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# SIMPLE PROOFS OF SOME THEOREMS IN BLOCK THEORY OF FINITE GROUPS

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**Abstract**

We give simple proofs of Laradji's theorem on blocks with central defect groups, Watanabe's theorem on the Glauberman–Watanabe correspondences of blocks and Robinson's theorem on defect groups of $p$-blocks of $p$-solvable groups attaining Brauer's upper bound for the number of irreducible characters.

## Introduction

In this paper all groups are finite groups. A block means a $p$-block for a fixed prime $p$. For a positive integer $n$, let $p^{\nu(n)}$ be the highest power of $p$ dividing $n$. Laradji [7] has proved:

**Theorem A.** *Let $Z$ be a central $p$-subgroup of a group $G$. Let $\chi$ be an irreducible character of $G$ such that $\nu(\chi(1)) = \nu(|G : Z|)$. Then $Z$ is a defect group of the block of $G$ containing $\chi$.*

Known proofs of this theorem ([7], [11], [13]) are rather complicated. Here we give a simple proof, which is analogous to the proof of Theorem 3.12 of [6].

Let $S$ and $G$ be groups such that $S$ acts on $G$ as automorphisms and that $(|S|, |G|) = 1$. Let $B$ be an $S$-invariant block of $G$ such that a defect group of $B$ is centralized by $S$. In this situation Watanabe [15] has proved:

**Theorem B.** *Any irreducible character in $B$ is $S$-invariant.*

Watanabe [15] has proved Theorem B by using a theorem of Dade [1]. Here we give a direct proof of Theorem B. Another direct proof, which uses the Glauberman correspondence, is found in Navarro [12].

Let $B$ be a block of a group with a defect group $D$. A well-known conjecture of R.Brauer asserts that $k(B) \leq |D|$. For $p$-solvable groups, this conjecture has been proved by [4]:

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**Theorem C.** *Let $B$ be a $p$-block of a $p$-solvable group with a defect group $D$. Then $k(B) \leq |D|$. In particular, for a $p$-solvable group $G$ with $O_{p'}(G) = 1$, we have $k(G) \leq |D|$, where $D$ is a Sylow $p$-subgroup of $G$.*

As to the equality in Theorem C, Robinson [14] has proved:

**Theorem D.** *Let $B$ be a $p$-block of a $p$-solvable group with a defect group $D$. If $k(B) = |D|$, then $D$ is abelian.*

We simplify Robinson's proof by using a theorem of Gallagher [3].

## 1. Proof of Theorem A

Proof of Theorem A. As in [7], we may assume that $\chi_Z$ is faithful. Let $Z$ act by multiplication on the set of all conjugacy classes of $G$. Let $\{K_i\}$ be a complete set of representatives of $Z$-orbits. As in the proof of Theorem 3.12 of [6], we obtain

$$\frac{|G : Z|}{\chi(1)} = \sum_i \omega_\chi(\hat{K}_i)\chi(x_i^{-1}),$$

where $x_i \in K_i$ for each $i$. This shows that $|G : Z|/\chi(1)$ is a rational integer, which is coprime to $p$ by our assumption. Let $K$ be a sufficiently large algebraic number field and let $P$ be a prime ideal of $K$ lying over $p$. Then there exists $i$ such that $\omega_\chi(\hat{K}_i)\chi(x_i^{-1}) \notin P$. This implies that $\chi$ has height 0 (cf. [2, IV 4.4]). So, if $D$ is a defect group of the block of $G$ containing $\chi$, then $|D| = |Z|$. Thus $D = Z$. This completes the proof. □

## 2. Proof of Theorem B

Watanabe [15, Proposition 1] proves essentially the following, which is stronger than Theorem B.

**Theorem B′.** *Suppose that a group $S$ acts on a group $G$ as automorphisms. Let $B$ be an $S$-invariant block of $G$ such that a defect group $D$ of $B$ is centralized by $S$. Assume that $(|S|, |N_G(D)/DC_G(D)|) = 1$. Then any irreducible character in $B$ is $S$-invariant.*

In this section we give a direct proof of Theorem B′. Let $\Gamma = SG$ be the semidirect product.

For a block $b$ of a normal subgroup $Y$ of a group $X$, let $\mathrm{BL}(X|b)$ be the set of blocks of $X$ covering $b$.

