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# THE EXPECTED VOLUME AND SURFACE AREA OF THE WIENER SAUSAGE IN ODD DIMENSIONS

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**Abstract**

We consider the Wiener sausage up to time $t$ associated with a closed ball. A formula for the expected volume of the Wiener sausage is obtained in odd dimensions. In these cases, we also find that the formula leads to the asymptotic expansion for large $t$ and each coefficient is represented by zeros of a modified Bessel function of the second kind. Moreover we obtain a formula for the expected surface area of the Wiener sausage.

## 1. Introduction

In connection with heat conduction problems, the volume of the Wiener sausage on the time interval $[0, t]$ for a Brownian motion associated with a non-polar compact set has been investigated for a long time. The expected volume of the Wiener sausage is interpreted as the total energy flow from the non-polar set. For large $t$ it is asymptotically equal to $2\pi t/\log t$ in the two dimensional case, which is given in [16], and $t$ multiple of the capacity of the non-polar set in higher dimensions, which can be found in [4] and [16]. In addition, Le Gall [11] provided several lower terms and Port [13] discussed the same problem for a stable sausage.

Some results on limit theorems for the volume of the Wiener sausage have been established. The law of large numbers was proved by Whitman in three or more dimensions, which is described in [8], and by Le Gall [9] in the two dimensional case. Le Gall [11] also established the central limit theorem. The results concerning large deviations are given in [1], [3] and [6]. Especially, the result on the Laplace transform of the volume of the Wiener sausage given in [3] are very useful for the investigation on random Schrödinger operators and Brownian motions in random environments. These are discussed in [15].

This article deals with the Wiener sausage associated with a closed ball with radius $r$ in odd dimensional cases. For $t > 0$ let $V_r(t)$ be the expected volume of the Wiener sausage up to time $t$. It is easy to evaluate it explicitly for dimension one. Then our interest turns to higher dimensional cases. The compact form of $V_r(t)$ was given in

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[16] for dimension three. Recently Hamana [5] represented $V_r(t)$ by the complementary error function for dimensions five and seven. Section 3 is devoted to the formula for $V_r(t)$ in other dimensional cases. We succeed in representing $V_r(t)$ by zeros of a modified Bessel function of the second kind.

**Theorem 1.1.** *If $d$ is odd and more than or equal to five, we have that*

$$V_r(t) = S_{d-1} r^{d-2} \left[ \frac{(d-2)t}{2} + \frac{2(d-2)r^2}{d(d-4)} + \frac{2r^3}{\sqrt{2\pi t}} \sum_{j=1}^{N} \frac{1}{(z_j^N)^3} - \frac{2r^5}{\sqrt{2\pi t^3}} \sum_{j=1}^{N} \frac{1}{(z_j^N)^3} \int_0^\infty x \exp\left( -\frac{r^2 x^2}{2t} + z_j^N x \right) dx \right]$$

*for $r > 0$ and $t > 0$, where $S_{d-1}$ is the surface area of $d-1$ dimensional unit sphere, $N = (d-3)/2$ and $z_1^N, z_2^N, \ldots, z_N^N$ are zeros of the modified Bessel function of the second kind of order $N + 1/2$.*

The main tool is the decomposition of the Laplace transform of $V_r$ into several rational functions. We remark that, if $w$ is one of zeros of $K_{N+1/2}$, the complex conjugate of $w$ is also a zero of $K_{N+1/2}$. This fact yields that two summations in the statement of Theorem 1.1 are real.

Section 4 deals with the asymptotic expansion of $V_r(t)$ as $t \to \infty$ in five or more dimensional cases. In addition, it can be proved that all coefficients are expressed by zeros of the modified Bessel function. The explicit form of $V_r(t)$ given in Section 3 plays an important role for calculations. Hamana [5] also proved that $V_r(t)$ can be represented as the absolutely convergent power series of $t^{1/2}$ for any $t > 0$. The coefficients are given inductively in [5] and this implies that we can evaluate them explicitly in principle. Actually they have very complicated forms. We obtain that the coefficients in the power series expansion are also represented by zeros of the modified Bessel function, which is also described in Section 4.

