Kato, S. and Muroya, H.
Osaka J. Math.
49 (2012), 931–992

MINIMAL SURFACES OF GENUS ONE WITH CATENOIDAL ENDS

SHIN KATO and HISAYOSHI MUROYA

(Received October 23, 2009, revised February 25, 2011)

Abstract

We give a necessary and sufficient condition for the existence of an n-end catenoid of genus one with prescribed flux. By using the condition, we construct new examples of families whose flux data go near to that of "the catenoid of genus one".

1. Introduction

Let $\overline{M}$ be a compact Riemann surface, and $X\colon M = \overline{M}\setminus\{q_1,\dots,q_n\} \to \mathbf{R}^3$ an *n-end catenoid*, that is, a conformal minimal immersion with catenoidal ends at $q_1, \dots, q_n \in \overline{M}$. Let γ_j be a loop surrounding q_j from the left, $\vec{n}$ a conormal such that $(\gamma_j, \vec{n})$ is positively oriented, and ds the line element of $X(M)$. Then the *flux vector* at the end q_j is defined by the integral $\varphi_j := \int_{\gamma_j} \vec{n}\,ds$. By the divergence formula, we get the *flux formula* $\sum_{j=1}^{n} \varphi_j = \mathbf{0}$. Let $G\colon \overline{M} \to \mathbf{S}^2 \subset \mathbf{R}^3$ be the extended Gauss map of X. Since we assume that the end q_j is catenoidal, $G(q_j)$ must be parallel to φ_j. We define the *weight* of the end q_j by $w(q_j) := \varphi_j/(4\pi G(q_j))$. Then the flux formula is rewritten as follows:

$$\sum_{j=1}^{n} w(q_j)G(q_j) = \mathbf{0}. \tag{1.1}$$

Conversely, we can consider a problem of finding n-end catenoids that realize given data $G(q_j)$ and $w(q_j)$ $(j = 1, \dots, n)$ satisfying (1.1). Umehara, Yamada and the first author [8, Theorem 3.6], [9, Theorem 3.1] reduced the problem to a system of algebraic equations, and proved that, for almost all flux data $v_1, \dots, v_n \in \mathbf{S}^2$ and $a_1, \dots, a_n \in \mathbf{R}\setminus\{0\}$ satisfying $\sum_{j=1}^{n} a_j v_j = \mathbf{0}$, there exists $X\colon M = \hat{\mathbf{C}}\setminus\{q_1,\dots,q_n\} \to \mathbf{R}^3$, an n-end catenoid of genus zero, that satisfies $G(q_j) = v_j$ and $w(q_j) = a_j$ $(j = 1, \dots, n)$, where $\hat{\mathbf{C}} := \mathbf{C}\cup\{\infty\}$. In the case that $\dim\langle v_1,\dots,v_n\rangle = 2$, Cosín and Ros [2] gave a necessary and sufficient condition for the existence of Alexandrov embedded n-end catenoids of genus zero with prescribed flux, by using flux polygons.

On the other hand, in the case of higher genus, most of the known examples are embedded and hence the flux vectors at the ends are parallel. For the case that the

2000 Mathematics Subject Classification. Primary 53C42; Secondary 58E12.

flux vectors span at least a two-dimensional vector space, Berglund–Rossman [1] and Rossman [14] constructed Jorge–Meeks type surfaces of genus one, and n-end catenoids of higher genus whose symmetries are those of the Platonic solids, etc., and it seems that there are few works for this case.

There are two possibilities for classes of n-end catenoids of genus one (see §4). In this paper, we consider one of the classes, that includes Costa's examples, Berglund and Rossman's examples, and, in a weak sense, catenoid fences also. In the class, we generalize results in [8], and give an equation with respect to elliptic functions, which describes a necessary and sufficient condition for the existence of n-end catenoids of genus one with prescribed flux. Applying our equations, we also give new examples, which enable us to observe the collapse of n-end catenoids of genus one to "the catenoid of genus one", which actually does not exist (cf. [16]).

The authors thank Professors W. Rossman and S. Fujimori for helpful advices. They also thank the referees for useful comments.

2. Flux of catenoidal or planar ends

Let $\hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$. Let $\Pi\colon \mathbf{S}^2 \to \hat{\mathbf{C}}$ be the stereographic projection from the north pole $\mathbf{e}_3 := {}^t(0, 0, 1)$. Then the inverse of this map is given by the following:

$$v(p) := \Pi^{-1}(p) = \frac{1}{|p|^2 + 1}\begin{pmatrix} 2 \operatorname{Re} p \\ 2 \operatorname{Im} p \\ |p|^2 - 1 \end{pmatrix}.$$

Let M be a Riemann surface. Then, by the *Weierstrass representation formula*, any conformal minimal immersion $X\colon M \to \mathbf{R}^3$ is given by

$$ {}^t X(z) = \operatorname{Re} \int_{z_0}^{z} (1 - g^2, \sqrt{-1}(1 + g^2), 2g)\eta, \tag{2.1}$$

where g is a meromorphic function on M, and η is a holomorphic 1-form on M such that the 1-forms $g\eta$ and $g^2\eta$ are also holomorphic on M, and η and $g^2\eta$ have no common zeroes. We call (g, η) the *Weierstrass data* of X. The function g is the stereographic image of the Gauss map $G\colon M \to \mathbf{S}^2$ of X, i.e. $g := \Pi \circ G$. The induced metrics on M are given by $X^*(g_{\mathbf{R}^3}) = (1 + |g|^2)^2|\eta|^2$.

Conversely, for any Riemann surface M, any meromorphic function g on M, and any holomorphic 1-form η on M such that $g\eta$ and $g^2\eta$ are also holomorphic on M, the map X given by (2.1) is a (branched) conformal minimal immersion on M.

The map X given by (2.1) is well-defined on M if and only if

$$\operatorname{Re} \int_{\gamma} (1 - g^2, \sqrt{-1}(1 + g^2), 2g)\eta = \mathbf{0} \tag{2.2}$$

holds for any loop γ in M. Set

$$R_i = R_i(\gamma) := \frac{1}{2\pi\sqrt{-1}} \int_\gamma g^i \eta \quad (i = 0, 1, 2). \tag{2.3}$$

Then the condition (2.2) is rewritten as

$$R_0 - R_2 \in \mathbf{R}, \quad R_0 + R_2 \in \sqrt{-1}\mathbf{R}, \quad R_1 \in \mathbf{R}, \tag{2.4}$$

and this is equivalent to

$$R_0 + \overline{R_2} = 0, \quad R_1 = \overline{R_1}. \tag{2.5}$$

Now, we have the following:

Theorem 2.1. *Let X be a conformal minimal immersion from the universal cover of a Riemann surface M to $\mathbf{R}^3$ given by* (2.1), *and let γ be a loop in M. Let p be a complex number satisfying*

$$p^2 R_0 - 2pR_1 + R_2 = 0. \tag{2.6}$$

Then X is well-defined on a neighbourhood of γ in M itself if and only if it holds that

$$\begin{cases} w := -pR_0 + R_1 \in \mathbf{R}, \\ w^* := -\dfrac{1}{2}(|p|^2 - 1)R_0 + \bar{p}R_1 = 0. \end{cases} \tag{2.7}$$

Proof. By the definitions of w and w^*, we have

$$\begin{cases} \dfrac{1}{2}(R_0 - R_2) = -\dfrac{p + \bar{p}}{|p|^2 + 1} w + \dfrac{-(p^2 - 1)}{|p|^2 + 1} w^*, \\ \dfrac{\sqrt{-1}}{2}(R_0 + R_2) = -\dfrac{-\sqrt{-1}(p - \bar{p})}{|p|^2 + 1} w + \dfrac{\sqrt{-1}(p^2 + 1)}{|p|^2 + 1} w^*, \\ R_1 = -\dfrac{|p|^2 - 1}{|p|^2 + 1} w + \dfrac{2p}{|p|^2 + 1} w^*. \end{cases} \tag{2.8}$$

If we assume (2.7), then, by (2.8), we have (2.4).

Conversely, if X is well-defined on a neighbourhood of γ in M, then, by (2.4), we have

$$\begin{pmatrix} -(p + \bar{p}) & -(p^2 - 1) & \bar{p}^2 - 1 \\ \sqrt{-1}(p - \bar{p}) & \sqrt{-1}(p^2 + 1) & \sqrt{-1}(\bar{p}^2 + 1) \\ -(|p|^2 - 1) & 2p & -2\bar{p} \end{pmatrix} \begin{pmatrix} w - \bar{w} \\ w^* \\ \bar{w}^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.9}$$

Since the determinant of the matrix of the left-hand-side of (2.9) is $2\sqrt{-1}(|p|^2+1)^3 \neq 0$, we get $w - \bar{w} = w^* = \bar{w}^* = 0$, namely, (2.7) holds. □

Note here that, for any curve $z = z(s)$ in M, the conormal is given by

$$ {}^t\vec{n} = -\operatorname{Im}(1 - g^2, \sqrt{-1}(1 + g^2), 2g)\eta(z'(s)), $$

where s is the arclength parameter with respect to $ds^2 = (1 + |g|^2)^2|\eta|^2$. Hence, it holds that

$$ {}^t\varphi = -\operatorname{Im}\int_\gamma (1 - g^2, \sqrt{-1}(1 + g^2), 2g)\eta. $$

From the equality above and (2.2), it follows that

$$ {}^t\varphi = -2\pi(R_0 - R_2, \sqrt{-1}(R_0 + R_2), 2R_1). \tag{2.10} $$

We call φ the *flux vector* of the loop γ. It depends only on the homology class of γ. By (2.10), we have

$$ \varphi = \frac{4\pi w}{|p|^2 + 1}\begin{pmatrix} p + \bar{p} \\ -\sqrt{-1}(p - \bar{p}) \\ |p|^2 - 1 \end{pmatrix} = 4\pi w v(p). \tag{2.11} $$

Let $\hat{M}$ be a Riemann surface, q an interior point of $\hat{M}$, and set $M := \hat{M} \setminus \{q\}$. Consider a conformal minimal immersion $X\colon M \to \mathbf{R}^3$ which cannot be extended to q. We call the image of a neighbourhood of q the *end* q. It is well known that the end q is embedded in a neighbourhood small enough, if its Weierstrass data (g, η) can be meromorphically extended to $\hat{M}$, and the order of the end q is at most 2, where we define the order of the end q by the maximum of the orders of the pole q of η, $g\eta$ and $g^2\eta$ (cf. [5, 10, 16]).

Consider R_0, R_1, R_2 as in (2.3) for a loop γ surrounding q once from the left. If a conformal minimal immersion X given by (2.1) has an embedded end at q and $g(q) = p \neq \infty$, then $(g - p)^2\eta$ does not have a pole at q. Hence we have

$$ 0 = \operatorname{Res}_q(g - p)^2\eta = R_2 - 2pR_1 + p^2R_0, $$

namely, (2.6) holds for $p = g(q)$.

Now, by Theorem 2.1, we have the following:

Corollary 2.2. *Let X be a conformal minimal immersion from the universal cover of $M = \hat{M} \setminus \{q\}$ to $\mathbf{R}^3$ given by* (2.1). *Set $p := g(q)$. If X has an end of order at most* 2 *at q, then X is well-defined on a neighbourhood of q in M itself if and only if the condition* (2.7) *holds.*

In the case of genus zero, this fact was shown in [8].

For the loop γ as above, we call $\varphi = \varphi(\gamma)$ the flux vector of the end q, as we have already mentioned in §1. We denote it by $\varphi = \varphi(q)$. We call the end q *catenoidal* (resp. *planar*) if the end q is of order at most 2 and $\varphi(q) \neq \mathbf{0}$ (resp. $= \mathbf{0}$).

By (2.11), $\varphi(q)$ is parallel to the limit normal $G(q) = v(p)$, and we call the value $w = w(q) = \varphi(q)/(4\pi G(q))$ the *weight* of the end q. The weight $w(q)$ is invariant under the action of conformal coordinate transformations of $\hat{M}$ and the orientation preserving congruent transformations of $\mathbf{R}^3$.

Let $\overline{M}$ be a compact Riemann surface, $q_1, \dots, q_n$ distinct points on $\overline{M}$, and set $M := \overline{M} \setminus \{q_1, \dots, q_n\}$. Then, for any conformal minimal immersion $X\colon M \to \mathbf{R}^3$, by (2.10) and the residue theorem, we have the balancing formula, also called the *flux formula*, $\sum_{j=1}^n \varphi_j = \mathbf{0}$, where $\varphi_j := \varphi(q_j)$. When X is non-branched, of finite total curvature and all the ends of X are embedded, we call X an *n-noid*. In particular, if all the ends are catenoidal, then we call X an *n-end catenoid*. For any n-noid, we have

$$\sum_{j=1}^n w(q_j)v(p_j) = \mathbf{0}, \tag{2.12}$$

where $p_j := g(q_j)$.

Now, the inverse problem of the flux formula is stated as follows:

PROBLEM 2.3. Let p_j be complex numbers or ∞. For any j, let a_j be a real number. Suppose that these numbers satisfy

$$\sum_{j=1}^n a_j v(p_j) = \mathbf{0}. \tag{2.13}$$

Does there exists an n-noid $X\colon M = \overline{M} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ satisfying the following condition?

$$g(q_j) = p_j, \quad w(q_j) = a_j, \quad \varphi(q_j) = 4\pi a_j v(p_j) \quad (j = 1, \dots, n). \tag{2.14}$$

By Theorem 2.1 and Corollary 2.2, Problem 2.3 is reduced to a problem of finding a conformal class of $\overline{M}$ and (g, η) satisfying (2.7) with $p = p_j$ $(j = 1, \dots, n)$, and satisfying (2.6) and (2.7) for a homology basis. For a general loop γ, it is difficult to determine p in advance. However, in the case that X has some symmetry, we can rewrite the condition (2.5) in a somewhat simpler form.

In this paper, we study Problem 2.3 in the case that $\overline{M}$ is a torus T^2.

3. The functions $h(z, q)$ and $h_1(z, q)$

In this section, we introduce the functions $h(z, q)$, $h_1(z, q)$, etc. We use $h(z, q)$ to describe the Weierstrass data (g, η) of n-noids in §4. To write down the global periods of the given data, we also need $h_1(z, q)$ in §6. We enumerate several properties of $h(z, q)$ and $h_1(z, q)$ in Lemmas 3.1–3.6, which we use repeatedly in the calculations in §§7–9.

Let $T^2 := \mathbf{C}/(\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2)$, where we assume that $\omega_1, \omega_2 \in \mathbf{C}$ satisfy $\mathrm{Im}(\omega_2/\omega_1) > 0$. Let $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$. Set

$$r := \exp\left(-2\pi\sqrt{-1}\frac{\omega_2}{\omega_1}\right), \quad r^{1/2} := \exp\left(-\pi\sqrt{-1}\frac{\omega_2}{\omega_1}\right), \tag{3.1}$$

and define an equivalence relation on $\mathbf{C}^*$ by

$$z \sim z' \Leftrightarrow z' = zr^l \quad \text{for some} \quad l \in \mathbf{Z}.$$

Consider the covering map

$$z\colon \mathbf{C} \to \mathbf{C}^* \quad u \mapsto z(u) := \exp\left(\frac{2\pi\sqrt{-1}}{\omega_1}u\right).$$

Then the map $z(u)$ naturally induces a biholomorphic map between $T^2 = \mathbf{C}/(\omega_1\mathbf{Z} + \omega_2\mathbf{Z})$ and $\mathbf{C}^*/\sim$. In some cases, it is more convenient to regard the torus T^2 as the quotient space $\mathbf{C}^*/\sim$.

Set

$$\begin{aligned} h_0(z, q) &:= \sum_{l=-\infty; l\neq 0}^{+\infty} \frac{r^{l/2}}{z - qr^l}, \\ h(z, q) &:= \sum_{l=-\infty}^{+\infty} \frac{r^{l/2}}{z - qr^l} = \frac{1}{z-q} + h_0(z, q), \\ h_1(z, q) &:= \sum_{l=-\infty; l\neq 0}^{+\infty} \frac{lr^{l/2}}{z - qr^l}, \end{aligned}$$

where $r^{1/2}$ is chosen as in (3.1). For simplicity, we denote $h_0(z, 1)$, $h(z, 1)$ and $h_1(z, 1)$ by $h_0(z)$, $h(z)$ and $h_1(z)$ respectively. Then it holds that

$$h_0(z, q) = \frac{1}{q}h_0\left(\frac{z}{q}\right), \quad h(z, q) = \frac{1}{q}h\left(\frac{z}{q}\right), \quad h_1(z, q) = \frac{1}{q}h_1\left(\frac{z}{q}\right). \tag{3.2}$$

We can express $h(z)$ in terms of elliptic functions. To see this, we mention that the Weierstrass $\wp$-function satisfies

$$\sqrt{\wp(u) - \wp\left(\frac{\omega_2}{2}\right)} = \frac{2\pi\sqrt{-1}}{\omega_1}\left\{\frac{1}{z^{1/2} - z^{-1/2}} + \sum_{l=1}^{\infty}\left(\frac{r^{l/2}z^{-1/2}}{1 - r^l z^{-1}} - \frac{r^{l/2}z^{1/2}}{1 - r^l z}\right)\right\}$$

(cf. [4, pp. 211 and 190]), and the Weierstrass σ-function satisfies

$$\wp(u) - \wp\left(\frac{\omega_2}{2}\right) = \exp(\eta_2 u)\frac{\sigma(u-\omega_2/2)^2}{\sigma(u)^2\sigma(\omega_2/2)^2}$$

(cf. [4, p. 181 Satz 3, p. 183 Satz 2]), where η_2 is the complex number associated with the Weierstrass ζ-function:

$$\eta_2 = \zeta(u+\omega_2) - \zeta(u).$$

Hence we have

$$z(u)h(z(u))^2 = \left(\frac{\omega_1}{2\pi\sqrt{-1}}\right)^2\left(\wp(u) - \wp\left(\frac{\omega_2}{2}\right)\right), \tag{3.3}$$

and

$$h(z(u)) = \frac{\omega_1}{2\pi\sqrt{-1}}\frac{(-1)}{\sigma(\omega_2/2)}\exp\left\{\left(\frac{\eta_2}{2} - \frac{\pi\sqrt{-1}}{\omega_1}\right)u\right\}\frac{\sigma(u-\omega_2/2)}{\sigma(u)}. \tag{3.4}$$

By straightforward calculations, we see that $h(z)$ and $h_0(z)$ have the following properties:

Lemma 3.1. *The functions $h(z)$ and $h_0(z)$ satisfy the following*:
(i) $h(rz) = r^{-1/2}h(z)$.
(ii) $h(z^{-1}) = -zh(z)$.
(iii) $h(r^{1/2}) = 0$.
(iv) $h_0(1) = 0$.
In particular, in the case that $r \in \mathbf{R}$, $h(z)$ satisfies also the following:
(v) $\overline{h(z)} = -zh(z)$ $(|z| = 1)$.
(vi) $h(z) \in z^{-1/2}\sqrt{-1}\mathbf{R}$ $(|z| = 1)$.

As a corollary to Lemma 3.1, we also have the following lemma:

Lemma 3.2. *The function $h(z, q)$ satisfies the following*:
(i) $h(z, q) = -h(q, z)$.
(ii) $h(rz, q) = h(z, rq) = r^{-1/2}h(z, q)$.
(iii) $h(z^{-1}, q) = -zh(qz) = -q^{-1}zh(z, q^{-1})$.
(iv) $h(z^{-1}, q^{-1}) = -qzh(z, q)$.
In particular, in the case that $r \in \mathbf{R}$, $h(z, q)$ satisfies also the following:
(v) $h(z, q) \in (zq)^{-1/2}\sqrt{-1}\mathbf{R}$ $(|z| = |q| = 1)$.

We also see that $h_1(z)$ has the following properties:

Lemma 3.3. *The function $h_1(z)$ satisfies the following*:
(i) $h_1(rz) = r^{-1/2}(h_1(z) + h(z))$.
(ii) $h_1(z^{-1}) = zh_1(z)$.
(iii) $h_1(-r^{1/2}) = (1/2)h(-r^{1/2})$.
(iv) $h_1(-1) = 0$.
In particular, in the case that $r \in \mathbf{R}$, $h_1(z)$ satisfies also the following:
(v) $\overline{h_1(z)} = zh_1(z)$ $(|z| = 1)$.
(vi) $h_1(z) \in z^{-1/2}\mathbf{R}$ $(|z| = 1)$.

As a corollary to Lemma 3.3, we also have the following lemma:

Lemma 3.4. *The function $h_1(z, q)$ satisfies the following*:
(i) $h_1(z, q) = h_1(q, z)$.
(ii) $h_1(rz, q) = r^{-1/2}(h_1(z, q) + h(z, q))$.
(iii) $h_1(z, rq) = r^{-1/2}(h_1(z, q) - h(z, q))$.
(iv) $h_1(z^{-1}, q) = zh_1(qz) = q^{-1}zh_1(z, q^{-1})$.
(v) $h_1(z^{-1}, q^{-1}) = qzh_1(z, q)$.
In particular, in the case that $r \in \mathbf{R}$, $h_1(z, q)$ satisfies also the following:
(vi) $h_1(z, q) \in (zq)^{-1/2}\mathbf{R}$ $(|z| = |q| = 1)$.

Lemma 3.5 (resp. 3.6) gives another expansion of $h(z)$ (resp. $h_1(z)$), which enables us to get various estimates for special values of the function.

Lemma 3.5. *For any z such that $|r|^{-1} < |z| < |r|$, $h(z)$ and $h_0(z)$ satisfy the following*:

$$h(z) = \frac{1}{z-1} + h_0(z) = \frac{1}{z-1} - \frac{1}{z}\sum_{m=1}^{+\infty}(z^m - z^{1-m})\frac{1}{r^{(2m-1)/2} - 1}.$$

Proof. For any z such that $|r|^{-1} < |z| < |r|$, we have

$$\begin{aligned} h_0(z) &= \sum_{l=1}^{+\infty}\left\{\frac{r^{l/2}}{z - r^l} + \frac{r^{-l/2}}{z - r^{-l}}\right\} = -\frac{1}{z}\sum_{l=1}^{+\infty}\left(r^{l/2}\frac{zr^{-l}}{1 - zr^{-l}} - zr^{l/2}\frac{z^{-1}r^{-l}}{1 - z^{-1}r^{-l}}\right) \\ &= -\frac{1}{z}\sum_{l=1}^{+\infty} r^{l/2}\sum_{m=1}^{+\infty}\frac{z^m - z^{1-m}}{(r^l)^m} = -\frac{1}{z}\sum_{m=1}^{+\infty}(z^m - z^{1-m})\sum_{l=1}^{+\infty}(r^{(1-2m)/2})^l \\ &= -\frac{1}{z}\sum_{m=1}^{+\infty}(z^m - z^{1-m})\frac{1}{r^{(2m-1)/2} - 1}. \end{aligned}$$

□

Lemma 3.6. *For any z such that $|r|^{-1} < |z| < |r|$, $h_1(z)$ satisfies the following*:

$$h_1(z) = -\frac{1}{z}\sum_{m=1}^{+\infty}(z^m + z^{1-m})\frac{r^{(2m-1)/2}}{(r^{(2m-1)/2}-1)^2}.$$

Proof. For any z such that $|r|^{-1} < |z| < |r|$, we have

$$\begin{aligned} h_1(z) &= \sum_{l=1}^{+\infty}\left\{\frac{lr^{l/2}}{z-r^l} + \frac{(-l)r^{-l/2}}{z-r^{-l}}\right\} = -\frac{1}{z}\sum_{l=1}^{+\infty}\left(lr^{l/2}\frac{zr^{-l}}{1-zr^{-l}} + zlr^{l/2}\frac{z^{-1}r^{-l}}{1-z^{-1}r^{-l}}\right) \\ &= -\frac{1}{z}\sum_{l=1}^{+\infty} lr^{l/2}\sum_{m=1}^{+\infty}\frac{z^m+z^{1-m}}{(r^l)^m} = -\frac{1}{z}\sum_{m=1}^{+\infty}(z^m+z^{1-m})\sum_{l=1}^{+\infty} l(r^{(1-2m)/2})^l \\ &= -\frac{1}{z}\sum_{m=1}^{+\infty}(z^m+z^{1-m})\frac{r^{(2m-1)/2}}{(r^{(2m-1)/2}-1)^2}. \end{aligned}$$

□

4. Weierstrass data of *n*-noids

Let $T^2 := \mathbf{C}/(\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2)$. We choose a fundamental period (ω_1, ω_2) so that $\operatorname{Im}(\omega_2/\omega_1) > 0$. Let $u_1, \dots, u_n$ be distinct points on T^2, and set $M := T^2 \setminus \{u_1, \dots, u_n\}$. Let $X\colon M \to \mathbf{R}^3$ be an n-noid of genus one, and (g, η) its Weierstrass data.

Assume $G(u_j) \neq v(\infty) = {}^t(0,0,1)$, i.e. $p_j = g(u_j) \neq \infty$, for any $j = 1,\dots,n$. Since X is well-defined on M, η must have a pole of order 2 at each end u_j $(j = 1, \dots, n)$. Then the sum of orders of poles of η is $2n$, and the sum of orders of zeroes of η is also $2n$. On the other hand, since X has no branch point, η and $g^2\eta$ have no common zero on M. Hence the zeroes of η must coincide with the poles of g, and the order of η at any zero is the double of the order of g at the same point as a pole. Now, we see that the degree of g must be equal to $2n/2 = n$, and that there exist $s_1, \dots, s_n$, a complete system of representatives of the poles of g, and $t_1, \dots, t_n$, that of the zeroes of g, which satisfy $s_1 + \cdots + s_n = t_1 + \cdots + t_n$, and

$$g(u) = C_1\frac{\sigma(u-t_1)\cdots\sigma(u-t_n)}{\sigma(u-s_1)\cdots\sigma(u-s_n)}$$

for some nonzero constant C_1. (Some of them may coincide with each other.)

Since all the poles $u_1, \dots, u_n$ of η and all the zeroes $s_1, \dots, s_n$ of η must be of order 2, they satisfy $2(u_1 + \cdots + u_n) \equiv 2(s_1 + \cdots + s_n) \bmod (\omega_1, \omega_2)$, and hence there exists an $\omega = m_1\omega_1 + m_2\omega_2 \in \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ satisfying

$$2(u_1 + \cdots + u_n) + \omega = 2(s_1 + \cdots + s_n). \tag{4.1}$$

Since we may choose $s_n + [m_1/2]\omega_1 + [m_2/2]\omega_2$ as s_n, we may assume

$$\omega \in \{0, \omega_1, \omega_2, \omega_1 + \omega_2\} \tag{4.2}$$

without loss of generality, where $[m_i/2]$ denotes the largest integer that does not exceed $m_i/2$. Here we also choose $t_n + [m_1/2]\omega_1 + [m_2/2]\omega_2$ as t_n.

Proposition 4.1. *In the case that $\omega = 0$, the Weierstrass data (g, η) of an n-noid X is given by*

$$g(u) = \frac{P(u)}{Q(u)}, \quad \eta = -Q(u)^2\, du, \tag{4.3}$$

with

$$P(u) = \sum_{j=1}^{n} c_j \zeta(u - u_j) + c_0, \quad Q(u) = \sum_{j=1}^{n} b_j \zeta(u - u_j) + b_0, \tag{4.4}$$

where $b_1, \ldots, b_n, b_0, c_1, \ldots, c_n, c_0$ are complex numbers satisfying $b_j \neq 0$, $c_j = p_j b_j$ $(j = 1, \ldots, n)$, and $\sum_{j=1}^{n} b_j = \sum_{j=1}^{n} c_j = 0$.

Proof. In the case that $\omega = 0$, we get

$$\eta = -\left(C_2 \frac{\sigma(u - s_1) \cdots \sigma(u - s_n)}{\sigma(u - u_1) \cdots \sigma(u - u_n)} \right)^2 du$$

for some nonzero constant C_2. Set

$$P(u) := C_1 C_2 \frac{\sigma(u - t_1) \cdots \sigma(u - t_n)}{\sigma(u - u_1) \cdots \sigma(u - u_n)}, \quad Q(u) := C_2 \frac{\sigma(u - s_1) \cdots \sigma(u - s_n)}{\sigma(u - u_1) \cdots \sigma(u - u_n)}.$$

Then both $P(u)$ and $Q(u)$ are meromorphic functions on T^2, and the data (g, η) is given by (4.3). Since $P(u)$ and $Q(u)$ are elliptic functions of period (ω_1, ω_2) and $\lim_{u \to u_j} (P(u)/Q(u)) = p_j$ $(j = 1, \ldots, n)$, they are described as (4.4). □

Now, let us consider the case that $\omega \neq 0$. In the case that $\omega = \omega_1$ (resp. $\omega_1 + \omega_2$), if we replace (ω_1, ω_2) by $(-\omega_2, \omega_1)$ (resp. $(\omega_1, \omega_1 + \omega_2)$), then ω is replaced by ω_2. Hence, when $\omega \neq 0$, we may assume $\omega = \omega_2$ without loss of generality.

