(\psi), d\psi(e_i)\big)\, d\psi(e_i).\end{aligned}$$

Similarly we have

$$\begin{aligned}&R^{N_1\times N_2}\big(\bar{\Delta}^{s}_{\phi}\tau(\phi), \bar{\Delta}^{t}_{\phi}\tau(\phi)\big)\\ &= R^{N_1}\big(\bar{\Delta}^{s}_{\varphi}\tau(\varphi), \bar{\Delta}^{t}_{\varphi}\tau(\varphi)\big) + R^{N_2}\big(\bar{\Delta}^{s}_{\psi}\tau(\psi), \bar{\Delta}^{t}_{\psi}\tau(\psi)\big),\\ &R^{N_1\times N_2}\big(\nabla^{\phi}_{d\phi(X)}\bar{\Delta}^{s}_{\phi}\tau(\phi), \bar{\Delta}^{t}_{\phi}\tau(\phi)\big)\\ &= R^{N_1}\big(\nabla^{\varphi}_{d\varphi(X)}\bar{\Delta}^{s}_{\varphi}\tau(\varphi), \bar{\Delta}^{t}_{\varphi}\tau(\varphi)\big) + R^{N_2}\big(\nabla^{\psi}_{d\psi(X)}\bar{\Delta}^{s}_{\psi}\tau(\psi), \bar{\Delta}^{t}_{\psi}\tau(\psi)\big),\\ &R^{N_1\times N_2}\big(\bar{\Delta}^{s}_{\phi}\tau(\phi), \nabla^{\phi}_{d\phi(X)}\bar{\Delta}^{t}_{\phi}\tau(\phi)\big)\\ &= R^{N_1}\big(\bar{\Delta}^{s}_{\varphi}\tau(\varphi), \nabla^{\varphi}_{d\varphi(X)}\bar{\Delta}^{t}_{\varphi}\tau(\varphi)\big) + R^{N_2}\big(\bar{\Delta}^{s}_{\psi}\tau(\psi), \nabla^{\psi}_{d\psi(X)}\bar{\Delta}^{t}_{\psi}\tau(\psi)\big),\end{aligned}$$

for all $t, s = 0, 1, 2, \dots$, and for all $X \in \Gamma(TM)$.

By using Theorem 2.5 and 2.6, we have the theorem. □

The following corollary is a generalization of Corollary 3.4 in [10]. This corollary for $k = 2$ is also proved in [1].

Corollary 4.3. *Let $\psi\colon (M, g) \to (N, h)$ be a smooth map. Then, the graph $\phi\colon (M, g) \to (M \times N, g \times h)$ with $\phi(x) = (x, \psi(x))$ is a k-harmonic map if and only if the map $\psi\colon (M, g) \to (N, h)$ is a k-harmonic map. Furthermore, if ψ is proper k-harmonic, then so is the graph.*

Proof. This follows from Theorem 4.2 with $\varphi\colon (M, g) \to (N, h)$ being identity map which is harmonic. □

5. *k*-harmonic curves into a Riemannian manifold with constant sectional curvature

Harmonic maps are always biharmonic maps. By Corollary 2.7, harmonic maps are always k-harmonic maps. In this section, we consider the following problem.

PROBLEM 5.1. Are biharmonic maps k-harmonic maps ($k = 3, 4, \ldots$)? More generally, for $s < k$, are s-harmonic maps k-harmonic maps?

Let $\{T, N\}$ be an orthonormal frame field tangent to N^2 along to γ, where $T = \gamma'$ is the unit vector field tangent to γ, N is the unit normal vector field in the direction of $\nabla_T T$.

Then, we have the following Frenet equations

$$\begin{cases} \gamma' = T, \quad \nabla^N_{\gamma'} T = \kappa N, \quad \nabla^N_{\gamma'} N = -\kappa T, \\ \langle T, N \rangle = 0, \quad \langle T, T \rangle = 1, \quad \langle N, N \rangle = 1, \end{cases} \tag{16}$$

where κ is the geodesic curvature and $\langle \cdot, \cdot \rangle = h$, the Riemannian metric on N^2. Then, we have the following proposition.

