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STABILITY CONDITIONS AND μ-STABLE SHEAVES ON K3 SURFACES WITH PICARD NUMBER ONE

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Abstract

In this article, we show that some semi-rigid μ-stable sheaves on a projective K3 surface X with Picard number 1 are stable under Bridgeland's stability condition. As a consequence of our work, we show that the special set $U(X) \subset \operatorname{Stab}(X)$ introduced by Bridgeland reconstructs X itself. This gives a sharp contrast to the case of an abelian surface.

1. Introduction and statement of results

In the paper [2], Bridgeland constructed the theory of stability conditions on triangulated categories $\mathcal{D}$. Roughly speaking a stability condition $\sigma = (\mathcal{A}, Z)$ is a pair consisting of the heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{D}$ and a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ where $K(\mathcal{A})$ is the Grothendieck group of $\mathcal{A}$. For σ, we can define the notion of σ-stability for objects $E \in \mathcal{D}$. Very roughly, E is said to be σ-stable if $\arg Z(A) < \arg Z(E)$ for any non-trivial "subobject" A of E. However, there is no notion of subobjects in $\mathcal{D}$. Thus the heart is necessary for us to define it.

Let us consider the case $\mathcal{D}$ is the bounded derived category $D(X)$ of a projective manifold X. Namely $D(X)$ is the bounded derived category of $\operatorname{Coh}(X)$, where $\operatorname{Coh}(X)$ is the abelian category of coherent sheaves on X.

One of the big problems is the non-emptiness of the space $\operatorname{Stab}(\mathcal{D})$ of stability conditions for an arbitrary triangulated category $\mathcal{D}$. However, when X is a projective K3 surface or an abelian surface, Bridgeland found a connected component $\operatorname{Stab}^{\dagger}(X)$ of the space $\operatorname{Stab}(X)$ of stability conditions on $D(X)$. $\operatorname{Stab}^{\dagger}(X)$ can be described by using the special locus "$U(X)$" given by (see also Sections 2 and 3)

$$\begin{aligned} U(X) := \{\sigma \in \operatorname{Stab}(X) \mid \forall x \in X,\ \mathcal{O}_x \text{ is } \sigma\text{-stable with the same phase} \\ \text{and } \sigma \text{ is good, locally finite and numerical}\}. \end{aligned}$$

Since $U(X)$ is connected by [3], we can define $\operatorname{Stab}^{\dagger}(X)$ by the connected component which contains $U(X)$. We also remark that $U(X)$ is a proper subset of $\operatorname{Stab}^{\dagger}(X)$ if X is a projective K3 surface by [3].

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Broadly speaking, the topic of our research is an analysis of the relation between $U(X)$ and Fourier–Mukai partners of X. Originally stability conditions are defined on $D(X)$ independently of X. Let us recall that for some K3 surface X, there is another K3 surface Y such that Y is not isomorphic to X but $D(Y)$ is equivalent to $D(X)$. Let $\Phi\colon D(Y) \to D(X)$ be an equivalence. Then Φ naturally induces an isomorphism $\Phi_*\colon \mathrm{Stab}(Y) \to \mathrm{Stab}(X)$. We shall treat the following problem:

PROBLEM. Suppose that Y is not isomorphic to X. Then does there exist an equivalence $\Phi\colon D(Y) \to D(X)$ so that $\Phi_*(U(Y)) = U(X)$?

We can see that the answer of this problem is negative by the following first main theorem.

Theorem 1.1 (Corollary 6.7). *Let X and Y be projective K3 surfaces with Picard number 1. Suppose that $\Phi\colon D(Y) \to D(X)$ is an equivalence with $\Phi_*(U(Y)) = U(X)$. Then Φ can be written as*:

$$\Phi(?) = M \otimes f_*(?)[n],$$

where M is a line bundle on X, f is an isomorphism $f\colon Y \to X$ and $n \in \mathbb{Z}$.

Recall that if X is a projective K3 surface of Picard number 1 and Y is a projective manifold such that $D(X) \sim D(Y)$ then Y is also a projective K3 surface of Picard number 1. Suitable reference is, for instance, [1] or [9]. Furthermore in Corollary 6.8, we give the interpretation of Theorem 1.1 from the viewpoint of the autoequivalence group $\mathrm{Aut}(D(X))$ of $D(X)$.

Theorem 1.1 implies that the special locus $U(X)$ is determined by X although $\mathrm{Stab}(X)$ is defined on the category $D(X)$. It is interesting to observe that, when X and Y are abelian surfaces, $\Phi_*(U(Y)) = U(X)$ for any equivalence $\Phi\colon D(Y) \to D(X)$ (cf. Remark 6.9). At first, we expected that there exists an equivalence $\Phi\colon D(Y) \to D(X)$ preserving $U(X)$ although Y is not isomorphic to X.

It is well known that any Fourier–Mukai partners of a projective K3 surface X are given by moduli spaces of Gieseker-stable sheaves. Hence our first approach was the investigation of σ-stability of μ-stable (or Gieseker stable) sheaves.

Before we state the second main theorem Theorem 1.2, we shall explain two notations which we use in the theorem (the details appear in Section 3). There is a subset $V(X)$ of $U(X)$ which is (roughly) parametrized by $\mathbb{R}$-divisors β and $\mathbb{R}$-ample divisors ω. So we write as $\sigma_{(\beta,\omega)} \in V(X)$. The set $V(X)$ contains the locus $V(X)_{>2}$ defined by

$$V(X)_{>2} := \{\sigma_{(\beta,\omega)} \in V(X) \mid \omega^2 > 2\}.$$

Theorem 1.2. *Let X be a projective $K3$ surface with* $\mathrm{NS}(X) = \mathbb{Z} \cdot L$. *We put* $d = L^2/2$. *Let E be a torsion free sheaf with* $v(E)^2 = 0$ (*see* Section 3.1 *for the definition of* $v(E)$) *and* $\operatorname{rank} E \leq \sqrt{d}$, *and let* $\sigma = (Z, \mathcal{P})$ *be in* $V(X)_{>2}$.
(1) *If E is Gieseker-stable and* $E \in \mathcal{P}((0,1])$ (*see* Section 2 *for the definition of* $\mathcal{P}((0,1])$), *then E is σ-stable.*
(2) *If E is μ-stable locally free and* $E \in \mathcal{P}((-1,0])$ (*see* Section 2 *for the definition of* $\mathcal{P}((-1,0])$), *then E is σ-stable.*
(3) *Let S be a spherical sheaf with* $\operatorname{rank} S \leq \sqrt{d}$. *Then S is σ-stable.*

The assertions (1) and (2) are proved in Theorem 4.6, and the assertion (3) is Proposition 5.4. The assumption "$\operatorname{rank} E \leq \sqrt{d}$ is the best possible in some sense (see Example 5.5), and we can not remove the assumption of local-freeness in (2) (see Corollary 5.7). We prove Theorem 1.1 applying Theorem 1.2.

Finally we explain the contents of this paper. Section 2 is a survey of the general theory of stability conditions on triangulated categories. In Section 3, we study the case when $\mathcal{D} = D(X)$ where X is a projective K3 surface. In the last half of Section 3, we shall recall the results on Gieseker stable sheaves and on Fourier–Mukai partners on K3 surfaces with Picard number 1.

In Section 4, we shall prove (1) and (2) of Theorem 1.2 (= Theorem 4.6). Hence the main part of this section is the comparison between the μ-stability (or Gieseker-stability) and the σ-stability. We remark that the σ-stability of $E \in D(X)$ depends on the argument of the complex number $Z(E)$. Hence we need an appropriate description of $Z(E)$ to compare the argument of $Z(E)$ and the slope $\mu_\omega(E)$. There are two keys for the comparison. One is the following expression of the stability function $Z_{(\beta,\omega)}$ (the definition of $Z_{(\beta,\omega)}$ is in Section 3):

$$Z_{(\beta,\omega)}(E) = \frac{v(E)^2}{2r_E} + \frac{r_E}{2}\left(\omega + \sqrt{-1}\left(\frac{\Delta_E}{r_E} - \beta\right)\right)^2.$$

The other is the assumption that the Picard number of X is one. If X satisfies the assumption, the right hand side of the above formula is just complex number. Thus we can compare the slope $\mu_\omega(E)$ and the argument of $Z(E)$.

In Section 5, we prove Theorem 1.2 (3) (= Proposition 5.4). The strategy of the proof is essentially the same as that of Theorem 4.6. We have two applications of Proposition 5.4. One is to prove that we cannot drop the assumption on rank and the condition of local-freeness in Theorem 4.6. The other is the determination of Harder–Narasimhan filtrations of some special objects $T_S(\mathcal{O}_x)$ (cf. Corollary 5.7 and 5.8). In general, it is very difficult to determine Harder–Narasimhan filtrations. So, these examples are valuable.

In Section 6, we shall treat two applications of Theorem 1.2. The first application is to find some pairs (E, σ) such that an object $E \in D(X)$ is a true complex and E is σ-stable for some $\sigma \in U(X)$. The second application is to prove Theorem 1.1.

2. Bridgeland’s stability condition

This section is a survey of the general theory of Bridgeland’s stability conditions on triangulated categories. Let $\mathcal{D}$ be a $\mathbb{C}$ linear triangulated category. The symbol [1] means the shift of $\mathcal{D}$ and $[n]$ means the n-times composition of [1].

DEFINITION 2.1. Let $\sigma = (Z, \mathcal{P})$ be a pair consisting of a group homomorphism $Z\colon K(\mathcal{D}) \to \mathbb{C}$ from the Grothendieck group of $\mathcal{D}$ to $\mathbb{C}$, and a collection $\mathcal{P} = \{\mathcal{P}(\phi)\}$ of additive full subcategories $\mathcal{P}(\phi)$ of $\mathcal{D}$ parametrized by the real numbers ϕ. This pair σ is a stability condition on $\mathcal{D}$ if it is satisfied the following condition:
(1) If $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E)\exp(\sqrt{-1}\pi\phi)$ where $m(E) > 0$.
(2) If $\phi > \psi$, then $\mathrm{Hom}_{\mathcal{D}}(E, F) = 0$ for all $E \in \mathcal{P}(\phi)$ and $F \in \mathcal{P}(\psi)$.
(3) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
(4) For all $0 \neq E \in \mathcal{D}$, there is a sequence of distinguished triangles satisfying the following condition:

$$\begin{tikzcd}
0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E, \arrow[dl] \\
& A_1 \arrow[ul, dashed, "{[1]}"] & & A_2 \arrow[ul, dashed, "{[1]}"] & & & & A_n \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd} \tag{2.1}$$

where each A_i is in $\mathcal{P}(\phi_i)$ $(i = 1, \dots, n)$ with $\phi_1 > \cdots > \phi_n$.

REMARK 2.2. (1) Each $\mathcal{P}(\phi)$ is an abelian category.
(2) By definition, for each $0 \neq E \in \mathcal{D}$, there is at most one $\phi \in \mathbb{R}$ such that $E \in \mathcal{P}(\phi)$. When $E \in \mathcal{P}(\phi)$, we define $\arg Z(E) := \phi$ and call ϕ the *phase* of E.
(3) $E \in \mathcal{D}$ is said to be σ*-semistable* when $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$. In particular, if E is minimal in $\mathcal{P}(\phi)$ (that is, E has no non-trivial subobjects) then E is said to be σ*-stable*.
(4) The sequence (2.1) is unique up to isomorphism. We can easily check this by using the property Definition 2.1 (2). Hence we define $\phi^+_\sigma(E) := \phi_1$, and $\phi^-_\sigma(E) := \phi_n$. We call the sequence the *Harder–Narasimhan filtration* (for short HN filtration) of E, and each A_i a *semistable factor* of E.
(5) Let $I \subset \mathbb{R}$ be an interval. For I, we define $\mathcal{P}(I)$ as the extension closed additive full subcategory of $\mathcal{D}$ generated by $\mathcal{P}(\phi)$ $(\phi \in I)$. If $E \in \mathcal{P}(I)$, then $\phi^+(E)$ and $\phi^-(E) \in I$.
(6) A stability condition σ is said to be *locally finite* if for all $\phi \in \mathbb{R}$, there is a positive number ϵ such that the quasi-abelian category $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is finite length, that is both increasing and decreasing sequences of subobjects of A will terminate (see also §4 of [2]). The property of local-finiteness guarantees the existence of Jordan–Hölder filtrations (for short JH filtrations), that is, for any $0 \neq A \in \mathcal{P}(\phi)$, there exists

a sequence of distinguished triangles

$$\begin{tikzcd}
0 \arrow[rr] & & A_1 \arrow[rr] \arrow[dl] & & A_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & A_{n-1} \arrow[rr] & & A_n = A \arrow[dl] \\
& S_1 \arrow[ul, dashed, "{[1]}"] & & S_2 \arrow[ul, dashed, "{[1]}"] & & & & S_n \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

such that each S_i is σ-stable with phase ϕ. We call each S_i a *stable factor* of A. We remark that JH filtrations may not be unique.