In this case, we have

$$\eta = -{C_3}^2 \frac{\sigma(u - s_1)^2 \cdots \sigma(u - s_n)^2}{\sigma(u - u_1)^2 \cdots \sigma(u - u_{n-1})^2 \cdot \sigma(u - u_n)\sigma(u - u_n - \omega_2)}\, du$$

for some nonzero constant C_3. Since

$$\frac{\sigma(u - u_n)}{\sigma(u - u_n - \omega_2)} = -\exp\left\{ \eta_2 \left(u - u_n - \frac{\omega_2}{2} \right) \right\},$$

where $\eta_2 = \zeta(u+\omega_2) - \zeta(u)$, we get

$$\eta = -\left[C_3 \frac{\sigma(u-s_1)\cdots\sigma(u-s_n)}{\sigma(u-u_1)\cdots\sigma(u-u_n)} \cdot \sqrt{-1}\exp\left\{\frac{\eta_2}{2}\left(u - u_n - \frac{\omega_2}{2}\right)\right\}\right]^2 du.$$

Let $z(u)$, r, $r^{1/2}$, $h(z)$ be as in the previous section. Then we get

$$\begin{aligned}\eta = -\Bigg[C_3 &\frac{\sigma(u-s_1)\cdots\sigma(u-s_n)}{\sigma(u-u_1)\cdots\sigma(u-u_n)} \\ &\times \sqrt{-1}\exp\left\{\frac{\eta_2}{2}\left(u-u_n-\frac{\omega_2}{2}\right) - \frac{\pi\sqrt{-1}}{\omega_1}u\right\} \cdot \sqrt{\frac{\omega_1}{2\pi\sqrt{-1}}}\Bigg]^2 dz \\ = -\left(C_4 \frac{\sigma(u-s_1)\cdots\sigma(u-s_n)}{\sigma(u-u_1)\cdots\sigma(u-u_n)} \cdot e^{C_5 u}\right)^2 dz,\end{aligned}$$

where we set

$$C_4 := C_3\sqrt{-1}\sqrt{\frac{\omega_1}{2\pi\sqrt{-1}}}\exp\left\{-\frac{1}{2}\eta_2\left(u_n + \frac{\omega_2}{2}\right)\right\}, \quad C_5 := \frac{\eta_2}{2} - \frac{\pi\sqrt{-1}}{\omega_1}.$$

Set

$$\tilde{P}(u) := C_1 C_4 \frac{\sigma(u-t_1)\cdots\sigma(u-t_n)}{\sigma(u-u_1)\cdots\sigma(u-u_n)} e^{C_5 u}, \quad \tilde{Q}(u) := C_4 \frac{\sigma(u-s_1)\cdots\sigma(u-s_n)}{\sigma(u-u_1)\cdots\sigma(u-u_n)} e^{C_5 u}.$$

Then both $\tilde{P}(u)$ and $\tilde{Q}(u)$ are meromorphic functions on $\mathbf{C}$, and the Weierstrass data (g, η) of X is given by

$$g(u) = \frac{\tilde{P}(u)}{\tilde{Q}(u)}, \quad \eta = -\tilde{Q}(u)^2\, dz.$$

Set $q_j := z(u_j)$,

$$\begin{aligned} b_j &:= \frac{2\pi\sqrt{-1}}{\omega_1} q_j \cdot \lim_{u \to u_j}\{\tilde{Q}(u)(u-u_j)\}, \\ c_j &:= \frac{2\pi\sqrt{-1}}{\omega_1} q_j \cdot \lim_{u \to u_j}\{\tilde{P}(u)(u-u_j)\} \end{aligned}$$

$(j = 1, \ldots, n-1)$, and

$$P(z) := \sum_{j=1}^{n} c_j h(z, q_j), \quad Q(z) := \sum_{j=1}^{n} b_j h(z, q_j).$$

Now, let us show that we can choose b_n and c_n such that $P(z(u)) = \tilde{P}(u)$ and $Q(z(u)) = \tilde{Q}(u)$.

Set

$$\tilde{Q}_1(u) := \tilde{Q}(u)e^{-C_5 u}\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)},$$
$$Q_1(u) := Q(z(u))e^{-C_5 u}\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}.$$

Since

$$\tilde{Q}_1(u) = C_4\frac{\sigma(u-s_1)\cdots\sigma(u-s_n)}{\sigma(u-u_1)\cdots\sigma(u-u_{n-1})\sigma(u-u_n-\omega_2/2)}$$

and $s_1+\cdots+s_n = u_1+\cdots+u_{n-1}+u_n+(\omega_2/2)$, $\tilde{Q}_1(u)$ is an elliptic function of period (ω_1,ω_2). On the other hand, by (3.2) and (3.4),

$$\begin{aligned}Q_1(u) &= \sum_{j=1}^{n}\frac{b_j}{q_j}h(z(u-u_j))e^{-C_5 u}\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}\\ &= \sum_{j=1}^{n}\frac{b_j}{q_j}\frac{\omega_1}{2\pi\sqrt{-1}}\frac{(-1)}{\sigma(\omega_2/2)}\frac{\sigma(u-u_j-\omega_2/2)}{\sigma(u-u_j)}\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)},\end{aligned}$$

and hence $Q_1(u)$ is also an elliptic function of period (ω_1,ω_2).

If $u_n+\omega_2/2 \not\equiv u_j \bmod (\omega_1,\omega_2)$ $(j=1,\dots,n-1)$, then both $\tilde{Q}_1(u)$ and $Q_1(u)$ have only poles of order 1 at $u_1,\dots,u_{n-1},u_n+\omega_2/2$. For any $j=1,\dots,n-1$,

$$\begin{aligned}&\lim_{u\to u_j}(u-u_j)Q_1(u)\\ &= \lim_{u\to u_j}(u-u_j)b_j h(z(u),q_j)\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}e^{-C_5 u}\\ &= b_j\lim_{u\to u_j}(u-u_j)\frac{1}{\exp\{(2\pi\sqrt{-1}/\omega_1)u\}-\exp\{(2\pi\sqrt{-1}/\omega_1)u_j\}}\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}e^{-C_5 u}\\ &= \frac{2\pi\sqrt{-1}}{\omega_1}q_j\lim_{u\to u_j}\{\tilde{Q}(u)(u-u_j)\}\cdot\lim_{u\to u_j}\frac{1}{(2\pi\sqrt{-1}/\omega_1)q_j}\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}e^{-C_5 u}\\ &= \lim_{u\to u_j}\tilde{Q}(u)(u-u_j)\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}e^{-C_5 u}\\ &= \lim_{u\to u_j}(u-u_j)\tilde{Q}_1(u),\end{aligned}$$

that is, the residues of $Q_1(u)$ and $\tilde{Q}_1(u)$ at u_j coincide with each other. Now, by the residue theorem, the residues at $u_n+\omega_2/2$ also coincide with each other.

If $u_n+\omega_2/2 \equiv u_j \bmod (\omega_1,\omega_2)$ for some $j\in\{1,\dots,n-1\}$, for instance, if such j is $n-1$, then both $\tilde{Q}_1(u)$ and $Q_1(u)$ have poles of order 1 at $u_1,\dots,u_{n-2}$, and a pole of order 2 at $u_{n-1}\equiv u_n+\omega_2/2 \bmod (\omega_1,\omega_2)$. For any $j=1,\dots,n-2$, by the same reason as above, the residues at u_j coincide with each other, and, by the residue theorem, the residues at u_{n-1} also coincide with each other. Moreover,

$$\begin{aligned}
&\lim_{u\to u_{n-1}}(u-u_{n-1})^2 Q_1(u)\\
&=\lim_{u\to u_{n-1}}(u-u_{n-1})^2 b_{n-1}h(z(u),q_{n-1})\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}e^{-C_5u}\\
&=b_{n-1}\lim_{u\to u_{n-1}}\frac{(u-u_{n-1})^2}{\exp\{(2\pi\sqrt{-1}/\omega_1)u\}-\exp\{(2\pi\sqrt{-1}/\omega_1)u_{n-1}\}}\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}e^{-C_5u}\\
&=\frac{2\pi\sqrt{-1}}{\omega_1}q_{n-1}\lim_{u\to u_{n-1}}\{\tilde{Q}(u)(u-u_{n-1})\}\cdot\lim_{u\to u_{n-1}}\frac{u-u_{n-1}}{(2\pi\sqrt{-1}/\omega_1)q_{n-1}}\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}e^{-C_5u}\\
&=\lim_{u\to u_{n-1}}\tilde{Q}(u)(u-u_{n-1})^2\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}e^{-C_5u}\\
&=\lim_{u\to u_{n-1}}(u-u_{n-1})^2\tilde{Q}_1(u),
\end{aligned}$$

that is, the coefficients of the term of order -2 of the Laurent expansion of $Q(u)$ and $\tilde{Q}(u)$ at u_{n-1} also coincide with each other.

Hence, in both cases, $Q_1(u)-\tilde{Q}_1(u)$ is a holomorphic function on T^2. Therefore $Q_1(u)-\tilde{Q}_1(u)$ must be a constant. Now, since

$$h(z(u),q_n)\frac{\sigma(u-u_n)}{\sigma(u-u_n-\omega_2/2)}e^{-C_5u}=\frac{1}{q_n}\frac{\omega_1}{2\pi\sqrt{-1}}\frac{(-1)}{\sigma(\omega_2/2)}$$

is a nonzero constant (cf. (3.2), (3.4)), we can choose b_n so that $Q_1(u)-\tilde{Q}_1(u)\equiv 0$ and hence $Q(z(u))=\tilde{Q}(u)$.

In the same way, we can choose c_n so that $P(z(u))=\tilde{P}(u)$. In particular, we have $c_j/b_j=\lim_{u\to u_j}(P(u)/Q(u))=p_j$.

If we regard X as a map defined on $\mathbf{C}^*/\sim$, then we get the following fact:

Proposition 4.2. *In the case that $\omega=\omega_2$, the Weierstrass data (g,η) of an n-noid X is given by*

$$g(z)=\frac{P(z)}{Q(z)},\quad \eta=-Q(z)^2\,dz, \tag{4.5}$$

with

$$P(z)=\sum_{j=1}^{n}c_jh(z,q_j),\quad Q(z)=\sum_{j=1}^{n}b_jh(z,q_j), \tag{4.6}$$

where $b_1,\ldots,b_n,c_1,\ldots,c_n$ are complex numbers satisfying $b_j\neq 0$, $c_j=p_jb_j$ ($j=1,\ldots,n$).

$P(z)$ and $Q(z)$ are not well-defined on $\mathbf{C}^*/\sim$, but g and η are well-defined.

We remark here that the assertion above is valid also when $p_j = \infty$ for some j. In this case, we have only to set $b_j = 0$ and $c_j \neq 0$.

We mention here that Kusner and Schmitt [11] observed a similar fact in terms of spin structures, in the case that all the ends are planar.

In this paper, we study the case that $\omega = \omega_2$. This class involves almost all known examples of minimal surfaces of genus one all of whose ends are embedded ends. We will discuss the case that $\omega = 0$ in the forthcoming paper [6].

As we mentioned in §1, Umehara, Yamada and the first author [8] reduced Problem 2.3 for genus zero to a certain system of algebraic equations. In §§5–6, we reduce Problem 2.3 in the case that $\overline{M} = T^2$ and $\omega = \omega_2$ to equations which are described by using elliptic functions.

5. Local period problems and relative weights

In the case that $\omega = \omega_2$ in (4.1) (see (4.2)), as we discussed in §4, the Weierstrass data (g, η) of any n-noid X is given by the form (4.5) with (4.6). This data automatically satisfies the condition (2.6).

Since $h_0(1) = 0$, the Laurent expansion of $P(z)Q(z)$ at $q_j r^l$ is given by

$$
\begin{aligned}
&P(z)Q(z)\\
&= \frac{c_j b_j r^l}{(z-q_j r^l)^2} + \frac{1}{z-q_j r^l}\Bigg(c_j r^{l/2} \sum_{m=-\infty;m\neq l}^{+\infty} b_j \frac{r^{m/2}}{q_j r^l - q_j r^m} + b_j r^{l/2} \sum_{m=-\infty;m\neq l}^{+\infty} c_j \frac{r^{m/2}}{q_j r^l - q_j r^m}\\
&\qquad + c_j r^{l/2} \sum_{k=1;k\neq j}^{n} b_k h(q_j r^l, q_k) + b_j r^{l/2} \sum_{k=1;k\neq j}^{n} c_k h(q_j r^l, q_k) \Bigg) + O(1)\\
&= \frac{c_j b_j r^l}{(z-q_j r^l)^2} + \frac{1}{z-q_j r^l}\Bigg(\frac{2c_j b_j}{q_j} \sum_{m=-\infty;m\neq l}^{+\infty} \frac{r^{(l+m)/2}}{r^l - r^m}\\
&\qquad + \sum_{k=1;k\neq j}^{n} (c_j b_k + b_j c_k) r^{l/2} h(q_j r^l, q_k) \Bigg) + O(1)\\
&= \frac{c_j b_j r^l}{(z-q_j r^l)^2} + \frac{1}{z-q_j r^l}\Bigg(\frac{2c_j b_j}{q_j} h_0(1) + \sum_{k=1;k\neq j}^{n} (c_j b_k + b_j c_k) h(q_j, q_k) \Bigg) + O(1)\\
&= \frac{c_j b_j r^l}{(z-q_j r^l)^2} + \frac{1}{z-q_j r^l} \sum_{k=1;k\neq j}^{n} (c_j b_k + b_j c_k) h(q_j, q_k) + O(1),
\end{aligned}
$$

In the same way, we also have

$$Q(z)^2 = \frac{{b_j}^2 r^l}{(z-q_j r^l)^2} + \frac{1}{z-q_j r^l} \sum_{k=1;\, k\neq j}^{n} 2b_j b_k h(q_j, q_k) + O(1),$$

$$P(z)^2 = \frac{{c_j}^2 r^l}{(z-q_j r^l)^2} + \frac{1}{z-q_j r^l} \sum_{k=1;\, k\neq j}^{n} 2c_j c_k h(q_j, q_k) + O(1).$$

Now, for each end q_j $(j = 1, \dots, n)$, denote the corresponding R_0, R_1, R_2, w, w^* as in (2.3) and (2.7) by R_{0j}, R_{1j}, R_{2j}, w_j, w_j^* respectively. Then we have

Lemma 5.1. *The integrals R_{0j}, R_{1j} and R_{2j} are given by the following equalities*:

$$R_{0j} = -\operatorname{Res}_{z=q_j} Q(z)^2\, dz = -\sum_{k=1;\, k\neq j}^{n} 2b_j b_k h(q_j, q_k),$$

$$\begin{aligned} R_{1j} &= -\operatorname{Res}_{z=q_j} P(z)Q(z)\, dz \\ &= -\sum_{k=1;\, k\neq j}^{n} (c_j b_k + b_j c_k) h(q_j, q_k) = -\sum_{k=1;\, k\neq j}^{n} (p_j + p_k) b_j b_k h(q_j, q_k), \end{aligned}$$

$$\begin{aligned} R_{2j} &= -\operatorname{Res}_{z=q_j} P(z)^2\, dz \\ &= -\sum_{k=1;\, k\neq j}^{n} 2c_j c_k h(q_j, q_k) = -\sum_{k=1;\, k\neq j}^{n} 2p_j p_k b_j b_k h(q_j, q_k). \end{aligned}$$

Henceforth, we use the notation "≡" to describe equalities given by definitions directly or formulas already given. By Corollary 2.2 and Lemma 5.1, any solution to Problem 2.3 must satisfy the following equation for the local period problem:

$$\text{(5.1)} \quad \begin{cases} w_j \equiv -p_j R_{0j} + R_{1j} \equiv \displaystyle\sum_{k=1;\, k\neq j}^{n} b_j b_k (p_j - p_k) h(q_j, q_k) = a_j, \\ w_j^* \equiv -\dfrac{1}{2}(|p_j|^2 - 1) R_{0j} + \overline{p_j} R_{1j} \\ \quad \equiv -\displaystyle\sum_{k=1;\, k\neq j}^{n} b_j b_k (\overline{p_j} p_k + 1) h(q_j, q_k) = 0 \qquad (j = 1, \dots, n). \end{cases}$$

For any data (g, η) as in (4.5) with (4.6), set

$$w_{jk} := b_j b_k (p_j - p_k) h(q_j, q_k), \quad w_{jk}^* := -b_j b_k (\overline{p_j} p_k + 1) h(q_j, q_k)$$
$$(j, k = 1, \dots, n;\ j \neq k).$$

When (g, η) realizes an n-noid X, we call w_{jk} the *relative weight* of the end-pair (q_j, q_k) $(j, k = 1, \dots, n;\ j \neq k)$ of X. As in the case of genus zero [7, Proposition 2.3], the values of w_{jk} are independent of the parametrizations. More precisely,

Proposition 5.2. *The relative weights w_{jk} are invariant under the conformal transformations of $\mathbf{C}^*/\sim$ and the orientation preserving congruent transformations of $\mathbf{R}^3$.*

Proof. (1) If we choose $\tilde{q}_j = q_j r^l$ (resp. $\tilde{q}_k = q_k r^m$) in spite of q_j (resp. q_k), then the corresponding coefficient b_j (resp. b_k) is replaced by $\tilde{b}_j = b_j r^{l/2}$ (resp. $\tilde{b}_k = b_k r^{m/2}$), and hence it holds that $\tilde{b}_j \tilde{b}_k h(\tilde{q}_j, \tilde{q}_k) = b_j b_k h(q_j, q_k)$.

(2) Consider a coordinate transformation $\tilde{z} := \alpha z$ for some $\alpha \in \mathbf{C} \setminus \{0\}$. Then each q_j and b_j are replaced by $\tilde{q}_j = \alpha q_j$ and $\tilde{b}_j = \sqrt{\alpha} b_j$ respectively. Hence it holds that $\tilde{b}_j \tilde{b}_k h(\tilde{q}_j, \tilde{q}_k) = b_j b_k h(q_j, q_k)$.

(3) If we choose another fundamental period $(\tilde{\omega}_1, \tilde{\omega}_2)$ such that $\tilde{\omega}_2 = c\omega_1 + d\omega_2$ for some even number c, and define $\tilde{z} := e^{2\pi\sqrt{-1}u/\tilde{\omega}_1}$ and $\tilde{h}(\tilde{z})$ by using $\tilde{r} = e^{-2\pi\sqrt{-1}\tilde{\omega}_2/\tilde{\omega}_1}$, then, since

$$\frac{\tilde{\omega}_1}{\tilde{z}}\, d\tilde{z} = 2\pi\sqrt{-1}\, du = \frac{\omega_1}{z}\, dz$$

and

$$\left(\sum_{j=1}^{n} \tilde{b}_j \tilde{h}(\tilde{z}, \tilde{q}_j)\right)^2 d\tilde{z} = -\eta = \left(\sum_{j=1}^{n} b_j h(z, q_j)\right)^2 dz,$$

we have

$$\left(\sum_{j=1}^{n} \tilde{b}_j \tilde{h}(\tilde{z}, \tilde{q}_j)\right)^2 \frac{\tilde{z}}{\tilde{\omega}_1} = \frac{-\eta}{2\pi\sqrt{-1}\, du} = \left(\sum_{j=1}^{n} b_j h(z, q_j)\right)^2 \frac{z}{\omega_1},$$

and hence

$$\left(\tilde{b}_j \frac{\tilde{\omega}_1}{2\pi\sqrt{-1}} \frac{1}{\tilde{q}_j}\right)^2 \frac{\tilde{z}}{\tilde{\omega}_1} = \lim_{u \to u_j} \frac{-(u-u_j)^2 \eta}{2\pi\sqrt{-1}\, du} = \left(b_j \frac{\omega_1}{2\pi\sqrt{-1}} \frac{1}{q_j}\right)^2 \frac{z}{\omega_1}.$$

Therefore we get

$$\tilde{b}_j = \sqrt{\frac{\omega_1}{\tilde{\omega}_1} \cdot \frac{\tilde{q}_j}{q_j}}\, b_j,$$

where we set $\sqrt{q_j} := e^{\pi\sqrt{-1}u_j/\omega_1}$ and $\sqrt{\tilde{q}_j} := e^{\pi\sqrt{-1}u_j/\tilde{\omega}_1}$. On the other hand, since

$$\begin{aligned}\frac{1}{\tilde{\omega}_1}\sqrt{\tilde{z}(u)}\tilde{h}(\tilde{z}(u)) &= \frac{1}{2\pi\sqrt{-1}}\sqrt{\wp(u) - \wp\left(\frac{\tilde{\omega}_2}{2}\right)} = \frac{1}{2\pi\sqrt{-1}}\sqrt{\wp(u) - \wp\left(\frac{\omega_2}{2}\right)} \\ &= \frac{1}{\omega_1}\sqrt{z(u)}h(z(u))\end{aligned}$$

holds as an equality with respect to u (see (3.3)), we also have

$$\frac{1}{\tilde{\omega}_1}\sqrt{\tilde{q}_j}\sqrt{\tilde{z}}\,\tilde{h}(\tilde{z},\tilde{q}_j)=\frac{1}{\omega_1}\sqrt{q_j}\sqrt{z}\,h(z,q_j),$$

and hence

$$\frac{1}{\tilde{\omega}_1}\sqrt{\tilde{q}_j}\sqrt{\tilde{q}_k}\,\tilde{h}(\tilde{q}_k,\tilde{q}_j)=\frac{1}{\omega_1}\sqrt{q_j}\sqrt{q_k}\,h(q_k,q_j).$$

Therefore we get $\tilde{b}_j\tilde{b}_k h(\tilde{q}_j,\tilde{q}_k)=b_jb_kh(q_j,q_k)$.

(4) Consider an orthogonal transformation P of $\mathbf{R}^3$ such that

$$F(\zeta)=\Pi\circ P|_{\mathbf{S}^2}\circ\Pi^{-1}(\zeta)=\frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}.$$

Then each p_j and b_j are replaced by $\tilde{p}_j=F(p_j)$ and $\tilde{b}_j=b_j/\sqrt{F'(p_j)}$ respectively. Hence it holds that $\tilde{b}_j\tilde{b}_k(\tilde{p}_j-\tilde{p}_k)=b_jb_k(p_j-p_k)$. □

We can rewrite the equation (5.1) by using the relative weights:

$$\text{(5.2)}\qquad \begin{cases}\displaystyle\sum_{k=1;k\neq j}^{n} w_{jk}=a_j,\\ \displaystyle\sum_{k=1;k\neq j}^{n} w_{jk}^{*}\left(=\sum_{k=1;k\neq j}^{n} w_{jk}\frac{\overline{p_j}\,p_k+1}{p_k-p_j}\right)=0\end{cases}\qquad (j=1,\dots,n).$$

It is remarkable that this equation is quite the same as in the case of genus zero.

6. Global period problems

In this section, we calculate the global period around the generators of the first homology group of T^2. First, by a direct computation, we have

$$\begin{aligned}P(z)Q(z)&=\sum_{j=1}^{n}c_jh(z,q_j)\sum_{k=1}^{n}b_kh(z,q_k)\\&=\sum_{j=1}^{n}\sum_{l=-\infty}^{+\infty}\frac{c_jr^{l/2}}{z-q_jr^l}\sum_{k=1}^{n}\sum_{m=-\infty}^{+\infty}\frac{b_kr^{m/2}}{z-q_kr^m}\\&=\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\frac{c_jr^{l/2}}{z-q_jr^l}\frac{b_kr^{m/2}}{z-q_kr^m}\\&=\sum_{j=1}^{n}\sum_{l=-\infty}^{+\infty}c_jb_j\left(\frac{r^{l/2}}{z-q_jr^l}\right)^2+\sum_{j=1}^{n}\sum_{l=-\infty}^{+\infty}\sum_{m=-\infty;\,m\neq l}^{+\infty}\frac{c_jr^{l/2}}{z-q_jr^l}\frac{b_jr^{m/2}}{z-q_jr^m}\\&\quad+\sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\sum_{l=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\frac{c_jr^{l/2}}{z-q_jr^l}\frac{b_kr^{m/2}}{z-q_kr^m}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \sum_{l=-\infty}^{+\infty} \frac{c_j b_j r^l}{(z - q_j r^l)^2} \\
&\quad + \sum_{j=1}^{n} \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty;\, m\neq l}^{+\infty} \frac{c_j b_j r^{(l+m)/2}}{q_j (r^l - r^m)} \left(\frac{1}{z - q_j r^l} - \frac{1}{z - q_j r^m} \right) \\
&\quad + \sum_{j=1}^{n} \sum_{k=1;\, k\neq j}^{n} \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{c_j b_k r^{(l+m)/2}}{q_j r^l - q_k r^m} \left(\frac{1}{z - q_j r^l} - \frac{1}{z - q_k r^m} \right).
\end{aligned}
$$

In the same way, we also have

$$
\begin{aligned}
Q(z)^2 &= \sum_{j=1}^{n} \sum_{l=-\infty}^{+\infty} \frac{{b_j}^2 r^l}{(z - q_j r^l)^2} \\
&\quad + \sum_{j=1}^{n} \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty;\, m\neq l}^{+\infty} \frac{{b_j}^2 r^{(l+m)/2}}{q_j (r^l - r^m)} \left(\frac{1}{z - q_j r^l} - \frac{1}{z - q_j r^m} \right) \\
&\quad + \sum_{j=1}^{n} \sum_{k=1;\, k\neq j}^{n} \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{b_j b_k r^{(l+m)/2}}{q_j r^l - q_k r^m} \left(\frac{1}{z - q_j r^l} - \frac{1}{z - q_k r^m} \right), \\
P(z)^2 &= \sum_{j=1}^{n} \sum_{l=-\infty}^{+\infty} \frac{{c_j}^2 r^l}{(z - q_j r^l)^2} \\
&\quad + \sum_{j=1}^{n} \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty;\, m\neq l}^{+\infty} \frac{{c_j}^2 r^{(l+m)/2}}{q_j (r^l - r^m)} \left(\frac{1}{z - q_j r^l} - \frac{1}{z - q_j r^m} \right) \\
&\quad + \sum_{j=1}^{n} \sum_{k=1;\, k\neq j}^{n} \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{c_j c_k r^{(l+m)/2}}{q_j r^l - q_k r^m} \left(\frac{1}{z - q_j r^l} - \frac{1}{z - q_k r^m} \right).
\end{aligned}
$$

We now assume $1 \leq |q_j| < |r|$ $(j = 1, \dots, n)$. Let R be a positive number such that $R < 1$ and $q_j r^{-1} \notin \{z \mid R \leq |z| < 1\}$ holds for any $j = 1, \dots, n$. Choose an argument of r (see (3.1)) so that $0 \leq \arg r < 2\pi$, independent of the choice of a fundamental period (ω_1, ω_2) in §4, and choose a unit complex number z_0 such that $q_j r^{-t} \neq z_0$ for $0 \leq t < 1$ and $j = 1, \dots, n$. We consider the following two loops on $\overline{M} = \mathbf{C}^*/\sim$:

$$
\begin{aligned}
&\gamma_1 : z(t) = R e^{\sqrt{-1}t} \quad (0 \leq t \leq 2\pi), \\
&\gamma_2 : z(t) = z_0 r^t = z_0 |r|^t e^{\sqrt{-1}t \arg r} \quad (0 \leq t \leq 1).
\end{aligned}
$$

In particular, in the case that r is a positive real number, γ_2 is defined by $z(t) = z_0 |r|^t$ independent of the choice of the signature of $r^{1/2}$. The loops γ_1 and γ_2 generate the first homology group of $\overline{M}$.

To describe the integrals of $Q(z)^2$, $P(z)^2$ and $P(z)Q(z)$ on γ_1 and γ_2, we use $h(z, q)$ and $h_1(z, q)$.