Proposition 5.2. *Let $\gamma \colon I \to (N^2, \langle \cdot\,, \cdot \rangle)$ be a smooth curve parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold $(N^2, \langle \cdot\,, \cdot \rangle)$ with constant sectional curvature K. Then, γ is a 3-harmonic curve if and only if*

$$\begin{cases} \kappa^{(4)} - 15\kappa(\kappa')^2 - 10\kappa^2\kappa'' + \kappa^5 + K(\kappa'' - 2\kappa^3) = 0, \\ \kappa\kappa^{(3)} - 2\kappa^3\kappa' + 2\kappa'\kappa'' = 0, \end{cases}$$

where κ is the geodesic curvature of γ.

Proof. We calculate $(\nabla^N_{\gamma'}\nabla^N_{\gamma'})^2\tau(\gamma)$ as follows.

$$\begin{aligned} (\nabla^N_{\gamma'}\nabla^N_{\gamma'})^2\tau(\gamma) = {} & (\kappa^{(4)} - 15\kappa(\kappa')^2 - 10\kappa^2\kappa'' + \kappa^5)N \\ & + (-5\kappa\kappa^{(3)} + 10\kappa^3\kappa' - 10\kappa'\kappa'')T. \end{aligned} \tag{17}$$

Therefore, γ is 3-harmonic if and only if

$$\begin{aligned} & (\kappa^{(4)} - 15\kappa(\kappa')^2 - 10\kappa^2\kappa'' + \kappa^5 + K(\kappa'' - 2\kappa^3))N \\ & + (-5\kappa\kappa^{(3)} + 10\kappa^3\kappa' - 10\kappa'\kappa'')T = 0. \end{aligned} \tag{18}$$

So we have Proposition 5.2. □

Corollary 5.3. *Let $\gamma \colon I \to (N^2, \langle \cdot, \cdot \rangle)$ be a 3-harmonic curve parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold $(N^2, \langle \cdot, \cdot \rangle)$ with constant sectional curvature $K \geq 0$. If the geodesic curvature κ is constant, then $\kappa = \sqrt{2K}$.*

Proof. We can show this corollary by a direct computation. The proof is omitted. □

Proposition 5.4. *Let $\gamma : I \to (N^n, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a smooth curve parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold $(N^n, \langle\,\cdot\,,\,\cdot\,\rangle)$ with constant sectional curvature K. Then, γ is $2s$-harmonic curve if and only if*

$$
\begin{aligned}
\tau_{2s}(\gamma) = {}& \big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{2s-1}\tau(\gamma) \\
& + K\big\{\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{2s-2}\tau(\gamma) - \big\langle\gamma', \big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{2s-2}\tau(\gamma)\big\rangle\gamma'\big\} \\
& - \sum_{l=1}^{s-1} K\big\{\big\langle\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-l-1}\tau(\gamma), \gamma'\big\rangle\nabla^N_{\gamma'}\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s+l-2}\tau(\gamma) \\
& \qquad - \big\langle\gamma', \nabla^N_{\gamma'}\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s+l-2}\tau(\gamma)\big\rangle\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-l-1}\tau(\gamma) \\
& \qquad - \big\langle\nabla^N_{\gamma'}\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-l-1}\tau(\gamma), \gamma'\big\rangle\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s+l-2}\tau(\gamma) \\
& \qquad + \big\langle\gamma', \big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s+l-2}\tau(\gamma)\big\rangle\nabla^N_{\gamma'}\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-l-1}\tau(\gamma)\big\} = 0.
\end{aligned} \tag{19}
$$

Proof. We only notice that

$$
\begin{aligned}
&\bar{\Delta} = -\nabla^N_{\gamma'}\nabla^N_{\gamma'}, \\
&R^N(V, W)Z = K(\langle W, Z\rangle V - \langle Z, V\rangle W), \\
&\langle\gamma', \gamma'\rangle = 1.
\end{aligned}
$$

