Cao, W.
Osaka J. Math.
49 (2012), 901–907

DISCRETENESS CRITERION IN SL(2, $\mathbb{C}$) BY A TEST MAP

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(Received October 27, 2010, revised February 17, 2011)

Abstract

In the paper [12], Yang conjectured that a nonelementary subgroup G of $SL(2,\mathbb{C})$ containing elliptic elements is discrete if for each elliptic element $g \in G$ the group $\langle f, g\rangle$ is discrete, where $f \in SL(2, \mathbb{C})$ is a test map being loxodromic or elliptic. By embedding $SL(2, \mathbb{C})$ into $U(1, 1; \mathbb{H})$, we give an affirmative answer to this question. As an application, we show that a nonelementary and nondiscrete subgroup of $\mathrm{Isom}(H^3)$ must contain an elliptic element of order at least 3.

1. Introduction

The discreteness of Möbius groups is a fundamental problem, which has been discussed by many authors. In 1976, Jørgensen established the following discreteness criterion by using the well-known Jørgensen's inequality [8].

Theorem J. *A nonelementary subgroup G of Möbius transformations acting on $\hat{\mathbb{C}}$ is discrete if and only if for each pair of elements $f, g \in G$, the group $\langle f, g\rangle$ is discrete.*

This result shows that the discreteness of a nonelementary Möbius group depends on the information of all its rank two subgroups. The above result has been generalized by many authors by using information of partial rank two subgroups. For example, Gilman [5] and Isochenko [7] used each pair of loxodromic elements, Tukia and Wang [10] used each pair of elliptic elements.

Sullivan [9] showed that a nonelementary and non-discrete subgroup is either dense in $SL(2,\mathbb{C})$ or conjugate to a dense subgroup of $SL(2,\mathbb{R})$. This result gives an approach to studying the discreteness of Möbius groups from the topological aspect. Mainly using Sullivan's result, Yang [11] obtained some generalizations by the information of the remaining four kinds of rank two subgroups.

Recently, Chen [3] proposed to use a fixed Möbius transformation as a test map to test the discreteness of a given Möbius group. His result suggests that the discreteness is not a totally interior affair of the involved group and provides a new point of

2000 Mathematics Subject Classification. Primary 30C62; Secondary 30F40, 20H10.
Supported by NSFS of China (10801107, 11071063), supported by NSF of China (11071063) and NSF of Guangdong Province (S2011010000735).

view to the discreteness problem. Yang [12] generalized some results by test maps (see Theorems 2.4–2.7) and proposed the following conjecture.

Conjecture 1.1. *Let G be a nonelementary subgroup of* $\mathrm{SL}(2,\mathbb{C})$ *containing elliptic elements and f a loxodromic* (*resp. an elliptic*) *transformation. If for each elliptic element $g \in G$ the group $\langle f, g\rangle$ is discrete, then G is discrete.*

In $\mathrm{SL}(2,\mathbb{R})$, since the trace is real, one can find a sequence $\{g_n\}$ of distinct elliptic elements in G such that $g_n \to I$. In fact, this is a special case (i.e. $\dim M(G) = 2$) of [4, Corollary 4.5.3]. Yang mainly used this fact to prove the following theorem (Theorems 2.9 in [12]).

Theorem Y1. *Let G be a nonelementary subgroup of* $\mathrm{SL}(2,\mathbb{R})$ *containing elliptic elements and f a loxodromic* (*resp. an elliptic*) *transformation. If for each elliptic element $g \in G$ the group $\langle f, g\rangle$ is discrete, then G is discrete.*

For the general case in $\mathrm{SL}(2,\mathbb{C})$, Greenberg [6] gave an example such that G is a loxodromic group and is not discrete with $\dim M(G) = 3$. This example indicates that it is nontrivial to construct a subgroup generated by f and an elliptic element in G which is nonelementary, in which one can apply Jørgensen's inequality to obtain a contradiction. However, in the case of $\mathrm{SL}(2,\mathbb{C})$, Yang also obtained the following theorem (Theorems 2.11 in [12]).