**Lemma 1.** *Let the notation be as in* Theorem B′. *Assume in addition that $S$ is cyclic.*

(i) *If $S$ is a $p'$-group, then* $|\mathrm{BL}(\Gamma|B)| = |S|$.
(ii) *If $S$ is a $p$-group, let $\hat{B}$ be a unique block of $\Gamma$ covering $B$. Then $\hat{B}$ has a defect group $R$ such that $R = DC_R(D)$.*

Proof. (i) Let $\tilde{B}$ be the Brauer correspondent of $B$ with respect to $D$ in $N_G(D)$. By the Harris–Knörr theorem [5], it suffices to show $|\mathrm{BL}(N_\Gamma(D) \mid \tilde{B})| = |S|$. We note that $N_\Gamma(D) = S \ltimes N_G(D)$, $\tilde{B}$ is $S$-invariant and $D$ is a defect group of $\tilde{B}$.

A slight modification of the proof of Proposition 1 of [15] shows that there exists an $S$-invariant block, $b$ say, of $C_G(D)$ covered by $\tilde{B}$. It is clear that a block of $N_\Gamma(D)$ covers $\tilde{B}$ if and only if it covers one of the blocks in $\mathrm{BL}(C_\Gamma(D)|b)$. For each block $\beta \in \mathrm{BL}(C_\Gamma(D)|b)$, there exists a unique block in $\mathrm{BL}(N_\Gamma(D)|\tilde{B})$ which covers $\beta$. Thus it suffices to show the following:
(1) $|\mathrm{BL}(C_\Gamma(D) \mid b)| = |S|$;
(2) No two distinct blocks in $\mathrm{BL}(C_\Gamma(D) \mid b)$ are $N_\Gamma(D)$-conjugate.

(1) Let $\zeta$ be the canonical character of $b$. Since $b$ is $S$-invariant, so is $\zeta$. Since $C_\Gamma(D) = S \ltimes C_G(D)$ and $S$ is cyclic, there exists an extension of $\zeta$ to $C_\Gamma(D)$. Let $\mathcal{E}$ be the set of extensions of $\zeta$ to $C_\Gamma(D)$. For any $\eta \in \mathcal{E}$, let $B(\eta)$ be the block of $C_\Gamma(D)$ containing $\eta$. Then $B(\eta)$ covers $b$. Since $C_\Gamma(D)/C_G(D)$ is a $p'$-group, $B(\eta)$ has defect group $Z(D)$. Therefore $\eta$ is the canonical character of $B(\eta)$. In particular, $B(\eta) \neq B(\eta')$ if $\eta, \eta' \in \mathcal{E}$ and $\eta \neq \eta'$. Clearly any block of $C_\Gamma(D)$ covering $b$ is of the form $B(\eta)$ for some $\eta \in \mathcal{E}$. Since $|\mathcal{E}| = |S|$, (1) follows.

(2) We claim first that any $\eta \in \mathcal{E}$ is $N_G(D)_\zeta$-invariant, where $N_G(D)_\zeta$ is the inertial group of $\zeta$ in $N_G(D)$. Indeed, for any $x \in N_G(D)_\zeta$, we have $\eta^x \in \mathcal{E}$. Thus $\eta^x = \eta \otimes \lambda_x$ for a unique $\lambda_x \in \mathrm{Irr}(C_\Gamma(D)/C_G(D)) = \mathrm{Irr}(S)$. Since $[C_\Gamma(D), N_G(D)_\zeta] \leq C_G(D)$, $\lambda_x$ is $N_G(D)_\zeta$-invariant. Therefore the map $x \mapsto \lambda_x$ is a group homomorphism from $N_G(D)_\zeta$ to $\mathrm{Irr}(S)$. Since this map is trivial on $C_G(D)$, it factors through $N_G(D)_\zeta/C_G(D)$. Since $(|S|, |N_G(D)_\zeta/C_G(D)|) = 1$, this map is a trivial homomorphism. Thus the claim is proved.

Now assume $B(\eta)^x = B(\eta')$ for $x \in N_\Gamma(D)$, $\eta, \eta' \in \mathcal{E}$. We may assume $x \in N_G(D)$. We have $\eta^x = \eta'$, so that $\zeta^x = \zeta$. Thus $x \in N_G(D)_\zeta$. Then $\eta = \eta'$ by the above, and (2) is proved. The proof of (i) is complete.