We get the proposition. □

Similarly we have

Proposition 5.5. *Let $\gamma : I \to (N^n, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a smooth curve parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold $(N^n, \langle\,\cdot\,,\,\cdot\,\rangle)$ with constant sectional curvature K. Then, γ is $(2s+1)$-harmonic curve if and only if*

$$
\begin{aligned}
\tau_{2s+1}(\gamma) = {}& -\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{2s}\tau(\gamma) \\
& - K\big\{\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{2s-1}\tau(\gamma) - \big\langle\gamma', \big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{2s-1}\tau(\gamma)\big\rangle\gamma'\big\} \\
& + \sum_{l=1}^{s-1} K\big\{\big\langle\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-l-1}\tau(\gamma), \gamma'\big\rangle\nabla^N_{\gamma'}\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s+l-1}\tau(\gamma) \\
& \qquad - \big\langle\gamma', \nabla^N_{\gamma'}\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s+l-1}\tau(\gamma)\big\rangle\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-l-1}\tau(\gamma) \\
& \qquad - \big\langle\nabla^N_{\gamma'}\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-l-1}\tau(\gamma), \gamma'\big\rangle\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s+l-1}\tau(\gamma) \\
& \qquad + \big\langle\gamma', \big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s+l-1}\tau(\gamma)\big\rangle\nabla^N_{\gamma'}\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-l-1}\tau(\gamma)\big\} \\
& + K\big\{\big(\big\langle\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-1}\tau(\gamma), \gamma'\big\rangle\nabla^N_{\gamma'}\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-1}\tau(\gamma)\big) \\
& \qquad - \big\langle\gamma', \nabla^N_{\gamma'}\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-1}\tau(\gamma)\big\rangle\big(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\big)^{s-1}\tau(\gamma))\big\} = 0.
\end{aligned} \tag{20}
$$

Using these propositions, we show the following propositions.

Proposition 5.6. *Let $\gamma : I \to (N^2, \langle\cdot, \cdot\rangle)$ be a $2s$-harmonic curve ($s = 1, 2, \ldots$) parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold $(N^2, \langle\cdot, \cdot\rangle)$ with constant sectional curvature $K \geq 0$. If the geodesic curvature κ is constant, then $\kappa = \sqrt{(2s-1)K}$.*

Proof. By assumption, for all $t = 0, 1, 2, \ldots,$

$$(\nabla_{\gamma'}\nabla_{\gamma'})^t\tau(\gamma) = (-1)^t\kappa^{2t+1}N,$$
$$\nabla_{\gamma'}(\nabla_{\gamma'}\nabla_{\gamma'})^t\tau(\gamma) = -(-1)^t\kappa^{2t+2}N.$$

Using these and Proposition 5.4, we have

$$\begin{aligned}\tau_{2s}(\gamma) &= -\kappa^{4s-1}N + K(\kappa^{4s-3}N + 2K(s-1)\kappa^{4s-3}N)\\ &= \kappa^{4s-3}(-\kappa^2 + (2s-1)K)N = 0.\end{aligned}$$

Therefore we have the proposition. □

Proposition 5.7. *Let $\gamma : I \to (N^2, \langle\cdot, \cdot\rangle)$ be a $(2s+1)$-harmonic curve ($s = 0, 1, 2, \ldots$) parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold $(N^2, \langle\cdot,\cdot\rangle)$ with constant sectional curvature $K \geq 0$. If the geodesic curvature κ is constant, then $\kappa = \sqrt{2sK}$.*

Proof. By assumption, for all $t = 0, 1, 2, \ldots,$

$$(\nabla_{\gamma'}\nabla_{\gamma'})^t\tau(\gamma) = (-1)^t\kappa^{2t+1}N,$$
$$\nabla_{\gamma'}(\nabla_{\gamma'}\nabla_{\gamma'})^t\tau(\gamma) = -(-1)^t\kappa^{2t+2}N.$$