Theorem Y2. *Let G be a nonelementary subgroup of* $\mathrm{SL}(2,\mathbb{C})$ *containing elliptic elements and f a loxodromic* (*resp. an elliptic*) *transformation with $|tr^2(f)-4| < 1$. If for each elliptic element $g \in G$ the group $\langle f, g\rangle$ is discrete, then G is discrete.*

In this paper, we mainly use an embedding of $\mathrm{SL}(2,\mathbb{C})$ into $U(1,1;\mathbb{H})$ and then apply Corollary 4.5.2 in [4] to prove Conjecture 1.1.

Theorem 1.1. Conjecture 1.1 *is positive.*

In [13, Remark 2.7], Yang observed the following proposition and gave an example [13, Example 2.1] to show that for $n \geq 4$, there does exist a nonelementary and non-discrete subgroup of $\mathrm{Isom}(H^n)$ with all elliptic elements having order 2.

Proposition 1.1. *A nonelementary and nondiscrete subgroup of* $\mathrm{Isom}(H^2)$ *must contain an elliptic element of order at least* 3.

Based on the above observations, he proposed the following problem in [13, Remark 2.7].

PROBLEM 1.1. Whether there is a nonelementary and nondiscrete subgroup of $\mathrm{Isom}(H^3) = \mathrm{PSL}(2, \mathbb{C})$ which contains an elliptic element such that each of them has order 2.

As an application of our embedding, we obtain the following theorem.

Theorem 1.2. *The answer to* Problem 1.1 *is negative.*

2. The unitary group and embedding principle

In this section, we will recall some facts about quaternion and the quaternionic hyperbolic geometry. The reader is referred to [1, 2, 4] for more information.

Let $\mathbb{H}$ denote the division ring of real quaternions. Elements of $\mathbb{H}$ have the form $q = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} \in \mathbb{H}$ where $q_i \in \mathbb{R}$ and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Let $\overline{q} = q_1 - q_2\mathbf{i} - q_3\mathbf{j} - q_4\mathbf{k}$ be the *conjugate* of q, and

$$|q| = \sqrt{\overline{q}q} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$$

be the *modulus* of q. We define $\Re(q) = (q + \overline{q})/2$ to be the *real part* of q, and $\Im(q) = (q - \overline{q})/2$ to be the *imaginary part* of q. Also $q^{-1} = \overline{q}|q|^{-2}$ is the *inverse* of q. We remark that for a complex number c, we have $\mathbf{j}c = \bar{c}\mathbf{j}$.

Let $\mathbb{H}^{1,1}$ be the vector space of dimension 2 over $\mathbb{H}$ with the unitary structure defined by the Hermitian form

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J\mathbf{z} = \overline{w_1}z_1 - \overline{w_2}z_2,$$

where $\mathbf{z}$ and $\mathbf{w}$ are the column vectors in $\mathbb{H}^{1,1}$ with entries (z_1, z_2) and (w_1, w_2) respectively, $\cdot^*$ denotes the conjugate transpose and J is the Hermitian matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define a *unitary transformation* g to be an automorphism $\mathbb{H}^{1,1}$, that is, a linear bijection such that

$$\langle g(\mathbf{z}), g(\mathbf{w}) \rangle = \langle \mathbf{z}, \mathbf{w} \rangle \tag{1}$$

for all $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{H}^{1,1}$. We denote the group of all unitary transformations by $\mathrm{U}(1,1;\mathbb{H})$.

Following [4, Section 2], let

$$V_0 = \{\mathbf{z} \in \mathbb{H}^{1,1} - \{0\} \colon \langle \mathbf{z}, \mathbf{z} \rangle = 0\}, \quad V_- = \{\mathbf{z} \in \mathbb{H}^{1,1} \colon \langle \mathbf{z}, \mathbf{z} \rangle < 0\}.$$