Lemma 6.1. *The integrals $R_0(\gamma_1)$, $R_1(\gamma_1)$ and $R_2(\gamma_1)$ as in* (2.3) *are given by the following equalities*:

$$(6.1)\quad \begin{cases} R_0(\gamma_1) = \dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_{\gamma_1}(-Q(z)^2)\,dz = -\sum_{j=1}^{n}\sum_{k=1}^{n} b_j b_k h_1(q_j, q_k), \\ R_1(\gamma_1) = \dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_{\gamma_1}(-P(z)Q(z))\,dz = -\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{2}(c_j b_k + b_j c_k)h_1(q_j, q_k) \\ \qquad = -\displaystyle\sum_{j=1}^{n}\sum_{k=1}^{n} b_j c_k h_1(q_j, q_k), \\ R_2(\gamma_1) = \dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_{\gamma_1}(-P(z)^2)\,dz = -\sum_{j=1}^{n}\sum_{k=1}^{n} c_j c_k h_1(q_j, q_k). \end{cases}$$

Proof. We prove our assertion for $R_1(\gamma_1)$. Since

$$\int_{\gamma_1}\frac{c_j b_j r^l}{(z - q_j r^l)^2}\,dz = 0$$

and

$$\int_{\gamma_1}\left(\frac{1}{z - q_j r^l} - \frac{1}{z - q_k r^m}\right)dz = \begin{cases} 2\pi\sqrt{-1} & (l < 0,\ m \geq 0), \\ 0 & (l < 0,\ m < 0), \\ 0 & (l \geq 0,\ m \geq 0), \\ -2\pi\sqrt{-1} & (l \geq 0,\ m < 0), \end{cases}$$

we have

$$\begin{aligned} &\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_1} P(z)Q(z)\,dz \\ &= \sum_{j=1}^{n}\left\{\sum_{l=-\infty}^{-1}\sum_{m=0}^{+\infty}\frac{c_j b_j r^{(m-l)/2}}{q_j(1 - r^{m-l})} - \sum_{l=0}^{+\infty}\sum_{m=-\infty}^{-1}\frac{c_j b_j r^{(m-l)/2}}{q_j(1 - r^{m-l})}\right\} \\ &\quad + \sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n}\left\{\sum_{l=-\infty}^{-1}\sum_{m=0}^{+\infty}\frac{c_j b_k r^{(m-l)/2}}{q_j - q_k r^{m-l}} - \sum_{l=0}^{+\infty}\sum_{m=-\infty}^{-1}\frac{c_j b_k r^{(m-l)/2}}{q_j - q_k r^{m-l}}\right\} \\ &= \sum_{j=1}^{n} c_j b_j h_1(q_j, q_j) + \sum_{j=1}^{n}\sum_{k=1;\,k\neq j}^{n} c_j b_k h_1(q_j, q_k) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} c_j b_k h_1(q_j, q_k) = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{2}(c_j b_k + b_j c_k)h_1(q_j, q_k). \end{aligned}$$

If we consider the case that $b_j = c_j$, then we get our assertions also for $R_0(\gamma_1)$ and $R_2(\gamma_2)$. □

Lemma 6.2. *The integrals* $R_0(\gamma_2)$, $R_1(\gamma_2)$ *and* $R_2(\gamma_2)$ *as in* (2.3) *are given by the following equalities*:

$$
(6.2)\quad
\begin{cases}
R_0(\gamma_2) = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{\gamma_2} (-Q(z)^2)\, dz \\
\qquad\quad = \dfrac{-1}{2\pi\sqrt{-1}} \left\{ -\displaystyle\sum_{j=1}^{n} \dfrac{{b_j}^2}{q_j} - \log r \cdot R_0(\gamma_1) + \displaystyle\sum_{j=1}^{n} \log q_j \cdot R_{0j} \right\}, \\
R_1(\gamma_2) = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{\gamma_2} (-P(z)Q(z))\, dz \\
\qquad\quad = \dfrac{-1}{2\pi\sqrt{-1}} \left\{ -\displaystyle\sum_{j=1}^{n} \dfrac{c_j b_j}{q_j} - \log r \cdot R_1(\gamma_1) + \displaystyle\sum_{j=1}^{n} \log q_j \cdot R_{1j} \right\}, \\
R_2(\gamma_2) = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{\gamma_2} (-P(z)^2)\, dz \\
\qquad\quad = \dfrac{-1}{2\pi\sqrt{-1}} \left\{ -\displaystyle\sum_{j=1}^{n} \dfrac{{c_j}^2}{q_j} - \log r \cdot R_2(\gamma_1) + \displaystyle\sum_{j=1}^{n} \log q_j \cdot R_{2j} \right\}.
\end{cases}
$$

Proof. We prove our assertion for $R_1(\gamma_2)$. Since

$$
\begin{aligned}
\int_{\gamma_2} \frac{c_j b_j r^l}{(z - q_j r^l)^2}\, dz &= \int_0^1 \frac{c_j b_j r^l}{(z(t) - q_j r^l)^2} z'(t)\, dt = \left[-\frac{c_j b_j r^l}{z(t) - q_j r^l} \right]_{t=0}^1 \\
&= -\frac{c_j b_j r^l}{z_0 r - q_j r^l} + \frac{c_j b_j r^l}{z_0 - q_j r^l} = -\frac{c_j b_j r^{l-1}}{z_0 - q_j r^{l-1}} + \frac{c_j b_j r^l}{z_0 - q_j r^l},
\end{aligned}
$$

we have

$$
\begin{aligned}
\sum_{l=-\infty}^{+\infty} \int_{\gamma_2} \frac{c_j b_j r^l}{(z - q_j r^l)^2}\, dz &= \sum_{l=-\infty}^{+\infty} \left(-\frac{c_j b_j r^{l-1}}{z_0 - q_j r^{l-1}} + \frac{c_j b_j r^l}{z_0 - q_j r^l} \right) \\
&= \lim_{N_+ \to +\infty, N_- \to -\infty} \sum_{l=N_-}^{N_+} \left(-\frac{c_j b_j r^{l-1}}{z_0 - q_j r^{l-1}} + \frac{c_j b_j r^l}{z_0 - q_j r^l} \right) \\
&= \lim_{N_+ \to +\infty, N_- \to -\infty} \left(-\frac{c_j b_j r^{N_- - 1}}{z_0 - q_j r^{N_- - 1}} + \frac{c_j b_j r^{N_+}}{z_0 - q_j r^{N_+}} \right) \\
&= -\frac{c_j b_j \cdot 0}{z_0 - q_j \cdot 0} + \frac{c_j b_j}{z_0 \cdot 0 - q_j} = -\frac{c_j b_j}{q_j}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\sum_{l=-\infty}^{+\infty} \sum_{m=-\infty;\, m\neq l}^{+\infty} \frac{c_j b_j r^{(l+m)/2}}{q_j (r^l - r^m)} \int_{\gamma_2} \left(\frac{1}{z - q_j r^l} - \frac{1}{z - q_j r^m} \right) dz \\
&= c_j b_j \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty;\, m\neq l}^{+\infty} \frac{r^{(m-l)/2}}{q_j (1 - r^{m-l})} \int_0^1 \left(\frac{1}{z(t) - q_j r^l} - \frac{1}{z(t) - q_j r^m} \right) z'(t)\, dt
\end{aligned}
$$

$$= c_j b_j \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty;\, m\neq 0}^{+\infty} \frac{r^{m/2}}{q_j(1-r^m)} \int_0^1 \left(\frac{1}{z(t)-q_j r^l} - \frac{1}{z(t)-q_j r^{l+m}} \right) z'(t)\, dt$$
$$= c_j b_j \sum_{m=-\infty;\, m\neq 0}^{+\infty} \frac{r^{m/2}}{q_j(1-r^m)} \sum_{l=-\infty}^{+\infty} \int_0^1 \left(\frac{1}{z(t)-q_j r^l} - \frac{1}{z(t)-q_j r^{l+m}} \right) z'(t)\, dt,$$

and

$$\sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{c_j b_k r^{(l+m)/2}}{q_j r^l - q_k r^m} \int_{\gamma_2} \left(\frac{1}{z-q_j r^l} - \frac{1}{z-q_k r^m} \right) dz$$
$$= c_j b_k \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{r^{(m-l)/2}}{q_j - q_k r^{m-l}} \int_0^1 \left(\frac{1}{z(t)-q_j r^l} - \frac{1}{z(t)-q_k r^m} \right) z'(t)\, dt$$
$$= c_j b_k \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \frac{r^{m/2}}{q_j - q_k r^{m}} \int_0^1 \left(\frac{1}{z(t)-q_j r^l} - \frac{1}{z(t)-q_k r^{l+m}} \right) z'(t)\, dt$$
$$= c_j b_k \sum_{m=-\infty}^{+\infty} \frac{r^{m/2}}{q_j - q_k r^{m}} \sum_{l=-\infty}^{+\infty} \int_0^1 \left(\frac{1}{z(t)-q_j r^l} - \frac{1}{z(t)-q_k r^{l+m}} \right) z'(t)\, dt,$$

since both of these series are absolutely convergent series. In particular, if we set $z(t) := z_0 r^t = z_0 |r|^t e^{\sqrt{-1}t \arg r}$ for $t \in \mathbf{R}$, then it holds that

$$\sum_{l=-\infty}^{+\infty} \int_0^1 \left(\frac{1}{z(t)-q_j r^l} - \frac{1}{z(t)-q_k r^{l+m}} \right) z'(t)\, dt$$
$$= \sum_{l=-\infty}^{+\infty} \int_0^1 \left(\frac{1}{z(t)r^{-l}-q_j} - \frac{1}{z(t)r^{-l}-q_k r^{m}} \right) r^{-l} z'(t)\, dt$$
$$= \sum_{l=-\infty}^{+\infty} \int_0^1 \left(\frac{1}{z(t-l)-q_j} - \frac{1}{z(t-l)-q_k r^{m}} \right) z'(t-l)\, dt$$
$$= \sum_{l=-\infty}^{+\infty} \int_{-l}^{-l+1} \left(\frac{1}{z(s)-q_j} - \frac{1}{z(s)-q_k r^{m}} \right) z'(s)\, ds$$
$$= \lim_{N_+\to+\infty, N_-\to-\infty} \sum_{l=N_-}^{N_+} \int_{-l}^{-l+1} \left(\frac{1}{z(s)-q_j} - \frac{1}{z(s)-q_k r^{m}} \right) z'(s)\, ds$$
$$= \lim_{N_+\to+\infty, N_-\to-\infty} \int_{N_-}^{N_+} \left(\frac{1}{z(s)-q_j} - \frac{1}{z(s)-q_k r^{m}} \right) z'(s)\, ds.$$

To compute this integral, let us define $\log(z - q_j r^l)$ on a simply connected domain $\mathbf{C} \setminus \{q_j r^t \mid t \geq l\}$.

In the case that $0 < \arg r < 2\pi$, for any $j = 1, \ldots, n$, set $t_j := \log_{|r|}|q_j|$, and choose $\arg q_j$ so that $0 < \arg q_j - \arg z_0 r^{t_j} < 2\pi$, that is $\arg z_0 + t_j \arg r < \arg q_j <$

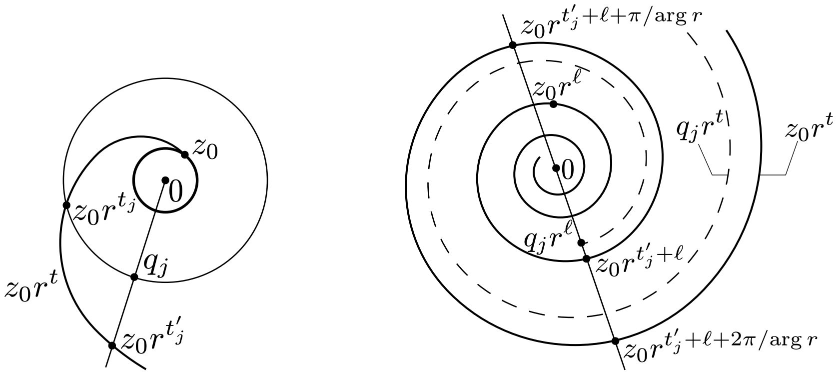


Fig. 6.1.

$\arg z_0 + t_j \arg r + 2\pi$. For any $l \in \mathbf{Z}$, it is natural to choose $\arg q_j r^l = \arg q_j + l \arg r$. Define $\arg(z - q_j r^l)$ on $\mathbf{C} \setminus \{q_j r^t \mid t \geq l\}$ so that $\arg(z_0 r^{t'_j + l} - q_j r^l) = \arg q_j r^l$ holds for $t'_j = (\arg q_j - \arg z_0)/\arg r$. Then we have

$$\begin{aligned}
&\arg q_j r^l - 2\pi < \arg(z_0 r^t - q_j r^l) < \arg q_j r^l \quad (-\infty < t < t'_j + l),\\
&\arg(0 - q_j r^l) = \arg q_j r^l - \pi,\\
&\arg q_j r^l + (N-1)\pi \leq \arg(z_0 r^t - q_j r^l) < \arg q_j r^l + N\pi\\
&(t'_j + l + (N-1)\pi/\arg r \leq t < t'_j + l + N\pi/\arg r,\ N \in \mathbf{N}).
\end{aligned}$$

Now, $\log(z - q_j r^l) := \log|z - q_j r^l| + \sqrt{-1}\arg(z - q_j r^l)$ is well-defined on $\mathbf{C} \setminus \{q_j r^t \mid t \geq l\}$ (see Fig. 6.1).

In the case that $\arg r = 0$, for any $j = 1, \dots, n$, choose $\arg q_j$ so that $0 < \arg q_j - \arg z_0 < 2\pi$, that is $\arg z_0 < \arg q_j < \arg z_0 + 2\pi$. For any $l \in \mathbf{Z}$, define $\arg(z - q_j r^l)$ on $\mathbf{C} \setminus \{q_j r^t \mid t \geq l\}$ so that $\arg(0 - q_j r^l) = \arg q_j r^l - \pi = \arg q_j - \pi$. Then we have $\arg q_j - 2\pi = \arg q_j r^l - 2\pi < \arg(z - q_j r^l) < \arg q_j r^l = \arg q_j$. Now, $\log(z - q_j r^l) := \log|z - q_j r^l| + \sqrt{-1}\arg(z - q_j r^l)$ is well-defined on $\mathbf{C} \setminus \{q_j r^t \mid t \geq l\}$.

By using $\log(z - q_j r^l)$ defined above, we see that

$$\begin{aligned}
&\lim_{N_+ \to +\infty, N_- \to -\infty} \int_{N_-}^{N_+} \left(\frac{1}{z(s) - q_j} - \frac{1}{z(s) - q_k r^m} \right) z'(s)\, ds\\
&= \lim_{N_+ \to +\infty, N_- \to -\infty} [\log(z(s) - q_j) - \log(z(s) - q_k r^m)]_{s=N_-}^{N_+}\\
&= \lim_{N_+ \to +\infty} \{\log(z(N_+) - q_j) - \log(z(N_+) - q_k r^m)\}\\
&\quad - \lim_{N_- \to -\infty} \{\log(z(N_-) - q_j) - \log(z(N_-) - q_k r^m)\}.
\end{aligned}$$

The first term satisfies

$$\begin{aligned}
&\lim_{N_+\to+\infty}\{\log(z(N_+)-q_j)-\log(z(N_+)-q_kr^m)\}\\
&=\lim_{N_+\to+\infty}[\{\log|z(N_+)-q_j|+\sqrt{-1}\arg(z(N_+)-q_j)\}\\
&\qquad\qquad-\{\log|z(N_+)-q_kr^m|+\sqrt{-1}\arg(z(N_+)-q_kr^m)\}]\\
&=\lim_{N_+\to+\infty}\log\left|\frac{z(N_+)-q_j}{z(N_+)-q_kr^m}\right|+\sqrt{-1}\lim_{N_+\to+\infty}\{\arg(z(N_+)-q_j)-\arg(z(N_+)-q_kr^m)\}\\
&=\lim_{N_+\to+\infty}\log\left|\frac{1-q_j/z(N_+)}{1-q_kr^m/z(N_+)}\right|\\
&\quad+\sqrt{-1}\Bigg[\lim_{N_+\to+\infty}\{\arg(z(N_+)-q_j)-\arg z(N_+)\}\\
&\qquad\qquad-\lim_{N_+\to+\infty}\{\arg(z(N_+)-q_kr^m)-\arg z(N_+)\}\Bigg].
\end{aligned}$$

Since

$$\begin{aligned}
&\cos|\arg(z(t)-q_jr^l)-\arg z(t)|=\frac{|z(t)|^2+|z(t)-q_jr^l|^2-|q_jr^l|^2}{2|z(t)|\cdot|z(t)-q_jr^l|}\\
&=\frac{1+|1-q_jr^l/z(t)|^2-|q_jr^l/z(t)|^2}{2\cdot1\cdot|1-q_jr^l/z(t)|}\to\frac{1+|1-0|^2-|0|^2}{2\cdot1\cdot|1-0|}=1\quad(t\to+\infty),
\end{aligned}$$

it holds that

$$\lim_{t\to+\infty}|\arg(z(t)-q_jr^l)-\arg z(t)|=0$$

for any $j=1,\dots,n$ and any $l\in\mathbf{Z}$. Hence we have

$$\lim_{N_+\to+\infty}\{\log(z(N_+)-q_j)-\log(z(N_+)-q_kr^m)\}=0.$$

On the other hand, we have

$$\begin{aligned}
&\lim_{N_-\to-\infty}\{\log(z(N_-)-q_j)-\log(z(N_-)-q_kr^m)\}\\
&=\log(0-q_j)-\log(0-q_kr^m)\\
&=\{\log|-q_j|+\sqrt{-1}\arg(-q_j)\}-\{\log|-q_kr^m|+\sqrt{-1}\arg(-q_kr^m)\}\\
&=\log|q_j|+\sqrt{-1}(\arg q_j-\pi)-\log|q_kr^m|-\sqrt{-1}(\arg q_kr^m-\pi)\\
&=\log\frac{q_j}{q_k}-m\log r.
\end{aligned}$$

Hence we get

$$
\begin{aligned}
&\int_{\gamma_2} P(z)Q(z)\,dz \\
&= \sum_{j=1}^{n}\left(-\frac{c_j b_j}{q_j}\right) + \sum_{j=1}^{n} c_j b_j \sum_{m=-\infty;\, m\neq 0}^{+\infty} \frac{r^{m/2}}{q_j(1-r^m)}\left(m\log r - \log\frac{q_j}{q_j}\right) \\
&\quad + \sum_{j=1}^{n}\sum_{k=1;\, k\neq j}^{n} c_j b_k \sum_{m=-\infty}^{+\infty} \frac{r^{m/2}}{q_j - q_k r^m}\left(m\log r - \log\frac{q_j}{q_k}\right) \\
&= -\sum_{j=1}^{n}\frac{c_j b_j}{q_j} + \log r\left\{\sum_{j=1}^{n} c_j b_j h_1(q_j, q_j) + \sum_{j=1}^{n}\sum_{k=1;\, k\neq j}^{n} c_j b_k h_1(q_j, q_k)\right\} \\
&\quad - \sum_{j=1}^{n}\sum_{k=1;\, k\neq j}^{n} c_j b_k h(q_j, q_k)\log\frac{q_j}{q_k} \\
&= -\sum_{j=1}^{n}\frac{c_j b_j}{q_j} + \log r \sum_{j=1}^{n}\sum_{k=1}^{n} c_j b_k h_1(q_j, q_k) \\
&\quad - \left\{\sum_{j=1}^{n}\log q_j \sum_{k=1;\, k\neq j}^{n} c_j b_k h(q_j, q_k) + \sum_{k=1}^{n}\log q_k \sum_{j=1;\, j\neq k}^{n} c_j b_k h(q_k, q_j)\right\} \\
&= -\sum_{j=1}^{n}\frac{c_j b_j}{q_j} + \log r \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{2}(c_j b_k + b_j c_k) h_1(q_j, q_k) \\
&\quad - \sum_{j=1}^{n}\log q_j \sum_{k=1;\, k\neq j}^{n} (c_j b_k + b_j c_k) h(q_j, q_k) \\
&= -\sum_{j=1}^{n}\frac{c_j b_j}{q_j} - \log r \cdot R_1(\gamma_1) + \sum_{j=1}^{n}\log q_j \cdot R_{1j}.
\end{aligned}
$$

If we consider the case that $b_j = c_j$, then we get our assertions also for $R_0(\gamma_2)$ and $R_2(\gamma_2)$. □

The Weierstrass data (g, η) of an n-noid of the form (4.5) with (4.6) must satisfy both the condition (5.1) and the condition (2.5) with (6.1) and (6.2). In the case that $\overline{M} = T^2$ and $\omega = \omega_2$, Problem 2.3 is reduced to a problem of finding q_j, b_j, c_j $(j = 1, \ldots, n)$ and r satisfying these conditions.

Theorem 6.3. *There exists an n-noid* $X\colon M=T^2\setminus\{q_1,\dots,q_n\}\to\mathbf{R}^3$ *of type* $\omega=\omega_2$ *satisfying* (2.14) *if and only if there exist* q_j, b_j, $c_j=p_jb_j$ $(j=1,\dots,n)$ *satisfying* (6.3)

$$\begin{cases}\left.\begin{aligned} w_j\equiv b_j\sum_{k=1;k\neq j}^{n}(p_j-p_k)b_kh(q_j,q_k)=a_j\\ w_j^*\equiv -b_j\sum_{k=1;k\neq j}^{n}(\overline{p_j}p_k+1)b_kh(q_j,q_k)=0\end{aligned}\right\}(j=1,\dots,n),\\ R_0(\gamma_1)+\overline{R_2(\gamma_1)}\equiv-\sum_{j=1}^{n}\sum_{k=1}^{n}(b_jb_kh_1(q_j,q_k)+\overline{c_jc_kh_1(q_j,q_k)})=0,\\ R_1(\gamma_1)\equiv-\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{2}(c_jb_k+b_jc_k)h_1(q_j,q_k)\in\mathbf{R},\\ R_0(\gamma_2)+\overline{R_2(\gamma_2)}\\ \equiv\frac{-1}{2\pi\sqrt{-1}}\Bigg\{-\sum_{j=1}^{n}\frac{{b_j}^2}{q_j}+\log r\cdot\sum_{j=1}^{n}\sum_{k=1}^{n}b_jb_kh_1(q_j,q_k)-\sum_{j=1}^{n}\log q_j\cdot\sum_{k=1;k\neq j}^{n}2b_jb_kh(q_j,q_k)\\ \qquad+\overline{\sum_{j=1}^{n}\frac{{c_j}^2}{q_j}}-\overline{\log r\cdot\sum_{j=1}^{n}\sum_{k=1}^{n}c_jc_kh_1(q_j,q_k)}+\overline{\sum_{j=1}^{n}\log q_j\cdot\sum_{k=1;k\neq j}^{n}2c_jc_kh(q_j,q_k)}\Bigg\}\\ =0,\\ R_1(\gamma_2)\equiv\frac{-1}{2\pi\sqrt{-1}}\Bigg\{-\sum_{j=1}^{n}\frac{c_jb_j}{q_j}+\log r\cdot\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{2}(c_jb_k+b_jc_k)h_1(q_j,q_k)\\ \qquad-\sum_{j=1}^{n}\log q_j\cdot\sum_{k=1;k\neq j}^{n}(c_jb_k+b_jc_k)h(q_j,q_k)\Bigg\}\in\mathbf{R},\end{cases}$$

and the degree of g given by (4.5) *is n.*

7. *n*-noids symmetric with respect to the x_1x_2-plane

We can show the following facts about symmetry of minimal surfaces, in the same way as the condition for a minimal surface to be a double cover of a nonorientable minimal surface (cf. [13]).

Proposition 7.1. *Let X be a conformal minimal immersion into* $\mathbf{R}^3$, *defined on a Riemann surface M with the Weierstrass data* (g,η). *Then X is symmetric with respect to the* x_1x_2*-plane* (*up to parallel transformations*) *if and only if* (g,η) *satisfies the condition*

$$g\circ I=\frac{1}{\overline{g}},\quad I^*\eta=-\overline{g^2\eta} \tag{7.1}$$

for some antiholomorphic involution $I\colon M\to M$, *that is,* $I^2(z)=z$ *and* $I_z=0$.

In the case of n-noids symmetric with respect to the x_1x_2-plane, the equation for the global period problem in the previous section can be rewritten to a simpler form.

Assume that $\overline{M} = T^2 = \mathbf{C}/(\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2)$ satisfies $\omega_1 \in \mathbf{R}_+$ and $\omega_2 \in \sqrt{-1}\mathbf{R}_+$ or $\omega_2 - \omega_1 \in \sqrt{-1}\mathbf{R}_+$. If $\omega_2 \in \sqrt{-1}\mathbf{R}_+$, then $r > 1$ and $r^{1/2} > 0$. If $\omega_2 - \omega_1 \in \sqrt{-1}\mathbf{R}_+$, then $r > 1$ and $r^{1/2} < 0$. In both cases, it holds that $\overline{h(z,q)} = h(\overline{z},\overline{q})$, and $\overline{h_1(z,q)} = h_1(\overline{z},\overline{q})$.