Using these and Proposition 5.5, we have

$$\begin{aligned}\tau_{2s+1}(\gamma) &= -\kappa^{4s+1}N + K\kappa^{4s-1}N + 2K(s-1)\kappa^{4s-1}N + K\kappa^{4s-1}N\\ &= \kappa^{4s-1}\{-\kappa^2 + 2sK\}N = 0.\end{aligned}$$

Therefore we have the proposition. □

Therefore, we obtain the answer of Problem 5.1. For $s < k$, a s-harmonic map is not always a k-harmonic map.

Next we consider 3-harmonic curves into a Riemannian manifold with constant sectional curvature.

DEFINITION 5.8. The Frenet frame $\{e_i\}_{i=1,\dots,n}$ associated to a curve $\gamma\colon I \in \mathbb{R} \to (N^n, \langle\cdot, \cdot\rangle)$, parametrized by arc length, is the orthonormalization of the $\{\nabla^{N(k)}_{d\gamma(\partial/\partial t)} d\gamma(\partial/\partial t)\}_{k=1,\dots,n}$, described by

$$\begin{aligned}
&e_1 = d\gamma\left(\frac{\partial}{\partial t}\right),\\
&\nabla^N_{d\gamma(\partial/\partial t)} e_1 = \kappa_1 e_2,\\
&\nabla^N_{d\gamma(\partial/\partial t)} e_i = -\kappa_{i-1} e_{i-1} + \kappa_i e_{i+1} \quad (i = 2, \dots, n-1),\\
&\nabla^N_{d\gamma(\partial/\partial t)} e_n = -\kappa_{n-1} e_{n-1},
\end{aligned}$$

where the functions $\kappa_1, \kappa_2, \dots, \kappa_{n-1}$ are called the curvatures of γ. Note that $e_1 = \gamma'$ is the unit tangent vector field along the curve.

Let $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ and κ_5, are constant.

Proposition 5.9. *Let $\gamma\colon I \to (N^n, \langle\cdot, \cdot\rangle)$ be a smooth curve parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold $(N^n, \langle\cdot,\cdot\rangle)$ with constant sectional curvature K. And κ_i $(i = 1, 2, \dots, 5)$ is constant. Then, γ is 3-harmonic curve if and only if*

$$\begin{cases}(\kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2) - K(2\kappa_1^3 + \kappa_1\kappa_2^2) = 0,\\ -\kappa_1\kappa_2\kappa_3(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2 - K) = 0,\\ \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5 = 0.\end{cases} \tag{21}$$

Proof.

$$\begin{aligned}
(-1)^2\left(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\right)^2\tau(\gamma) = {}&(\kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2)e_2\\
&+ (-\kappa_1^3\kappa_2\kappa_3 - \kappa_1\kappa_2^3\kappa_3 - \kappa_1\kappa_2\kappa_3^3 - \kappa_1\kappa_2\kappa_3\kappa_4^2)e_4\\
&+ \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5 e_6,\\
-\left(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\right)\tau(\gamma) = {}&(\kappa_1^3 + \kappa_1\kappa_2^2)e_2 - \kappa_1\kappa_2\kappa_3 e_4,\\
\left(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\right)\tau(\gamma) - \left\langle\gamma', \left(\nabla^N_{\gamma'}\nabla^N_{\gamma'}\right)\tau(\gamma)\right\rangle\gamma' = {}&(\kappa_1^3 + \kappa_1\kappa_2^2)e_2 - \kappa_1\kappa_2\kappa_3 e_4,\\
\langle\tau(\gamma), \gamma'\rangle\nabla^N_{\gamma'}\tau(\gamma) - \left\langle\gamma', \nabla^N_{\gamma'}\tau(\gamma)\right\rangle\tau(\gamma) = {}&\kappa_1^3 e_2.
\end{aligned}$$