It is obvious that V_0 and V_- are invariant under $\mathrm{U}(1,1;\mathbb{H})$. We define V^s to be $V^s = V_- \cup V_0$. Let $P\colon V^s \to P(V^s) \subset \mathbb{H}$ be the projection map defined by

$$P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 {z_2}^{-1}.$$

We define $\mathbb{B} = P(V_-)$, the ball model of 1-dimensional quaternionic hyperbolic space. It is easy to see that $\mathbb{B}$ can be identified with the quaternionic unit ball $\{z \in \mathbb{H}\colon |z| < 1\}$. Also the unit sphere in $\mathbb{H}$ is $\partial\mathbb{B} = P(V_0)$ and the center of the ball is $0 = P\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}(1,1;\mathbb{H})$ then, by definition, g preserves the Hermitian form. Hence

$$\mathbf{w}^* J \mathbf{z} = \langle \mathbf{z}, \mathbf{w} \rangle = \langle g\mathbf{z}, g\mathbf{w} \rangle = \mathbf{w}^* g^* J g \mathbf{z}$$

for all $\mathbf{z}$ and $\mathbf{w}$ in V. Letting $\mathbf{z}$ and $\mathbf{w}$ vary over a basis for V we see that $J = g^* J g$. From this we find $g^{-1} = J^{-1} g^* J$. That is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \overline{a} & -\overline{c} \\ -\overline{b} & \overline{d} \end{pmatrix}$$

and consequently,

$$|a| = |d|, \quad |b| = |c|, \quad |a|^2 - |c|^2 = 1, \quad \bar{a}b = \bar{c}d, \quad a\bar{c} = b\bar{d}. \tag{2}$$

As in [1, 2], we can regard $\mathrm{U}(1,1;\mathbb{H})$ as the isometries of real hyperbolic 4-space, whose model is the unit ball in the quaternions $\mathbb{H}$. $\mathrm{SL}(2,\mathbb{C})$, the isometries of real hyperbolic 3-space, can be embedded as a subgroup of $\mathrm{U}(1,1;\mathbb{H})$ as following:

$$f \in \mathrm{SL}(2,\mathbb{C}) \hookrightarrow T f T^{-1} \in U(1,1;\mathbb{H}),$$

where

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathbf{j} \\ -\mathbf{j} & 1 \end{pmatrix}.$$

Let $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$. Then

$$\hat{f} = T f T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -\mathbf{j} \\ -\mathbf{j} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \mathbf{j} \\ \mathbf{j} & 1 \end{pmatrix} \in \mathrm{U}(1,1;\mathbb{H}).$$

We mention that our model is slight different from the model in [4], where the Hermitian matrix is $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. It follows from (1) that both models define the same unitary group. This difference just exchanges the inner and outer of the same unit sphere of those two models.

The following lemma is crucial to us.

Lemma 2.1 (cf. [4, Corollary 4.5.2]). *Let G be a subgroup of* $\mathrm{U}(1,n;\mathbb{H})$ *such that*
(a) *G does not leave invariant a point in $\partial H^n_{\mathbb{H}}$ or a proper totally geodesic submanifold of $H^n_{\mathbb{H}}$*
(b) *the identity is not an accumulation point of the elliptic elements in G. Then G is discrete.*

Using the same notation as in [4], for any totally geodesic submanifold $M \in H^n_{\mathbb{H}}$, we denote by $I(M)$ the subgroup of $\mathrm{U}(1,n;\mathbb{H})$ which leaves M invariant. By [4, Proposition 2.5.1], the proper totally geodesic submanifolds of $H^1_{\mathbb{H}}$ are equivalent to one of the four types: $H^1_{\mathbb{R}}$, $H^1_{\mathbb{C}}$ and $H^1(\mathbb{I})$.

By [4, Lemmas 4.2.1,2], we have the following lemma.

Lemma 2.2. *Let $g \in U(1,1;\mathbb{H})$. Then*
(i) *the elements $g \in I(H^1_{\mathbb{R}})$ are of the form*

$$g = A\lambda, \quad A \in U(1,1;\mathbb{R}), \quad \lambda \in \mathbb{H}, \quad |\lambda| = 1;$$

(ii) *the elements $g \in I(H^1_{\mathbb{C}})$ are of the form*

$$g = A, \quad A \in U(1,1;\mathbb{C});$$

(iii) *the elements $g \in I(H^1(\mathbb{I}))$ are of the form*

$$g = \begin{pmatrix} a & b \\ -\varepsilon b & \varepsilon a \end{pmatrix} \in \mathrm{U}(1,1;\mathbb{H}), \quad \varepsilon = \pm 1. \tag{3}$$