Let X be an n-noid of genus one whose Weierstrass data (g, η) is of the form (4.5) with (4.6). The data (g, η) satisfies the condition (7.1) with $I(z) = \overline{z}$ if and only if $Q \circ I(z) = \pm\sqrt{-1}\overline{P(z)}$, that is

$$\sum_{j=1}^{n} b_j h(\overline{z}, q_j) = \pm\sqrt{-1}\sum_{j=1}^{n} \overline{c_j} h(\overline{z}, \overline{q_j}).$$

To realize such an n-noid, we may assume that $r > 1$ and $q_j,\ p_j,\ b_j,\ c_j$ $(j = 1, \ldots, n)$ satisfy

$$\text{(7.2)}\qquad \begin{cases} 1 \le |q_j| < r \quad (j = 1, \ldots, n), \\ q_j \in \mathbf{R}_+, \quad |p_j| = 1 \quad (j = 1, \ldots, N_1), \\ q_j \in \mathbf{R}_-, \quad |p_j| = 1 \quad (j = N_1 + 1, \ldots, N_1 + N_2), \\ q_j \notin \mathbf{R}, \quad \overline{q_j} = q_{j'}, \quad p_j\overline{p_{j'}} = 1 \quad (j = N_1 + N_2 + 1, \ldots, n), \\ \pm b_j = \begin{cases} \sqrt{-1}\overline{c_j} & (j = 1, \ldots, N_1 + N_2) \\ \sqrt{-1}\overline{c_{j'}} & (j = N_1 + N_2 + 1, \ldots, n), \end{cases} \\ \text{where} \quad N_1 + N_2 + 2N_3 = n, \\ j' = \begin{cases} j + N_3 & (j = N_1 + N_2 + 1, \ldots, N_1 + N_2 + N_3) \\ j - N_3 & (j = N_1 + N_2 + N_3 + 1, \ldots, n). \end{cases} \end{cases}$$

In this case, Problem 2.3 is reduced to the following:

Theorem 7.2. *There exists an n-noid $X\colon M = T^2 \setminus \{q_1, \ldots, q_n\} \to \mathbf{R}^3$ satisfying* (2.14) *and* (7.1) *with $I(z) = \overline{z}$ if and only if there exist $q_j,\ b_j,\ c_j = p_jb_j$ $(j = 1, \ldots, n)$ satisfying* (7.2) *and*

(7.3)

$$\begin{cases} \left.\begin{aligned} w_j &\equiv b_j \sum_{k=1;k\ne j}^{n} (p_j - p_k) b_k h(q_j, q_k) = a_j \\ w_j^* &\equiv -b_j \sum_{k=1;k\ne j}^{n} (\overline{p_j}p_k + 1) b_k h(q_j, q_k) = 0 \end{aligned}\right\} (j = 1, \ldots, N_1 + N_2 + N_3), \\ P_1 := -2R_1(\gamma_1) \equiv \sum_{j=1}^{n}\sum_{k=1}^{n} (p_j + p_k) b_j b_k h_1(q_j, q_k) = 0, \\ P_2 := -\pi\sqrt{-1}(R_0(\gamma_2) + \overline{R_2(\gamma_2)}) \\ \quad \equiv -\sum_{j=1}^{n} \frac{{b_j}^2}{q_j} + \log r \sum_{j=1}^{n}\sum_{k=1}^{n} b_j b_k h_1(q_j, q_k) - 2\sum_{j=1}^{n} \log|q_j| \sum_{k=1;k\ne j}^{n} b_j b_k h(q_j, q_k) = 0, \end{cases}$$

and the degree of g given by (4.5) *is n.*

Proof. By the assumption (7.2) and Lemmas 5.1 and 6.1, it holds that

$$\begin{cases} -\overline{\sum_{j=1}^{n} \frac{{c_j}^2}{q_j}} = \sum_{j=1}^{n} \frac{{b_j}^2}{q_j}, \quad -\overline{\sum_{j=1}^{n} \frac{c_j b_j}{q_j}} = \sum_{j=1}^{n} \frac{c_j b_j}{q_j}, \\ \overline{R_2(\gamma_1)} = -R_0(\gamma_1), \quad \overline{R_1(\gamma_1)} = -R_1(\gamma_1), \\ \overline{R_{2j}} = -R_{0j}, \quad \overline{R_{1j}} = -R_{1j} \quad (j = 1, \dots, N_1 + N_2), \\ \overline{R_{2j}} = -R_{0j'}, \quad \overline{R_{1j}} = -R_{1j'} \quad (j = N_1 + N_2 + 1, \dots, n), \end{cases}$$

from which, and from Lemma 6.2, it also holds that

$$R_1(\gamma_2) - \overline{R_1(\gamma_2)} = \frac{-1}{2\pi\sqrt{-1}} \sum_{j=1}^{n} (\log q_j \cdot R_{1j} + \overline{\log q_j} \cdot \overline{R_{1j}}).$$

Now, if $w_j = a_j \in \mathbf{R}$ and $w_j^* = 0$ hold for $j = 1, \dots, N_1 + N_2 + N_3$, then they also hold for $j = N_1 + N_2 + N_3 + 1, \dots, n$, and hence $R_{1j} = \overline{R_{1j}}$ holds for $j = 1, \dots, n$. Therefore we have

$$\begin{cases} R_{1j} = 0 & (j = 1, \dots, N_1 + N_2), \\ R_{1j} + R_{1j'} = 0 & (j = N_1 + N_2 + 1, \dots, n). \end{cases}$$

Since $\log|q_j| = \log|q_{j'}|$ $(j = N_1 + N_2 + 1, \dots, n)$, we get

$$R_1(\gamma_2) - \overline{R_1(\gamma_2)} = \frac{-1}{2\pi\sqrt{-1}} \sum_{j=N_1+N_2+1}^{n} \log|q_j| \cdot (R_{1j} + R_{1j'}) = 0. \qquad \square$$

The data (g, η) satisfies the condition (7.1) with $I(z) = 1/\bar{z}$ if and only if $Q \circ I(z) = \pm \bar{z}\overline{P(z)}$, that is

$$\sum_{j=1}^{n} \frac{b_j}{q_j} h\left(\bar{z}, \frac{1}{q_j}\right) = \pm \sum_{j=1}^{n} \overline{c_j} h(\bar{z}, \overline{q_j}).$$

To realize such an n-noid, we may assume that $r > 1$ and q_j, p_j, b_j, c_j $(j = 1, \dots, n)$ satisfy

(7.4)

$$\begin{cases} 1 \le |q_j| < r \quad (j = 1, \dots, n), \\ |q_j| = 1, \quad |p_j| = 1 \quad (j = 1, \dots, N_1), \\ |q_j| = |r^{1/2}|, \quad |p_j| = 1 \quad (j = N_1 + 1, \dots, N_1 + N_2), \\ |q_j| \ne 1, \quad |q_j| \ne |r^{1/2}|, \quad q_j \overline{q_{j'}} = r, \quad p_j \overline{p_{j'}} = 1 \quad (j = N_1 + N_2 + 1, \dots, n), \\ \pm \frac{b_j}{q_j} = \begin{cases} \overline{c_j} & (j = 1, \dots, N_1), \\ \overline{c_{j'}} r^{-1/2} & (j = N_1 + 1, \dots, n), \end{cases} \\ \text{where} \quad N_1 + N_2 + 2N_3 = n, \\ j' = \begin{cases} j & (j = N_1 + 1, \dots, N_1 + N_2), \\ j + N_3 & (j = N_1 + N_2 + 1, \dots, N_1 + N_2 + N_3), \\ j - N_3 & (j = N_1 + N_2 + N_3 + 1, \dots, n). \end{cases} \end{cases}$$

In this case, Problem 2.3 is reduced to the following:

Theorem 7.3. *There exists an n-noid* $X\colon M = T^2 \setminus \{q_1, \ldots, q_n\} \to \mathbf{R}^3$ *satisfying* (2.14) *and* (7.1) *with* $I(z) = 1/\bar{z}$ *if and only if there exist* q_j, b_j, $c_j = p_j b_j$ $(j = 1, \ldots, n)$ *satisfying* (7.4) *and*

(7.5)

$$\begin{cases} \left. \begin{aligned} w_j &\equiv b_j \sum_{k=1;k\neq j}^{n} (p_j - p_k) b_k h(q_j, q_k) = a_j \\ w_j^* &\equiv -b_j \sum_{k=1;k\neq j}^{n} (\overline{p_j} p_k + 1) b_k h(q_j, q_k) = 0 \end{aligned} \right\} \quad (j = 1, \ldots, N_1 + N_2 + N_3), \\ P_1' := -\dfrac{1}{2}(R_0(\gamma_1) + \overline{R_2(\gamma_1)}) \equiv \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{n} b_j b_k h_1(q_j, q_k) + \sum_{j=1}^{N_1}\sum_{k=N_1+1}^{n} b_j b_k h(q_j, q_k) = 0, \\ P_2' := -\pi\sqrt{-1}(R_1(\gamma_2) - \overline{R_1(\gamma_2)}) \quad (= -2\pi\sqrt{-1} R_1(\gamma_2)) \\ \qquad \equiv \log r \left\{ \dfrac{1}{2} \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{n} (p_j + p_k) b_j b_k h_1(q_j, q_k) \right. \\ \qquad\qquad \left. + \displaystyle\sum_{j=N_1+N_2+1}^{N_1+N_2+N_3} \sum_{k=1;k\neq j}^{n} (p_j + p_k) b_j b_k h(q_j, q_k) \right\} \\ \qquad - \displaystyle\sum_{j=1}^{n} \frac{p_j {b_j}^2}{q_j} - 2 \sum_{j=N_1+N_2+1}^{N_1+N_2+N_3} \log|q_j| \sum_{k=1;k\neq j}^{n} (p_j + p_k) b_j b_k h(q_j, q_k) = 0, \end{cases}$$

and the degree of g *given by* (4.5) *is* n.

Proof. By the assumption (7.4) and Lemmas 3.2, 3.4, 5.1 and 6.1, it holds that

$$\begin{cases} \overline{\displaystyle\sum_{j=1}^{n} \frac{{c_j}^2}{q_j}} = \displaystyle\sum_{j=1}^{n} \frac{{b_j}^2}{q_j}, \quad \overline{\displaystyle\sum_{j=1}^{n} \frac{c_j b_j}{q_j}} = \displaystyle\sum_{j=1}^{n} \frac{c_j b_j}{q_j}, \\ \overline{R_2(\gamma_1)} = R_0(\gamma_1) - \displaystyle\sum_{j=N_1+1}^{n} R_{0j}, \quad \overline{R_1(\gamma_1)} = R_1(\gamma_1) - \displaystyle\sum_{j=N_1+1}^{n} R_{1j}, \\ \overline{R_{2j}} = -R_{0j}, \quad \overline{R_{1j}} = -R_{1j} \quad (j = 1, \ldots, N_1 + N_2), \\ \overline{R_{2j}} = -R_{0j'}, \quad \overline{R_{1j}} = -R_{1j'} \quad (j = N_1 + N_2 + 1, \ldots, n), \end{cases}$$

from which, and from Lemma 6.2, it also holds that

$$R_0(\gamma_2) + \overline{R_2(\gamma_2)} = \frac{-1}{2\pi\sqrt{-1}} \left\{ -\log r \sum_{j=N_1+1}^{n} R_{0j} + \sum_{j=1}^{n} (\log q_j \cdot R_{0j} - \overline{\log q_j} \cdot \overline{R_{2j}}) \right\}.$$

Now, if $w_j = a_j \in \mathbf{R}$ and $w_j^* = 0$ hold for $j = 1, \dots, N_1 + N_2 + N_3$, then they also hold for $j = N_1 + N_2 + N_3 + 1, \dots, n$, and hence $R_{0j} + \overline{R_{2j}} = 0$ and $R_{1j} = \overline{R_{1j}}$ hold for $j = 1, \dots, n$. Therefore we have

$$\begin{cases} R_{1j} = 0 & (j = 1, \dots, N_1 + N_2), \\ R_{0j} = R_{0j'} & (j = N_1 + N_2 + 1, \dots, n). \end{cases}$$

Since

$$\log|q_j| = \begin{cases} 0 & (j = 1, \dots, N_1), \\ \dfrac{1}{2}\log r & (j = N_1 + 1, \dots, N_1 + N_2), \end{cases}$$

$$\log|q_j| + \log|q_{j'}| = \log r \quad (j = N_1 + N_2 + 1, \dots, n),$$

we get

$$R_0(\gamma_2) + \overline{R_2(\gamma_2)} = \frac{-1}{2\pi\sqrt{-1}} \sum_{j=N_1+N_2+1}^{n} \log|q_j| \cdot (R_{0j} - R_{0j'}) = 0.$$

On the other hand, we also get

$$R_1(\gamma_1) - \overline{R_1(\gamma_1)} = \sum_{j=N_1+1}^{n} R_{1j} = -\sum_{j=1}^{N_1} R_{1j} = 0. \qquad \square$$

8. Examples 1

Jorge–Meeks type n-noids of genus one invariant under the action of the dihedral group D_n ($n \geq 3$) were constructed by Berglund–Rossman [1]. In this section, we construct Jorge–Meeks type $2N$-noids of genus one with alternating sizes of ends, and $2N$-noid fences, by applying Theorem 7.3. In particular, we construct examples such that the ratio of the two weights of the alternating sizes of ends is negative. Throughout this section, we use the notation $\zeta_{2N} := e^{2\pi\sqrt{-1}/2N}$.

EXAMPLE 8.1. Let N be an integer larger than 1, and set $n := 2N$. Consider the following flux data:

$$\begin{cases} p_j := \zeta_{2N}{}^{j-1} & (j = 1, \dots, 2N), \\ a_j := \begin{cases} a \in \mathbf{R} \setminus \{0\} & (j\colon \text{even}), \\ \tilde{a} \in \mathbf{R} & (j\colon \text{odd}). \end{cases} \end{cases}$$

To find a surface realizing these data, it is natural to assume $r > 1$ and set

$$\begin{cases} q_j := p_j = \zeta_{2N}{}^{j-1} & (j = 1, \dots, 2N), \\ b_j := \begin{cases} b \neq 0 & (j\colon \text{even}), \\ \tilde{b} & (j\colon \text{odd}). \end{cases} \end{cases}$$

For j even, by using (3.2) and Lemma 3.1 (ii), we have

$$
\begin{aligned}
w_j^* &= -b^2 \sum_{k=2:\ \text{even};\ k\neq j}^{2N} (\overline{{\zeta_{2N}}^{j-1}}{\zeta_{2N}}^{k-1}+1)h({\zeta_{2N}}^{j-1},{\zeta_{2N}}^{k-1}) \\
&\quad - b\tilde{b} \sum_{k=1:\ \text{odd}}^{2N-1} (\overline{{\zeta_{2N}}^{j-1}}{\zeta_{2N}}^{k-1}+1)h({\zeta_{2N}}^{j-1},{\zeta_{2N}}^{k-1}) \\
&= -{\zeta_{2N}}^{1-j}\left\{b^2 \sum_{k=2:\ \text{even};\ k\neq j}^{2N} (1+{\zeta_{2N}}^{j-k})h({\zeta_{2N}}^{j-k}) \right. \\
&\qquad\qquad \left. + b\tilde{b} \sum_{k=1:\ \text{odd}}^{2N-1} (1+{\zeta_{2N}}^{j-k})h({\zeta_{2N}}^{j-k})\right\} \\
&= -{\zeta_{2N}}^{1-j}\left\{b^2 \sum_{k=2:\ \text{even}}^{2N-2} ({\zeta_{2N}}^{k}+1)h({\zeta_{2N}}^{k}) + b\tilde{b} \sum_{k=1:\ \text{odd}}^{2N-1} ({\zeta_{2N}}^{k}+1)h({\zeta_{2N}}^{k})\right\} \\
&= -{\zeta_{2N}}^{1-j}\left\{b^2 \sum_{k=2:\ \text{even}}^{2N-2} (-h({\zeta_{2N}}^{-k})+h({\zeta_{2N}}^{k})) + b\tilde{b} \sum_{k=1:\ \text{odd}}^{2N-1} (-h({\zeta_{2N}}^{-k})+h({\zeta_{2N}}^{k}))\right\} \\
&= 0.
\end{aligned}
$$

In the same way, for j odd, we also have $w_j^* = 0$. For the loop γ_1, we have

$$
\begin{aligned}
P_1' &= b^2 \sum_{j=2:\ \text{even}}^{2N} \sum_{k=2:\ \text{even}}^{2N} h_1({\zeta_{2N}}^{j-1},{\zeta_{2N}}^{k-1}) + b\tilde{b} \sum_{j=2:\ \text{even}}^{2N} \sum_{k=1:\ \text{odd}}^{2N-1} h_1({\zeta_{2N}}^{j-1},{\zeta_{2N}}^{k-1}) \\
&\quad + \tilde{b}b \sum_{j=1:\ \text{odd}}^{2N-1} \sum_{k=2:\ \text{even}}^{2N} h_1({\zeta_{2N}}^{j-1},{\zeta_{2N}}^{k-1}) + \tilde{b}^2 \sum_{j=1:\ \text{odd}}^{2N-1} \sum_{k=1:\ \text{odd}}^{2N-1} h_1({\zeta_{2N}}^{j-1},{\zeta_{2N}}^{k-1}) \\
&= b^2 \sum_{j=2:\ \text{even}}^{2N} \sum_{k=2:\ \text{even}}^{2N} {\zeta_{2N}}^{1-k}h_1({\zeta_{2N}}^{j-k}) + b\tilde{b} \sum_{j=2:\ \text{even}}^{2N} \sum_{k=1:\ \text{odd}}^{2N-1} {\zeta_{2N}}^{1-k}h_1({\zeta_{2N}}^{j-k}) \\
&\quad + \tilde{b}b \sum_{j=1:\ \text{odd}}^{2N-1} \sum_{k=2:\ \text{even}}^{2N} {\zeta_{2N}}^{1-k}h_1({\zeta_{2N}}^{j-k}) + \tilde{b}^2 \sum_{j=1:\ \text{odd}}^{2N-1} \sum_{k=1:\ \text{odd}}^{2N-1} {\zeta_{2N}}^{1-k}h_1({\zeta_{2N}}^{j-k}) \\
&= b^2 \sum_{k=2:\ \text{even}}^{2N} {\zeta_{2N}}^{1-k} \sum_{j=2:\ \text{even}}^{2N} h_1({\zeta_{2N}}^{j}) + b\tilde{b} \sum_{k=1:\ \text{odd}}^{2N-1} {\zeta_{2N}}^{1-k} \sum_{j=2:\ \text{odd}}^{2N} h_1({\zeta_{2N}}^{j}) \\
&\quad + \tilde{b}b \sum_{k=2:\ \text{even}}^{2N} {\zeta_{2N}}^{1-k} \sum_{j=1:\ \text{odd}}^{2N-1} h_1({\zeta_{2N}}^{j}) + \tilde{b}^2 \sum_{k=1:\ \text{odd}}^{2N-1} {\zeta_{2N}}^{1-k} \sum_{j=1:\ \text{even}}^{2N-1} h_1({\zeta_{2N}}^{j}) \\
&= b^2\cdot 0 + b\tilde{b}\cdot 0 + \tilde{b}b\cdot 0 + \tilde{b}^2\cdot 0 = 0.
\end{aligned}
$$

Hence we have only to consider the period problem for w_j and P_2'.

For j even, by using (3.2) and Lemma 3.1 (ii) again, we have

$$\begin{aligned} w_j &= b^2 \sum_{k=2:\text{ even};\, k \neq j}^{2N} ({\zeta_{2N}}^{j-1} - {\zeta_{2N}}^{k-1}) h({\zeta_{2N}}^{j-1}, {\zeta_{2N}}^{k-1}) \\ &\quad + b\tilde{b} \sum_{k=1:\text{ odd}}^{2N-1} ({\zeta_{2N}}^{j-1} - {\zeta_{2N}}^{k-1}) h({\zeta_{2N}}^{j-1}, {\zeta_{2N}}^{k-1}) \\ &= b^2 \sum_{k=2:\text{ even};\, k \neq j}^{2N} ({\zeta_{2N}}^{j-k} - 1) h({\zeta_{2N}}^{j-k}) + b\tilde{b} \sum_{k=1:\text{ odd}}^{2N-1} ({\zeta_{2N}}^{j-k} - 1) h({\zeta_{2N}}^{j-k}) \\ &= b^2 \sum_{k=2:\text{ even}}^{2N-2} ({\zeta_{2N}}^{k} - 1) h({\zeta_{2N}}^{k}) + b\tilde{b} \sum_{k=1:\text{ odd}}^{2N-1} ({\zeta_{2N}}^{k} - 1) h({\zeta_{2N}}^{k}) \\ &= b^2 \sum_{k=2:\text{ even}}^{2N-2} (-h({\zeta_{2N}}^{-k}) - h({\zeta_{2N}}^{k})) + b\tilde{b} \sum_{k=1:\text{ odd}}^{2N-1} (-h({\zeta_{2N}}^{-k}) - h({\zeta_{2N}}^{k})) \\ &= -2\left(b^2 \sum_{k=2:\text{ even}}^{2N-2} h({\zeta_{2N}}^{k}) + b\tilde{b} \sum_{k=1:\text{ odd}}^{2N-1} h({\zeta_{2N}}^{k}) \right). \end{aligned}$$

In the same way, for j odd, we also have

$$w_j = -2\left(\tilde{b}^2 \sum_{k=2:\text{ even}}^{2N-2} h({\zeta_{2N}}^{k}) + \tilde{b}b \sum_{k=1:\text{ odd}}^{2N-1} h({\zeta_{2N}}^{k}) \right).$$

Set $\alpha := \tilde{a}/a$, $\beta := \tilde{b}/b$, and

(8.1)
$$C_N := \frac{\sum_{k=1:\text{ odd}}^{2N-1} h({\zeta_{2N}}^{k})}{\sum_{k=2:\text{ even}}^{2N-2} h({\zeta_{2N}}^{k})}.$$

If $w_j = a$ (j: even), $\tilde{a}$ (j: odd), then it holds that

(8.2)
$$\alpha = \frac{\beta^2 + \beta C_N}{1 + \beta C_N}.$$

and hence

$$\beta^2 + (1-\alpha) C_N \beta - \alpha = 0.$$

Solving this equation, we get a solution

$$\beta = \frac{1}{2}\left\{ -(1-\alpha) C_N + \sqrt{(1-\alpha)^2 {C_N}^2 + 4\alpha} \right\}.$$

In particular, if $\alpha > 0$, then we have a positive solution β. Now, we can choose b so that $w_j = a$ (j: even), $\tilde{a}$ (j: odd).

For the loop γ_2, we have

$$\begin{aligned}
P_2' = {} & b^2\left(\log r \sum_{j=2:\,\text{even}}^{2N} \sum_{k=2:\,\text{even}}^{2N} {\zeta_{2N}}^{k-1} h_1({\zeta_{2N}}^{j-1}, {\zeta_{2N}}^{k-1}) - N\right) \\
& + b\tilde{b}\log r\left(\sum_{j=2:\,\text{even}}^{2N} \sum_{k=1:\,\text{odd}}^{2N-1} {\zeta_{2N}}^{k-1} h_1({\zeta_{2N}}^{j-1}, {\zeta_{2N}}^{k-1}) \right.\\
& \qquad \left. + \sum_{j=1:\,\text{odd}}^{2N-1} \sum_{k=2:\,\text{even}}^{2N} {\zeta_{2N}}^{k-1} h_1({\zeta_{2N}}^{j-1}, {\zeta_{2N}}^{k-1})\right) \\
& + \tilde{b}^2\left(\log r \sum_{j=1:\,\text{odd}}^{2N-1} \sum_{k=1:\,\text{odd}}^{2N-1} {\zeta_{2N}}^{k-1} h_1({\zeta_{2N}}^{j-1}, {\zeta_{2N}}^{k-1}) - N\right) \\
= {} & b^2\left(\log r \sum_{j=2:\,\text{even}}^{2N} \sum_{k=2:\,\text{even}}^{2N} h_1({\zeta_{2N}}^{j-k}) - N\right) \\
& + \tilde{b}^2\left(\log r \sum_{j=1:\,\text{odd}}^{2N-1} \sum_{k=1:\,\text{odd}}^{2N-1} h_1({\zeta_{2N}}^{j-k}) - N\right) \\
& + b\tilde{b}\log r\left(\sum_{j=2:\,\text{even}}^{2N} \sum_{k=1:\,\text{odd}}^{2N-1} h_1({\zeta_{2N}}^{j-k}) + \sum_{j=1:\,\text{odd}}^{2N-1} \sum_{k=2:\,\text{even}}^{2N} h_1({\zeta_{2N}}^{j-k})\right) \\
= {} & (b^2+\tilde{b}^2)\left(\log r \cdot N \sum_{k=2:\,\text{even}}^{2N} h_1({\zeta_{2N}}^{k}) - N\right) + b\tilde{b}\log r \cdot 2N \sum_{k=1:\,\text{odd}}^{2N-1} h_1({\zeta_{2N}}^{k}) \\
= {} & N(b^2+\tilde{b}^2)\left\{\log r\left(\sum_{k=2:\,\text{even}}^{2N} h_1({\zeta_{2N}}^{k}) + \frac{2b\tilde{b}}{b^2+\tilde{b}^2}\sum_{k=1:\,\text{odd}}^{2N-1} h_1({\zeta_{2N}}^{k})\right) - 1\right\} \\
= {} & Nb^2(1+\beta^2)\left\{\log r\left(\sum_{k=2:\,\text{even}}^{2N} h_1({\zeta_{2N}}^{k}) + \frac{2\beta}{1+\beta^2}\sum_{k=1:\,\text{odd}}^{2N-1} h_1({\zeta_{2N}}^{k})\right) - 1\right\}.
\end{aligned}$$

Note here that, for any $m \in \mathbf{Z}$,

$$\sum_{k=2:\,\text{even}}^{2N} {\zeta_{2N}}^{mk} = \begin{cases} 0 & \text{if} \quad {\zeta_{2N}}^{2m} \neq 1, \\ N & \text{if} \quad {\zeta_{2N}}^{2m} = 1 \quad \text{i.e.} \quad m = lN \quad \text{for some} \quad l \in \mathbf{Z}, \end{cases}$$

$$\sum_{k=1:\,\text{odd}}^{2N-1} {\zeta_{2N}}^{mk} = \begin{cases} 0 & \text{if} \quad {\zeta_{2N}}^{m} \neq \pm 1, \\ -N & \text{if} \quad {\zeta_{2N}}^{m} = -1 \quad \text{i.e.} \quad m = (2l-1)N \quad \text{for some} \quad l \in \mathbf{Z}, \\ N & \text{if} \quad {\zeta_{2N}}^{m} = 1 \quad \text{i.e.} \quad m = 2lN \quad \text{for some} \quad l \in \mathbf{Z}. \end{cases}$$

By Lemma 3.5 and $h_0(1) = 0$, it holds that

$$\begin{aligned}\sum_{k=2:\text{ even}}^{2N-2} h({\zeta_{2N}}^k) &= \sum_{k=2:\text{ even}}^{2N-2} \frac{1}{{\zeta_{2N}}^k - 1} + \sum_{k=2:\text{ even}}^{2N} h_0({\zeta_{2N}}^k) - h_0(1) \\ &= \sum_{k=2:\text{ even}}^{2N-2} \frac{1}{{\zeta_{2N}}^k - 1} - \sum_{k=2:\text{ even}}^{2N} {\zeta_{2N}}^{-k} \sum_{m=1}^{+\infty} ({\zeta_{2N}}^{km} - {\zeta_{2N}}^{k(1-m)}) \frac{1}{r^{(2m-1)/2} - 1} \\ &= -\frac{N-1}{2} - \sum_{m=1}^{+\infty} \left(\sum_{k=2:\text{ even}}^{2N} {\zeta_{2N}}^{(m-1)k} - \sum_{k=2:\text{ even}}^{2N} {\zeta_{2N}}^{-mk} \right) \frac{1}{r^{(2m-1)/2} - 1} \\ &= -\frac{N-1}{2} - N \left(\sum_{l=0}^{+\infty} \frac{1}{r^{\{2(lN+1)-1\}/2} - 1} - \sum_{l=1}^{+\infty} \frac{1}{r^{(2lN-1)/2} - 1} \right) \\ &= -\frac{N-1}{2} - N \sum_{l=1}^{+\infty} \left(\frac{1}{r^{\{2lN-(2N-1)\}/2} - 1} - \frac{1}{r^{(2lN-1)/2} - 1} \right) \\ &= -\frac{N-1}{2} - N \sum_{l=1}^{+\infty} \frac{r^{\{2lN-(2N-1)\}/2}(r^{N-1} - 1)}{(r^{\{2lN-(2N-1)\}/2} - 1)(r^{(2lN-1)/2} - 1)}.\end{aligned}$$