By using Proposition 5.5, γ is 3-harmonic curve if and only if

$$\begin{aligned}
&(\kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2)e_2\\
&+ (-\kappa_1^3\kappa_2\kappa_3 - \kappa_1\kappa_2^3\kappa_3 - \kappa_1\kappa_2\kappa_3^3 - \kappa_1\kappa_2\kappa_3\kappa_4^2)e_4
\end{aligned}$$

$$+\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5 e_6$$
$$-K\{(\kappa_1^3+\kappa_1\kappa_2^2)e_2-\kappa_1\kappa_2\kappa_3 e_4+\kappa_1^3 e_2\}=0.$$

Thus we have,

$$\begin{cases}(\kappa_1^5+2\kappa_1^3\kappa_2^2+\kappa_1\kappa_2^4+\kappa_1\kappa_2^2\kappa_3^2)-K(2\kappa_1^3+\kappa_1\kappa_2^2)=0,\\ -\kappa_1\kappa_2\kappa_3(\kappa_1^2+\kappa_2^2+\kappa_3^2+\kappa_4^2-K)=0,\\ \kappa_1\kappa_2\kappa_3\kappa_4\kappa_5=0.\end{cases} \tag{22}$$ □

Since κ_i, $(i=1,2,3,4,5)$ is constant, we can write κ_i as,

$$\kappa_2=\alpha\kappa_1,\quad \kappa_3=\beta\kappa_1,\quad \kappa_4=\delta\kappa_1,\quad \kappa_5=\theta\kappa_1,$$

where α, β, δ and θ are constant.

Proposition 5.10. *Let $\gamma\colon I\to(N^n,\langle\cdot,\cdot\rangle)$ be a smooth curve parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold $(N^n,\langle\cdot,\cdot\rangle)$ with constant sectional curvature K. And κ_i $(i=1,2,\dots,5)$ is constant. Then, γ is a 3-harmonic curve if and only if*

(1) *When $n=2$, $\kappa_1=\sqrt{2K}$.*
(2) *When $n=3$, $\kappa_1=\sqrt{2K}$, or $\kappa_1=\sqrt{(2+\alpha^2)K}/(1+\alpha^2)$.*
(3) *When $n\geq 4$, $\kappa_1=\sqrt{2K}$, or $\kappa_1=\sqrt{(2+\alpha^2)K}/(1+\alpha^2)$ and $\kappa_3=0$.*

Proof. When $n=2$, by Proposition 5.7, $\kappa_1=\sqrt{2K}$.

When, $\dim N=3$, namely $\kappa_3=\kappa_4=\kappa_5=0$, γ is 3-harmonic if and only if

$$0=\kappa_1^4+2\alpha^2\kappa_1^4+\alpha^4\kappa_2^4-K(2\kappa_1^2+\alpha^2\kappa_1^2).$$

Thus, we have

$$\kappa_1=\frac{\sqrt{(2+\alpha^2)K}}{1+\alpha^2}\leq\sqrt{2K}.$$

When, $\dim N=4$, namely $\kappa_4=\kappa_5=0$, γ is 3-harmonic if and only if

$$(\kappa_1^5+2\kappa_1^3\kappa_2^2+\kappa_1\kappa_2^4+\kappa_1\kappa_2^2\kappa_3^2)-K(2\kappa_1^3+\kappa_1\kappa_2^2)=0, \tag{23}$$

$$\kappa_1\kappa_2\kappa_3=0,\quad\text{or}\quad \kappa_1^2+\kappa_2^2+\kappa_3^2=K. \tag{24}$$

If $\kappa_2=0$, $\kappa_1^2=2K$.

If $\kappa_3=0$, $\kappa_1=\sqrt{(2+\alpha^2)K}/(1+\alpha^2)$.

If $\kappa_1^2+\kappa_2^2+\kappa_3^2=K$, there are no solution.