In general it is difficult to construct stability conditions on $\mathcal{D}$. However, by using Proposition 2.4 (below), we can explicitly construct them in some cases. Before we state the proposition, we introduce the notion of a stability condition on abelian categories.

DEFINITION 2.3. Let $\mathcal{A}$ be an abelian category, and $Z\colon K(\mathcal{A}) \to \mathbb{C}$ a group homomorphism from the Grothendieck group $K(\mathcal{A})$ of $\mathcal{A}$ to $\mathbb{C}$, satisfying

$$Z(E) = m_E \exp(\sqrt{-1}\pi\phi_E) \quad \text{for} \quad 0 \neq E \in \mathcal{A}, \quad \text{where} \quad \phi_E \in (0, 1] \quad \text{and} \quad m_E > 0.$$

We call Z a *stability function* on $\mathcal{A}$. An object $E \in \mathcal{A}$ is called a (*semi*)*stable object* for Z when, for any non-trivial subobjects F of E, the following inequality holds:

$$\phi_F < \phi_E, \quad (\phi_F \leq \phi_E).$$

If Z has the following property, we call Z a stability function equipped with the *Harder–Narasimhan* (*for short HN*) *property*:

$$0 \neq \forall E \in \mathcal{A}, \quad \exists \text{a filtration} \quad 0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E \quad \text{such that}$$
$$A_i = E_i/E_{i-1} \quad \text{is semistable and} \quad \phi_{A_1} > \cdots > \phi_{A_n}.$$

Proposition 2.4 ([2, Proposition 5.3]). *Let $\mathcal{D}$ be a triangulated category. Then the following are equivalent*:
(1) *To give a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$.*
(2) *To give a pair $(\mathcal{A}, Z_{\mathcal{A}})$ consisting of the heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{D}$ and a stability function $Z_{\mathcal{A}}$ on $\mathcal{A}$ which has the HN property.*

For the convenience of readers, we give a sketch of the proof.

From (1) to (2). For the pair $\sigma = (Z, \mathcal{P})$, $\mathcal{P}((0, 1])$ is the heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{D}$. We define a stability function $Z_{\mathcal{A}}$ as Z. Then the pair $(\mathcal{P}((0, 1]), Z)$ is what we need.

From (2) to (1). For a real number $\phi \in (0, 1]$ we define $\mathcal{P}(\phi)$ by

$$\mathcal{P}(\phi) := \{A \in \mathcal{A} \mid A \text{ is semistable for } Z \text{ with } \phi_A = \phi\} \cup \{0\}.$$

If $\psi \in \mathbb{R}\backslash(0,1]$, we define $\mathcal{P}(\psi)$ by $\mathcal{P}(\psi_0)[k]$ where $\psi = \psi_0 + k$ with $\psi_0 \in (0,1]$ and $k \in \mathbb{Z}$. Since $K(\mathcal{A}) = K(\mathcal{D})$, we can define Z by $Z_{\mathcal{A}}$. Then the pair $(Z, \mathcal{P})$ gives a stability condition on $\mathcal{D}$. $\square$

In the following lemma, we introduce two actions of groups on $\mathrm{Stab}(X)$.

Lemma 2.5 ([2, Lemma 8.2]). *Let* $\mathrm{Stab}(\mathcal{D})$ *be the space of stability condition on* $\mathcal{D}$, $\widetilde{GL}^+(2,\mathbb{R})$ *the universal covering space of* $GL^+(2,\mathbb{R})$, *and* $\mathrm{Aut}(\mathcal{D})$ *the autoequivalence group of* $\mathcal{D}$. $\mathrm{Stab}(\mathcal{D})$ *carries a right action of* $\widetilde{GL}^+(2,\mathbb{R})$, *and a left action of* $\mathrm{Aut}(\mathcal{D})$. *In addition, these two actions commute.*

REMARK 2.6. By the definition of the action of $\widetilde{GL}^+(2,\mathbb{R})$, we can easily see that for any $\sigma \in \mathrm{Stab}(\mathcal{D})$ and any $\tilde{g} \in \widetilde{GL}^+(2,\mathbb{R})$, $E \in \mathcal{D}$ is σ-(semi)stable if and only if E is $\sigma \cdot \tilde{g}$-(semi)stable.

3. Stability conditions on K3 surfaces

In this section X is a projective K3 surface over $\mathbb{C}$, $\mathrm{Coh}(X)$ is the abelian category of coherent sheaves on X, and $D(X)$ is the bounded derived category of $\mathrm{Coh}(X)$. The purpose of this section is to give a description of $\mathrm{Stab}(X)$.

We first introduce some notations. Let A and B be in $D(X)$. If the i-th cohomology $H^i(A)$ is concentrated only at degree $i = 0$, we call A a *sheaf*. We put $\mathrm{Hom}^n_X(A,B) := \mathrm{Hom}_{D(X)}(A, B[n])$. If both A and B are sheaves, then $\mathrm{Hom}^n_X(A,B)$ is just $\mathrm{Ext}^n_{\mathcal{O}_X}(A,B)$. We also put $\hom^n_X(A,B) := \dim_{\mathbb{C}} \mathrm{Hom}^n_X(A,B)$ and $\mathrm{ext}^n_X(A,B) := \dim \mathrm{Ext}^n_{\mathcal{O}_X}(A,B)$. Sometimes we omit X of $\mathrm{Hom}^n_X(A,B)$ and so on. We remark that

$$\mathrm{Hom}^n_X(A,B) = \mathrm{Hom}^{2-n}_X(B,A)^*$$

by the Serre duality.

We secondly recall the notion of the μ-stability. For a torsion free sheaf F and an ample divisor ω, the *slope* $\mu_\omega(F)$ is defined by $(c_1(F)\cdot\omega)/\mathrm{rank}\, F$ where $c_1(F)$ is the first Chern class of F. If the inequality $\mu_\omega(A) \leq \mu_\omega(F)$ holds for any non-trivial subsheaf A of F, then F is said to be μ*-semistable*. Moreover if the strict inequality $\mu_\omega(A) < \mu_\omega(F)$ holds for any non-trivial subsheaf A with $\mathrm{rank}\, A < \mathrm{rank}\, F$, then F is said to be μ*-stable*. The notion of the μ-stability admits the Harder–Narasimhan filtration of F (details in [6]). We define $\mu^+_\omega(F)$ by the maximal slope of semistable factors of F, and $\mu^-_\omega(F)$ by the minimal slope of semistable factors of F.

3.1. On numerical stability conditions on $D(X)$.

Let $K(X)$ be the Grothendieck group of $D(X)$. $K(X)$ has the natural $\mathbb{Z}$ bilinear form χ:

$$\chi\colon K(X)\times K(X) \to \mathbb{Z}, \quad \chi(E,F) := \sum_i (-1)^i \hom^i_X(E,F).$$

Let $\mathcal{N}(X)$ be the quotient of $K(X)$ by numerical equivalent classes with respect to χ. Then $\mathcal{N}(X)$ is $H^0(X,\mathbb{Z})\oplus \mathrm{NS}(X)\oplus H^4(X,\mathbb{Z})$, where $\mathrm{NS}(X)$ is the Néron-Severi lattice of X. A stability condition $\sigma = (Z,\mathcal{P})$ on $D(X)$ is said to be *numerical* if Z factors through $\mathcal{N}(X)$:

$$\begin{tikzcd} K(X) \arrow[r] \arrow[dr, "Z"'] & \mathcal{N}(X) \arrow[d, "Z_{\mathcal{N}}"] \\ & \mathbb{C}. \end{tikzcd}$$

Let $\chi_{\mathcal{N}}$ be the descent of χ. Since $\chi_{\mathcal{N}}$ is non-degenerate on $\mathcal{N}(X)\otimes_{\mathbb{Z}}\mathbb{C}$, $Z_{\mathcal{N}}$ is canonically in $\mathcal{N}(X)\otimes\mathbb{C}$:

$$\mathrm{Hom}_{\mathbb{C}}(\mathcal{N}(X)\otimes\mathbb{C},\mathbb{C}) \to \mathcal{N}(X)\otimes\mathbb{C}, \quad Z_{\mathcal{N}} \mapsto Z^{\vee},$$

where $Z(E) = \chi_{\mathcal{N}}(Z^{\vee}, E)$. Thus we define $\mathrm{Stab}(X)$ by

$$\mathrm{Stab}(X) := \{\sigma \in \mathrm{Stab}(D(X)) \mid \sigma \text{ is locally finite and numerical}\}.$$

Then we have the following natural map:

$$\pi : \mathrm{Stab}(X) \to \mathcal{N}(X)\otimes\mathbb{C}, \quad \pi((Z,\mathcal{P})) = Z^{\vee}.$$

We remark that π is a locally homeomorphism (The details are in [2, Corollary 1.3]). Hence the map π gives a complex structure on $\mathrm{Stab}(X)$. In particular $\mathrm{Stab}(X)$ is a complex manifold.

Let $\langle -,-\rangle$ be the Mukai pairing on $\mathcal{N}(X)$:

$$\langle r\oplus\Delta\oplus s, r'\oplus\Delta'\oplus s'\rangle = \Delta\Delta' - rs' - r's,$$

where both $r\oplus\Delta\oplus s$ and $r'\oplus\Delta'\oplus s'$ are in $H^0(X,\mathbb{Z})\oplus\mathrm{NS}(X)\oplus H^4(X,\mathbb{Z})$. For an objects $E\in D(X)$, we put $v(E) = ch(E)\sqrt{td_X}\in\mathcal{N}(X)$ and call it the *Mukai vector* of E. Then we have $\chi(E,F) = -\langle v(E), v(F)\rangle$ for E and $F\in D(X)$ by the Riemann-Roch theorem. We have the following famous consequence:

Lemma 3.1. *Let X be a projective $K3$ surface and $E\in D(X)$. Assume that* $\hom^0_X(E,E) = 1$. *Then we have*

$$\langle v(E)\rangle^2 + 2 = \hom^1_X(E,E).$$

Thus we have $\langle v(E)\rangle^2 \geq -2$ and the equality holds if and only if $\hom^1(E,E) = 0$.

If, for $E\in D(X)$, $\hom^1(E,E) = 2$, E is said to be *semi-rigid*. Assume that $\hom^0(E,E) = 1$. Then by the above lemma, $\langle v(E)\rangle^2 = 0$ if and only if E is semi-rigid.