By Lemmas 3.5 and 3.6, it also holds that

$$\begin{aligned}\sum_{k=1:\text{ odd}}^{2N-1} h({\zeta_{2N}}^k) &= \sum_{k=1:\text{ odd}}^{2N-1} \frac{1}{{\zeta_{2N}}^k - 1} - \sum_{k=1:\text{ odd}}^{2N-1} {\zeta_{2N}}^{-k} \sum_{m=1}^{+\infty} ({\zeta_{2N}}^{km} - {\zeta_{2N}}^{k(1-m)}) \frac{1}{r^{(2m-1)/2} - 1} \\ &= -\frac{N}{2} - \sum_{m=1}^{+\infty} \left(\sum_{k=1:\text{ odd}}^{2N-1} {\zeta_{2N}}^{(m-1)k} - \sum_{k=1:\text{ odd}}^{2N-1} {\zeta_{2N}}^{-mk} \right) \frac{1}{r^{(2m-1)/2} - 1} \\ &= -\frac{N}{2} - N \left(\sum_{l=0}^{+\infty} \frac{1}{r^{\{2(2lN+1)-1\}/2} - 1} - \sum_{l=1}^{+\infty} \frac{1}{r^{[2\{(2l-1)N+1\}-1]/2} - 1} \right. \\ &\qquad \left. + \sum_{l=1}^{+\infty} \frac{1}{r^{\{2(2l-1)N-1\}/2} - 1} - \sum_{l=1}^{+\infty} \frac{1}{r^{(2\cdot 2lN-1)/2} - 1} \right) \\ &= -\frac{N}{2} - N \sum_{l=1}^{+\infty} \left(\frac{1}{r^{\{4lN-(4N-1)\}/2} - 1} - \frac{1}{r^{\{4lN-(2N-1)\}/2} - 1} \right. \\ &\qquad \left. + \frac{1}{r^{\{4lN-(2N+1)\}/2} - 1} - \frac{1}{r^{(4lN-1)/2} - 1} \right) \\ &= -\frac{N}{2} - N \sum_{l=1}^{+\infty} \left\{ \frac{r^{\{4lN-(4N-1)\}/2}(r^N - 1)}{(r^{\{4lN-(4N-1)\}/2} - 1)(r^{\{4lN-(2N-1)\}/2} - 1)} \right. \\ &\qquad \left. + \frac{r^{\{4lN-(2N+1)\}/2}(r^N - 1)}{(r^{\{4lN-(2N+1)\}/2} - 1)(r^{(4lN-1)/2} - 1)} \right\},\end{aligned}$$

$$\sum_{k=2:\text{ even}}^{2N} h_1({\zeta_{2N}}^k) = -\sum_{k=2:\text{ even}}^{2N} {\zeta_{2N}}^{-k} \sum_{m=1}^{+\infty} ({\zeta_{2N}}^{km} + {\zeta_{2N}}^{k(1-m)}) \frac{r^{(2m-1)/2}}{(r^{(2m-1)/2}-1)^2}$$

$$= -\sum_{m=1}^{+\infty} \left(\sum_{k=2:\text{ even}}^{2N} {\zeta_{2N}}^{(m-1)k} + \sum_{k=2:\text{ even}}^{2N} {\zeta_{2N}}^{-mk} \right) \frac{r^{(2m-1)/2}}{(r^{(2m-1)/2}-1)^2}$$

$$= -N\left(\sum_{l=0}^{+\infty} \frac{r^{\{2(lN+1)-1\}/2}}{(r^{\{2(lN+1)-1\}/2}-1)^2} + \sum_{l=1}^{+\infty} \frac{r^{(2lN-1)/2}}{(r^{(2lN-1)/2}-1)^2} \right)$$

$$= -N\left(\sum_{l=0}^{+\infty} \frac{r^{(2lN+1)/2}}{(r^{(2lN+1)/2}-1)^2} + \sum_{l=1}^{+\infty} \frac{r^{(2lN-1)/2}}{(r^{(2lN-1)/2}-1)^2} \right),$$

$$\sum_{k=1:\text{ odd}}^{2N-1} h_1({\zeta_{2N}}^k) = -\sum_{k=1:\text{ odd}}^{2N-1} {\zeta_{2N}}^{-k} \sum_{m=1}^{+\infty} ({\zeta_{2N}}^{km} + {\zeta_{2N}}^{k(1-m)}) \frac{r^{(2m-1)/2}}{(r^{(2m-1)/2}-1)^2}$$

$$= -\sum_{m=1}^{+\infty} \left(\sum_{k=1:\text{ odd}}^{2N-1} {\zeta_{2N}}^{(m-1)k} + \sum_{k=1:\text{ odd}}^{2N-1} {\zeta_{2N}}^{-mk} \right) \frac{r^{(2m-1)/2}}{(r^{(2m-1)/2}-1)^2}$$

$$= -N\left(\sum_{l=0}^{+\infty} \frac{r^{\{2(2lN+1)-1\}/2}}{(r^{\{2(2lN+1)-1\}/2}-1)^2} - \sum_{l=1}^{+\infty} \frac{r^{[2\{(2l-1)N+1\}-1]/2}}{(r^{[2\{(2l-1)N+1\}-1]/2}-1)^2} \right.$$
$$\left. - \sum_{l=1}^{+\infty} \frac{r^{\{2(2l-1)N-1\}/2}}{(r^{\{2(2l-1)N-1\}/2}-1)^2} + \sum_{l=1}^{+\infty} \frac{r^{(2\cdot 2lN-1)/2}}{(r^{(2\cdot 2lN-1)/2}-1)^2} \right)$$

$$= -N\left(\sum_{l=0}^{+\infty} \frac{r^{(4lN+1)/2}}{(r^{(4lN+1)/2}-1)^2} - \sum_{l=1}^{+\infty} \frac{r^{\{4lN-(2N-1)\}/2}}{(r^{\{4lN-(2N-1)\}/2}-1)^2} \right.$$
$$\left. - \sum_{l=1}^{+\infty} \frac{r^{\{4lN-(2N+1)\}/2}}{(r^{\{4lN-(2N+1)\}/2}-1)^2} + \sum_{l=1}^{+\infty} \frac{r^{(4lN-1)/2}}{(r^{(4lN-1)/2}-1)^2} \right).$$

Combining these equalities, we have

$$\frac{1}{Nb^2(1+\beta^2)} P_2'$$
$$= -\left[N \log r \left\{ \frac{(1+\beta)^2}{1+\beta^2} \left(\sum_{l=0}^{+\infty} \frac{r^{(4lN+1)/2}}{(r^{(4lN+1)/2}-1)^2} + \sum_{l=1}^{+\infty} \frac{r^{(4lN-1)/2}}{(r^{(4lN-1)/2}-1)^2} \right) \right. \right.$$
$$\left. \left. + \frac{(1-\beta)^2}{1+\beta^2} \left(\sum_{l=1}^{+\infty} \frac{r^{\{4lN-(2N-1)\}/2}}{(r^{\{4lN-(2N-1)\}/2}-1)^2} + \sum_{l=1}^{+\infty} \frac{r^{\{4lN-(2N+1)\}/2}}{(r^{\{4lN-(2N+1)\}/2}-1)^2} \right) \right\} + 1 \right].$$

Now, set $\rho := |r^{1/2}|$.

First, let us consider the case that $\rho = r^{1/2} > 0$. In this case, it holds that

$$\begin{aligned}
&\frac{1}{Nb^2(1+\beta^2)} P_2' \\
&= -\Bigg[2N \log\rho \Bigg\{ \frac{(1+\beta)^2}{1+\beta^2} \Bigg(\sum_{l=0}^{+\infty} \frac{\rho^{4lN+1}}{(\rho^{4lN+1}-1)^2} + \sum_{l=1}^{+\infty} \frac{\rho^{4lN-1}}{(\rho^{4lN-1}-1)^2} \Bigg) \\
&\qquad + \frac{(1-\beta)^2}{1+\beta^2} \Bigg(\sum_{l=1}^{+\infty} \frac{\rho^{4lN-(2N-1)}}{(\rho^{4lN-(2N-1)}-1)^2} + \sum_{l=1}^{+\infty} \frac{\rho^{4lN-(2N+1)}}{(\rho^{4lN-(2N+1)}-1)^2} \Bigg) \Bigg\} + 1 \Bigg] \\
&< 0,
\end{aligned}$$

and hence, we cannot find a well-defined $2N$-noid of genus one satisfying the given flux data. Each of these data realizes a fence of Jorge–Meeks type $2N$-noids.

Secondly, let us consider the case that $-\rho = r^{1/2} < 0$. In this case, it holds that

$$\begin{aligned}
&\frac{1}{Nb^2(1+\beta^2)} P_2' \\
&= 2N \log\rho \Bigg\{ \frac{(1+\beta)^2}{1+\beta^2} \Bigg(\sum_{l=0}^{+\infty} \frac{\rho^{4lN+1}}{(\rho^{4lN+1}+1)^2} + \sum_{l=1}^{+\infty} \frac{\rho^{4lN-1}}{(\rho^{4lN-1}+1)^2} \Bigg) \\
&\qquad + \frac{(1-\beta)^2}{1+\beta^2} \Bigg(\sum_{l=1}^{+\infty} \frac{\rho^{4lN-(2N-1)}}{(\rho^{4lN-(2N-1)}+1)^2} + \sum_{l=1}^{+\infty} \frac{\rho^{4lN-(2N+1)}}{(\rho^{4lN-(2N+1)}+1)^2} \Bigg) \Bigg\} - 1.
\end{aligned}$$

For any $\rho \geq 2^{1/4N}$, since $1/(1-\rho^{-4N}) \leq 2$, it holds that

$$\begin{aligned}
&\frac{1}{Nb^2(1+\beta^2)} P_2' \\
&< 2N \log\rho \Bigg\{ \frac{(1+\beta)^2}{1+\beta^2} \Bigg(\sum_{l=0}^{+\infty} \frac{1}{\rho^{4lN+1}} + \sum_{l=1}^{+\infty} \frac{1}{\rho^{4lN-1}} \Bigg) \\
&\qquad + \frac{(1-\beta)^2}{1+\beta^2} \Bigg(\sum_{l=1}^{+\infty} \frac{1}{\rho^{4lN-(2N-1)}} + \sum_{l=1}^{+\infty} \frac{1}{\rho^{4lN-(2N+1)}} \Bigg) \Bigg\} - 1 \\
&= 2N \log\rho \Bigg\{ \frac{(1+\beta)^2}{1+\beta^2} \Bigg(\frac{1}{\rho} \frac{1}{1-\rho^{-4N}} + \frac{1}{\rho^{4N-1}} \frac{1}{1-\rho^{-4N}} \Bigg) \\
&\qquad + \frac{(1-\beta)^2}{1+\beta^2} \Bigg(\frac{1}{\rho^{2N+1}} \frac{1}{1-\rho^{-4N}} + \frac{1}{\rho^{2N-1}} \frac{1}{1-\rho^{-4N}} \Bigg) \Bigg\} - 1 \\
&\leq 2N \log\rho \Bigg\{ \frac{(1+\beta)^2}{1+\beta^2} \Bigg(\frac{2}{\rho} + \frac{2}{\rho^{4N-1}} \Bigg) + \frac{(1-\beta)^2}{1+\beta^2} \Bigg(\frac{2}{\rho^{2N+1}} + \frac{2}{\rho^{2N-1}} \Bigg) \Bigg\} - 1 \\
&\leq 2N \log\rho \Bigg\{ \frac{(1+\beta)^2}{1+\beta^2} \frac{4}{\rho} + \frac{(1-\beta)^2}{1+\beta^2} \frac{4}{\rho} \Bigg\} - 1 = 2N \log\rho \cdot \frac{8}{\rho} - 1.
\end{aligned}$$

Hence, for any $\rho > 512N^2$, we have

$$\frac{1}{Nb^2(1+\beta^2)}P_2' < 2N\log\rho\cdot\frac{8}{\rho} - 1 < 2N\sqrt{2\rho}\frac{8}{\rho} - 1 < 0. \tag{8.3}$$

Set $c_N := (16N\log 2)/25$. Since $c_N > 1$ for $N \geq 3$, it also holds that

$$\left.\frac{1}{Nb^2(1+\beta^2)}P_2'\right|_{\rho=4} > 2N\log 4\frac{(1+\beta)^2}{1+\beta^2}\frac{4}{(4+1)^2} - 1 = c_N\frac{(1+\beta)^2}{1+\beta^2} - 1 > 0 \tag{8.4}$$

if $\beta > \beta_N := (-c_N + \sqrt{2c_N - 1})/(c_N - 1)$. For example, β_3 is close to $-1/8$. For any $N \geq 3$, $\beta_N < 0$.

Regard C_N defined in (8.1) as a function of ρ. Now, since

$$\sum_{k=2:\text{ even}}^{2N-2} h({\zeta_{2N}}^k) = -N\left\{\frac{1}{2} - \frac{1}{2N} + \sum_{m=1}^{+\infty}\left(-\frac{1}{\rho^{2N(m-1)+1}+1} + \frac{1}{\rho^{2Nm-1}+1}\right)\right\},$$

$$\sum_{k=1:\text{ odd}}^{2N-1} h({\zeta_{2N}}^k) = -N\left\{\frac{1}{2} + \sum_{m=1}^{+\infty}(-1)^m\left(\frac{1}{\rho^{2N(m-1)+1}+1} + \frac{1}{\rho^{2Nm-1}+1}\right)\right\}$$

(cf. Lemma 3.5), we have

$$\frac{1}{2} - \frac{1}{2N} - \frac{1}{\rho+1} < -\frac{1}{N}\sum_{k=2:\text{ even}}^{2N-2} h({\zeta_{2N}}^k) < \frac{1}{2} - \frac{1}{2N} - \frac{1}{\rho+1} + \frac{1}{\rho^{2N-1}+1},$$

$$-\frac{1}{N}\sum_{k=1:\text{ odd}}^{2N-1} h({\zeta_{2N}}^k) > \frac{1}{2} - \frac{1}{\rho+1} - \frac{1}{\rho^{2N-1}+1}.$$

If $N \geq 3$ and $\rho > 2$, then it holds that

$$\frac{1}{2} - \frac{1}{2N} - \frac{1}{\rho+1} > \frac{1}{2} - \frac{1}{6} - \frac{1}{3} = 0,$$

and

$$\left(\frac{1}{2} - \frac{1}{\rho+1} - \frac{1}{\rho^{2N-1}+1}\right) - \left(\frac{1}{2} - \frac{1}{2N} - \frac{1}{\rho+1} + \frac{1}{\rho^{2N-1}+1}\right)$$
$$= \frac{1}{2N} - \frac{2}{\rho^{2N-1}+1} > 0,$$

since

$$\begin{aligned}\rho^{2N-1} + 1 - 4N &> 2^{2N-1} + 1 - 4N\\ &> 1 + (2N-1) + \frac{(2N-1)(2N-2)}{2} + 1 - 4N = 2N^2 - 5N + 2\\ &> 0.\end{aligned}$$

Hence, by (8.1), we have $C_N(\rho) > 1$,

As we mentioned before, for any $\alpha > 0$, there exists a unique $\beta > 0$ satisfying (8.2). If $\alpha = 0$, then $\beta = 0$. On the other hand, for any $\alpha < 0$, there exists a unique $\beta \in (-1/C_N(\rho), 0)$ satisfying (8.2). In particular, it holds that $\alpha = \beta + \beta(1-\beta)(C_N(\rho)-1)/(1+\beta C_N(\rho)) < \beta$. Therefore, for any $\alpha > \beta_N$, there exists a β satisfying (8.2), (8.3), (8.4), and hence, by the intermediate value theorem, there exists a $\rho > 4$ such that $P_2' = 0$.

On the other hand, it does not hold for $N = 2$. We discuss the case that $N = 2$ in the next section.

Now, let us show that the $2N$-noids (or N-noids) constructed here have no branch points. The Riemannian metric of each surface is given by $ds^2 = (|\eta| + |g^2\eta|)^2$ with

$$\eta = -b^2(\beta h_{(3)}(z) + h_{(4)}(z))^2\, dz,$$
$$g^2\eta = -b^2(\beta h_{(1)}(z) + h_{(2)}(z))^2\, dz,$$
$$\begin{aligned} h_{(1)}(z) &:= \sum_{k=2:\ \text{even}}^{2N} h(z{\zeta_{2N}}^k) \\ &= N\left\{\frac{1}{z^N - 1} + \sum_{m=1}^{+\infty}\left(z^{N(m-1)}\frac{1}{\rho^{2N(m-1)+1}+1} - \frac{1}{z^{Nm}}\frac{1}{\rho^{2Nm-1}+1}\right)\right\}, \\ h_{(2)}(z) &:= \sum_{k=1:\ \text{odd}}^{2N-1} h(z{\zeta_{2N}}^k) = h_{(1)}(z\zeta_{2N}) \\ &= N\left\{-\frac{1}{z^N + 1} + \sum_{m=1}^{+\infty}(-1)^{m-1}\left(z^{N(m-1)}\frac{1}{\rho^{2N(m-1)+1}+1} + \frac{1}{z^{Nm}}\frac{1}{\rho^{2Nm-1}+1}\right)\right\}, \\ h_{(3)}(z) &:= \sum_{k=2:\ \text{even}}^{2N} {\zeta_{2N}}^k h(z{\zeta_{2N}}^k) \\ &= N\left\{\frac{z^{N-1}}{z^N - 1} + \sum_{m=1}^{+\infty}\left(-\frac{1}{z^{N(m-1)+1}}\frac{1}{\rho^{2N(m-1)+1}+1} + z^{Nm-1}\frac{1}{\rho^{2Nm-1}+1}\right)\right\}, \\ h_{(4)}(z) &:= \sum_{k=1:\ \text{odd}}^{2N-1} {\zeta_{2N}}^k h(z{\zeta_{2N}}^k) = \zeta_{2N} h_{(3)}(z\zeta_{2N}) \\ &= N\left\{\frac{z^{N-1}}{z^N + 1} + \sum_{m=1}^{+\infty}(-1)^m\left(\frac{1}{z^{N(m-1)+1}}\frac{1}{\rho^{2N(m-1)+1}+1} + z^{Nm-1}\frac{1}{\rho^{2Nm-1}+1}\right)\right\}, \end{aligned}$$

where we use the equality

$$h(z) = \frac{1}{z-1} + \sum_{m=1}^{+\infty}\left(z^{m-1} - \frac{1}{z^m}\right)\frac{1}{\rho^{2m-1}+1} \quad (1/r < |z| < r)$$

(cf. Lemma 3.5). By the equalities $h(z) = -z^{-1}h(z^{-1})$, $h(z) = -\rho h(\rho^2 z)$ (cf. Lemma 3.1 (ii), (i)), the estimates above, and the assumptions $N \geq 3$, $\rho > 2$, we get the following estimates:

$$\begin{aligned}
h_{(1)}(\rho) &= -\rho^{-1}h_{(3)}(\rho^{-1}) = h_{(3)}(\rho) \\
&= N\left\{\frac{\rho^{N-1}}{\rho^N - 1} + \sum_{m=1}^{+\infty}\left(-\frac{1}{\rho^{N(m-1)+1}}\frac{1}{\rho^{2N(m-1)+1}+1} + \frac{\rho^{Nm-1}}{\rho^{2Nm-1}+1}\right)\right\} \\
&> N\left\{\frac{\rho^{N-1}}{\rho^N - 1} - \frac{1}{\rho}\frac{1}{\rho+1}\right\} = \frac{N}{\rho}\left\{1 + \frac{1}{\rho^N - 1} - \frac{1}{\rho+1}\right\} > 0, \\
h_{(2)}(\rho) &= -\rho^{-1}h_{(4)}(\rho^{-1}) = h_{(4)}(\rho) \\
&= N\left\{\frac{\rho^{N-1}}{\rho^N + 1} + \sum_{m=1}^{+\infty}(-1)^m\left(\frac{1}{\rho^{N(m-1)+1}}\frac{1}{\rho^{2N(m-1)+1}+1} + \frac{\rho^{Nm-1}}{\rho^{2Nm-1}+1}\right)\right\} \\
&> N\left\{\frac{\rho^{N-1}}{\rho^N + 1} - \left(\frac{1}{\rho}\frac{1}{\rho+1} + \frac{\rho^{N-1}}{\rho^{2N-1}+1}\right)\right\} \\
&= \frac{N}{\rho}\left\{1 - \frac{1}{\rho^N + 1} - \frac{1}{\rho+1} - \frac{\rho^N}{\rho^{2N-1}-1}\right\} \\
&> \frac{N}{\rho}\left\{1 - \frac{1}{\rho^N} - \frac{1}{\rho+1} - \frac{1}{\rho^{N-1}}\right\} > \frac{N}{\rho}\left(1 - \frac{1}{8} - \frac{1}{3} - \frac{1}{4}\right) > 0,
\end{aligned}$$

where we use

$$\frac{\rho^{Nm-1}}{\rho^{2Nm-1}+1} > \frac{\rho^{Nm-1}}{2\rho^{2Nm-1}} > \frac{\rho^{Nm-1}}{\rho^{2Nm}} > \frac{\rho^{N(m+1)-1}}{\rho^{2N(m+1)-1}} > \frac{\rho^{N(m+1)-1}}{\rho^{2N(m+1)-1}+1}.$$

We also have $h_{(1)}(\rho) > h_{(2)}(\rho)$, since

$$\begin{aligned}
&h_{(1)}(\rho) - h_{(2)}(\rho) \\
&= N\left\{\frac{1}{\rho^N - 1} - \frac{1}{\rho^N + 1} + 2\sum_{m=1}^{+\infty}\left(-\frac{1}{\rho^{N(2m-1)}}\frac{1}{\rho^{2N(2m-1)-1}+1} + \frac{\rho^{N(2m-1)}}{\rho^{2N(2m-1)+1}+1}\right)\right\} \\
&> N\left\{\frac{2\rho^N}{\rho^{2N}-1} - \frac{2}{\rho^N}\frac{1}{\rho^{2N-1}+1}\right\} > \frac{N}{\rho^N}\left(2 - \frac{2}{\rho^{2N-1}+1}\right) > 0.
\end{aligned}$$

In the case that $\beta = 0$, by the symmetry of the N-noid, it cannot be branched at any point.

In the case that $\beta > 0$, by the symmetry of the $2N$-noid, it cannot be branched at $z \neq \rho{\zeta_{2N}}^{2k}r^l, \rho{\zeta_{2N}}^{2k-1}r^l$ $(k, l \in \mathbf{Z})$. Hence we have only to show that $z = \rho, \rho\zeta_{2N}$ are not branch points. Since $\beta h_{(3)}(\rho) + h_{(4)}(\rho) > 0$ and $\zeta_{2N}(\beta h_{(3)}(\rho\zeta_{2N}) + h_{(4)}(\rho\zeta_{2N})) = \beta h_{(4)}(\rho) + h_{(3)}(\rho) > 0$, we get $\eta = -b^2(\beta h_{(3)}(z) + h_{(4)}(z))^2\, dz \neq 0$ at $z = \rho, \rho\zeta_{2N}$.

Also in the case that $-1 < -1/C_N(\rho) < \beta < 0$, by the symmetry of the $2N$-noid, it cannot be branched at $z \neq \rho{\zeta_{2N}}^{2k}r^l, \rho{\zeta_{2N}}^{2k-1}r^l$ $(k, l \in \mathbf{Z})$. Hence we have only to show that $z = \rho$, $\rho\zeta_{2N}$ are not branch points. Since $\beta h_{(1)}(\rho\zeta_{2N}) + h_{(2)}(\rho\zeta_{2N}) = \beta h_{(2)}(\rho) + h_{(1)}(\rho) > (1+\beta)h_{(2)}(\rho) > 0$, we get $g^2\eta = -b^2(\beta h_{(1)}(z) + h_{(2)}(z))^2\, dz \neq 0$ at $z = \rho\zeta_{2N}$. Note here that $\eta|_{z=\rho} = -b^2(\beta h_{(3)}(\rho) + h_{(4)}(\rho))^2\, dz = -b^2(\beta h_{(1)}(\rho) + h_{(2)}(\rho))^2\, dz = g^2\eta|_{z=\rho}$. Hence, $ds^2|_{z=\rho} = 2|\eta||_{z=\rho}$. Set $\beta'_N := -h_{(2)}(\rho)/h_{(1)}(\rho)$. Then, $\eta \neq 0$ at $z = \rho$ if $\beta > \beta'_N$.

Let us show that $\beta'_N < -1/C_N(\rho)$, that is, $-\beta'_N C_N(\rho) > 1$. By the definitions of β'_N and $C_N(\rho)$,

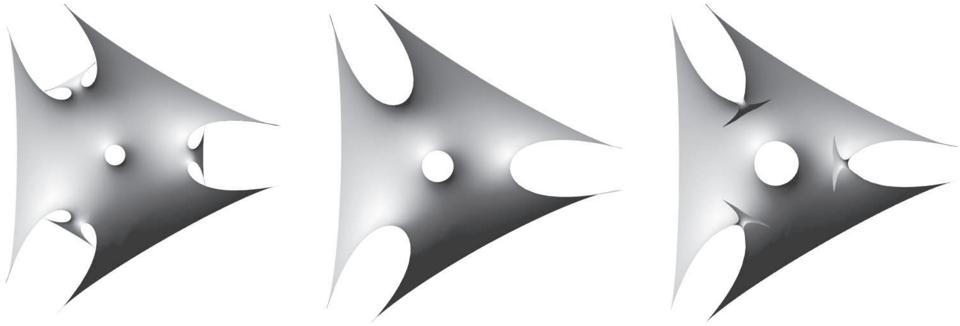
$$-\beta'_N C_N(\rho) = \frac{h_{(2)}(\rho)}{h_{(1)}(\rho)} \frac{h_{(2)}(1)}{h_{(1')}(1)},$$

where we set $h_{(1')}(z) := \sum_{k=1:\,\text{even}}^{2N-2} h(z{\zeta_{2N}}^k)$. Since

$$\begin{aligned}
\frac{1}{N}h_{(2)}(\rho) &> \frac{\rho^{N-1}}{\rho^N+1} - \frac{1}{\rho}\frac{1}{\rho+1} - \frac{\rho^{N-1}}{\rho^{2N-1}+1} = \frac{1}{\rho+1} - \frac{1}{\rho(\rho^N+1)} - \frac{\rho^{N-1}}{\rho^{2N-1}+1} \\
&> \frac{1}{\rho+1} - \frac{1}{\rho(\rho^N-1)} - \frac{1}{\rho^N-1} = \frac{1}{\rho+1} - \frac{\rho+1}{\rho(\rho^N-1)} > 0, \\
-\frac{1}{N}h_{(2)}(1) &= \frac{1}{2} + \sum_{m=1}^{+\infty}(-1)^m\left(\frac{1}{\rho^{2N(m-1)+1}+1} + \frac{1}{\rho^{2Nm-1}+1}\right) \\
&> \frac{1}{2} - \frac{1}{\rho+1} - \frac{1}{\rho^{2N-1}+1} > \frac{1}{2} - \frac{1}{\rho+1} - \frac{1}{\rho^{2N-1}} > 0, \\
0 < \frac{1}{N}h_{(1)}(\rho) &= \frac{1}{\rho^N-1} + \sum_{m=1}^{+\infty}\left(\frac{\rho^{N(m-1)}}{\rho^{2N(m-1)+1}+1} - \frac{1}{\rho^{Nm}}\frac{1}{\rho^{2Nm-1}+1}\right) \\
&< \frac{1}{\rho^N-1} + \frac{1}{\rho+1} + \sum_{m=2}^{+\infty}\frac{1}{\rho^{N(m-1)+1}} = \frac{1}{\rho+1} + \frac{\rho+1}{\rho(\rho^N-1)}, \\
0 < -\frac{1}{N}h_{(1')}(1) &= \frac{1}{2} - \frac{1}{2N} + \sum_{m=1}^{+\infty}\left(-\frac{1}{\rho^{2N(m-1)+1}+1} + \frac{1}{\rho^{2Nm-1}+1}\right) \\
&< \frac{1}{2} - \frac{1}{2N} - \frac{1}{\rho+1} + \frac{1}{\rho^{2N-1}+1} < \frac{1}{2} - \frac{1}{2N} - \frac{1}{\rho+1} + \frac{1}{\rho^{2N-1}},
\end{aligned}$$

we get, for $\rho > 4$,

$$\begin{aligned}
&\left(\frac{1}{N}h_{(2)}(\rho)\right)\left(-\frac{1}{N}h_{(2)}(1)\right) - \left(\frac{1}{N}h_{(1)}(\rho)\right)\left(-\frac{1}{N}h_{(1')}(1)\right) \\
&> \left(\frac{1}{\rho+1} - \frac{\rho+1}{\rho(\rho^N-1)}\right)\left(\frac{1}{2} - \frac{1}{\rho+1} - \frac{1}{\rho^{2N-1}}\right) \\
&\quad - \left(\frac{1}{\rho+1} + \frac{\rho+1}{\rho(\rho^N-1)}\right)\left(\frac{1}{2} - \frac{1}{2N} - \frac{1}{\rho+1} + \frac{1}{\rho^{2N-1}}\right) \\
&= \frac{1}{2N(\rho+1)} + \frac{\rho+1}{2N\rho(\rho^N-1)} - \frac{\rho-1}{\rho(\rho^N-1)} - \frac{2}{\rho^{2N-1}(\rho+1)}
\end{aligned}$$



$\alpha \approx 1/7$ $\alpha = 0$ $\alpha \approx -1/9$

Fig. 8.1.

$$
\begin{aligned}
&> \frac{1}{2N(\rho+1)} + \frac{\rho+1}{2N\rho(\rho^N-1)} - \frac{\rho-1}{\rho(\rho^N-1)} - \frac{2}{\rho^{N-1}(\rho+1)(\rho^N-1)} \\
&= \frac{\rho^{N+1} + (\rho^2+\rho+1) - 2N\rho^2 + 2N(1-2\rho^{2-N})}{2N\rho(\rho+1)(\rho^N-1)} \\
&> \frac{\rho^2(\rho^{N-1}-2N)}{2N\rho(\rho+1)(\rho^N-1)} > \frac{\rho^2(4^{N-1}-2N)}{2N\rho(\rho+1)(\rho^N-1)} > 0.
\end{aligned}
$$

We conclude that, *for any* $N \in \mathbf{N}$, $N \geq 3$, *and any* $\alpha \in (\beta_N, 0) \cup (0, +\infty)$, *there exists a Jorge–Meeks type* $2N$*-noid of genus one whose ratio of alternating weights of ends is* α. Fig. 8.1 shows some examples for $N = 3$.

We note here that the holes, the handles on the plane of symmetry, in the case that $\alpha < 0$ are larger than those in the case that $\alpha > 0$.

9. Examples 2

Throughout this section, we assume $r > 1$ and $r^{1/2} < 0$, and set $\rho := |r^{1/2}| = -r^{1/2}$.

First, we describe the data of Costa's family of 3-end catenoids (cf. [3]) by using our notation. This family collapses to three catenoids as a limit.