When, dim $N = 5$, namely $\kappa_5 = 0$, γ is 3-harmonic if and only if

$$(\kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2) - K(2\kappa_1^3 + \kappa_1\kappa_2^2) = 0, \tag{25}$$

$$\kappa_1\kappa_2\kappa_3 = 0, \quad \text{or} \quad \kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2 = K. \tag{26}$$

If $\kappa_2 = 0$, $\kappa_1^2 = 2K$.
If $\kappa_3 = 0$, $\kappa_1 = \sqrt{(2 + \alpha^2)K}/(1 + \alpha^2)$.
If $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2 = K$, there are no solution.
When, dim $N \geq 6$, γ is 3-harmonic if and only if

$$(\kappa_1^5 + 2\kappa_1^3\kappa_2^2 + \kappa_1\kappa_2^4 + \kappa_1\kappa_2^2\kappa_3^2) - K(2\kappa_1^3 + \kappa_1\kappa_2^2) = 0, \tag{27}$$

$$-\kappa_1\kappa_2\kappa_3(\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2 - K) = 0, \tag{28}$$

$$\kappa_1\kappa_2\kappa_3\kappa_4\kappa_5 = 0. \tag{29}$$

If $\kappa_2 = 0$, $\kappa_1^2 = 2K$.
If $\kappa_3 = 0$, $\kappa_1 = \sqrt{(2 + \alpha^2)K}/1 + \alpha^2$.
If $\kappa_4 = 0$, there are no solution.
If $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 + \kappa_4^2 = K$, there are no solution. □

Finally, we determine that the ODEs of the 3-harmonic curve equations into a sphere. This result was proved for $k = 2$ in [2] and for S^3 in [3].

Proposition 5.11. *Let $\gamma\colon I \to S^n \subset \mathbb{R}^{n+1}$ be a smooth curve parametrized by arc length. Then γ is 3-harmonic curve if and only if*

$$-\gamma^{(6)} - 2\gamma^{(4)} - (2g_{13} + 3)\gamma'' + 4g_{23}\gamma' + (1 + 9g_{24} + 8g_{33})\gamma = 0, \tag{30}$$

where $g_{ij} = g_0(\gamma^{(i)}, \gamma^{(j)})$, $(i, j = 0, 1, \dots)$, and g_0 is the standard metric on the Euclidean space $\mathbb{R}^{n+1}$.

Proof.
$$\nabla^0_{\gamma'}\gamma' = B(\gamma', \gamma') + \nabla_{\gamma'}\gamma',$$

which yields that
$$\nabla_{\gamma'}\gamma' = \nabla^0_{\gamma'}\gamma' + g(\gamma', \gamma')\gamma.$$

Therefore, we have $\nabla_{\gamma'}\gamma' = \gamma'' + \gamma$. Similarly, we have

$$(\nabla_{\gamma'}\nabla_{\gamma'})(\nabla_{\gamma'}\gamma') = \gamma^{(4)} + \gamma'' + (g_{13} + 1)\gamma,$$

$$\begin{aligned}(\nabla_{\gamma'}\nabla_{\gamma'})^2(\nabla_{\gamma'}\gamma') &= \gamma^{(6)} + \gamma^{(4)} + (g_{13} + 1)\gamma'' + (2g_{23} + 3g_{14})\gamma' \\ &\quad + (1 + g_{33} + 3g_{24} + 3g_{15} + g_{22} + 3g_{13})\gamma,\end{aligned} \tag{31}$$

$$R^N(\bar{\Delta}\tau(\gamma), \gamma')\gamma' = -\gamma^{(4)} - \gamma'' - (g_{13}+1)\gamma + g_{14}\gamma', \tag{32}$$

$$R^N(\nabla_{\gamma'}\tau(\gamma), \gamma')\gamma' = -(g_{13}+1)\gamma'' - (g_{13}+1)\gamma, \tag{33}$$

where we used $g_{13} = -g_{22}$, $g_{14} = -3g_{23}$, $g_{15} = -3g_{33} - 4g_{24}$. So we have Proposition 5.11. □

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