3.2. Construction of $U(X)$. Next, following Bridgeland, we define a special subset $U(X)$ of $\mathrm{Stab}(X)$ and give two descriptions of $U(X)$. Put

$$\mathrm{NS}(X)_{\mathbb{R}} := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad \mathrm{Amp}(X)_{\mathbb{R}} := \{\omega \in \mathrm{NS}(X)_{\mathbb{R}} \mid \omega \text{ is ample}\}.$$

We first define the subset $\mathcal{V}(X)$ of $\mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}$ by

$$\begin{aligned}\mathcal{V}(X) := \{(\beta, \omega) \in \mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}} \mid \\ \forall \delta \in \Delta^{+}(X),\ \langle \exp(\beta + \sqrt{-1}\omega), \delta\rangle \notin \mathbb{R}_{\leq 0}\},\end{aligned}$$

where $\Delta^{+}(X) = \{r \oplus \Delta \oplus s \in \mathcal{N}(X) \mid \langle r \oplus \Delta \oplus s\rangle^2 = -2 \text{ and } r > 0\}$. If $\omega^2 > 2$ then $(\beta, \omega) \in \mathcal{V}(X)$ for all $\beta \in \mathrm{NS}(X)_{\mathbb{R}}$. Hence $\mathcal{V}(X) \neq \emptyset$. Thus we define

$$\mathcal{V}(X)_{>2} := \{(\beta, \omega) \in \mathcal{V}(X) \mid \omega^2 > 2\}.$$

We can define a torsion pair $(\mathcal{T}_{(\beta,\omega)}, \mathcal{F}_{(\beta,\omega)})$ (see below) of $\mathrm{Coh}(X)$ by using a pair $(\beta, \omega) \in \mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}$. As a consequence we have a new heart of the bounded t-structure which comes from the torsion pair $(\mathcal{T}_{(\beta,\omega)}, \mathcal{F}_{(\beta,\omega)})$.

Lemma 3.2 ([3, Lemma 6.1]). *Let $\beta \in \mathrm{NS}(X)_{\mathbb{R}}$ and $\omega \in \mathrm{Amp}(X)_{\mathbb{R}}$. We define respectively $\mathcal{T}_{(\beta,\omega)}$, $\mathcal{F}_{(\beta,\omega)}$ and $\mathcal{A}_{(\beta,\omega)}$ by*

$$\begin{aligned}\mathcal{T}_{(\beta,\omega)} &:= \{E \in \mathrm{Coh}(X) \mid E \textit{ is a torsion sheaf or } \mu_{\omega}^{-}(E/\mathrm{torsion}) > \beta\omega\},\\ \mathcal{F}_{(\beta,\omega)} &:= \{E \in \mathrm{Coh}(X) \mid E \textit{ is torsion free and } \mu_{\omega}^{+}(E) \leq \beta\omega\},\end{aligned}$$

and

$$\mathcal{A}_{(\beta,\omega)} := \left\{ E^{\bullet} \in D(X) \;\middle|\; H^{i}(E^{\bullet}) \begin{cases} \in \mathcal{T}_{(\beta,\omega)} & (i = 0),\\ \in \mathcal{F}_{(\beta,\omega)} & (i = -1),\\ = 0 & (i \neq 0, -1)\end{cases} \right\}.$$

(1) *The pair $(\mathcal{T}_{(\beta,\omega)}, \mathcal{F}_{(\beta,\omega)})$ is a torsion pair of* $\mathrm{Coh}(X)$.
(2) *$\mathcal{A}_{(\beta,\omega)}$ is the heart of the bounded t-structure determined by the torsion pair $(\mathcal{T}_{(\beta,\omega)}, \mathcal{F}_{(\beta,\omega)})$.*

The condition that $(\beta, \omega) \in \mathcal{V}(X)$ is necessary when we construct a stability function $Z_{(\beta,\omega)}$ on $\mathcal{A}_{(\beta,\omega)}$.

Proposition 3.3 ([3]). *For $(\beta, \omega) \in \mathcal{V}(X)$, we define the group homomorphism $Z_{(\beta,\omega)} \colon K(X) \to \mathbb{C}$ by*

$$Z_{(\beta,\omega)}(E) := \langle \exp(\beta + \sqrt{-1}\omega), v(E)\rangle.$$

Then $Z_{(\beta,\omega)}$ is a stability function on $\mathcal{A}_{(\beta,\omega)}$ with the HN-property. Hence the pair $(\mathcal{A}_{(\beta,\omega)}, Z_{(\beta,\omega)})$ defines a stability condition $\sigma_{(\beta,\omega)}$ on $D(X)$. In particular $\sigma_{(\beta,\omega)}$ is numerical and locally finite.

Here we put

$$V(X) := \{\sigma_{(\beta,\omega)} \mid (\beta,\omega) \in \mathcal{V}(X)\} \quad \text{and} \quad V(X)_{>2} := \{\sigma_{(\beta,\omega)} \mid (\beta,\omega) \in \mathcal{V}(X)_{>2}\}.$$

The most important property of $\sigma \in V(X)$ is the σ-stability of the structure sheaves $\mathcal{O}_x$ of closed points x of X.

Proposition 3.4 ([3, Lemma 6.3]). *Let $x \in X$. Then $\mathcal{O}_x$ is minimal in $\mathcal{A}_{(\beta,\omega)}$ for any $(\beta,\omega) \in \mathcal{V}(X)$. Namely $\mathcal{O}_x$ does not have non-trivial subobjects in $\mathcal{A}_{(\beta,\omega)}$. In particular $\mathcal{O}_x$ is σ-stable with phase 1 for any $\sigma \in V(X)$.*

REMARK 3.5. Let $\sigma_{(\beta,\omega)} = (Z, \mathcal{P}) \in V(X)$.
(1) By Proposition 3.4 and [3, Lemma 10.1], any sheaf $F \in \mathrm{Coh}(X)$ is in $\mathcal{P}((-1,1])$. In addition to Proposition 3.4, if $E \in D(X)$ is $\sigma_{(\beta,\omega)}$-stable with phase 1 then E is $\mathcal{O}_x$ for some $x \in X$ or $\mathcal{E}[1]$ where $\mathcal{E}$ is a locally free sheaf. In particular, there is no torsion free σ-semistable sheaf of phase 1.
(2) As we stated, $\mathrm{Coh}(X)$ is a full subcategory of $\mathcal{P}((-1,1])$. Moreover by Proposition 3.4, we have

$$\mathcal{T}_{(\beta,\omega)} = \mathcal{P}((0,1]) \cap \mathrm{Coh}(X), \quad \text{and} \quad \mathcal{F}_{(\beta,\omega)} = \mathcal{P}((-1,0]) \cap \mathrm{Coh}(X). \tag{3.1}$$

This fact is proved in Step 2 of the proof of [3, Proposition 10.3]. Now, assume that a torsion free sheaf E is μ-semistable for ω. Then by (3.1):

$$E \in \begin{cases} \mathcal{T}_{(\beta,\omega)} & (\text{if } \mu_\omega(E) > \beta\omega), \\ \mathcal{F}_{(\beta,\omega)} & (\text{if } \mu_\omega(E) \le \beta\omega). \end{cases}$$

We define

$$U(X) := V(X) \cdot \widetilde{GL}^+(2,\mathbb{R}) \quad \text{and} \quad U(X)_{>2} := V(X)_{>2} \cdot \widetilde{GL}^+(2,\mathbb{R}).$$

We remark that the action of $\widetilde{GL}^+(2,\mathbb{R})$ on $U(X)$ is transitive. Since $V(X)$ is connected, $U(X)$ is also connected. This is the concrete definition of $U(X)$. Conversely we shall give an abstract definition of $U(X)$. To do this, we define the notion of good stability conditions.

For $\mho \in \mathcal{N}(X) \otimes \mathbb{C}$, we have $\mho = \mho_R + \sqrt{-1}\mho_I$ where $\mho_R$ and $\mho_I$ are in $\mathcal{N}(X) \otimes \mathbb{R}$. Let $P(X)$ be the set of vectors $\mho \in \mathcal{N}(X) \otimes \mathbb{C}$ such that Mukai pairing is positive definite on the real 2-plane spanned by $\mho_R$ and $\mho_I$. Let $\Delta(X)$ be the subset of $\mathcal{N}(X)$ defined by

$$\Delta(X) := \{\delta \in \mathcal{N}(X) \mid \langle\delta\rangle^2 = -2\}.$$

We define $P_0(X)$ by

$$P_0(X) := P(X) - \bigcup_{\delta \in \Delta(X)} \delta^{\perp},$$

where $\delta^{\perp} = \{\mho \in \mathcal{N}(X) \otimes \mathbb{C} \mid \langle \mho, \delta \rangle = 0\}$.

DEFINITION 3.6. A stability condition $\sigma \in \mathrm{Stab}(X)$ is said to be *good*, if $\pi(\sigma) \in P_0(X)$.

Proposition 3.7 ([3, Proposition 10.3]). *We have*

$$U(X) = \{\sigma \in \mathrm{Stab}(X) \mid \sigma \text{ is good and } \forall \mathcal{O}_x \text{ is } \sigma\text{-stable in a common phase}\}.$$

In [3], $U(X)$ is defined by the right hand side of Proposition 3.7. Define $\mathrm{Stab}^{\dagger}(X)$ by the unique connected component containing $U(X)$.

3.3. Gieseker stability and Fourier–Mukai partners. The last topic of Section 3 is a review of Gieseker stability. The details are in [6]. Let E be a torsion free sheaf on a K3 surface X and $p(E)$ the reduced Hilbert polynomial for an ample divisor L:

$$p(E) = \frac{\chi(\mathcal{O}_X, E \otimes nL)}{\text{rank } E} = \frac{\chi(-nL, E)}{\text{rank } E} \in \mathbb{Q}[n].$$

Using the Mukai vector $v(E) = r_E \oplus \Delta_E \oplus s_E$ of E, we write down $p(E)$:

$$\begin{aligned} p(E) &= -\frac{\langle v(-nL), v(E) \rangle}{r_E} \\ &= \frac{L^2}{2}n^2 + \frac{\Delta.L}{r_E}n + \frac{s_E}{r_E} + 1. \end{aligned} \tag{3.2}$$

A torsion free sheaf E is called a *Gieseker semistable* sheaf if, for any non-trivial subsheaf A, $p(A) \leq p(E)$ as polynomial. In particular, E is called a *Gieseker stable* sheaf when the strict inequality $p(A) < p(E)$ holds. For a torsion free sheaf E, we can easily check the following well known fact by the formula (3.2):

$$\mu\text{-stable} \Rightarrow \text{Gieseker stable} \Rightarrow \text{Gieseker semistable} \Rightarrow \mu\text{-semistable}.$$

Let $\mathcal{M}_L(v)$ be the moduli space of Gieseker stable torsion free sheaves with Mukai vector $v = r \oplus \Delta \oplus s$. If v is primitive in $\mathcal{N}(X)$, then $\mathcal{M}_L(v)$ is projective.

By the result of [5] or [11], we have a beautiful description of Fourier–Mukai partners of X when the Picard number of X is 1. Let us recall it.

Theorem 3.8 ([5, Theorem 2.1], [11]). *Let X be a projective K3 surface with $\mathrm{NS}(X) = \mathbb{Z} \cdot L$ where L is an ample line bundle on X, and let $\mathrm{FM}(X)$ be the set of isomorphic classes of Fourier–Mukai partners of X:*

$$\mathrm{FM}(X) = \{Y \mid Y \textit{ is a projective K3 surface and } D(Y) \sim D(X)\}/\sim_{\mathrm{isom}}.$$

Then $\mathrm{FM}(X)$ is given by

$$\mathrm{FM}(X) = \{\mathcal{M}_L(r \oplus L \oplus s) \mid 2rs = L^2,\ \gcd(r, s) = 1,\ r \le s\}.$$

We remark that $\mathcal{M}_L(r \oplus L \oplus s)$ is the fine moduli space of μ-stable sheaves.

4. σ-stability of μ-stable semi-rigid sheaves

From this section we mainly consider projective K3 surfaces with Picard number 1. In this article, a pair (X, L) is said to be a *generic K*3, if X is a projective K3 surface and L is an ample line bundle which generates $\mathrm{NS}(X)$. We define $\deg X$ by L^2 and call it *degree of X*. We also write the Mukai vector $v(E)$ of $E \in D(X)$ by $r_E \oplus \Delta_E \oplus s_E$. Then we have $r_E = \operatorname{rank} E$, $\Delta_E = c_1(E)$ and $s_E = \chi(\mathcal{O}_X, E) - \operatorname{rank} E$. Since $\mathrm{NS}(X) = \mathbb{Z} \cdot L$, we can write $\Delta_E = n_E L$ for some integer $n_E \in \mathbb{Z}$. So we also write $v(E) = r_E \oplus n_E L \oplus s_E$.