EXAMPLE 9.1. Consider the following flux data:

$$n := 3, \quad p_1 := 1, \quad p_2 = p_3 := -1, \quad a_1 = a_2 + a_3.$$

Set

$$
\begin{cases}
q_1 := 1, \quad q_2 := \rho, \quad q_3 := -1, \\
b_1 := \beta e^{\pi\sqrt{-1}/4}, \quad b_2 := -\beta\beta_2\rho^{1/2}e^{-\pi\sqrt{-1}/4}, \quad b_3 := -\beta\beta_3 e^{-\pi\sqrt{-1}/4}, \\
\beta, \beta_2, \beta_3 > 0.
\end{cases}
$$

Then the surface given by these data is symmetric with respect to both x_1x_2-plane and x_1x_3-plane, and we can apply Theorem 7.2. By the standard calculation, we see that there exists a complex number β such that the conditions $w_j \in \mathbf{R}$ and $w_j^* = 0$ ($j = 1, 2, 3$) are automatically satisfied, and the nontrivial condition is rewritten as follows:

$$\begin{cases} \dfrac{1}{2}\beta^{-2}P_1 \equiv (1+{\beta_2}^2-{\beta_3}^2)h_1(1) - 2\beta_2\beta_3\rho^{1/2}h_1(-\rho) = 0, \\ -\sqrt{-1}\beta^{-2}P_2 \equiv (1-{\beta_2}^2+{\beta_3}^2)(2h_1(1)\log\rho - 1) + 4\beta_2\beta_3\rho^{1/2}h_1(-\rho)\log\rho = 0. \end{cases}$$

It is equivalent with

$$\begin{cases} {\beta_2}^2 - {\beta_3}^2 = 1 - 4h_1(1)\log\rho, \\ \beta_2\beta_3 = \dfrac{h_1(1)(1-2h_1(1)\log\rho)}{\rho^{1/2}h_1(-\rho)}. \end{cases}$$

For any $r = \rho^2 > e^{2\pi}$ (resp. $r = \rho^2 = e^{2\pi}$), it holds that $1 - 2h_1(1)\log\rho > 1 - 4h_1(1)\log\rho > 0$ (resp. $1 - 2h_1(1)\log\rho > 1 - 4h_1(1)\log\rho = 0$) and $h_1(-\rho) > 0$, and hence there exists a $(\beta_2, \beta_3) \in (0, +\infty) \times (0, +\infty)$ satisfying the equation above, and

$$\begin{cases} a_1 = 2\{\rho^{1/2}h(\rho)\beta_2 - (-h(-1))\beta_3\}\beta^2 > 0, \\ a_2 = 2\rho^{1/2}h(\rho)\beta_2\beta^2 > 0, \\ a_3 = -2(-h(-1))\beta_3\beta^2 < 0. \end{cases}$$

In particular, $a_3/a_2 \in (-1, -1/2)$ (cf. [3, Lemma 3]). Costa's 1-parameter family of 3-end catenoids collapses to three catenoids as $a_3/a_2 \to -1/2$.

In Example 9.2 below, we treat, by applying Theorem 7.2, the remaining case of Example 8.1. In Example 9.3, we give a complete proof of the existence of two families of 3-end catenoids, which were first observed in [1] by using the MESH program. Schoen [16] proved that there is no catenoid of genus one. Hence, if the data of n-noid of genus one goes near to that of "the catenoid of genus one", then the surface must collapse. The families of n-end catenoids of genus one we construct in Examples 9.2 and 9.3 enable us to observe such a phenomenon.

EXAMPLE 9.2. To construct Jorge–Meeks type 4-noids of genus one with alternating sizes of ends, let us consider the following flux data:

$$\begin{cases} n := 4, \\ p_j := e^{2\pi\sqrt{-1}(j-1)/4} = (\sqrt{-1})^{j-1} \quad (j = 1, \dots, 4), \\ a_j := \begin{cases} a & \in \mathbf{R}\setminus\{0\} \quad (j = 2, 4), \\ \tilde{a} & \in \mathbf{R} \qquad\quad (j = 1, 3). \end{cases} \end{cases}$$

Set

$$\begin{cases} q := \rho^{1/2}, \quad q_j := q^{j-1} \quad (j = 1, \ldots, 4), \\ b_1 := \tilde{b} \neq 0, \quad b_2 := b \neq 0, \quad b_3 := -\sqrt{-1}q\tilde{b}, \quad b_4 := -\sqrt{-1}qb. \end{cases}$$

Then, by (3.2) and Lemmas 3.1–3.4, it holds that

$$\begin{aligned} -h(q^k, q^j) &= h(q^j, q^k) = \frac{1}{q^k}h(q^{j-k}), \\ h(q^3) &= h\left(\frac{r}{q}\right) = r^{-1/2}h\left(\frac{1}{q}\right) = -\frac{1}{q^2}(-qh(q)) = \frac{1}{q}h(q), \\ h_1(q^k, q^j) &= h_1(q^j, q^k) = \frac{1}{q^k}h_1(q^{j-k}), \\ h_1(q^3) &= h_1\left(\frac{r}{q}\right) = r^{-1/2}\left(h_1\left(\frac{1}{q}\right) + h\left(\frac{1}{q}\right)\right) \\ &= -\frac{1}{q^2}(qh_1(q) - qh(q)) = \frac{1}{q}(-h_1(q) + h(q)), \\ h_1(q^2) &= h_1(-r^{1/2}) = \frac{1}{2}h(-r^{1/2}) = \frac{1}{2}h(q^2), \end{aligned}$$

and hence w_j^* $(j = 1, 2, 3, 4)$ and P_2 automatically vanish. Therefore we have only to consider the period problem for w_j $(j = 1, 2, 3, 4)$ and P_1 with

$$\begin{aligned} &w_1 = w_3 = \tilde{b}^2 \cdot 2\sqrt{-1}qh(q^2) - \tilde{b}b \cdot 2\sqrt{2}e^{-\pi\sqrt{-1}/4}h(q), \\ &w_2 = w_4 = -b^2 \cdot 2h(q^2) - b\tilde{b} \cdot 2\sqrt{2}e^{-\pi\sqrt{-1}/4}h(q), \\ &\frac{1}{2}P_1 = -R_1(\gamma_1) = -\sqrt{-1}\frac{1}{q}(\sqrt{-1}q\tilde{b}^2 - b^2) \cdot 2h_1(1) + \tilde{b}b \cdot \sqrt{2}e^{\pi\sqrt{-1}/4}(4h_1(q) - h(q)). \end{aligned}$$

Set $\alpha := \tilde{a}/a$ and $\beta := e^{-\pi\sqrt{-1}/4}\tilde{b}/b$. If $w_1 = w_3 = \tilde{a}$ and $w_2 = w_4 = a$, then it holds that

$$\alpha = \frac{\beta^2 qh(q^2) + \sqrt{2}\beta h(q)}{h(q^2) + \sqrt{2}\beta h(q)}. \tag{9.1}$$

Solving (9.1) as an equation of β, we get a solution

$$\beta = \frac{\sqrt{2}\alpha(h(q^2)/h(q))}{(1-\alpha) + \sqrt{(1-\alpha)^2 + 2\alpha q(h(q^2)/h(q))^2}}.$$

If $\alpha \in (0,1)$, then $\beta > 0$. Now, we can choose b so that $w_1 = w_3 = \tilde{a}$ and $w_2 = w_4 = a$.

Set

$$\tilde{P}_1 := \frac{1}{2b^2\sqrt{-1}}P_1 = -\frac{1}{b^2\sqrt{-1}}R_1(\gamma_1) = 2\left(\beta^2 + \frac{1}{q}\right)h_1(1) + \sqrt{2}\beta(4h_1(q) - h(q)).$$

Now, let us show that, for any $\alpha \in (0,1)$, there exists a q satisfying $\tilde{P}_1 = 0$.

First we give an estimate for $\tilde{P}_1$ from below at $q = 2$. Note here that

$$h_1(1) = 2\sum_{k=1}^{\infty} \frac{q^{2(2k-1)}}{(q^{2(2k-1)}+1)^2},$$

$$h_1(q) = \sum_{k=1}^{\infty}\left(q^{k-1} + \frac{1}{q^k}\right)\frac{q^{2(2k-1)}}{(q^{2(2k-1)}+1)^2},$$

$$\begin{aligned} h(q) &= \frac{1}{q-1} + \sum_{k=1}^{\infty}\left(q^{k-1} - \frac{1}{q^k}\right)\frac{1}{q^{2(2k-1)}+1} \\ &< \frac{1}{q-1} + 2\sum_{k=1}^{\infty}\left(q^{k-1} - \frac{1}{q^k}\right)\frac{q^{2(2k-1)}}{(q^{2(2k-1)}+1)^2} \quad (q > 1) \end{aligned}$$

(cf. Lemmas 3.5 and 3.6). It holds that

$$\begin{aligned} \tilde{P}_1 &\geq 4\frac{\beta}{\sqrt{q}}h_1(1) + \sqrt{2}\beta(4h_1(q) - h(q)) \\ &> \sqrt{2}\beta\left\{\sum_{k=1}^{\infty}\left(\frac{4\sqrt{2}}{\sqrt{q}} + 2q^{k-1} + \frac{6}{q^k}\right)\frac{q^{2(2k-1)}}{(q^{2(2k-1)}+1)^2} - \frac{1}{q-1}\right\} \\ &> \sqrt{2}\beta\left\{\left(\frac{4\sqrt{2}}{\sqrt{q}} + 2 + \frac{6}{q}\right)\frac{q^2}{(q^2+1)^2} - \frac{1}{q-1}\right\}. \end{aligned}$$

Hence we have

$$\tilde{P}_1|_{q=2} > \sqrt{2}\beta\left\{(4+2+3)\frac{4}{25} - 1\right\} = \sqrt{2}\beta \cdot \frac{11}{25} > 0.$$

Secondly, we give the asymptotic behaviour of $\tilde{P}_1$ as $q \to +\infty$. Note here that

$$h(q) = \frac{1}{q-1} - \sum_{k=1}^{\infty}\left(q^{k-1} - \frac{1}{q^k}\right)\sum_{m=1}^{\infty}\frac{(-1)^m}{q^{2m(2k-1)}} = \frac{1}{q} + O\left(\frac{1}{q^2}\right),$$

$$h(q^2) = \frac{1}{q^2-1} - \sum_{k=1}^{\infty}\left(q^{2k-2} - \frac{1}{q^{2k}}\right)\sum_{m=1}^{\infty}\frac{(-1)^m}{q^{2m(2k-1)}} = \frac{2}{q^2} + O\left(\frac{1}{q^4}\right),$$

$$h_1(q) = -\sum_{k=1}^{\infty}\left(q^{k-1} + \frac{1}{q^k}\right)\sum_{m=1}^{\infty}\frac{(-1)^m m}{q^{2m(2k-1)}} = \frac{1}{q^2} + O\left(\frac{1}{q^3}\right),$$

$$h_1(1) = -2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^m m}{q^{2m(2k-1)}} = O\left(\frac{1}{q^2}\right).$$

By using these estimates, we also have

$$\frac{h(q^2)}{h(q)} = \frac{2}{q} + O\left(\frac{1}{q^2}\right),$$
$$q\left(\frac{h(q^2)}{h(q)}\right)^2 = O\left(\frac{1}{q}\right),$$
$$\beta = \frac{\sqrt{2}\alpha}{1-\alpha}\frac{1}{1+\sqrt{1+(2\alpha/(1-\alpha)^2)q(h(q^2)/h(q))^2}}\frac{h(q^2)}{h(q)} = \frac{\sqrt{2}\alpha}{1-\alpha}\frac{1}{q} + O\left(\frac{1}{q^2}\right),$$
$$4h_1(q) - h(q) = -\frac{1}{q} + O\left(\frac{1}{q^2}\right),$$

from which it follows that

$$\tilde{P}_1 = -\frac{2\alpha}{1-\alpha}\frac{1}{q^2} + O\left(\frac{1}{q^3}\right).$$

Hence, for any $\alpha \in (0, 1)$, there exists a $q_\alpha > 2$ such that $\tilde{P}_1|_{q=q_\alpha} < 0$.

Now, by the intermediate value theorem, there exists a $q > 2$ such that $\tilde{P}_1 = 0$.

Any surface we construct here has no branch point. Indeed, by the symmetry of the surface, it cannot be branched at $z \neq -r^l, -qr^l$ $(l \in \mathbf{Z})$. Hence we have only to show that $z = -1$ and $-q$ are not branch points. Recall here that $\eta = -Q(z)^2\,dz$ and

$$Q(z) = \sum_{j=1}^{4} b_j h(z, q_j) = b_1 h(z) + \frac{b_2}{q}h\left(\frac{z}{q}\right) - \frac{\sqrt{-1}b_1}{q}h\left(\frac{z}{q^2}\right) - \frac{\sqrt{-1}b_2}{q^2}h\left(\frac{z}{q^3}\right).$$

In particular, by $h(-1/q^2) = h(-1/\rho) = 0$,

$$Q(-1) = e^{\pi\sqrt{-1}/4} b_2(\beta h(-1) + \sqrt{2}h(-q)),$$
$$Q(-q) = b_2\left(\sqrt{2}\beta h(-q) + \frac{1}{q}h(-1)\right).$$

Since $\rho > 4$, $q = \rho^{1/2} > 2$ and hence

$$h(-1) = -\frac{1}{2} + 2\sum_{k=1}^{+\infty}(-1)^{k-1}\frac{1}{q^{2(2k-1)}+1} < -\frac{1}{2} + 2\frac{1}{q^2+1} < 0,$$
$$\begin{aligned} h(-q) &= -\left\{\frac{1}{q+1} + \sum_{k=1}^{+\infty}(-1)^k\left(q^{k-1} + \frac{1}{q^k}\right)\frac{1}{q^{2(2k-1)}+1}\right\} \\ &< -\left\{\frac{1}{q+1} - \left(1+\frac{1}{q}\right)\frac{1}{q^2+1}\right\} \\ &< -\frac{q^2}{(q+1)(q^2+1)}\left(1 - \frac{1}{q} - \frac{1}{q^2} - \frac{1}{q^3}\right) < 0, \end{aligned}$$

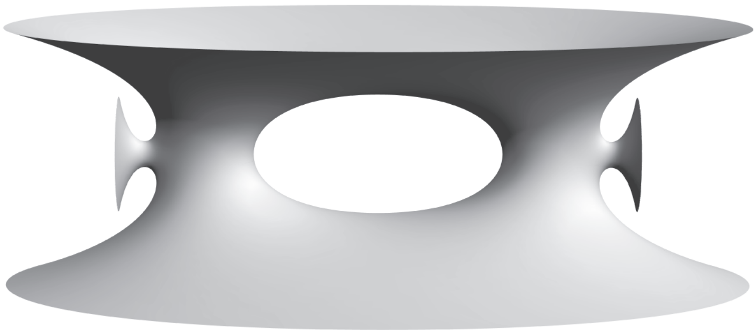


Fig. 9.1.

where we use the inequality

$$\frac{q^{k-1}}{q^{2(2k-1)}+1} > \frac{q^k}{2q^{2(2k-1)+1}} > \frac{q^k}{q^{2(2k+1)}} > \frac{q^k}{q^{2(2k+1)}+1}.$$

Since $\beta > 0$, we get $ds^2 = (|\eta| + |g^2\eta|)^2 \neq 0$ at $z = -1$ and $-q$.

We conclude that, *for any* $\alpha \in (0, 1)$, *there exists a Jorge–Meeks type* 4*-noid of genus one whose ratio of alternating weights of ends is* α. Fig. 9.1 shows the case that $\alpha \approx 1/7$.

EXAMPLE 9.3. In the case that $n = 3$ and the surface is symmetric with respect to a plane on which the flux vectors are arranged, if we assume that $q_j \in \mathbf{R}$ and $|p_j| = 1$ $(j = 1, 2, 3)$, then, by $w_j^* = 0$ $(j = 1, 2, 3)$, we have

$$b_j = b(p_k + p_l)h(q_k, q_l)$$

for some nonzero complex number b, where $(j, k, l) = (1, 2, 3)$ or $(2, 3, 1)$ or $(3, 1, 2)$. Substituting this, and by using Lemma 3.2 (i), we have

$$w_j = -2b^2 h(q_1, q_2)h(q_2, q_3)h(q_3, q_1)p_j({p_k}^2 - {p_l}^2) \quad (j = 1, 2, 3).$$

If a_j $(j = 1, 2, 3)$ satisfy $\sum_{j=1}^{3} a_j v(p_j) = \mathbf{0}$ and $a_j \neq 0$, then it holds that

$$a_1 : a_2 : a_3 = p_1({p_2}^2 - {p_3}^2) : p_2({p_3}^2 - {p_1}^2) : p_3({p_1}^2 - {p_2}^2).$$

Hence, we can choose b so that $w_j = a_j$ $(j = 1, 2, 3)$.

In this case, the periods P_1 and P_2 in Theorem 7.2 are given as follows:

$$
\begin{aligned}
\frac{1}{2}P_1 &= -R_1(\gamma_1) \\
&= \frac{b^2}{q_1q_2q_3}\Bigg[-\bigg\{p_1(p_2+p_3)^2 h\left(\frac{q_2}{q_3}\right)h\left(\frac{q_3}{q_2}\right) + p_2(p_3+p_1)^2 h\left(\frac{q_3}{q_1}\right)h\left(\frac{q_1}{q_3}\right) \\
&\qquad\qquad + p_3(p_1+p_2)^2 h\left(\frac{q_1}{q_2}\right)h\left(\frac{q_2}{q_1}\right)\bigg\}h_1(1) \\
&\qquad + (p_1+p_2)(p_2+p_3)(p_3+p_1) \\
&\qquad\quad \times \bigg\{h\left(\frac{q_2}{q_3}\right)h\left(\frac{q_3}{q_1}\right)h_1\left(\frac{q_1}{q_2}\right) \\
&\qquad\qquad + h\left(\frac{q_3}{q_1}\right)h\left(\frac{q_1}{q_2}\right)h_1\left(\frac{q_2}{q_3}\right) + h\left(\frac{q_1}{q_2}\right)h\left(\frac{q_2}{q_3}\right)h_1\left(\frac{q_3}{q_1}\right)\bigg\}\Bigg], \\
-P_2 &= \pi\sqrt{-1}(R_0(\gamma_2) + \overline{R_2(\gamma_2)}) \\
&= \frac{b^2}{q_1q_2q_3}\Bigg[\bigg\{(p_2+p_3)^2 h\left(\frac{q_2}{q_3}\right)h\left(\frac{q_3}{q_2}\right) + (p_3+p_1)^2 h\left(\frac{q_3}{q_1}\right)h\left(\frac{q_1}{q_3}\right) \\
&\qquad + (p_1+p_2)^2 h\left(\frac{q_1}{q_2}\right)h\left(\frac{q_2}{q_1}\right)\bigg\}(\log r\cdot h_1(1) - 1) \\
&\qquad - 2\log r\bigg\{(p_2+p_3)(p_3+p_1)h\left(\frac{q_2}{q_3}\right)h\left(\frac{q_3}{q_1}\right)h_1\left(\frac{q_1}{q_2}\right) \\
&\qquad\qquad + (p_3+p_1)(p_1+p_2)h\left(\frac{q_3}{q_1}\right)h\left(\frac{q_1}{q_2}\right)h_1\left(\frac{q_2}{q_3}\right) \\
&\qquad\qquad + (p_1+p_2)(p_2+p_3)h\left(\frac{q_1}{q_2}\right)h\left(\frac{q_2}{q_3}\right)h_1\left(\frac{q_3}{q_1}\right)\bigg\} \\
&\qquad - 2\bigg\{\log|q_1|({p_2}^2 - {p_3}^2) + \log|q_2|({p_3}^2 - {p_1}^2) \\
&\qquad\qquad + \log|q_3|({p_1}^2 - {p_2}^2)\bigg\}h\left(\frac{q_1}{q_2}\right)h\left(\frac{q_2}{q_3}\right)h\left(\frac{q_3}{q_1}\right)\Bigg],
\end{aligned}
$$

where we use (3.2), Lemma 3.1 (ii) and Lemma 3.3 (ii).

Here we consider the case that two of the flux vectors have a common weight:

$$p_1 := 1,\quad p_2 := p,\quad p_3 := \bar{p},\quad a_1 : a_2 : a_3 = -2\,\mathrm{Re}\,p : 1 : 1,\quad \frac{1}{2}\pi < \arg p < \frac{2}{3}\pi.$$

Set

$$q_1 := 1,\quad q_2 := r^{\epsilon},\quad q_3 := r^{1-\epsilon},\quad 0 < \epsilon < \frac{1}{2}.$$

Then P_1 and P_2 satisfy the following:

$$\begin{aligned}\frac{1}{2}P_1 &= b^2\cdot 4r^{2\epsilon-1}[\{(\mathrm{Re}\,p)^2 h(r^{2\epsilon})^2+(\mathrm{Re}\,p+1)r^{-\epsilon}h(r^{\epsilon})^2\}h_1(1)\\&\qquad+\mathrm{Re}\,p(\mathrm{Re}\,p+1)\{h(r^{\epsilon})^2h_1(r^{2\epsilon})-2h(r^{\epsilon})h(r^{2\epsilon})h_1(r^{\epsilon})\}],\\-P_2 &= b^2\cdot 4r^{2\epsilon-1}[-\{(\mathrm{Re}\,p)^2h(r^{2\epsilon})^2+\mathrm{Re}\,p(\mathrm{Re}\,p+1)r^{-\epsilon}h(r^{\epsilon})^2\}(\log r\cdot h_1(1)-1)\\&\qquad-\log r(\mathrm{Re}\,p+1)\{h(r^{\epsilon})^2h_1(r^{2\epsilon})-2\,\mathrm{Re}\,p h(r^{\epsilon})h(r^{2\epsilon})h_1(r^{\epsilon})\}\\&\qquad-2\log r\cdot\epsilon\{(\mathrm{Re}\,p)^2-1\}h(r^{\epsilon})^2h(r^{2\epsilon})].\end{aligned}$$

Since we assume $r^{1/2}<0$ and $\rho=-r^{1/2}$ here, it holds that

$$h(z)=\frac{1}{z-1}+\sum_{m=1}^{+\infty}\left(z^{m-1}-\frac{1}{z^m}\right)\frac{1}{\rho^{2m-1}+1}\quad(1/r<|z|<r),$$

$$h_1(z)=\sum_{m=1}^{+\infty}\left(z^{m-1}+\frac{1}{z^m}\right)\frac{\rho^{2m-1}}{(\rho^{2m-1}+1)^2}\quad(1/r<|z|<r)$$

(cf. Lemmas 3.5 and 3.6). Set $H_0(\epsilon,\rho):=h_0(\rho^{2\epsilon})\rho^{\epsilon}$, $H(\epsilon,\rho):=h(\rho^{2\epsilon})\rho^{\epsilon}$, and $H_1(\epsilon,\rho):=h_1(\rho^{2\epsilon})\rho^{\epsilon}$. Then

$$\begin{aligned}H_0(\epsilon,\rho)&=\sum_{m=1}^{\infty}(\rho^{(2m-1)\epsilon}-\rho^{-(2m-1)\epsilon})\frac{1}{\rho^{2m-1}+1},\\H(\epsilon,\rho)&=\frac{1}{\rho^{\epsilon}-\rho^{-\epsilon}}+H_0(\epsilon,\rho),\\H_1(\epsilon,\rho)&=\sum_{m=1}^{\infty}(\rho^{(2m-1)\epsilon}+\rho^{-(2m-1)\epsilon})\frac{\rho^{2m-1}}{(\rho^{2m-1}+1)^2}.\end{aligned}$$

In particular, by Lemma 3.1 (i), (ii) and Lemma 3.3 (i), (ii), we have

$$H(\epsilon,\rho)=-H(-\epsilon,\rho)=H(1-\epsilon,\rho)=H_1(\epsilon,\rho)+H_1(1-\epsilon,\rho).$$

Set $\tilde{P}_1:=P_1\cdot r/8b^2$, and $\tilde{P}_2:=P_2\cdot r/4b^2\log r$. Then

$$\begin{aligned}\tilde{P}_1&=\{(\mathrm{Re}\,p)^2H(2\epsilon,\rho)^2+(\mathrm{Re}\,p+1)H(\epsilon,\rho)^2\}H_1(0,\rho)\\&\qquad+\mathrm{Re}\,p(\mathrm{Re}\,p+1)\{H(\epsilon,\rho)^2H_1(2\epsilon,\rho)-2H(\epsilon,\rho)H(2\epsilon,\rho)H_1(\epsilon,\rho)\},\\\tilde{P}_2&=\mathrm{Re}\,p\{\mathrm{Re}\,pH(2\epsilon,\rho)^2+(\mathrm{Re}\,p+1)H(\epsilon,\rho)^2\}\left(H_1(0,\rho)-\frac{1}{2\log\rho}\right)\\&\qquad+(\mathrm{Re}\,p+1)\{H(\epsilon,\rho)^2H_1(2\epsilon,\rho)-2\,\mathrm{Re}\,p\cdot H(\epsilon,\rho)H(2\epsilon,\rho)H_1(\epsilon,\rho)\}\\&\qquad+2\epsilon\{(\mathrm{Re}\,p)^2-1\}H(\epsilon,\rho)^2H(2\epsilon,\rho).\end{aligned}$$

Note here that both $\tilde{P}_1$ and $\tilde{P}_2$ take real values.

Now, we will show that, for any p such that $|p| = 1$ and $\operatorname{Re} p \in (-1/2, 0)$, there exist $\epsilon \in (0, 1/2)$ and $\rho \in (1, +\infty)$ satisfying $\tilde{P}_1 = \tilde{P}_2 = 0$. To show this, we regard $\tilde{P}_1$ and $\tilde{P}_2$ as functions defined on a simply connected domain $D := \{(\epsilon, \rho) \mid \epsilon \in (0, 1/2), \rho \in (1, +\infty)\}$, and consider the map $\mathcal{P} := (\tilde{P}_1, \tilde{P}_2)\colon D \to \mathbf{R}^2$. By the homotopy argument (cf. Wohlgemuth [17], Sato [15]), for any loop l in D, if the winding number of the image $\mathcal{P}(l) \subset \mathbf{R}^2$ around $(0,0) \in \mathbf{R}^2$ is not 0, then there exists a (ϵ, ρ) in the domain surrounded by l such that $\mathcal{P}(\epsilon, \rho) = 0$. To apply this argument, we prove the following claims:

Claim 1. *$\tilde{P}_1 > 0$ or $\tilde{P}_2 > 0$ holds for $\epsilon = 1/3$ and $\rho \in (1, +\infty)$.*

Claim 2. *$\tilde{P}_1 > 0$ holds if $\rho^\epsilon = 2$ and $\rho \in (4, +\infty)$.*

Claim 3. *For any $\epsilon_1 \in (0, 1/9)$ small enough, there exists a $\rho_1 = \rho_1(\epsilon_1)$ such that $\tilde{P}_1 < 0$ holds for $\epsilon \in (\epsilon_1, 1/2 - \epsilon_1)$ and $\rho \in (\rho_1, +\infty)$.*

Claim 4. *There exist T_1^-, T_1^+, T_2^- such that $1 < T_1^- < T_1^+ < T_2^-$ and that, for any $\epsilon_2 \in (1/9, 1/6)$, there exists a $\rho_2 = \rho_2(\epsilon_2, \operatorname{Re} p, T_1^-, T_1^+, T_2^-) \in (\rho_1, +\infty)$ such that*

$$\begin{cases} \tilde{P}_1 > 0 & \text{if } \rho^\epsilon < T_1^-, \\ \tilde{P}_1 < 0 & \text{if } \rho^\epsilon > T_1^+, \\ \tilde{P}_2 < 0 & \text{if } \rho^\epsilon < T_2^- \end{cases}$$

holds for $\epsilon \in (0, \epsilon_2)$ and $\rho \in (\rho_2, +\infty)$.