Lemma 2.3. *Let G be a subgroup of* $\mathrm{SL}(2,\mathbb{C})$. *Then TGT^{-1} is a subgroup of* $\mathrm{U}(1,1;\mathbb{H})$. *If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $TGT^{-1} \subset I(H^1(\mathbb{I}))$ then either*
(i) *$a, d \in \mathbb{R}$ and $b, c \in \mathbf{i}\mathbb{R}$, or*
(ii) *$a, d \in \mathbf{i}\mathbb{R}$ and $b, c \in \mathbb{R}$.*

Proof. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $TGT^{-1} \subset I(H^1(\mathbb{I}))$, then TgT^{-1} is of form (3). By our embedding and the fact $\mathbf{j}c = \bar{c}\mathbf{j}$, $\forall c \in \mathbb{C}$, we can verify that the cases $\varepsilon = 1$ and $\varepsilon = -1$ correspond to cases (i) and (ii), respectively. □

By Lemma 2.3, we have the following corollary.

Corollary 2.1. *If G is dense in* $\mathrm{SL}(2,\mathbb{C})$, *then the smallest totally geodesic submanifold which is invariant under $G_1 = TGT^{-1}$ can not be $H^1(\mathbb{I})$.*

3. The proofs of Theorems 1.1 and 1.2

We also need the following lemma, which is a direct consequence of the well-known proposition in [9, Section 1].

Lemma 3.1. *Let G be a nonelementary subgroup of* $\mathrm{SL}(2,\mathbb{C})$. *Then either*
(i) *G is discrete, or*
(ii) *G is dense in* $\mathrm{SL}(2,\mathbb{C})$, *or*
(iii) *G is conjugate to a dense group of* $\mathrm{SL}(2,\mathbb{R})$.

The proof of Theorem 1.1. Suppose that G is nonelementary and not discrete. We may assume that G is dense in $\mathrm{SL}(2,\mathbb{C})$ by Theorem Y1 and Lemma 3.1, where $G_1 = TGT^{-1}$.

Let $M(G_1)$ be the smallest totally geodesic submanifold which is invariant under G_1. By our embedding, G_1 is a nonelementary and non-discrete subgroup of $\mathrm{U}(1,1;\mathbb{H})$. Applying conjugation if necessary, we may assume that $0 \in M(G_1)$. Since G_1 is nonelementary, $M(G_1) \neq H^1_{\mathbb{R}}$. Since G is dense in $\mathrm{SL}(2,\mathbb{C})$, $M(G_1) \neq H^1(\mathbb{I})$. By [4, Proposition 2.5.1], $M(G_1)$ is one of the two types: $H^1_{\mathbb{C}}$ and $H^1_{\mathbb{H}}$.

Suppose that $M(G_1) = H^1_{\mathbb{C}}$. By Lemma 2.2 and the fact that $\mathrm{PU}(1,1;\mathbb{C})$ is isomorphism to $\mathrm{PSL}(2,\mathbb{R})$, we can get the desired contradiction similarly as in the proof of Theorem Y1.

Suppose that $M(G_1) = H^1_{\mathbb{H}}$. By Lemma 2.1, we can find a sequence $\{g_n\}$ of distinct elliptic elements in G_1 such that

$$g_n \to I.$$

Since $g_n \in G_1$ and $T^{-1}g_nT \in G \subset \mathrm{SL}(2,\mathbb{C})$ has the same order, we get a sequence $\{T^{-1}g_nT\}$ of distinct elliptic elements in G such that $T^{-1}g_nT \to I$. By the same reasoning as in Theorem Y1, we can get the desired contradiction.

The proof is complete. □

The proof of Theorem 1.2. Suppose that G is nonelementary and not discrete. We may assume that G is dense in $\mathrm{SL}(2,\mathbb{C})$ by Proposition 1.1.

Taking the same notations as in the proof of Theorem 1.1, we are left to consider the case $M(G_1) = H^1_{\mathbb{H}}$. By Lemma 2.1, the identity is an accumulation point of the elliptic elements in G_1. Therefore we get a sequence $\{g_n\}$ of distinct elliptic elements in G such that $g_n \to I$. This implies that there exist an elliptic element with order greater than three.

The proof is complete. □

ACKNOWLEDGMENT. The author would like to express much gratitude to the referee for a careful reading of this paper as well as many useful suggestions.

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