(ii) If $S = 1$, there is nothing to prove. So we assume $S > 1$. Let $B'$ be the Harris–Knörr correspondent of $\hat{B}$ over $B$ in $N_\Gamma(D)$. Then $B'$ and $\hat{B}$ have a defect group in common. We have $N_\Gamma(D) = SN_G(D) = DC_\Gamma(D)N_G(D)$. So $N_\Gamma(D)/DC_\Gamma(D) \simeq N_G(D)/DC_G(D)$, which is a $p'$-group by assumption. Thus if $\beta$ is a block of $DC_\Gamma(D)$ covered by $B'$, then a defect group $R$ of $\beta$ is a defect group of $B'$. Hence $R$ is a defect group of $\hat{B}$. Now $D \leq R \leq DC_\Gamma(D)$, so that $R = DC_R(D)$. The proof is complete. □

REMARK 1. As in the proof of Proposition 2 of [15], Lemma 1 (i) follows from [1].

REMARK 2. In Lemma 1, the conclusions of (i) and (ii) are in fact equivalent to the equality $S = S[B]$, where $S[B]$ is defined as in Proposition 1 of [15]. A proof will be given in a separate paper.

Proof of Theorem B′. We may assume that either $S$ is a cyclic $p'$-group or a cyclic $p$-group.

Assume that $S$ is a cyclic $p'$-group. Let $\zeta$ be any irreducible character in $B$. Let $T$ be the inertial group of $\zeta$ in $\Gamma$. Since any block of $\Gamma$ covering $B$ contains an irreducible character lying over $\zeta$, any block in $\mathrm{BL}(\Gamma|B)$ is induced from a block in $\mathrm{BL}(T|B)$ (cf. [10, Lemma 5.3.1 (ii)]). So $|\mathrm{BL}(\Gamma|B)| \leq |\mathrm{BL}(T|B)|$. Also, $|\mathrm{BL}(T|B)| \leq k(T|\zeta) = |T/G| \leq |S|$, where $k(T|\zeta)$ denotes the number of irreduciblle characters of $T$ lying over $\zeta$. Thus Lemma 1 (i) yields $|T/G| = |S|$. Hence $T = \Gamma$ and $\zeta$ is $S$-invariant.

Assume that $S$ is a cyclic $p$-group. Let $\hat{B}$ and $R$ be as in Lemma 1 (ii). Then by Lemma 4.14 (ii) of [8], any irreducible character in $B$ is $R$-invariant. On the other hand, since $\hat{B}$ is weakly regular with respect to $G$ and $B$ is $\Gamma$-invariant, $RG/G$ is a Sylow $p$-subgroup of $\Gamma/G$ by Fong's theorem. Thus $\Gamma = RG$. So any irreducible character in $B$ is $S$-invariant. This completes the proof. □

### 3. Proof of Theorem D

Theorem D is equivalent to the following theorem, cf. Remarks of [14].

**Theorem D′.** *Let $G$ be a $p$-solvable group with $O_{p'}(G) = 1$ and $k(G) = |D|$, where $D$ is a Sylow $p$-subgroup of $G$. Then $D$ is abelian.*

**Lemma 2** (Gallagher [3]). *Let $N$ be a normal subgroup of a group $G$. Then $k(G) \leq k(G/N)k(N)$ and equality holds if and only if $C_G(x \bmod N) = C_G(x)N$ for all $x \in G$. Furthermore if equality holds, then every irreducible character of $N$ is $G$-invariant.*

Proof. The first statement is (3) of [3]. If equality holds, then, as shown in the proof of (3) of [3, p. 176], every conjugacy class of $N$ is $G$-invariant. As is well-known, this implies that every irreducible character of $N$ is $G$-invariant. □

Proof of Theorem D′.. Let $N = O_{p,p'}(G)$. Then, since $O_{p'}(N) = O_{p'}(G/N) = 1$, by Theorem C and Lemma 2,

$$|D| = k(G) \leq k(G/N)k(N) \leq |G/N|_p |N|_p = |D|.$$

Thus equality holds throughout. So every irreducible character of $N$ is $G$-invariant by Lemma 2. Then as in the proof of Lemma 3 of [14], we see $G = N$. Thus $D$ is normal in $G$.

Let $\Phi$ be the Frattini subgroup of $D$. Then, as in Nagao [9], $O_{p'}(G/\Phi) = 1$. Thus by Theorem C and Lemma 2,

$$|D| = k(G) \leq k(G/\Phi)k(\Phi) \leq |G/\Phi|_p|\Phi| = |D|.$$

Thus equality holds throughout. Let $x \in D$. By Lemma 2, we have $C_G(x \mod \Phi) = C_G(x)\Phi$. Since $D \leq C_G(x \mod \Phi)$, we obtain $D \leq C_G(x)\Phi$. Thus $D = C_D(x)\Phi$ and $D = C_D(x)$. Since $x \in D$ is arbitrary, $D$ is abelian. This completes the proof. □

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