Claim 5. *For any $\epsilon_3 \in (1/3, 1/2 - 1/9)$, there exists a $\rho_3 = \rho_3(\epsilon_3, \operatorname{Re} p) \in (\rho_1, +\infty)$ such that $\tilde{P}_2 < 0$ holds for $\epsilon \in (\epsilon_3, 1/2)$ and $\rho \in (\rho_3, +\infty)$.*

Claim 6. *For any $\rho_4 \in (\max\{\rho_3, 4\}, +\infty)$, there exists an $\epsilon_4 = \epsilon_4(\rho_4) \in [1/3, 1/2)$ such that $\tilde{P}_1 > 0$ holds for $\epsilon \in [\epsilon_4, 1/2)$ and $\rho \in [4, \rho_4]$.*

Choose $\rho_5 > \rho_2$. Let l_1 be the loop defined by joining the curves $\{(\epsilon, \rho) \mid \epsilon = 1/3,\ \rho \in [8, \rho_5]\}$, $\{(\epsilon, \rho) \mid \rho^\epsilon = 2,\ \rho \in [8, \rho_5]\}$, $\{(\epsilon, \rho) \mid \epsilon \in [\log 2/\log \rho_5, 1/3],\ \rho = \rho_5\}$. Then the winding number of $\Pi(l_1)$ around $(0, 0)$ is -1.

Choose $\rho_6 > \rho_3$. Let l_2 be the loop defined by joining the curves $\{(\epsilon, \rho) \mid \epsilon = 1/3,\ \rho \in [8, \rho_6]\}$, $\{(\epsilon, \rho) \mid \epsilon \in [1/3, \epsilon_4],\ \rho = \rho_6\}$, $\{(\epsilon, \rho) \mid \epsilon = \epsilon_4,\ \rho \in [2^{1/\epsilon_4}, \rho_6]\}$, $\{(\epsilon, \rho) \mid \epsilon \in [1/3, \epsilon_4], \rho^\epsilon = 2\}$. Then the winding number of $\Pi(l_2)$ around $(0, 0)$ is 1.

Hence, by the homotopy argument, we conclude that there exist two (ϵ, ρ) satisfying $\tilde{P}_1(\epsilon, \rho) = \tilde{P}_2(\epsilon, \rho) = 0$.

Now, let us prove the claims.

Proof of Claim 1. Note here that $H(2/3, \rho) = H(1/3, \rho)$ and $H_1(2/3, \rho) = H(1/3, \rho) - H_1(1/3, \rho)$. Then we have, for $\epsilon = 1/3$,

$$\tilde{P}_1 = H\left(\frac{1}{3}, \rho\right)^2 \Bigg[\{(\mathrm{Re}\, p)^2 + \mathrm{Re}\, p + 1\} H_1(0, \rho) + \mathrm{Re}\, p(\mathrm{Re}\, p + 1)\left(H\left(\frac{1}{3}, \rho\right) - 3H_1\left(\frac{1}{3}, \rho\right)\right)\Bigg],$$

$$\tilde{P}_2 = \frac{1}{3}(2\,\mathrm{Re}\, p + 1) H\left(\frac{1}{3}, \rho\right)^2 \Bigg[3\,\mathrm{Re}\, p\left(H_1(0, \rho) - \frac{1}{2\log\rho}\right) + (\mathrm{Re}\, p + 1)\left(H\left(\frac{1}{3}, \rho\right) - 3H_1\left(\frac{1}{3}, \rho\right)\right)\Bigg].$$

Since $H(1/3, \rho) > 0$ and $\mathrm{Re}\, p \in (-1/2, 0)$, it holds that

$$H\left(\frac{1}{3}, \rho\right)^{-2} \tilde{P}_1 - \mathrm{Re}\, p\left(\frac{1}{3}(2\,\mathrm{Re}\, p + 1) H\left(\frac{1}{3}, \rho\right)^2\right)^{-1} \tilde{P}_2$$
$$= \{-2(\mathrm{Re}\, p)^2 + \mathrm{Re}\, p + 1\} H_1(0, \rho) + \frac{3}{2}(\mathrm{Re}\, p)^2 \frac{1}{\log\rho} > 0.$$

Hence at least one of $\tilde{P}_1 > 0$ and $\tilde{P}_2 > 0$ must hold. □

Proof of Claim 2. In the case that $\rho \in (4, 8]$, note that $\mathrm{Re}\, p + 1 > (\mathrm{Re}\, p + 1)^2$, $H_1(2\epsilon, \rho) = H(2\epsilon, \rho) - H_1(1 - 2\epsilon, \rho) < H(2\epsilon, \rho)$, and $H(\epsilon, \rho) = H_1(\epsilon, \rho) + H_1(1 - \epsilon, \rho)$. Then we have

$$\begin{aligned}
\tilde{P}_1 &= \{(\mathrm{Re}\, p)^2 H(2\epsilon, \rho)^2 + (\mathrm{Re}\, p + 1) H(\epsilon, \rho)^2\} H_1(0, \rho) \\
&\quad + (-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1)\{2H(\epsilon, \rho) H(2\epsilon, \rho) H_1(\epsilon, \rho) - H(\epsilon, \rho)^2 H_1(2\epsilon, \rho)\} \\
&> \{(\mathrm{Re}\, p)^2 H(2\epsilon, \rho)^2 + (\mathrm{Re}\, p + 1)^2 H(\epsilon, \rho)^2\} H_1(0, \rho) \\
&\quad + (-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1)\{2H(\epsilon, \rho) H(2\epsilon, \rho) H_1(\epsilon, \rho) - H(\epsilon, \rho)^2 H(2\epsilon, \rho)\} \\
&\geq (-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1) H(\epsilon, \rho) H(2\epsilon, \rho)(2H_1(0, \rho) + 2H_1(\epsilon, \rho) - H(\epsilon, \rho)) \\
&= (-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1) H(\epsilon, \rho) H(2\epsilon, \rho)(2H_1(0, \rho) + H_1(\epsilon, \rho) - H_1(1 - \epsilon, \rho)).
\end{aligned}$$

By the assumption, we have $(-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1) H(\epsilon, \rho) H(2\epsilon, \rho) > 0$. On the other hand, if $\rho^\epsilon = 2$ and $\rho \in (4, 8]$, we also have,

$$\begin{aligned}
&2H_1(0, \rho) + H_1(\epsilon, \rho) - H_1(1 - \epsilon, \rho) \\
&= \sum_{m=1}^{+\infty} \left\{4 + 2^{2m-1} + 2^{-(2m-1)} - \left(\frac{\rho}{2}\right)^{2m-1} - \left(\frac{2}{\rho}\right)^{2m-1}\right\} \frac{\rho^{2m-1}}{(\rho^{2m-1} + 1)^2} \\
&> \left(4 + 2 + \frac{1}{2} - \frac{\rho}{2} - \frac{2}{\rho}\right) \frac{\rho}{(\rho + 1)^2} - \sum_{m=2}^{+\infty} \left(\frac{\rho}{2}\right)^{2m-1} \frac{\rho^{2m-1}}{(\rho^{2m-1} + 1)^2}
\end{aligned}$$

$$> \frac{-\rho^2 + 13\rho - 4}{2(\rho+1)^2} - \sum_{m=2}^{+\infty}\left(\frac{1}{2}\right)^{2m-1} = \frac{-4\rho^2 + 37\rho - 13}{6(\rho+1)^2} \geq \frac{27}{6(\rho+1)^2} > 0.$$

Now we get $\tilde{P}_1 > 0$ for $\rho^\epsilon = 2$ and $\rho \in (4, 8]$.

In the case that $\rho \in [8, +\infty)$, we have

$$\begin{aligned}\tilde{P}_1 &= \{(\mathrm{Re}\, p)^2 H(2\epsilon, \rho)^2 + (\mathrm{Re}\, p + 1)H(\epsilon, \rho)^2\} H_1(0, \rho) \\ &\quad + (-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1)\{2H(\epsilon, \rho)H(2\epsilon, \rho)H_1(\epsilon, \rho) - H(\epsilon, \rho)^2 H_1(2\epsilon, \rho)\} \\ &> (\mathrm{Re}\, p + 1)H(\epsilon, \rho)^2 \left\{H_1(0, \rho) - \frac{2}{3}(-\mathrm{Re}\, p)H_1(2\epsilon, \rho)\right\} \\ &\quad + 2(-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1)H(\epsilon, \rho)\left\{H(2\epsilon, \rho)H_1(\epsilon, \rho) - \frac{1}{6}H(\epsilon, \rho)H_1(2\epsilon, \rho)\right\}.\end{aligned}$$

By the assumption, we have $(\mathrm{Re}\, p+1)H(\epsilon,\rho)^2 > 0$ and $2(-\mathrm{Re}\ p)(\mathrm{Re}\, p+1)H(\epsilon,\rho) > 0$. On the other hand, if $\rho^\epsilon = 2$ and $\rho \in [8, +\infty)$, then we have

$$\begin{aligned}&H_1(0, \rho) - \frac{2}{3}(-\mathrm{Re}\, p)H_1(2\epsilon, \rho) > H_1(0, \rho) - \frac{1}{3}H_1(2\epsilon, \rho) \\ &= \sum_{m=1}^{+\infty}\left\{2 - \frac{1}{3}(4^{2m-1} + 4^{-(2m-1)})\right\}\frac{\rho^{2m-1}}{(\rho^{2m-1}+1)^2} \\ &> \left\{2 - \frac{1}{3}\left(4 + \frac{1}{4}\right)\right\}\frac{\rho}{(\rho+1)^2} - \frac{1}{3}\sum_{m=2}^{+\infty}\left(\frac{4}{\rho}\right)^{2m-1} \\ &= \frac{7\rho^4 - 368\rho^2 - 512\rho - 256}{12\rho(\rho+1)^2(\rho^2 - 16)} \geq \frac{3\rho^4}{16 \cdot 12\rho(\rho+1)^2(\rho^2-16)} > 0.\end{aligned}$$

We also have

$$\begin{aligned}&H(2\epsilon, \rho)H_1(\epsilon, \rho) - \frac{1}{6}H(\epsilon, \rho)H_1(2\epsilon, \rho) \\ &> \left(\frac{4}{15} + \frac{15}{4}\frac{1}{\rho+1}\right)\frac{5}{2}\frac{\rho}{(\rho+1)^2} - \frac{1}{6}\left\{\frac{2}{3} + \sum_{m=1}^{+\infty}\left(\frac{2}{\rho}\right)^{2m-1}\right\}\sum_{m=1}^{+\infty}\left\{\left(\frac{4}{\rho}\right)^{2m-1} + \frac{1}{\rho^{2m-1}}\right\} \\ &= \frac{\rho(8\rho^2 + 451\rho - 2932)}{72(\rho+1)^3(\rho-4)} \geq \frac{1188\rho}{72(\rho+1)^3(\rho-4)} > 0.\end{aligned}$$

Now we get $\tilde{P}_1 > 0$ for $\rho^\epsilon = 2$ and $\rho \in [8, +\infty)$. □

Proof of Claim 3. We have the following upper estimates for $H(\epsilon, \rho)$ etc.:

$$\frac{1}{\rho^{\epsilon}-\rho^{-\epsilon}} = \frac{1}{1-\rho^{-2\epsilon}}\frac{1}{\rho^{\epsilon}},$$

$$H_0(\epsilon, \rho) < \sum_{m=1}^{+\infty}\frac{\rho^{(2m-1)\epsilon}}{\rho^{2m-1}} = \frac{1}{1-\rho^{-2(1-\epsilon)}}\frac{1}{\rho^{1-\epsilon}} \quad (\epsilon \in (0,1)),$$

$$H(\epsilon, \rho) < \frac{1}{1-\rho^{-2\epsilon}}\frac{1}{\rho^{\epsilon}} + \frac{1}{1-\rho^{-2(1-\epsilon)}}\frac{1}{\rho^{1-\epsilon}} < \frac{2}{1-\rho^{-2\epsilon}}\frac{1}{\rho^{\epsilon}} \quad (\epsilon \in (0,1)),$$

$$H(2\epsilon, \rho) < \frac{1}{1-\rho^{-4\epsilon}}\frac{1}{\rho^{2\epsilon}} + \frac{1}{1-\rho^{-2(1-2\epsilon)}}\frac{1}{\rho^{1-2\epsilon}} \quad (\epsilon \in (0,1/2)),$$

$$H_1(\epsilon, \rho) < 2\sum_{m=1}^{+\infty}\frac{\rho^{(2m-1)\epsilon}}{\rho^{2m-1}} = \frac{2}{1-\rho^{-2(1-\epsilon)}}\frac{1}{\rho^{1-\epsilon}} \quad (\epsilon \in [0,1)),$$

$$H_1(0, \rho) < \frac{2}{1-\rho^{-2}}\frac{1}{\rho}.$$

In particular, if $\epsilon \in (\epsilon_1, 1/2-\epsilon_1)$ and $\rho^{\epsilon_1} > 2$, then we have

$$H(\epsilon, \rho) < \frac{4}{\rho^{\epsilon}} < \frac{4}{\rho^{\epsilon_1}}, \quad H(2\epsilon, \rho) < \frac{2}{\rho^{2\epsilon}} + \frac{2}{\rho^{1-2\epsilon}} < \frac{4}{\rho^{2\epsilon_1}},$$
$$H_1(\epsilon, \rho) < \frac{4}{\rho^{1-\epsilon}}, \quad H_1(0, \rho) < \frac{4}{\rho}.$$

We also have the following lower estimates:

$$H(\epsilon, \rho) > \frac{1}{\rho^{\epsilon}-\rho^{-\epsilon}} = \frac{1}{1-\rho^{-2\epsilon}}\frac{1}{\rho^{\epsilon}} > \frac{1}{\rho^{\epsilon}},$$
$$H_1(2\epsilon, \rho) > \rho^{2\epsilon}\frac{\rho}{4\rho^2} = \frac{1}{4}\frac{1}{\rho^{1-2\epsilon}}.$$

Now, if $\epsilon \in (\epsilon_1, 1/2-\epsilon_1)$ and $\rho^{\epsilon_1} > 2$, then we have

$$\begin{aligned}
\tilde{P}_1 &= \{(\mathrm{Re}\, p)^2 H(2\epsilon, \rho)^2 + (\mathrm{Re}\, p+1)H(\epsilon, \rho)^2\}H_1(0, \rho) \\
&\quad + (-\mathrm{Re}\, p)(\mathrm{Re}\, p+1)\{2H(\epsilon, \rho)H(2\epsilon, \rho)H_1(\epsilon, \rho) - H(\epsilon, \rho)^2 H_1(2\epsilon, \rho)\} \\
&< \left\{(\mathrm{Re}\, p)^2\frac{16}{\rho^{4\epsilon_1}} + (\mathrm{Re}\, p+1)\frac{16}{\rho^{2\epsilon_1}}\right\}\frac{4}{\rho} \\
&\quad + (-\mathrm{Re}\, p)(\mathrm{Re}\, p+1)\left\{2\frac{4}{\rho^{\epsilon}}\frac{4}{\rho^{2\epsilon_1}}\frac{4}{\rho^{1-\epsilon}} - \frac{1}{\rho^{2\epsilon}}\frac{1}{4}\frac{1}{\rho^{1-2\epsilon}}\right\} \\
&= -\frac{1}{4\rho^{1+4\epsilon_1}}\{(-\mathrm{Re}\, p)(\mathrm{Re}\, p+1)\rho^{4\epsilon_1} - 256(1-2\,\mathrm{Re}\, p)(\mathrm{Re}\, p+1)\rho^{2\epsilon_1} - 256(\mathrm{Re}\, p)^2\}.
\end{aligned}$$

Hence, for any $\epsilon_1 > 0$, if we set

$$\rho_1 = \rho_1(\epsilon_1, \mathrm{Re}\, p) := \left[\frac{128(1-2\,\mathrm{Re}\, p)(\mathrm{Re}\, p+1) + \sqrt{\{128(1-2\,\mathrm{Re}\, p)(\mathrm{Re}\, p+1)\}^2 + 256(-\mathrm{Re}\, p)^3(\mathrm{Re}\, p+1)}}{(-\mathrm{Re}\, p)(\mathrm{Re}\, p+1)} \right]^{1/2\epsilon_1},$$

then $\tilde{P}_1 < 0$ holds for any $\epsilon \in (\epsilon_1, 1/2 - \epsilon_1)$ and $\rho > \rho_1$. □

Proof of Claim 4. Set $C := \rho^\epsilon$. Then $\epsilon = \log C / \log \rho$. Assume $\epsilon \in (0, 1/4)$ and $\rho \in (4, +\infty)$, and fix $\mathrm{Re}\, p$. Set $c(t) := 1/(1 - 1/t) = t/(t-1)$. For any $k \in [0, 4)$, since $\rho^2/C^{2k} = \rho^{2(1-k\epsilon)} > 2^{4-k}$, it holds that $c(\rho^2/C^{2k}) < c(2^{4-k})$. Now, we have the following upper estimates for $H(\epsilon, \rho)$ etc.:

$$\begin{aligned} H(k\epsilon, \rho) &= \frac{1}{C^k - C^{-k}} + H_0(k\epsilon, \rho) \\ &= \frac{1}{C^k - C^{-k}} + \sum_{m=1}^{+\infty} (C^{k(2m-1)} - C^{-k(2m-1)}) \frac{1}{\rho^{2m-1} + 1} \\ &< \frac{1}{C^k - C^{-k}} + \sum_{m=1}^{+\infty} \left(\frac{C^k}{\rho} \right)^{2m-1} < \frac{1}{C^k - C^{-k}} + c\left(\frac{\rho^2}{C^{2k}} \right) \frac{C^k}{\rho} \\ &< \frac{1}{C^k - C^{-k}} \left(1 + c\left(\frac{\rho^2}{C^{2k}} \right) \frac{C^{2k}}{\rho} \right) < \frac{1}{C^k - C^{-k}} \left(1 + c(2^{4-k}) \frac{C^{2k}}{\rho} \right) \quad (k \neq 0), \end{aligned}$$

$$\begin{aligned} H_1(k\epsilon, \rho) &= (C^k + C^{-k}) \frac{\rho}{(\rho+1)^2} + \sum_{m=2}^{+\infty} (C^{k(2m-1)} + C^{-k(2m-1)}) \frac{\rho^{2m-1}}{(\rho^{2m-1}+1)^2} \\ &< (C^k + C^{-k}) \frac{\rho}{(\rho+1)^2} + 2 \sum_{m=2}^{+\infty} \left(\frac{C^k}{\rho} \right)^{2m-1} \\ &= (C^k + C^{-k}) \frac{\rho}{(\rho+1)^2} + 2c\left(\frac{\rho^k}{C^{2k}} \right) \frac{C^{3k}}{\rho^3} \\ &< (C^k + C^{-k}) \frac{\rho}{(\rho+1)^2} \left\{ 1 + 2c\left(\frac{\rho^k}{C^{2k}} \right) \frac{(\rho+1)^2}{\rho^2} \frac{C^{2k}}{\rho^2} \right\} \\ &< (C^k + C^{-k}) \frac{\rho}{(\rho+1)^2} \left(1 + \frac{25}{8} c(2^{4-k}) \frac{C^{2k}}{\rho^2} \right), \end{aligned}$$

$$\begin{aligned} H_1(k\epsilon, \rho) &= \sum_{m=1}^{+\infty} (C^{k(2m-1)} + C^{-k(2m-1)}) \frac{\rho^{2m-1}}{(\rho^{2m-1}+1)^2} < 2 \sum_{m=1}^{+\infty} \left(\frac{C^k}{\rho} \right)^{2m-1} \\ &= 2c\left(\frac{\rho^2}{C^{2k}} \right) \frac{C^k}{\rho} < 2c(2^{4-k}) \frac{C^k}{\rho}. \end{aligned}$$

We also have the following lower estimates:

$$H(k\epsilon,\rho) > \frac{1}{C^k - C^{-k}} \quad (k \neq 0),$$
$$H_1(k\epsilon,\rho) > (C^k + C^{-k})\frac{\rho}{(\rho+1)^2}.$$

We will use these estimates with $k = 0, 1, 2$. Set

$$A_1 := c(8) = \frac{8}{7}, \quad A_2 := c(4) = \frac{4}{3}, \quad A_3 := \frac{25}{8}c(16) = \frac{10}{3}, \quad A_4 := 2c(16) = \frac{32}{15},$$
$$A_5 := \frac{25}{8}c(8) = \frac{25}{7}, \quad A_6 := 2c(8) = \frac{16}{7}, \quad A_7 := \frac{25}{8}c(4) = \frac{25}{6}, \quad A_8 := 2c(4) = \frac{8}{3}.$$

First, we give an upper estimate for $\tilde{P}_1$.

$$\begin{aligned}
\tilde{P}_1 &= \{(\operatorname{Re} p)^2 H(2\epsilon,\rho)^2 + (\operatorname{Re} p+1)H(\epsilon,\rho)^2\}H_1(0,\rho) \\
&\quad + (-\operatorname{Re} p)(\operatorname{Re} p+1)\{-H(\epsilon,\rho)^2 H_1(2\epsilon,\rho) + 2H(\epsilon,\rho)H(2\epsilon,\rho)H_1(\epsilon,\rho)\} \\
&< \frac{1}{(C-C^{-1})^2(C+C^{-1})^2}\frac{\rho}{(\rho+1)^2} \\
&\quad \times\Bigg[\Bigg\{(\operatorname{Re} p)^2\left(1+A_2\frac{C^4}{\rho}\right)^2 + (\operatorname{Re} p+1)(C+C^{-1})^2\left(1+A_1\frac{C^2}{\rho}\right)^2\Bigg\}\cdot 2\left(1+A_3\frac{1}{\rho^2}\right) \\
&\qquad + (-\operatorname{Re} p)(\operatorname{Re} p+1)\Bigg\{-(C+C^{-1})^2(C^2+C^{-2}) \\
&\qquad\qquad + 2(C+C^{-1})^2\left(1+A_1\frac{C^2}{\rho}\right)^2\left(1+A_2\frac{C^4}{\rho}\right)^2\left(1+A_5\frac{C^2}{\rho^2}\right)^2\Bigg\}\Bigg] \\
&< \frac{1}{(C-C^{-1})^2(C+C^{-1})^2}\frac{\rho}{(\rho+1)^2}(P_{1T}+P_{1+}),
\end{aligned}$$

where we set

$$P_{1T} := \operatorname{Re} p(\operatorname{Re} p+1)(C+C^{-1})^4 + 2(\operatorname{Re} p+1)(1-2\operatorname{Re} p)(C+C^{-1})^2 + 2(\operatorname{Re} p)^2,$$

$$\begin{aligned}
P_{1+} := 2&\left\{(\operatorname{Re} p)^2 A_2(2+A_2)\frac{C^4}{\rho} + 4(\operatorname{Re} p+1)A_1(2+A_1)\frac{C^4}{\rho}\right\} \\
&+ 2A_3\left\{(\operatorname{Re} p)^2(1+A_2)^2\frac{1}{\rho^2} + 4(\operatorname{Re} p+1)(1+A_1)^2\frac{C^2}{\rho^2}\right\} \\
&+ (-\operatorname{Re} p+1)(\operatorname{Re} p+1)\Bigg\{8A_1(1+A_2)(1+A_5)\frac{C^4}{\rho} \\
&\qquad + 8A_2(1+A_5)\frac{C^6}{\rho} + 8A_5\frac{C^4}{\rho^2}\Bigg\}.
\end{aligned}$$

Choose $\epsilon_2 \in (0, 1/6)$. Since

$$
\begin{aligned}
P_{1+} < \Bigg[& 2\{(\mathrm{Re}\, p)^2 A_2(2 + A_2) + 4(\mathrm{Re}\, p + 1)A_1(2 + A_1)\} \\
& + 2A_3\left\{(\mathrm{Re}\, p)^2(1 + A_2)^2 \frac{1}{4} + 4(\mathrm{Re}\, p + 1)(1 + A_1)^2 \frac{1}{4}\right\} \\
& + (-\,\mathrm{Re}\, p + 1)(\mathrm{Re}\, p + 1)\bigg\{8A_1(1 + A_2)(1 + A_5) \\
& \qquad + 8A_2(1 + A_5) + 8A_5 \frac{1}{4}\bigg\}\Bigg] \times \frac{1}{\rho^{1-6\epsilon_2}}
\end{aligned} \tag{9.2}
$$

holds for $\epsilon \in (0, \epsilon_2)$, $P_{1+}(\epsilon, \rho)$ converges to 0 as $\rho \to +\infty$ uniformly on $(0, \epsilon_2)$.

Secondly, we give a lower estimate for $\tilde{P}_1$.

$$
\begin{aligned}
\tilde{P}_1 &= \{(\mathrm{Re}\, p)^2 H(2\epsilon, \rho)^2 + (\mathrm{Re}\, p + 1)H(\epsilon, \rho)^2\} H_1(0, \rho) \\
&\quad + (-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1)\{-H(\epsilon, \rho)^2 H_1(2\epsilon, \rho) + 2H(\epsilon, \rho)H(2\epsilon, \rho)H_1(\epsilon, \rho)\} \\
&> \frac{1}{(C - C^{-1})^2 (C + C^{-1})^2} \frac{\rho}{(\rho + 1)^2} \\
&\quad \times \Bigg[2\{(\mathrm{Re}\, p)^2 + (\mathrm{Re}\, p + 1)(C + C^{-1})^2\} \\
&\qquad + (-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1)\bigg\{-(C + C^{-1})^2(C^2 + C^{-2})\left(1 + A_1 \frac{C^2}{\rho}\right)^2 \left(1 + A_7 \frac{C^4}{\rho^2}\right) \\
&\qquad\qquad + 2(C + C^{-1})^2\bigg\}\Bigg] \\
&> \frac{1}{(C - C^{-1})^2 (C + C^{-1})^2} \frac{\rho}{(\rho + 1)^2} (P_{1T} - P_{1-}),
\end{aligned}
$$

where we set

$$
P_{1-} := (-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1)\left\{8A_1(2 + A_1)(1 + A_7)\frac{C^6}{\rho} + 8A_7 \frac{C^8}{\rho^2}\right\}.
$$

Since

$$
P_{1-} < (-\mathrm{Re}\, p)(\mathrm{Re}\, p + 1)\left\{8A_1(2 + A_1)(1 + A_7) + 8A_7 \frac{1}{2}\right\} \frac{1}{\rho^{1-6\epsilon_2}} \tag{9.3}
$$

holds for $\epsilon \in (0, \epsilon_2)$, $P_{1-}(\epsilon, \rho)$ converges to 0 as $\rho \to +\infty$ uniformly on $(0, \epsilon_2)$.

Thirdly, we give an upper estimate for $\tilde{P}_2$.

$$\begin{aligned}
\tilde{P}_2 = {} & (-\mathrm{Re}\, p)\{(-\mathrm{Re}\, p)H(2\epsilon,\rho)^2 - (\mathrm{Re}\, p+1)H(\epsilon,\rho)^2\}\left(H_1(0,\rho) - \frac{1}{2\log\rho}\right) \\
& + (\mathrm{Re}\, p+1)\{H(\epsilon,\rho)^2 H_1(2\epsilon,\rho) + 2(-\mathrm{Re}\, p)H(\epsilon,\rho)H(2\epsilon,\rho)H_1(\epsilon,\rho)\} \\
& - 2\epsilon\{1-(\mathrm{Re}\, p)^2\}H(\epsilon,\rho)^2 H(2\epsilon,\rho) \\
< {} & \frac{1}{2(C-C^{-1})^2(C+C^{-1})^2\log\rho} \\
& \times \Bigg[2(-\mathrm{Re}\, p)^2\left(1+A_2\frac{C^4}{\rho}\right)A_4\frac{\log\rho}{\rho} \\
& \quad + (-\mathrm{Re}\, p)\left\{-(-\mathrm{Re}\, p) + (\mathrm{Re}\, p+1)(C+C^{-1})^2\left(1+A_1\frac{C^2}{\rho}\right)^2\right\} \\
& \quad + 2(\mathrm{Re}\, p+1)\Bigg\{(C+C^{-1})^2\left(1+A_1\frac{C^2}{\rho}\right)^2 A_8\frac{C^2}{\rho} \\
& \qquad\qquad + 2(-\mathrm{Re}\, p)(C+C^{-1})\left(1+A_1\frac{C^2}{\rho}\right)\left(1+A_2\frac{C^4}{\rho}\right)^2 A_6\frac{C}{\rho}\Bigg\}\log\rho \\
& \quad - 4\{1-(\mathrm{Re}\, p)^2\}\frac{C+C^{-1}}{C-C^{-1}}\log\rho\Bigg] \\
< {} & \frac{1}{2(C-C^{-1})^2(C+C^{-1})^2\log\rho}(P_{2T}+P_{2+}),
\end{aligned}$$

where we set

$$\begin{aligned}
P_{2T} &:= (-\mathrm{Re}\, p)(\mathrm{Re}\, p+1)(C+C^{-1})^2 + 3(\mathrm{Re}\, p)^2 - 4, \\
P_{2+} &:= 2(-\mathrm{Re}\, p)^2(1+A_2)A_4\frac{\log\rho}{\rho} + 4(-\mathrm{Re}\, p)(\mathrm{Re}\, p+1)A_1(2+A_1)\frac{C^4}{\rho} \\
&\quad + 2(\mathrm{Re}\, p+1)\Bigg\{4(1+A_1)^2 A_8\frac{C^4\log\rho}{\rho} \\
&\qquad\qquad + 4(-\mathrm{Re}\, p)(1+A_1)(1+A_2)A_6\frac{C^2\log\rho}{\rho}\Bigg\}.
\end{aligned}$$

Since

(9.4)

$$\begin{aligned}
P_{2+} < {} & \Bigg[2(-\mathrm{Re}\, p)^2(1+A_2)A_4 + 4(-\mathrm{Re}\, p)(\mathrm{Re}\, p+1)A_1(2+A_1)\frac{1}{\log 4} \\
& + 2(\mathrm{Re}\, p+1)\{4(1+A_1)^2A_8 + 4(-\mathrm{Re}\, p)(1+A_1)(1+A_2)A_6\}\Bigg] \times \frac{\log\rho}{\rho^{1-4\epsilon_2}}
\end{aligned}$$

holds for $\epsilon \in (0,\epsilon_2)$, $P_{2+}(\epsilon,\rho)$ converges to 0 as $\rho \to +\infty$ uniformly on $(0,\epsilon_2)$.