Our research and results are based on another expression of the function $Z_{(\beta,\omega)}$, where $\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathcal{P}_{(\beta,\omega)}) \in V(X)$. For $E \in D(X)$, assume that $r_E \neq 0$. Then we can rewrite the stability function $Z_{(\beta,\omega)}$ in the following way[1]:

$$Z_{(\beta,\omega)}(E) = \frac{v(E)^2}{2r_E} + \frac{r_E}{2}\left(\omega + \sqrt{-1}\left(\frac{\Delta_E}{r_E} - \beta\right)\right)^2. \tag{4.1}$$

We introduce a function which will appear in the proofs of Lemmas 4.5 and 5.3, and in Example 5.5. For a generic K3 (X, L) with degree $2d$, assume that $\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathcal{P}_{(\beta,\omega)}) \in V(X)$. We put $(\beta, \omega) = (xL, yL)$. Then, for $E \in D(X)$, the imaginary part of $Z_{(\beta,\omega)}(E)$ is $2\sqrt{-1}yd\lambda_E$ where $\lambda_E = n_E - r_E x$. For $E, A \in D(X)$, we define $N_{A,E}(x, y)$ by

$$N_{A,E}(x, y) := \lambda_E \cdot \mathfrak{Re}\, Z_{(\beta,\omega)}(A) - \lambda_A \cdot \mathfrak{Re}\, Z_{(\beta,\omega)}(E), \tag{4.2}$$

where $\mathfrak{Re}$ means taking the real part.

Recall the notion $\arg Z(A)$ for a σ-semistable object A and $\sigma \in \mathrm{Stab}(X)$ (cf. Remark 2.2 (2)). In general, we can not determine the argument of the complex number $Z(E)$ for an object $E \in D(X)$. However if $E \in \mathcal{P}((a, a+1])$ (for some $a \in \mathbb{R}$) then we can determine the argument of $Z(E)$. So we denote also it by $\arg Z(E)$, that is, $\phi = \arg Z(E) \overset{\text{def}}{\iff} Z(E) = m \exp(\sqrt{-1}\pi\phi)$ for some $m \in \mathbb{R}_{>0}$.

[1] We wrote the symbols $\langle\ ,\ \rangle$ till last section. From here we will omit them.

We shall use Lemma 4.1 and Proposition 4.2 to analyze of the maximal (semi)stable factor of Gieseker stable sheaves E when $E \in \mathcal{P}((0,1])$ for $\sigma = (Z, \mathcal{P}) \in V(X)$.

Lemma 4.1. *Let (X, L) be a generic K3 and $\sigma_{(\beta,\omega)} = (Z, \mathcal{P}) \in V(X)$. Assume that $A \to E \to F \to A[1]$ is a non-trivial distinguished triangle in $\mathcal{P}((0,1])$, that is, A, E and F are in $\mathcal{P}((0,1])$.*
(1) *If E is a torsion free sheaf then A is also a torsion free sheaf.*
(2) *In addition to (1), assume that E is a Gieseker stable sheaf. If $\arg Z(E) \le \arg Z(A) < 1$, then $\mu_\omega(A) < \mu_\omega(E)$.*

Proof. We first prove the assertion (1). If $G \in \mathcal{P}((0,1]) = \mathcal{A}_{(\beta,\omega)}$, then the i-th cohomology $H^i(G)$ is concentrated at $i = 0$ and -1. Then we see that A is a sheaf by the exact sequence

$$0 = H^{-2}(F) \to H^{-1}(A) \to H^{-1}(E) = 0$$

where we use the fact that E is a sheaf for the last equality. Since E and A are sheaves, we have the following exact sequence of sheaves:

$$0 \to H^{-1}(F) \to A \xrightarrow{f} E \to H^0(F) \to 0.$$

The sheaf $H^{-1}(F)$ is torsion free since it is in $\mathcal{F}_{(\beta,\omega)}$. Thus A is an extension of torsion free sheaves. Hence A is torsion free.

Let us prove the assertion (2).

CASE I. Assume $H^{-1}(F) = 0$. Then A is a subsheaf of E. So we have

$$p(A) < p(E). \tag{4.3}$$

Thus $\mu_\omega(A) \le \mu_\omega(E)$. Assume that $\mu_\omega(A) = \mu_\omega(E)$. By the formula (3.2) and the inequality (4.3) we have

$$\frac{s_A}{r_A} < \frac{s_E}{r_E},$$

where $v(A) = r_A \oplus \Delta_A \oplus s_A$ and $v(E) = r_E \oplus \Delta_E \oplus s_E$. Hence we have $v(A)^2/r_A^2 > v(E)^2/r_E^2$. Here we also used the fact that the Picard number is 1. Combining this with $\mu_\omega(A) = \mu_\omega(E)$, we have $\arg Z(A)/r_A < \arg Z(E)/r_E$ by the formula (4.1). This contradicts the fact that $\arg Z(E) \le \arg Z(A)$.

CASE II. Assume $H^{-1}(F) \ne 0$. Recall that $H^{-1}(F)$ is torsion free. We have the following inequalities:

$$\mu_\omega(H^{-1}(F)) \le \mu_\omega^+(H^{-1}(F)) \le \beta\omega < \mu_\omega^-(A) \le \mu_\omega(A).$$

Hence we have $\mu_\omega(H^{-1}(F)) < \mu_\omega(A) < \mu_\omega(\mathrm{Im}(f))$, where $\mathrm{Im}(f)$ is the image of $f\colon A \to E$. Since $\mathrm{Im}(f)$ is a subsheaf of E, $\mu_\omega(\mathrm{Im}(f)) \leq \mu_\omega(E)$. Hence we have $\mu_\omega(A) < \mu_\omega(E)$. $\square$

As a consequence of Lemma 4.1, we prove the following proposition.

Proposition 4.2. *Let (X, L) be a generic $K3$, let $\sigma = \sigma_{(\beta,\omega)} = (Z, \mathcal{P})$ be in $V(X)$, and let E be a Gieseker stable torsion free sheaf with $v(E)^2 \leq 0$ and $E \in \mathcal{P}((0, 1])$.*
(1) *Assume that E is not σ-semistable. Then there is a torsion free σ-stable sheaf S such that $\beta\omega < \mu_\omega(S) < \mu_\omega(E)$, $v(S)^2 = -2$ and* $\arg Z(S) = \phi_\sigma^+(E)$. *In particular* $\arg Z(E) < \arg Z(S)$.
(2) *Assume that E is not σ-stable but σ-semistable. Then there is a torsion free σ-stable sheaf S such that $\beta\omega < \mu_\omega(S) < \mu_\omega(E)$, $v(S)^2 = -2$ and* $\arg Z(S) = \arg Z(E)$.

Proof. We prove (1). Since E is not σ-semistable, there is the non-trivial HN filtration of E:

$$\begin{tikzcd}
0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E. \arrow[dl] \\
& A_1 \arrow[ul, dashed, "{[1]}"] & & A_2 \arrow[ul, dashed, "{[1]}"] & & & & A_n \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

Let S be a stable subobject of A_1. We show that S satisfies our requirement. By the composition of natural two morphisms, we have the following distinguished triangle in $\mathcal{P}((0, 1])$:

$$S \to E \to F \to S[1]. \tag{4.4}$$

Then S is a torsion free sheaf by Lemma 4.1 (1). By Remark 3.5, we have $\arg Z(S) = \arg Z(A_1) < 1$. Thus $\beta\omega < \mu_\omega(S)$. By Lemma 4.1, $\mu_\omega(S) < \mu_\omega(E)$. Hence $v(S)^2$ should be negative by the assumption $v(E)^2 \leq 0$ and the formula (4.1). Since S is stable, we have $v(S)^2 = -2$.

Next we prove (2). If E satisfies the assumption, E has a σ-stable subobject S with $\arg Z(S) = \arg Z(E)$. Thus we have the same triangle as (4.4). Hence we have proved the assertion. $\square$

Next we prepare, in some sense, dual assertions of Lemma 4.1 and Proposition 4.2 for the case $E \in \mathcal{P}((-1, 0])$.

Lemma 4.3. *Let (X, L) be a generic $K3$ and $\sigma_{(\beta,\omega)} = (Z, \mathcal{P}) \in V(X)$. Assume that $F \to E \to A \to F[1]$ is a non-trivial distinguished triangle in $\mathcal{P}((-1, 0])$.*
(1) *If E is a torsion free sheaf then A is also a torsion free sheaf.*

(2) *If E is a μ-stable locally free sheaf, then A is a torsion free sheaf and the strict inequality $\mu_\omega(E) < \mu_\omega(A)$ holds.*

Proof. We first prove (1). Since $\mathcal{P}((-1,0]) = \mathcal{P}((0,1])[-1] = \mathcal{A}_{(\beta,\omega)}[-1]$, the i-th cohomology $H^i(G)$ of $G \in \mathcal{P}((-1,0])$ is concentrated at $i = 0$ and 1. Note that $H^1(A) =$ is 0 by the fact $H^2(F) = H^1(E) = 0$. Since E and A are sheaves, we have the following exact sequence of sheaves:

$$0 \to H^0(F) \to E \xrightarrow{f} A \to H^1(F) \to 0.$$

Since $A \in \mathcal{F}_{(\beta,\omega)}$, A is torsion free. We remark that $H^0(F)$ is also torsion free.

Next we prove the inequality in (2).

CASE I. Assume $H^0(F) \neq 0$. Then $\operatorname{rank}(\operatorname{Im}(f)) < \operatorname{rank} E$ where $\operatorname{Im}(f)$ is the image of f. Since E is μ-stable, we have $\mu_\omega(E) < \mu_\omega(\operatorname{Im}(f))$.

(I-i) Assume that $H^1(F) = 0$. Then $\operatorname{Im}(f) = A$. So we have $\mu_\omega(E) < \mu_\omega(A)$.

(I-ii) Assume that $H^1(F)$ is torsion. Then $\omega\Delta_{H^1(F)} \geq 0$. Since $\operatorname{rank}\operatorname{Im}(f) = \operatorname{rank} A$ and $\Delta_A = \Delta_{\operatorname{Im}(f)} + \Delta_{H^1(F)}$, we have $\mu_\omega(\operatorname{Im}(f)) \leq \mu_\omega(A)$. Hence we get the inequality.

(I-iii) Assume that $H^1(F) \supsetneqq T$, where T is the maximal torsion subsheaf of $H^1(F)$. Then we have the following diagram of exact sequences:

$$\begin{tikzcd}
 & & A \arrow[d] & & \\
0 \arrow[r] & T \arrow[r] & H^1(F) \arrow[r] \arrow[d] & H^1(F)/T \arrow[r] & 0. \\
 & & 0 & &
\end{tikzcd}$$

Recall the following inequalities:

$$\mu_\omega(A) \leq \mu_\omega^+(A) \leq \beta\omega < \mu_\omega^-(H^1(F)/T) \leq \mu_\omega(H^1(F)/T).$$

By the argument of (I-ii), we have $\mu_\omega(H^1(F)/T) \leq \mu_\omega(H^1(F))$. So $\mu_\omega(A) < \mu_\omega(H^1(F))$. Since the following sequence is exact, we have $\mu_\omega(\operatorname{Im}(f)) < \mu_\omega(A)$:

$$0 \to \operatorname{Im}(f) \to A \to H^1(F) \to 0.$$

Thus we have proved the inequality $\mu_\omega(E) < \mu_\omega(A)$.

CASE II. Assume $H^0(F) = 0$. The sequence

$$0 \to E \to A \to H^1(F) \to 0 \tag{4.5}$$

is an exact sequences of sheaves. Hence we use F instead of $H^1(F)$. Notice that both A and E are in $\mathcal{F}_{(\beta,\omega)}$ and that F is in $\mathcal{T}_{(\beta,\omega)}$.