Now, we get

$$P_{1T} - P_{1-} < (C - C^{-1})^2 (C + C^{-1})^2 \frac{(\rho + 1)^2}{\rho} \tilde{P}_1 < P_{1T} + P_{1+},$$
$$2(C - C^{-1})^2 (C + C^{-1})^2 \log \rho \cdot \tilde{P}_2 < P_{2T} + P_{2+}.$$

Note here that $(C + C^{-1})^2$ is monotone increasing on $C \in (1, +\infty)$.

Since P_{1T} is a polynomial of $(C + C^{-1})^2$ of degree 2, its top coefficient satisfies $\mathrm{Re}\, p(\mathrm{Re}\, p + 1) < 0$, and $P_{1T}|_{(C+C^{-1})^2=4} = 2(\mathrm{Re}\, p + 2)^2 > 0$, there exists a unique $(C + C^{-1})^2 > 4$ satisfying $P_{1T} = 0$. Denote the value of $C > 1$ satisfying $P_{1T} = 0$ by T_1.

On the other hand, since P_{2T} is a polynomial of $(C + C^{-1})^2$ of degree 1, its top coefficient satisfies $(-\mathrm{Re}\, p)(\mathrm{Re}\, p+1) > 0$, and $P_{2T}|_{(C+C^{-1})^2=4} = -(\mathrm{Re}\, p)^2 - 4(\mathrm{Re}\, p+1) < 0$, there exists a unique $(C + C^{-1})^2 > 4$ satisfying $P_{2T} = 0$. Denote the least value of $C > 1$ satisfying $P_{2T} = 0$ by T_2.

Moreover, since

$$P_{1T}|_{C=T_2} = \frac{\{2 - (\mathrm{Re}\, p)^2\}(\mathrm{Re}\, p + 2)^2}{\mathrm{Re}\, p(\mathrm{Re}\, p + 1)} < 0,$$

we have $T_1 < T_2$.

Choose T_1^-, T_1^+, T_2^- so that $1 < T_1^- < T_1 < T_1^+ < T_2^- < T_2$ and $P_{1T}|_{C=T_1^-} < 2(\mathrm{Re}\, p + 2)^2 = P_{1T}|_{C=1}$. Then, for any $\epsilon_2 \in (0, 1/6)$, there exists a

$$\rho_2 = \rho_2(\epsilon_2, \mathrm{Re}\, p, T_1^-, T_1^+, T_2^-) \in (\rho_1, +\infty)$$

such that

$$\begin{cases} \tilde{P}_1 > 0 & \text{if } \rho^\epsilon < T_1^-, \\ \tilde{P}_1 < 0 & \text{if } \rho^\epsilon > T_1^+, \\ \tilde{P}_2 < 0 & \text{if } \rho^\epsilon < T_2^- \end{cases}$$

holds for $\epsilon \in (0, \epsilon_2)$ and $\rho \in (\rho_2, +\infty)$.

Indeed, since there exists a $T_0 \in (1, T_1)$ such that P_{1T} is monotone increasing for $C \in [1, T_0]$, and monotone decreasing for $C \in [T_0, +\infty)$, it holds that $P_{1T} > P_{1T}|_{C=T_1^-}$ for $C \in [1, T_1^-)$, and that $|P_{1T}| > \left|P_{1T}|_{C=T_1^+}\right|$ for $C \in [T_1^+, +\infty)$. By $\epsilon_2 < 1/6$ and (9.2), (9.3), there exists a $\rho_{T_1^-}$ such that $P_{1-} < P_{1T}|_{C=T_1^-}$ holds for $\rho > \rho_{T_1^-}$, and there exists a $\rho_{T_1^+}$ such that $P_{1+} < \left|P_{1T}|_{C=T_1^+}\right|$ holds for $\rho > \rho_{T_1^+}$.

On the other hand, since $P_{2T} < -\mathrm{Re}\, p(\mathrm{Re}\, p+1)\{T_2^- + (T_2^-)^{-1}\}^2 + 3(\mathrm{Re}\, p)^2 - 4$, and the right-hand-side is monotone increasing for C, it holds that $|P_{2T}| > |-\mathrm{Re}\, p(\mathrm{Re}\, p + 1)\{T_2^- + (T_2^-)^{-1}\}^2 + 3(\mathrm{Re}\, p)^2 - 4|$. Hence, by $\epsilon_2 < 1/6$ and (9.4), there exists a $\rho_{T_2^-}$ such that $P_{2+} < |-\mathrm{Re}\, p(\mathrm{Re}\, p + 1)\{T_2^- + (T_2^-)^{-1}\}^2 + 3(\mathrm{Re}\, p)^2 - 4|$ holds for $\rho > \rho_{T_2^-}$. Therefore if we set $\rho_2 := \max\{\rho_{T_1^-}, \rho_{T_1^+}, \rho_{T_2^-}\}$, then we get our assertion. □

Proof of Claim 5. Set $C := \rho^{1/2-\epsilon}$. Then $\epsilon = 1/2 - \log C/\log\rho$. Assume $\epsilon \in (1/3, 1/2)$ and $\rho \in (4, +\infty)$, and fix $\operatorname{Re} p$. Let $c(t)$ be as before. Note here that $\rho/C^2 = \rho^{2\epsilon} > 2\cdot 2^{1/3} > 2$, $\rho^2/C^4 = \rho^{4\epsilon} > 4\cdot 2^{2/3} > 4$, and $C^2\rho = \rho^{2(1-\epsilon)} > 4$. Set

$$B_1 := 4c(2) = 8, \quad B_2 := c(4) = \frac{4}{3}, \quad B_3 := 2c(4) = \frac{8}{3}.$$

Now, we have the following upper estimates for $H(\epsilon, \rho)$ etc.:

$$\begin{aligned}
H(\epsilon,\rho) &= H_1(1-\epsilon,\rho) + H_1(\epsilon,\rho)\\
&= \sum_{m=1}^{+\infty}\Bigg\{\left(\frac{\rho^{1/2}}{C}\right)^{2m-1} + \left(\frac{C}{\rho^{1/2}}\right)^{2m-1}\\
&\qquad + (C\rho^{1/2})^{2m-1} + \left(\frac{1}{C\rho^{1/2}}\right)^{2m-1}\Bigg\}\frac{\rho^{2m-1}}{(\rho^{2m-1}+1)^2}\\
&< 4\sum_{m=1}^{+\infty}(C\rho^{1/2})^{2m-1}\frac{1}{\rho^{2m-1}} = 4\sum_{m=1}^{+\infty}\left(\frac{C}{\rho^{1/2}}\right)^{2m-1} = 4c\left(\frac{\rho}{C^2}\right)\frac{C}{\rho^{1/2}} < B_1\frac{C}{\rho^{1/2}},\\
H(2\epsilon,\rho) &= H(1-2\epsilon,\rho) = \frac{1}{C^2 - C^{-2}} + H_0(1-2\epsilon,\rho)\\
&= \frac{1}{C^2-C^{-2}} + \sum_{m=1}^{+\infty}(C^{2(2m-1)} - C^{-(2m-1)})\frac{1}{\rho^{2m-1}+1}\\
&< \frac{1}{C^2-C^{-2}}\left(1 + c\left(\frac{\rho^2}{C^4}\right)\frac{C^4}{\rho}\right) < \frac{1}{C^2-C^{-2}}(1 + B_2\rho^{1-4\epsilon})\\
&< \frac{1}{C^2-C^{-2}}(1+B_2),\\
H_1(0,\rho) &= 2\sum_{m=1}^{+\infty}\frac{\rho^{2m-1}}{(\rho^{2m-1}+1)^2} < 2c(\rho^2)\frac{1}{\rho} < A_4\frac{1}{\rho},\\
H_1(\epsilon,\rho) &= \sum_{m=1}^{+\infty}\left\{\left(\frac{\rho^{1/2}}{C}\right)^{2m-1} + \left(\frac{C}{\rho^{1/2}}\right)^{2m-1}\right\}\frac{\rho^{2m-1}}{(\rho^{2m-1}+1)^2} < 2\sum_{m=1}^{+\infty}\left(\frac{\rho^{1/2}}{C}\right)^{2m-1}\frac{1}{\rho^{2m-1}}\\
&= 2\sum_{m=1}^{+\infty}\left(\frac{1}{C\rho^{1/2}}\right)^{2m-1} = 2c(C^2\rho)\frac{1}{C\rho^{1/2}} < B_3\frac{1}{C\rho^{1/2}},\\
H_1(2\epsilon,\rho) &= H(1-2\epsilon,\rho) - H_1(1-2\epsilon,\rho) < H(1-2\epsilon,\rho) < \frac{1}{C^2-C^{-2}}(1+B_2).
\end{aligned}$$

We also have the following lower estimate:

$$H(2\epsilon,\rho) > \frac{1}{C^2-C^{-2}}.$$

Now, we give an upper estimate for $\tilde{P}_2$.

$$\begin{aligned}\tilde{P}_2 &= (-\mathrm{Re}\,p)\{(-\mathrm{Re}\,p)H(2\epsilon,\rho)^2 - (\mathrm{Re}\,p+1)H(\epsilon,\rho)^2\}\left(H_1(0,\rho) - \frac{1}{2\log\rho}\right)\\
&\quad + (\mathrm{Re}\,p+1)\{H(\epsilon,\rho)^2 H_1(2\epsilon,\rho) + 2(-\mathrm{Re}\,p)H(\epsilon,\rho)H(2\epsilon,\rho)H_1(\epsilon,\rho)\}\\
&\quad - 2\epsilon\{1-(\mathrm{Re}\,p)^2\}H(\epsilon,\rho)^2 H(2\epsilon,\rho)\\
&< \frac{1}{2(C-C^{-1})^2(C+C^{-1})^2\log\rho}\\
&\quad \times\Bigg[-(\mathrm{Re}\,p)^2 + 2(\mathrm{Re}\,p)^2(1+B_2)^2 A_4\frac{\log\rho}{\rho}\\
&\qquad + (-\mathrm{Re}\,p)(\mathrm{Re}\,p+1)(C^2-C^{-2})^2 {B_1}^2\frac{C^2}{\rho}\\
&\qquad + 2(\mathrm{Re}\,p+1)\Bigg\{(C^2-C^{-2}){B_1}^2(1+B_2)\frac{C^2\log\rho}{\rho}\\
&\qquad\qquad + 2(-\mathrm{Re}\,p)(C^2-C^{-2})B_1(1+B_2)B_3\frac{\log\rho}{\rho}\Bigg\}\Bigg]\\
&< \frac{1}{2(C-C^{-1})^2(C+C^{-1})^2\log\rho}\{-(\mathrm{Re}\,p)^2 + P'_{2+}\},\end{aligned}$$

where we set

$$\begin{aligned}P'_{2+} &:= 2(\mathrm{Re}\,p)^2(1+B_2)^2 A_4\frac{\log\rho}{\rho} + (-\mathrm{Re}\,p)(\mathrm{Re}\,p+1)B_1^2\frac{C^6}{\rho}\\
&\quad + 2(\mathrm{Re}\,p+1)\left\{{B_1}^2(1+B_2)\frac{C^4\log\rho}{\rho} + 2(-\mathrm{Re}\,p)B_1(1+B_2)B_3\frac{C^2\log\rho}{\rho}\right\}.\end{aligned}$$

Choose $\epsilon_3 \in (1/3, 1/2)$. Since

$$\begin{aligned}P'_{2+} &< \Bigg[2(\mathrm{Re}\,p)^2(1+B_2)^2 A_4 + (-\mathrm{Re}\,p)(\mathrm{Re}\,p+1){B_1}^2\frac{1}{\log 4}\\
&\quad + 2(\mathrm{Re}\,p+1)\{{B_1}^2(1+B_2) + 2(-\mathrm{Re}\,p)B_1(1+B_2)B_3\}\Bigg]\times\rho^{-6\epsilon_3+2}\log\rho\end{aligned}$$

holds for $\epsilon \in (\epsilon_3, 1/2)$, $P'_{2+}(\epsilon,\rho)$ converges to 0 as $\rho \to +\infty$ uniformly on $(\epsilon_3, 1/2)$.

Therefore, for any $\epsilon_3 \in (1/3, 1/2)$, there exists a $\rho_3 = \rho_3(\epsilon_3, \mathrm{Re}\,p) \in (\rho_1, +\infty)$ such that $\tilde{P}_2 < 0$ holds for $\epsilon \in (\epsilon_3, 1/2)$ and $\rho \in (\rho_3, +\infty)$. □

Proof of Claim 6. Note here that

$$\tilde{P}_1 > (-\mathrm{Re}\,p)(\mathrm{Re}\,p+1)H(\epsilon,\rho)H_1(2\epsilon,\rho)(2H_1(0,\rho) + H_1(\epsilon,\rho) - H_1(1-\epsilon,\rho)).$$

By the assumption, we have $(-\mathrm{Re}\,p)(\mathrm{Re}\,p+1)H(\epsilon,\rho)H_1(2\epsilon,\rho) > 0$. Fix $\rho_4 \geq 4$. Since $2H_1(0,\rho) + H_1(\epsilon,\rho) - H_1(1-\epsilon,\rho)$ is a continuous function on $\{(\epsilon,\rho) \mid \epsilon \in$

$[1/3, 1/2]$, $\rho \in [4, \rho_4]\}$, and $2H_1(0, \rho) > 0$ on $\{(1/2, \rho) \mid \rho \in [4, \rho_4]\}$, there exists an $\epsilon_4 = \epsilon_4(\rho_4) \in [1/3, 1/2)$ such that $\tilde{P}_1 > 0$ holds for $\epsilon \in [\epsilon_4, 1/2)$ and $\rho \in [4, \rho_4]$. $\square$

Any surface we construct here has no branch point. To see this, we have only to show that $\# g^{-1}(1) = 3$. Recall here that g is given by

$$g(z) = \frac{\sum_{j=1}^3 p_j b_j h(z, q_j)}{\sum_{j=1}^3 b_j h(z, q_j)} = \sqrt{-1}\frac{\sum_{j=1}^3 \overline{b_j} h(z, q_j)}{\sum_{j=1}^3 b_j h(z, q_j)},$$

where

$$q_1 = 1, \quad q_2 = \rho^{2\epsilon}, \quad q_3 = \rho^{2-2\epsilon},$$
$$b_1 = \sqrt{-1}\overline{b_1}, \quad b_2 = -\frac{(\bar{p}+1)h(\rho^{2\epsilon})}{2\,\mathrm{Re}\,ph(\rho^{4\epsilon})}b_1, \quad b_3 = \frac{(p+1)h(\rho^{2\epsilon})}{2\,\mathrm{Re}\,ph(\rho^{4\epsilon})}\rho^{1-2\epsilon}b_1.$$

Since

$$g(z) = \sqrt{-1}\frac{\overline{b_1}/(z-1) + \overline{b_1}h_0(z, q_1) + \sum_{j=2}^3 \overline{b_j}h(z, q_j)}{b_1/(z-1) + b_1 h_0(z, q_1) + \sum_{j=2}^3 b_j h(z, q_j)},$$

it holds that $g(1) = \sqrt{-1}\overline{b_1}/b_1 = p_1 = 1$. The denominator of g is

$$\sum_{j=1}^3 b_j h(z, q_j) = b_1\left\{h(z) - \frac{(\bar{p}+1)h(\rho^{2\epsilon})}{2\,\mathrm{Re}\,ph(\rho^{4\epsilon})\rho^{2\epsilon}}h\left(\frac{z}{\rho^{2\epsilon}}\right) + \frac{(p+1)h(\rho^{2\epsilon})}{2\,\mathrm{Re}\,ph(\rho^{4\epsilon})\rho}h\left(\frac{z}{\rho^{2-2\epsilon}}\right)\right\},$$

and the numerator of g is

$$\sum_{j=1}^3 \overline{b_j}h(z, q_j) = \overline{b_1}\left\{h(z) - \frac{(p+1)h(\rho^{2\epsilon})}{2\,\mathrm{Re}\,ph(\rho^{4\epsilon})\rho^{2\epsilon}}h\left(\frac{z}{\rho^{2\epsilon}}\right) + \frac{(\bar{p}+1)h(\rho^{2\epsilon})}{2\,\mathrm{Re}\,ph(\rho^{4\epsilon})\rho}h\left(\frac{z}{\rho^{2-2\epsilon}}\right)\right\}.$$

Since

$$\begin{aligned} b_1^{-1}\sum_{j=1}^3 b_j h(\rho, q_j) &= h(\rho) - \frac{(\bar{p}+1)h(\rho^{2\epsilon})}{2\,\mathrm{Re}\,ph(\rho^{4\epsilon})\rho^{2\epsilon}}h(\rho^{1-2\epsilon}) + \frac{(p+1)h(\rho^{2\epsilon})}{2\,\mathrm{Re}\,ph(\rho^{4\epsilon})\rho}h(\rho^{-1+2\epsilon}) \\ &= h(\rho) + \frac{(\mathrm{Re}\,p+1)h(\rho^{2\epsilon})}{(-\mathrm{Re}\,p)h(\rho^{4\epsilon})\rho^{2\epsilon}}h(\rho^{1-2\epsilon}) > 0, \end{aligned}$$

and

$$\overline{b_1}^{-1}\sum_{j=1}^3 \overline{b_j}h(\rho, q_j) = h(\rho) + \frac{(\mathrm{Re}\,p+1)h(\rho^{2\epsilon})}{(-\mathrm{Re}\,p)h(\rho^{4\epsilon})\rho^{2\epsilon}}h(\rho^{1-2\epsilon}) > 0,$$

it holds that $g(\rho) = \sqrt{-1}\overline{b_1}/b_1 = 1$. Here we used the fact that $h(\rho) > 0$ and $h(\rho^{1-2\epsilon}) > 0$. On the other hand, since

$$\begin{aligned}
&{b_1}^{-1} \sum_{j=1}^{3} b_j h(-1, q_j) \\
&= h(-1) - \frac{(\bar{p}+1)h(\rho^{2\epsilon})}{2\,\mathrm{Re}\, p h(\rho^{4\epsilon})\rho^{2\epsilon}} h(-\rho^{-2\epsilon}) + \frac{(p+1)h(\rho^{2\epsilon})}{2\,\mathrm{Re}\, p h(\rho^{4\epsilon})\rho} h(-\rho^{-2+2\epsilon}) \\
&= h(-1) + \frac{(\mathrm{Re}\, p + 1)h(\rho^{2\epsilon})}{(-\mathrm{Re}\, p)h(\rho^{4\epsilon})} h(-\rho^{2\epsilon}) < 0,
\end{aligned}$$

and

$$\overline{b_1}^{-1} \sum_{j=1}^{3} \overline{b_j} h(-1, q_j) = h(-1) + \frac{(\mathrm{Re}\, p + 1)h(\rho^{2\epsilon})}{(-\mathrm{Re}\, p)h(\rho^{4\epsilon})} h(-\rho^{2\epsilon}) < 0,$$

it also holds that $g(-1) = \sqrt{-1}\overline{b_1}/b_1 = 1$. Here we used the fact that $h(-1) < 0$ for $\rho > 4$, and $h(-\rho^{2\epsilon}) < 0$ for $\epsilon \in (0, 1/2)$ and $\rho^{\epsilon} > 2$. Indeed, if $\rho > 3$, then it holds that

$$h(-1) = -\frac{1}{2} + 2\sum_{k=1}^{+\infty} (-1)^{k-1} \frac{1}{\rho^{2k-1}+1} < -\frac{1}{2} + 2\frac{1}{\rho+1} < 0.$$

On the other hand, $h(-\rho^{2\epsilon})$ is expressed as follows.

$$\begin{aligned}
h(-\rho^{2\epsilon}) &= -\frac{1}{\rho^{2\epsilon}+1} + \sum_{k=1}^{+\infty} \left\{ (-1)^{k-1}\rho^{2\epsilon(k-1)} - \frac{(-1)^k}{\rho^{2\epsilon k}} \right\} \frac{1}{\rho^{2k-1}+1} \\
&= \sum_{k=1}^{+\infty} \frac{(-1)^k}{\rho^{2\epsilon k}} - \sum_{k=1}^{+\infty} (-1)^k \left(\rho^{2\epsilon(k-1)} + \frac{1}{\rho^{2\epsilon k}} \right) \frac{1}{\rho^{2k-1}+1} \\
&= \sum_{k=1}^{+\infty} (-1)^k \left\{ 1 - \frac{\rho^{2\epsilon(2k-1)}+1}{\rho^{2k-1}+1} \right\} \frac{1}{\rho^{2\epsilon k}} = \sum_{k=1}^{+\infty} (-1)^k \frac{1-\rho^{-(1-2\epsilon)(2k-1)}}{1+\rho^{-(2k-1)}} \frac{1}{\rho^{2\epsilon k}}.
\end{aligned}$$

Since this is an alternating series and the first term is negative, it is enough for $h(-\rho^{2\epsilon}) < 0$ that the following inequality holds for any positive odd number k.

$$\begin{aligned}
&\frac{1-\rho^{-(1-2\epsilon)(2k-1)}}{1+\rho^{-(2k-1)}} \frac{1}{\rho^{2\epsilon k}} - \frac{1-\rho^{-(1-2\epsilon)(2k+1)}}{1+\rho^{-(2k+1)}} \frac{1}{\rho^{2\epsilon(k+1)}} \\
&= \frac{\rho^{2\epsilon}(1-\rho^{-(1-2\epsilon)(2k-1)})(1+\rho^{-(2k+1)}) - (1-\rho^{-(1-2\epsilon)(2k+1)})(1+\rho^{-(2k-1)})}{\rho^{2\epsilon(k+1)}(1+\rho^{-(2k-1)})(1+\rho^{-(2k+1)})} > 0.
\end{aligned}$$

$\epsilon < 1/3$ $\epsilon > 1/3$

Fig. 9.2.

If $k = 1$ and $\rho^{2\epsilon} > 4$, then it holds that

$$\begin{aligned}&\rho^{2\epsilon}(1-\rho^{-(1-2\epsilon)(2k-1)})(1+\rho^{-(2k+1)}) - (1-\rho^{-(1-2\epsilon)(2k+1)})(1+\rho^{-(2k-1)})\\&= (1-\rho^{-(1-2\epsilon)})(1+\rho^{-1})(\rho^{2\epsilon}+\rho^{2\epsilon-2}-1-2\rho^{-(1-2\epsilon)}-\rho^{-2(1-2\epsilon)})\\&> (1-\rho^{-(1-2\epsilon)})(1+\rho^{-1})(4-4) = 0.\end{aligned}$$

If $k \geq 3$ and $\rho^{2\epsilon} > 4$, then it holds that

$$\begin{aligned}&\rho^{2\epsilon}(1-\rho^{-(1-2\epsilon)(2k-1)})(1+\rho^{-(2k+1)}) - (1-\rho^{-(1-2\epsilon)(2k+1)})(1+\rho^{-(2k-1)})\\&> 4(1-\rho^{-(1-2\epsilon)(2k-1)}) - 2(1-\rho^{-(1-2\epsilon)(2k+1)})\\&= 4\left(\frac{1+\rho^{-(1-2\epsilon)(2k+1)}}{2} - \rho^{-(1-2\epsilon)(2k-1)}\right)\\&\geq 4(\rho^{-(1-2\epsilon)(2k+1)/2} - \rho^{-(1-2\epsilon)(2k-1)}) > 0.\end{aligned}$$

We conclude that, *for any* $\alpha \in (0,1)$, *there exist two* 3*-end catenoids of genus one whose ratio of weights of ends is* $w_1 : w_2 : w_3 = \alpha : 1 : 1$. Both surfaces are symmetric with respect to the common two planes orthogonal to each other. Fig. 9.2 shows the case that $\alpha = 1/5$.

References

[1] J. Berglund and W. Rossman: *Minimal surfaces with catenoid ends*, Pacific J. Math. **171** (1995), 353–371.

[2] C. Cosín and A. Ros: *A Plateau problem at infinity for properly immersed minimal surfaces with finite total curvature*, Indiana Univ. Math. J. **50** (2001), 847–879.

[3] C.J. Costa: *Classification of complete minimal surfaces in* $\mathbf{R}^3$ *with total curvature* 12π, Invent. Math. **105** (1991), 273–303.

[4] A. Hurwitz and R. Courant: Vorlesungen über allgemine Funktionentheorie und elliptische Functionen, Springer-Verlag, 1964.

[5] L.P. Jorge and W.H. Meeks, III: *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology **22** (1983), 203–221.
[6] S. Kato and H. Muroya: *Minimal surfaces of genus one with catenoidal ends* II, in preparation.
[7] S. Kato and K. Nomura: *On the weights of end-pairs in n-end catenoids of genus zero*, Osaka J. Math. **41** (2004), 507–532.
[8] S. Kato, M. Umehara and K. Yamada: *An inverse problem of the flux for minimal surfaces*, Indiana Univ. Math. J. **46** (1997), 529–559.
[9] S. Kato, M. Umehara and K. Yamada: *General existence of minimal surfaces of genus zero with catenoidal ends and prescribed flux*, Comm. Anal. Geom. **8** (2000), 83–114.
[10] M. Kokubu, M. Umehara and K. Yamada: *Minimal surfaces that attain equality in the Chern–Osserman inequality*; in Differential Geometry and Integrable Systems (Tokyo, 2000), Contemp. Math. **308**, Amer. Math. Soc., Providence, RI, 2002, 223–228.
[11] R. Kusner and N. Schmitt: *The spinor representation of surfaces in space*, arXiv: dg-ga/9610005v1, (1996).
[12] F.J. López: *The classification of complete minimal surfaces with total curvature greater than* -12π, Trans. Amer. Math. Soc. **334** (1992), 49–74.
[13] W.H. Meeks, III: *The classification of complete minimal surfaces in* $\mathbf{R}^3$ *with total curvature greater than* -8π, Duke Math. J. **48** (1981), 523–535.
[14] W. Rossman: *Minimal surfaces in* $\mathbf{R}^3$ *with dihedral symmetry*, Tohoku Math. J. (2) **47** (1995), 31–54.
[15] K. Sato: *Construction of higher genus minimal surfaces with one end and finite total curvature*, Tohoku Math. J. (2) **48** (1996), 229–246.
[16] R.M. Schoen: *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), 791–809.
[17] M. Wohlgemuth: *Minimal surfaces of higher genus with finite total curvature*, Arch. Rational Mech. Anal. **137** (1997), 1–25.

Shin Kato
Department of Mathematics
Osaka City University
3-3-138 Sugimoto, Sumiyoshi-ku
Osaka, 558-8585
Japan
e-mail: shinkato@sci.osaka-cu.ac.jp

Hisayoshi Muroya
Department of Mathematics
Osaka City University
3-3-138 Sugimoto, Sumiyoshi-ku
Osaka, 558-8585
Japan
e-mail: muroya@sci.osaka-cu.ac.jp