(II-i) Assume that $F \supsetneq tor$ where tor is the maximal torsion subsheaf of F. By the argument of (I-iii), we have the inequality.
(II-ii) Assume that F is torsion with $\dim \mathrm{Supp}(F) = 1$. Then $\mathrm{rank}\, A = \mathrm{rank}\, E$ and $\Delta_F \omega > 0$. So we have the inequality.
(II-iii) Assume that F is torsion with $\dim \mathrm{Supp}(F) = 0$. Let x be a closed point in $\mathrm{Supp}(F)$. By (4.5), we have the exact sequence of $\mathbb{C}$ vector spaces:

$$\mathrm{Ext}^1_{\mathcal{O}_X}(E, \mathcal{O}_x) \to \mathrm{Ext}^2_{\mathcal{O}_X}(F, \mathcal{O}_x) \to \mathrm{Ext}^2_{\mathcal{O}_X}(A, \mathcal{O}_x) \to \mathrm{Ext}^2_{\mathcal{O}_X}(E, \mathcal{O}_x).$$

Since E is locally free and $\dim X = 2$, $\mathrm{Ext}^1_{\mathcal{O}_X}(E, \mathcal{O}_x) = \mathrm{Ext}^2_{\mathcal{O}_X}(E, \mathcal{O}_x) = 0$. By the Serre duality we have

$$\mathrm{Ext}^2_{\mathcal{O}_X}(F, \mathcal{O}_x) = \mathrm{Hom}^0_X(\mathcal{O}_x, F)^* \quad \text{and} \quad \mathrm{Ext}^2_{\mathcal{O}_X}(A, \mathcal{O}_x) = \mathrm{Hom}^0_X(\mathcal{O}_x, A)^*.$$

Since $x \in \mathrm{Supp}(F)$, $\mathrm{Hom}^0_X(\mathcal{O}_x, F) \neq 0$. So $\mathrm{Hom}^0_X(\mathcal{O}_x, A)$ also is not 0. This contradicts the torsion-freeness of A. Thus we complete the proof. $\square$

Proposition 4.4. *Let (X, L) be a generic K3, let $\sigma = (Z, \mathcal{P})$ be in $V(X)$, and let E be a μ-stable locally free sheaf with $v(E)^2 \leq 0$ and $E \in \mathcal{P}((-1, 0])$.*
(1) *Assume that E is not σ-semistable. Then there is a σ-stable torsion free sheaf S such that $\mu_\omega(E) < \mu_\omega(S)$, $v(S)^2 = -2$ and $\arg Z(S) = \phi^-_\sigma(E)$. In particular $\arg Z(S) < \arg Z(E)$ and $\mu_\omega(S) < \beta\omega$.*
(2) *Assume that E is not σ-stable but σ-semistable. Then there is a σ-stable torsion free sheaf S such that $\mu_\omega(E) < \mu_\omega(S)$, $v(S)^2 = -2$ and $\arg Z(E) = \arg Z(S)$. Moreover we have $\mu_\omega(S) < \beta\omega$.*

Proof. Let us prove (1). Since E is not σ-semistable, E has the HN filtration:

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr] & & E_1 \arrow[dl] \arrow[rr] & & E_2 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E. \arrow[dl] \\
& A_1 \arrow[ul, dashed, "{[1]}"] & & A_2 \arrow[ul, dashed, "{[1]}"] & & & & A_n \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

Let S be a stable quotient of A_n in $\mathcal{P}((-1, 0])$. Then we show that S is what we need. By the composition of natural morphisms, we have the following distinguished triangle in $\mathcal{P}((-1, 0])$:

$$F \to E \to S \to F[1]. \tag{4.6}$$

By Lemma 4.3, S is a torsion free sheaf and we have $\mu_\omega(E) < \mu_\omega(S)$. Since $v(E)^2 \leq 0$, $v(S)^2$ should be negative. Since S is σ-stable, we have $v(S)^2 = -2$. Finally we prove the inequality $\mu_\omega(S) < \beta\omega$. Since $S \in \mathcal{P}((-1, 0])$ we have $\mu_\omega(S) \leq \mu_\omega(S)^+ \leq \beta\omega$. So, If the equality $\mu_\omega(S) = \beta\omega$ holds then we have $\arg Z(S) = 0$. This contradicts the fact that $\arg Z(S) < \arg Z(E) \leq 0$.

(2) By the assumption, E has a stable quotient $E \to S$. Then we have the same triangle as (4.6). Similarly to (1) we see that S is a σ-stable torsion free sheaf with $v(S)^2 = -2$ and $\mu_\omega(E) < \mu_\omega(S)$. Finally we consider the inequality $\mu_\omega(S) < \beta\omega$. Similarly to (1), we have $\mu_\omega(S) \le \beta\omega$. If $\mu_\omega(S) = \beta\omega$ then $\arg Z(S) = 0$. On the other hand, we have $\mu_\omega(E) < \mu_\omega(S) = \beta\omega$. Thus $\arg Z(E)$ should be negative. This contradicts the fact that $\arg Z(E) = \arg Z(S)$. Thus we have got the assertion. $\square$

The following lemma is very important since it implies the non-existence of σ-stable factors in the proof of Theorem 4.6.

Lemma 4.5. *Let (X, L) be a generic K3 with* $\deg X = 2d$. *Assume that E is a sheaf with $0 < \operatorname{rank} E \le \sqrt{d}$ and $v(E)^2 = 0$, and A is a sheaf with $v(A)^2 = -2$. For $\sigma_{(\beta,\omega)} = (Z, \mathcal{P}) \in V(X)_{>2}$, the following holds.*
(1) *If $\beta\omega < \mu_\omega(A) < \mu_\omega(E)$, then $0 < \arg Z(A) < \arg Z(E) < 1$.*
(2) *If $\mu_\omega(E) < \mu_\omega(A) < \beta\omega$, then $-1 < \arg Z(E) < \arg Z(A) < 0$.*

Proof. Since $\mathrm{NS}(X) = \mathbb{Z} \cdot L$, we put

$$\beta = xL, \quad \omega = yL, \quad v(E) = r_E \oplus n_E L \oplus s_E \quad \text{and} \quad v(A) = r_A \oplus n_A L \oplus s_A.$$

Since $v(A)^2 = -2$, r_A is positive. By the formula (4.1) and by the fact $v(E)^2 = 0$, we have

$$\begin{aligned} Z(E) &= \frac{r_E}{2}\left(\omega + \sqrt{-1}\left(\frac{n_E L}{r_E} - \beta\right)\right)^2 \\ &= dr_E\left(y^2 - \frac{\lambda_E^2}{r_E^2}\right) + 2\sqrt{-1}\, dy\lambda_E, \end{aligned}$$

where $\lambda_E = n_E - r_E x$, and

$$\begin{aligned} Z(A) &= \frac{v(A)^2}{2r_A} + \frac{r_A}{2}\left(\omega + \sqrt{-1}\left(\frac{n_A L}{r_A} - \beta\right)\right)^2 \\ &= -\frac{1}{r_A} + dr_A\left(y^2 - \frac{\lambda_A^2}{r_A^2}\right) + 2\sqrt{-1}\, dy\lambda_A, \end{aligned}$$

where $\lambda_A = n_A - r_A x$.

Proof of (1). By the assumption, we have $x < n_A/r_A < n_E/r_E$. So both λ_A and λ_E are positive, and the strict inequality $r_A n_E - r_E n_A > 0$ holds. Hence

$$\begin{aligned} \arg Z(A) < \arg Z(E) &\iff \frac{\mathfrak{Re}\, Z(E)}{\lambda_E} < \frac{\mathfrak{Re}\, Z(A)}{\lambda_A} \\ &\iff 0 < N_{A,E}(x, y). \end{aligned}$$

Then

$$
\begin{aligned}
N_{A,E}(x,y) &= \lambda_E\left(-\frac{1}{r_A}+dr_Ay^2-\frac{d\lambda_A^2}{r_A}\right)-\lambda_A\left(dr_Ey^2-\frac{d\lambda_E^2}{r_E}\right)\\
&= dy^2(r_A\lambda_E-r_E\lambda_A)+d\lambda_A\lambda_E\left(\frac{\lambda_E}{r_E}-\frac{\lambda_A}{r_A}\right)-\frac{\lambda_E}{r_A}\\
&= dy^2(r_An_E-r_En_A)+d(n_A-r_Ax)(n_E-r_Ex)\left(\frac{n_E}{r_E}-\frac{n_A}{r_A}\right)\\
&\quad -\frac{n_E-r_Ex}{r_A}\\
&= d(r_An_E-r_En_A)y^2+d(r_An_E-r_En_A)(x-\mathfrak{a})^2\\
&\quad -d(r_An_E-r_En_A)\mathfrak{a}^2+d\frac{n_An_E}{r_Ar_E}(r_An_E-r_En_A)-\frac{n_E}{r_A},
\end{aligned}
\tag{4.7}
$$

where

$$
\mathfrak{a}:=\frac{1}{2}\left(\frac{n_A}{r_A}+\frac{n_E}{r_E}-\frac{r_E}{dr_A(r_An_E-r_En_A)}\right).
$$

We shall prove $N_{A,E}(x,y)>N_{A,E}(n_A/r_A,1/\sqrt{d})$ (notice that $y^2=1/d \iff \omega^2=2$) for any (β,ω) satisfying the assumption. We first prove $n_A/r_A\le\mathfrak{a}$. In fact,

$$
\begin{aligned}
\frac{n_A}{r_A}\le\mathfrak{a} &\iff \frac{n_A}{r_A}-\frac{n_E}{r_E}\le\frac{r_E}{dr_A(r_En_A-r_An_E)}\\
&\iff \frac{r_En_A-r_An_E}{r_E}\le\frac{r_E}{d(r_En_A-r_An_E)}.
\end{aligned}
\tag{4.8}
$$

Since the integer $r_En_A-r_An_E$ is smaller than 0, the inequality (4.8) is equivalent to the following:

$$
\frac{(r_En_A-r_An_E)^2}{r_E^2}\ge\frac{1}{d}.
\tag{4.9}
$$

Since $(r_En_A-r_An_E)^2>0$ and $\sqrt{d}\ge r_E$, the inequality (4.9) holds. Hence we have $n_A/r_A\le\mathfrak{a}$.

Since $(r_An_E-r_En_A)>0$, $N_{A,E}(x,y)$ is strict increasing with respect to $y>1/\sqrt{d}$. Since $(r_An_E-r_En_A)>0$ and $x<n_A/r_A\le\mathfrak{a}$, $N_{A,E}(x,y)$ is strict decreasing with respect to $x<n_A/r_A$. Hence we have $N_{A,E}(x,y)>N_{A,E}(n_A/r_A,1/\sqrt{d})$.

If we prove $N_{A,E}(n_A/r_A,y)>0$, the proof will be complete. If $x=n_A/r_A$, we have $N_{A,E}(x,y)=\lambda_E\cdot\mathfrak{Re}\,Z(A)$. Recall that the pair (β,ω) is in $\mathcal{V}(X)$ by $\omega^2>2$. Thus we have $\mathfrak{Re}\,Z(A)>0$. We have proved the assertion.

Proof of (2). By the assumption, we have $n_E/r_E < n_A/r_A < x$ and $r_An_E - r_En_A < 0$. In addition, both λ_E and λ_A are negative. Similarly to the case (1), we have

$$\begin{aligned} \arg Z(E) < \arg Z(A) &\iff \frac{\mathfrak{Re}\, Z(E)}{\lambda_E} < \frac{\mathfrak{Re}\, Z(A)}{\lambda_A} \\ &\iff 0 > N_{A,E}(x, y). \end{aligned}$$

We have the same formula as (4.7) for $N_{A,E}(x,y)$ with two differences. One is $(r_An_E - r_En_A) < 0$ (this is obvious). The other is $\mathfrak{a} \le n_A/r_A$. So we shall prove the second inequality $\mathfrak{a} \le n_A/r_A$. In fact

$$\begin{aligned} \frac{n_A}{r_A} \ge \mathfrak{a} &\iff \frac{n_A}{r_A} - \frac{n_E}{r_E} \ge \frac{r_E}{dr_A(r_En_A - r_An_E)} \\ &\iff \frac{(r_En_A - r_An_E)^2}{r_E^2} \ge \frac{1}{d}. \end{aligned} \tag{4.10}$$

The inequality (4.10) holds by $\sqrt{d} \ge r_E$.

Since $r_An_E - r_En_A$ is negative, $N_{A,E}(x, y)$ is strict decreasing to $y > 1/\sqrt{d}$. Similarly to (1), since the inequality $\mathfrak{a} \le n_A/r_A$ holds, $N_{A,E}(x, y)$ is strict decreasing with respect to $x > n_A/r_A$. Thus we have $N_{A,E}(x, y) < N_{A,E}(n_A/r_A, 1/\sqrt{d})$. Hence it is enough to show $N_{A,E}(n_A/r_A, y) < 0$. This follows from $\omega^2 > 2$. So we have proved the assertion (2). □

Now we are ready to prove the main theorem of this section.

Theorem 4.6. *Let (X, L) be a generic $K3$ with $\deg X = 2d$, $\sigma_{(\beta,\omega)}$ in $V(X)_{>2}$ and E a torsion free sheaf with $v(E)^2 = 0$ and $\operatorname{rank} E \le \sqrt{d}$.*
(1) *Assume that E is Gieseker stable and $\beta\omega < \mu_\omega(E)$. Then E is $\sigma_{(\beta,\omega)}$-stable.*
(2) *Assume that E is μ-stable locally free and $\mu_\omega(E) \le \beta\omega$. Then E is $\sigma_{(\beta,\omega)}$-stable.*

Proof. We put $\sigma_{(\beta,\omega)} = (Z, \mathcal{P})$. The assumption of (1) implies $E \in \mathcal{P}((0, 1])$ and that of (2) implies $E \in \mathcal{P}((-1, 0])$.

Proof of (1). Suppose to the contrary that E is not $\sigma_{(\beta,\omega)}$-stable. By Proposition 4.2, there is a $\sigma_{(\beta,\omega)}$-stable sheaf S with $v(S)^2 = -2$, $\mu_\omega(S) < \mu_\omega(E)$ and $\arg Z(S) \ge \arg Z(E)$. This contradicts Lemma 4.5 (1). Hence E is $\sigma_{(\beta,\omega)}$-stable.

Proof of (2). Suppose to the contrary that E is not $\sigma_{(\beta,\omega)}$-stable. Then by Lemma 4.4, there is a $\sigma_{(\beta,\omega)}$-stable sheaf S with $\mu_\omega(E) < \mu_\omega(S)$, $v(S)^2 = -2$ and $\arg Z(S) \le \arg Z(E)$. This contradicts Lemma 4.5 (2). Hence E is $\sigma_{(\beta,\omega)}$-stable. □

Corollary 4.7. *Let (X, L) be a generic $K3$ with $\deg X = 2d$ and let E be a μ-stable locally free sheaf with $\operatorname{rank} E \le \sqrt{d}$. Then for all $\sigma \in U(X)_{>2}$, E is σ-stable.*

Proof. Let $\sigma \in U(X)$ and $\tilde{g} \in \widetilde{GL}^+(2, \mathbb{R})$. E is σ-stable if and only if E is $\sigma \cdot \tilde{g}$-stable. Thus we have finished the proof by Theorem 4.6. □

The assumption rank $E \le \sqrt{d}$ may seem to be artificial but it is just the same as the condition $r \le s$ in Theorem 3.8. In Example 5.5 we shall show that the assumption is optimal.

5. σ-stability of spherical sheaves

Let the notations be as in Section 4. In this section, for a generic K3 (X, L), we prove that some spherical sheaves are σ-stable for all $\sigma \in U(X)_{>2}$. We start in this section with a brief review of spherical objects. An object $S \in D(X)$ is called a *spherical object*[2] if the morphism space $\mathrm{Hom}^i_X(S, S)$ is

$$\mathrm{Hom}^i_X(S, S) = \begin{cases} \mathbb{C} & (i = 0, 2), \\ 0 & (\text{otherwise}). \end{cases}$$

By virtue of [10], we can define an autoequivalence T_S called a *spherical twist*. For $E \in D(X)$ the complex $T_S(E)$ is isomorphic to

$$T_S(E) \simeq \text{the mapping cone of } (\mathrm{Hom}_X(S, E[*]) \otimes S \xrightarrow{ev} E), \tag{5.1}$$

where *ev* is the evaluation map.

In general it is difficult to compute $T_S(E)$, but much easier to compute the Mukai vector $v(T_S(E))$. In fact, we have

$$v(T_S(E)) = v(E) + \langle v(E), v(S) \rangle v(S). \tag{5.2}$$

Recall that any equivalence Φ: $D(Y) \to D(X)$ induces an isometry Φ^H: $\mathcal{N}(Y) \to \mathcal{N}(X)$. Since $v(\Phi(E)) = \Phi^H(v(E))$, we have $T^H_S \circ T^H_S = id_{\mathcal{N}(X)}$ by (5.2).

EXAMPLE 5.1. Let X be a projective K3 surface. Then any line bundle M is spherical. The spherical twist $T_M(\mathcal{O}_x)$ of $\mathcal{O}_x$ by M is $\mathcal{I}_x \otimes M[1]$ where $\mathcal{I}_x$ is the ideal sheaf of the closed point $x \in X$. This follows from the formula (5.1)

Proposition 5.2. *Let (X, L) be a generic K3 and S a spherical sheaf. Then S is a μ-stable locally free sheaf.*

Proof. We first show that S is locally free. Let $t(S)$ be the maximal torsion subsheaf of S. Then we have the following exact sequence of sheaves:

$$0 \to t(S) \to S \to S/t(S) \to 0.$$

[2] This definition is "K3" version. More generalized definition of spherical object appears in [8, Chapter 8] or [10].

Since $\mathrm{Hom}(t(S), S/t(S)) = 0$, the result [4, Corollary 2.8] gives us the following inequality:

$$0 \le \hom^1(t(S), t(S)) + \hom^1(S/t(S), S/t(S)) \le \hom^1(S, S) = 0.$$

Thus $v(t(S))^2 < 0$ unless $t(S) = 0$. However $v(t(S))^2 \ge 0$ for $t(S)$ is torsion and S is of Picard number 1. Hence $t(S) = 0$. Thus S is torsion free. Then the local-freeness of S comes from [4, Proposition 3.3].

Finally we show that S is μ-stable. Since $v(S)^2 = -2$, the greatest common divisor of (r_S, n_S) is 1. Then the μ-stability of S follows from [6, Lemma 1.2.14] under the assumption that the Picard number is one. □

The following lemma is a modified version of Lemma 4.5.

Lemma 5.3. *Let* (X, L) *be a generic* $K3$ *with* $\deg X = 2d$, $\sigma_{(\beta,\omega)} \in V(X)_{>2}$ *and both* A *and* E *spherical sheaves with* $\operatorname{rank} E \le \sqrt{d}$.

(1) *Assume that* $\beta\omega < \mu_\omega(A) < \mu_\omega(E)$. *Then* $0 < \arg Z(A) < \arg Z(E) < 1$.

(2) *Assume that* $\mu_\omega(E) < \mu_\omega(A) < \beta\omega$. *Then* $-1 < \arg Z(E) < \arg Z(A) < 0$.

Proof. Since $\mathrm{NS}(X) = \mathbb{Z} \cdot L$, we can put

$$\beta = xL, \quad \omega = yL, \quad v(E) = r_E \oplus n_E L \oplus s_E, \quad \text{and} \quad v(A) = r_A \oplus n_A L \oplus s_A.$$

Then, by the formula (4.1) in Section 4, we have

$$Z(E) = -\frac{1}{r_E} + dr_E\left(y^2 - \frac{\lambda_E^2}{r_E^2}\right) + 2\sqrt{-1}dy\lambda_E \quad \text{and}$$
$$Z(A) = -\frac{1}{r_A} + dr_A\left(y^2 - \frac{\lambda_A^2}{r_A^2}\right) + 2\sqrt{-1}dy\lambda_A,$$

where $\lambda_E = n_E - r_E x$ and $\lambda_A = n_A - r_A x$.

We only prove (1), because the proof of (2) is essentially the same as not only the proof of (1) but also it of Lemma 4.5.

Since both λ_A and λ_E are positive by the assumption, we know that

$$\arg Z(A) < \arg Z(E) \iff N_{A,E}(x, y) > 0.$$

Similarly to Lemma 4.5, we have

$$\begin{aligned} N_{A,E}(x, y) &= dy^2(r_A\lambda_E - r_E\lambda_A) + d\lambda_A\lambda_E\left(\frac{\lambda_E}{r_E} - \frac{\lambda_A}{r_A}\right) + \frac{\lambda_A}{r_E} - \frac{\lambda_E}{r_A} \\ &= d(r_A n_E - r_E n_A)y^2 + d(r_A n_E - r_E n_A)(x - \mathfrak{a})^2 \\ &\qquad + (\text{other terms}), \end{aligned}$$

where $\mathfrak{a}$ is

$$\mathfrak{a} = \frac{1}{2}\left(\frac{n_A}{r_A} + \frac{n_E}{r_E} + \frac{1}{d(r_A n_E - r_E n_A)}\left(\frac{r_A}{r_E} - \frac{r_E}{r_A}\right)\right).$$

Then we shall show that $n_A/r_A < \mathfrak{a}$. Since the integer $r_E n_A - r_A n_E$ is negative, we have

$$\begin{aligned} \frac{n_A}{r_A} < \mathfrak{a} &\iff \frac{n_A}{r_A} - \frac{n_E}{r_E} < \frac{1}{d(r_A n_E - r_E n_A)}\left(\frac{r_A}{r_E} - \frac{r_E}{r_A}\right) \\ &\iff (r_E n_A - r_A n_E)^2 > \frac{r_E^2 - r_A^2}{d}. \end{aligned} \tag{5.3}$$

By the assumption $0 < \operatorname{rank} E \le \sqrt{d}$ we have $(r_E n_A - r_A n_E)^2 \ge r_E^2/d$. Thus the last inequality (5.3) holds.

Since $n_A/r_A < \mathfrak{a}$, $N_{A,E}(x,y)$ is strict decreasing with respect to $x < n_A/r_A$. Moreover by $r_A n_E - r_E n_A > 0$, $N_{A,E}(x.y)$ is strict increasing with respect to $y > 1/\sqrt{d}$. Thus we have $N_{A,E}(x,y) > N_{A,E}(n_A/r_A, 1/\sqrt{d})$. Thus it is enough to show that $N_{A,E}(n_A/r_A, y) > 0$. This follows from $\omega^2 > 2$. Hence we have $N_{A,E}(x, y) > 0$ for all (β, ω) satisfying the assumption. □

In the same way as Theorem 4.6, we have the following proposition.

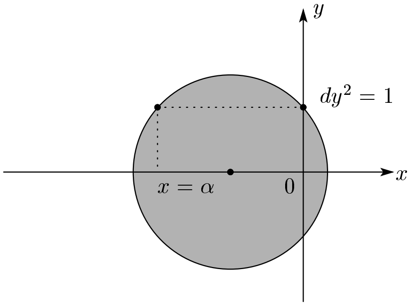
Proposition 5.4. *Let (X, L) be a generic $K3$ with $\deg X = 2d$ and E a spherical sheaf on X with $\operatorname{rank} E \le \sqrt{d}$. Then E is σ-stable for all $\sigma \in U(X)_{>2}$.*

Proof. We first remark that the proof is essentially the same as that of Theorem 4.6. We can assume that $\sigma = \sigma_{(\beta,\omega)} = (Z, \mathcal{P}) \in V(X)_{>2}$. Since E is μ-stable by Proposition 5.2, $E \in \mathcal{P}((0, 1])$ or $E \in \mathcal{P}((-1, 0])$.

Let $E \in \mathcal{P}((0, 1])$. Assume to the contrary that E is not σ-stable. From Proposition 4.2 we know that there is a σ-stable torsion free sheaf $S \in \mathcal{P}((0,1])$ with $v(S)^2 = -2$, $\mu_\omega(S) < \mu_\omega(E)$ and $\arg Z(E) \le \arg Z(S)$. However, by Lemma 5.3, we have $\arg Z(S) < \arg Z(E)$. This is contradiction.

Let $E \in \mathcal{P}((-1, 0])$. Assume to the contrary that E is not σ-stable. Then, by Proposition 4.4, there is a σ-stable sheaf S' with $\mu_\omega(E) < \mu_\omega(S')$, $v(S')^2 = -2$ and $\arg Z(S') \le \arg Z(E)$. However, by Lemma 5.3, we have $\arg Z(S') > \arg Z(E)$. So E is σ-stable. □

In Example 5.5, we show that the assumption on the rank of E in Theorem 4.6 is optimal. Namely we give an example of a Gieseker stable sheaf E with $\operatorname{rank} E > \sqrt{d}$ which is not σ-stable for some $\sigma \in V(X)_{>2}$.



EXAMPLE 5.5. Let (X, L) be a generic K3 with $\deg X = 2d$, and E a Gieseker stable locally free sheaf with $\langle v(E)\rangle^2 = \langle r_E \oplus L \oplus s_E\rangle^2 = 0$ where $v(E) = r_E \oplus L \oplus s_E$ with $r_E > \sqrt{d}$. Then we claim that there is a $\sigma \in V(X)_{>2}$ such that E is not σ-semistable. To prove our claim, it is enough to find $\sigma_{(\beta,\omega)} = (Z, \mathcal{P}) \in V(X)_{>2}$ such that

$$\arg Z(\mathcal{O}_X) > \arg Z(E). \tag{5.4}$$

In fact, assume that such a stability condition $\sigma_0 \in V(X)_{>2}$ exists. By Lemma 5.6 (below), we have $\chi(\mathcal{O}_X, E) > 0$. Since $\mu_\omega(\mathcal{O}_X) < \mu_\omega(E)$, $\mathrm{Hom}_X^2(\mathcal{O}_X, E)^* = \mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X) = 0$. Thus we have

$$0 < \chi(\mathcal{O}_X, E) = \hom^0(\mathcal{O}_X, E) - \hom^1(\mathcal{O}_X, E) \leq \hom^0(\mathcal{O}_X, E). \tag{5.5}$$

Recall that $\mathcal{O}_X$ is σ_0-stable by Proposition 5.4. If E is σ_0-semistable, we have $\mathrm{Hom}_X(\mathcal{O}_X, E) = 0$ by the assumption (5.4). This contradicts (5.5). Hence E is not σ_0-semistable.

We finally show that there is a $\sigma_{(\beta,\omega)} \in V(X)_{>2}$ satisfying the condition (5.4). We put $(\beta, \omega) = (xL, yL)$. Let $N_{A,E}(x, y)$ be the function defined by (4.2). Since $v(\mathcal{O}_X) = 1 \oplus 0 \oplus 1$ and $v(E) = r_E \oplus L \oplus s_E$, we have

$$N_{\mathcal{O}_X,E}(x, y) = dx^2 + \left(r_E - \frac{d}{r_E}\right)x + dy^2 - 1.$$

Take $x < 0$. Then the condition (5.4) is equivalent to

$$N_{\mathcal{O}_X,E}(x, y) < 0.$$

Let us consider the special case $dy^2 = 1$. This means $\omega^2 = 2$. If $dy^2 = 1$, the solutions of $N_{\mathcal{O}_X,E}(x, \sqrt{1/d}) = 0$ are

$$x = 0, \alpha, \quad \text{where} \quad \alpha = \frac{d - r_E^2}{r_E d}.$$

The region defined by $N_{\mathcal{O}_X,E}(x, y) < 0$ is the inside of the above circle: Hence we can choose $\sigma_{(\beta,\omega)} \in V(X)_{>2}$ so that $x < 0$ and $N_{\mathcal{O}_X,E}(x, y) < 0$.

Lemma 5.6. *Let (X, L) be a generic $K3$, let E be a sheaf with $v(E)^2 \leq 0$ and* $\operatorname{rank} E > 0$, *and let A be a sheaf with $v(A)^2 < 0$. Then we have $\chi(A, E) > 0$.*

Proof. We put

$$v(A) = r_A \oplus n_A L \oplus s_A, \quad \text{and} \quad v(E) = r_E \oplus n_E L \oplus s_E.$$

Since $v(A)^2 < 0$ and the Picard number is one, r_A should be positive. So we have

$$\frac{s_A}{r_A} = \frac{L^2}{2}\left(\frac{n_A}{r_A}\right)^2 - \frac{v(A)^2}{2r_A^2} \quad \text{and} \quad \frac{s_E}{r_E} = \frac{L^2}{2}\left(\frac{n_E}{r_E}\right)^2 - \frac{v(E)^2}{2r_E^2}.$$

Then

$$\begin{aligned}\frac{\chi(A, E)}{r_A r_E} &= \frac{-\langle v(A), v(E)\rangle}{r_A r_E} \\ &= \frac{L^2}{2}\left(\frac{n_A}{r_A} - \frac{n_E}{r_E}\right)^2 - \left(\frac{v(E)^2}{2r_E^2} + \frac{v(A)^2}{2r_A^2}\right) \\ &> 0.\end{aligned}$$

Hence $\chi(A, E) > 0$. $\square$

By virtue of Proposition 5.4 we can determine the HN filtrations of some special complexes for $\sigma \in V(X)_{>2}$. We remark that there is a similar assertion to the following two corollaries in [7, Proposition 2.15] when X is a K3 surface with $\operatorname{NS}(X) = 0$.

Corollary 5.7. *Let (X, L) be a generic $K3$ with $\deg X = 2d$, $\sigma = \sigma_{(\beta,\omega)} = (Z, \mathcal{P})$ in $V(X)_{>2}$ and S a spherical sheaf on X with $\operatorname{rank} S \leq \sqrt{d}$. We put $\beta = bL$ and $v(S) = r \oplus nL \oplus s$.*
(1) *If $b > n/r$, then $T_S(\mathcal{O}_x)$ is not σ-semistable. The HN filtration of $T_S(\mathcal{O}_x)$ is given by*

(5.6)
$$\begin{tikzcd} 0 \arrow[rr] & & \mathcal{O}_x \arrow[rr] \arrow[dl] & & T_S(\mathcal{O}_x). \arrow[dl] \\ & \mathcal{O}_x \arrow[ul, dashed, "{[1]}"] & & S^{\oplus r}[1] \arrow[ul, dashed, "{[1]}"] & \end{tikzcd}$$

(2) *If $b = n/r$, then $T_S(\mathcal{O}_x)$ is σ-semistable. The JH filtration of $T_S(\mathcal{O}_x)$ is given by the sequence* (5.6).
(3) *If $b < n/r$ and $r \leq d^{1/4}$, then $T_S(\mathcal{O}_x)$ is σ-stable.*

Proof. We first remark that the sequence of distinguished triangles (5.6) comes from the formula (5.1).

(1) Assume that $b > n/r$. Then $S^{\oplus r}$ is in $\mathcal{P}((-1,0])$ and it is σ-semistable by Proposition 5.4. Hence $\arg Z(\mathcal{O}_x) > \arg Z(S^{\oplus r}[1]) > 0$. Thus the sequence (5.6) is the HN filtration of $T_S(\mathcal{O}_x)$.

(2) If $b = n/r$ then $\arg Z(\mathcal{O}_x) = \arg Z(S^{\oplus r}[1])$. By Proposition 5.4, S is σ-stable. Thus (5.6) is a JH filtration of $T_S(\mathcal{O}_x)$.

(3) We put $\tilde{S}_x = \mathrm{Ker}(S^{\oplus r} \to \mathcal{O}_x)$. Note that rank $\tilde{S}_x = r^2$. Then $T_S(\mathcal{O}_x) = \tilde{S}_x[1]$. So it is enough to show that $\tilde{S}_x$ is σ-stable. Since T_S is an equivalence we have

$$\hom_X^0(\tilde{S}_x, \tilde{S}_x) = 1, \quad \hom_X^1(\tilde{S}_x, \tilde{S}_x) = 2 \quad \text{and } v(\tilde{S}_x) \text{ is primitive.}$$

Thus $\tilde{S}_x$ is Gieseker stable by [4, Proposition 3.14]. Then $\tilde{S}_x$ is σ-stable by Theorem 4.6 (1). □

By Corollary 5.7 (1), we can see that it is impossible to remove the assumption of local-freeness in Theorem 4.6 (2).

Corollary 5.8. *Let the notations be as in* Corollary 5.7.
(1) *If $b \le n/r$ and $r \le d^{1/4}$ then the HN filtration of $T_S^n(\mathcal{O}_x)$ $(n > 1)$ is given by*

$$\begin{tikzcd}
0 \arrow[rr] & & T_S(\mathcal{O}_x) \arrow[rr] \arrow[dl] & & T_S^2(\mathcal{O}_x) \arrow[r] \arrow[dl] & \cdots \arrow[r] & T_S^{n-1}(\mathcal{O}_x) \arrow[rr] & & T_S^n(\mathcal{O}_x). \arrow[dl] \\
& T_S(\mathcal{O}_x) \arrow[ul, dashed, "{[1]}"] & & S^{\oplus r} \arrow[ul, dashed, "{[1]}"] & & & & S^{\oplus r}[2-n] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

(2) *If $b > n/r$, then the HN filtration of $T_S^n(\mathcal{O}_x)$ is*

$$\begin{tikzcd}
0 \arrow[rr] & & \mathcal{O}_x \arrow[rr] \arrow[dl] & & T_S(\mathcal{O}_x) \arrow[r] \arrow[dl] & \cdots \arrow[r] & T_S^{n-1}(\mathcal{O}_x) \arrow[rr] & & T_S^n(\mathcal{O}_x). \arrow[dl] \\
& \mathcal{O}_x \arrow[ul, dashed, "{[1]}"] & & S^{\oplus r}[1] \arrow[ul, dashed, "{[1]}"] & & & & S^{\oplus r}[2-n] \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

Proof. By (5.1), we obtain the following distinguished triangle:

$$S^{\oplus r} \to \mathcal{O}_x \to T_S(\mathcal{O}_x) \to S^{\oplus r}[1].$$

Since $T_S(S) \simeq S[-1]$[3], we can easily show that the two sequences of triangles exist. By Corollary 5.7, both sequences are the HN filtrations of $T_S^n(\mathcal{O}_x)$. □

[3] One can prove this fact $T_S(S) \simeq S[-1]$ easily in the following way. We have the natural exact sequence of sheaves by taking cohomologies of the distinguished triangle arising from (5.1). Then the fact follows from the exact sequence of sheaves. See also [8, Exercise 8.5].

6. Applications of Theorem 1.2

In this section we deal with two applications of Theorem 1.2. We first observe the morphism Φ_* between the space of stability conditions induced by an equivalence Φ of triangulated categories.

Let X and Y be projective K3 surfaces, and $\Phi\colon D(Y) \to D(X)$ an equivalence. Then Φ induces a natural morphism $\Phi_*\colon \mathrm{Stab}(Y) \to \mathrm{Stab}(X)$ as follows:

$$\Phi_*\colon \mathrm{Stab}(Y) \to \mathrm{Stab}(X), \quad \Phi_*((Z_Y, \mathcal{P}_Y)) = (Z_X, \mathcal{P}_X)$$
$$\text{where} \quad Z_X(E) = Z_Y(\Phi^{-1}(E)), \quad \text{and} \quad \mathcal{P}_X(\phi) = \Phi(\mathcal{P}_Y(\phi)).$$

Then the following proposition is almost obvious.

Proposition 6.1. *Let X and Y be projective K3 surfaces, and $\Phi\colon D(Y) \to D(X)$ an equivalence. For $\sigma \in U(X)$, σ is in $\Phi_*(U(Y))$ if and only if $\Phi(\mathcal{O}_y)$ is σ-stable with the same phase for all closed points $y \in Y$.*

Proof. By the definition of $\Phi_*\colon \mathrm{Stab}(Y) \to \mathrm{Stab}(X)$, $\Phi_*(U(Y))$ is given by:

$$\begin{aligned}\Phi_*(U(Y)) &= \Phi_*(\{\sigma \in \mathrm{Stab}(Y) \mid \sigma \text{ is good, } \mathcal{O}_y \text{ is } \sigma\text{-stable } (\forall y \in Y)\}) \\ &= \{\tau \in \mathrm{Stab}(X) \mid \tau \text{ is good, } \Phi(\mathcal{O}_y) \text{ is } \tau\text{-stable } (\forall y \in Y)\}.\end{aligned}$$

Recall that the Φ induces the isometry $\Phi^H\colon \mathcal{N}(Y) \to \mathcal{N}(X)$. So if $\sigma \in \mathrm{Stab}(Y)$ is good, then $\Phi_*(\sigma)$ is also good. This completes the proof. $\square$

Let us consider the first application of Theorem 4.6.

EXAMPLE 6.2. In this example we claim that there is a pair (E, τ) such that a true complex $E \in D(X)$ is τ-stable for $\tau \in V(X) \setminus V(X)_{>2}$.

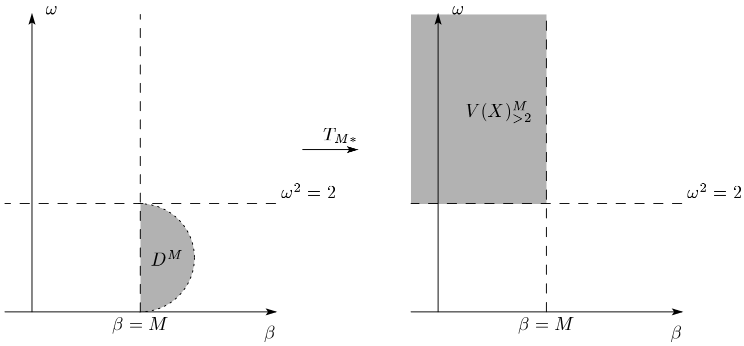
We first define a special subset D^M of $V(X)\setminus V(X)_{>2}$ depending on a line bundle M in the following way. We put $V(X)^M_{>2}$ for M by

$$V(X)^M_{>2} := \{\sigma_{(\beta,\omega)} \in V(X)_{>2} \mid \beta\omega < \mu_\omega(M)\}.$$

By Proposition 6.1 and Corollary 5.7 (3), we see $V(X)^M_{>2} \subset (T_M)_*(U(X)) \cap V(X)$. We also put $U(X)^M_{>2} := V(X)^M_{>2} \cdot \widetilde{GL}^+(2, \mathbb{R})$. By Remark 2.6, we see $U(X)^M_{>2} \subset (T_M)_*(U(X)) \cap U(X)$. Then we define

$$D^M := T^{-1}_{M*}(U(X)^M_{>2}) \cap V(X).$$

Since $T_M = \left(\bigotimes M\right) \circ T_{\mathcal{O}_X} \circ \left(\bigotimes M^{-1}\right)$ we see that D^M is the following half circle:



Thus $D^M \subset V(X)\backslash V(X)_{>2}$.

Next we show that there is a true complex $E \in D(X)$ which is τ-stable for $\tau \in D^M$. In fact, by Proposition 6.1, $E \in D(X)$ is σ-stable for any $\sigma \in V(X)^M_{>2}$ (for example E is a torsion free sheaf in Theorem 1.2 or $\mathcal{O}_x$), if and only if $T_M^{-1}(E)$ is τ-stable for any $\tau \in D^M$. For instance, $T_M^{-1}(\mathcal{O}_x)$ is truly complex which is τ-stable for any $\tau \in D^M$. By the definition of T_M, we can easily compute the i-th cohomology H^i of $T_M^{-1}(\mathcal{O}_x)$. In fact we have

$$H^i = \begin{cases} \mathcal{O}_x & (i = 0), \\ M & (i = -1), \\ 0 & (\text{otherwise}). \end{cases}$$

The crucial part of Example 6.2 is that the spherical twist T_M enables us to exchange the unbounded region $V(X)^M_{>2}$ into the bounded region D^M. We use this idea in the proof of Theorem 1.1.

Next we shall explain the second application. In general spherical twists send sheaves to complexes. We first show this easy statement in a special case.

Lemma 6.3. *Let (X, L) be a generic $K3$, and E a Gieseker stable torsion free sheaf with $v(E)^2 \leq 0$. Then there is a line bundle M such that the spherical twist $T_M(E)$ of E is a true complex with $r' \neq 0$ where $v(T_M(E)) = r' \oplus \Delta' \oplus s'$.*

Proof. Let $v(E) = r_E \oplus n_E L \oplus s_E$ and let $M = mL$ be a line bundle with

$$\frac{n_E}{r_E} < m. \tag{6.1}$$

Here we compute $v(T_M(E))$:

$$\begin{aligned} v(T_M(E)) &= v(E) + \langle v(E), v(M)\rangle v(M) \\ &= r' \oplus n'L \oplus s'. \end{aligned}$$

The condition $r' = 0$ is a closed condition and the condition (6.1) is open. Hence we can choose M so that $r' \neq 0$ and M satisfies the condition (6.1).

Let H^i be the i-th cohomology of $T_M(E)$. By the definition of spherical twists, we obtain the following exact sequence of sheaves:

$$\begin{aligned} 0 &\to \mathrm{Hom}^0_X(M, E) \otimes M \to E \to H^0 \\ &\to \mathrm{Hom}^1_X(M, E) \otimes M \to 0 \to H^1 \\ &\to \mathrm{Hom}^2_X(M, E) \otimes M \to 0 \to H^2 \to 0. \end{aligned}$$

Since both M and E are Gieseker stable, $\mathrm{Hom}^0_X(M, E) = 0$ by (6.1). Hence H^0 is not 0. By Lemma 5.6, we have $\mathrm{Hom}^2_X(M, E) \neq 0$. So $H^1 \neq 0$. Thus $T_M(E)$ is a complex. □

The following lemma is due to [3] and [12].

Lemma 6.4 ([3, Proposition 14.2], [12, Proposition 6.4]). *Let X be a projective $K3$ surface, $\sigma_{(\beta,\omega)} = (Z, \mathcal{P}) \in V(X)$ and E in $\mathcal{P}((0, 1])$. We put $v(E) = r \oplus \Delta \oplus s$.*
(1) *Assume that $r > 0$. If E is $\sigma_{(\beta,n\omega)}$-semistable for any sufficiently large $n \gg 0$, then E is a torsion free sheaf.*
(2) *Assume that $r = 0$. If E is $\sigma_{(\beta,n\omega)}$-semistable for any sufficiently large $n \gg 0$, then E is a torsion sheaf.*

The first assertion of Lemma 6.4 are proved by [3] and the second one proved by [12]. We can prove the second assertion in a similar way to [3].

In the next proposition, we show that it is impossible to extend Theorem 1.2 to $V(X)$ by using Lemma 6.4 and the idea of Example 6.2.

Proposition 6.5. *Let (X, L) be a generic $K3$ and E a Gieseker stable torsion free sheaf with $v(E)^2 \leq 0$. Then there is a σ in $V(X)$ such that E is not σ-semistable.*

Proof. Assume that E is σ-semistable for all $\sigma \in V(X)$. By Lemma 6.3, there is a line bundle M such that $T_M(E)$ is a complex with $r' \neq 0$ where $v(T_M(E)) = r' \oplus \Delta' \oplus s'$. By a shift of $T_M(E)$ we can assume that $r' > 0$ if necessary. By the assumption $T_M(E)$ is σ-semistable for all σ not only in $(T_M)_* V(X)$ but also in $(T_M)_* U(X)$.

Recall that, $(T_M)_*(U(X)) \cap V(X)$ contains the set $V(X)^M_{>2}$ defined in Example 6.2. Hence, there is a $\tau_{(\beta,\omega)} = (Z, \mathcal{P}) \in V(X)^M_{>2}$ such that

$$\beta\omega < \frac{\Delta'}{r'}\omega.$$

This implies that $T_M(E)[2n]$ is in $\mathcal{P}((0,1])$ for some $n \in \mathbb{Z}$. By Lemma 6.4 (1), $T_M(E)[2n]$ should be a sheaf. This contradicts the fact that $T_M(E)$ is a true complex. □

Theorem 6.6. *Let (X, L) be a generic $K3$ and $E \in D(X)$. We assume that* $\mathrm{Hom}^0_X(E, E) = \mathbb{C}$, *$v(E)$ is primitive and $v(E)^2 = 0$. If E is σ-semistable for all $\sigma \in V(X)$, then E is $\mathcal{O}_x$ for some $x \in X$ up to shifts.*

Proof. We put $v(E) = r_E \oplus n_E L \oplus s_E$. Assume that $r_E \neq 0$. If E is σ-semistable, then $E[1]$ is also σ-semistable. Thus we can assume that $r_E > 0$. Let ϕ be the phase of E. Then we can assume $\phi \in (-1, 1]$ by even shifts. There is an $\mathbb{R}$ divisor $\beta = bL$ such that $b < n_E/r_E$. Let us consider $\sigma_{(\beta,\omega)} = (Z, \mathcal{P})$ for all ample divisors ω with $\omega^2 > 2$. Notice that E is in $\mathcal{P}((0,1])$. By Lemma 6.4, E should be a torsion free sheaf. In addition, E is a Gieseker stable sheaf by [4, Proposition 3.14]. This contradicts Proposition 6.5.

Assume that $r_E = 0$. Since $v(E)^2 = 0$, we have $n_E = 0$. Since there is an $\mathbb{R}$ divisor $\beta = bL$ such that $b < 0$, E is a torsion sheaf by Lemma 6.4 (2). Since $n_E = 0$, $\dim \mathrm{Supp}(E) = 0$. By the assumption $\mathrm{Hom}^0_X(E, E) = \mathbb{C}$, E is $\mathcal{O}_x$ for some $x \in X$. □

Now we are ready to prove an easy consequence of Theorem 6.6.

Corollary 6.7 (= Theorem 1.1). *Let (X, L_X) and (Y, L_Y) be generic $K3$ and let $\Phi\colon D(Y) \to D(X)$ be an equivalence. If $\Phi_*(U(Y)) = U(X)$, then Φ can be written in the following way*:

$$\Phi(?) = M \otimes f_*(?)[n],$$

where M is a line bundle on X, f is an isomorphism $f : Y \to X$ and $n \in \mathbb{Z}$.

Proof. Let E_y be $\Phi(\mathcal{O}_y)$ for an arbitrary closed point $y \in Y$. Since $\Phi_*(U(Y)) = U(X)$, E_y is $\mathcal{O}_x[n_y]$ $(n_y \in \mathbb{Z})$ for some $x \in X$ by Theorem 6.6. In addition the phase of E_y is constant. So $[n_y]$ is also constant. Thus E_y is given by $\mathcal{O}_{f(y)}[n]$. By [8, Corollary 5.23], we complete the proof. □

Here we define the subgroup $\mathrm{Aut}(D(X), U(X))$ of $\mathrm{Aut}(D(X))$:

$$\mathrm{Aut}(D(X), U(X)) := \{\Phi \in \mathrm{Aut}(\mathcal{D}) \mid \Phi_*(U(X)) = U(X)\}.$$

Thus we obtain the following statement:

Corollary 6.8. *Notations being as above, we have*

$$\mathrm{Aut}(D(X), U(X)) = \mathrm{Tri}(X),$$

where $\mathrm{Tri}(X)$ *is the subgroup generated by shifts, tensor products of line bundles and automorphisms.*

We remark that $\mathrm{Tri}(X)$ is actually written by $(\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)) \times \mathbb{Z}[1]$.

Proof of Corollary 6.8. If Φ is in the right hand side, $\Phi(\mathcal{O}_x) = \mathcal{O}_y[n]$ for some $y \in X$ and $n \in \mathbb{Z}$. Thus $\Phi_*(U(X)) = U(X)$. Conversely, if Φ is in the left hand side, Φ is in the right hand side by Corollary 6.7. □

REMARK 6.9. Throughout this remark, we assume that A and A' are abelian surfaces. Similarly to the case of K3 surfaces, we can construct $U(A)$. Hence $\mathrm{Stab}(A)$ is nonempty. In particular $\mathrm{Stab}^\dagger(A) = U(A)$ since $D(A)$ has no spherical objects (cf. [3, Section 15]). In addition, the set of good stability conditions is equal to $U(A)$ (and thus is connected) by the result of [7, Theorem 3.15]. The property "good" preserved by any equivalence $\Phi\colon D(A') \to D(A)$. Hence for any equivalence $\Phi\colon D(A') \to D(A)$, $\Phi_*(U(A')) = U(A)$. Thus we have

$$\mathrm{Aut}(D(A), U(A)) = \mathrm{Aut}(D(A)).$$

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