On the other hand, Rataj, Schmidt and Spodarev [14] proved that the expected surface area of the Wiener sausage coincides with the first derivative of $V_r(T)$ with respect to the radius $r$ for fixed $T > 0$. Similarly to the expected volume, we obtain the explicit form and the power series expansion of the expected surface area in Section 5. Unfortunately we could not succeed in giving the asymptotic expansion for large $t$.

## 2. Notation and preliminaries

Let $r$ be a positive number and $B = \{B(t)\}_{t \geq 0}$ be a Brownian motion on $\mathbb{R}^d$. The Wiener sausage for $B$ with radius $r$ is the process defined as

$$C_r(t) = \{x \in \mathbb{R}^d;\ x + B(s) \in D_r \text{ for some } s \in [0, t]\}$$

for $t \geqq 0$, where $D_r$ is the closed ball with center 0 and radius $r$. It is easy to see that the expected volume of $C_r(t)$, which was denoted by $V_r(t)$ in the previous section, is represented as

$$V_r(t) = \int_{\mathbb{R}^d} P_x[\tau_r \leqq t]\, dx,$$

where $\tau_r = \inf\{t \geqq 0;\ B(t) \in D_r\}$ and $P_x$ is the probability measure of events related to the Brownian motion starting from $x \in \mathbb{R}^d$. For $t > 0$ let

$$L_r(t) = \int_{\mathbb{R}^d \setminus D_r} P_x[\tau_r \leqq t]\, dx.$$

It is obvious that

$$V_r(t) = L_r(t) + \frac{S_{d-1} r^d}{d} \tag{2.1}$$

since the volume of $D_r$ is equal to $S_{d-1} r^d / d$.

According to the result in [7], $P_x[\tau_r \leqq t]$ is the unique solution of the heat conduction problem

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \Delta u(t, x)$$

for $t > 0$ and $x \in \mathbb{R}^d \setminus D_r$ subject to the initial condition $u(0, x) = 0$ for $x \in \mathbb{R}^d \setminus D_r$ and the boundary condition $u(t, y) = 1$ for $t > 0$ and $y \in D_r$. Hence $P_x[\tau_r \leqq t]$ is interpreted as the temperature at time $t$ at the point $x \in \mathbb{R}^d$. Then $L_r(t)$ is the total energy flow in time $t$ from $D_r$ into $\mathbb{R}^d \setminus D_r$.

If $d = 1$, of course, we can evaluate $L_r(t)$ with the help of the formula

$$P_x[\tau_r \leqq t] = \int_0^t \frac{|x| - r}{\sqrt{2\pi s^3}} \exp\left[ -\frac{(|x| - r)^2}{2s} \right] ds$$

for $|x| > r$, which is given in [8]. Hence we obtain that $V_r(t) = 2\sqrt{2t/\pi} + 2r$. For dimension three Spitzer [16] showed that $V_r(t) = 2\pi r t + 4r^2 \sqrt{2\pi t} + 4\pi r^3/3$. This can be also derived directly by the following well-known formula:

$$P_x[\tau_r \leqq t] = \frac{r(\|x\| - r)}{\|x\|} \int_0^t \frac{1}{\sqrt{2\pi s^3}} \exp\left[ -\frac{(\|x\| - r)^2}{2s} \right] ds$$

for $\|x\| > r$, which is described in [11] for example. The notation $\|x\|$ has been used to denote the Euclidean norm of $x \in \mathbb{R}^d$.

In higher dimensions, we have no useful formula for the distribution of $\tau_r$ except for the Laplace transform. It follows that

$$E_x[e^{-\lambda \tau_r}] = \frac{\|x\|^{-d/2+1} K_{d/2-1}(\|x\| \sqrt{2\lambda})}{r^{-d/2+1} K_{d/2-1}(r\sqrt{2\lambda})} \tag{2.2}$$

for $\lambda > 0$ and $\|x\| \geqq r$, where $E_x$ denotes the expectation under the probability measure $P_x$ and $K_\nu$ is the modified Bessel function of the second kind of order $\nu$. This formula can be found in [2] and [8].

Throughout this paper, for a suitable function $f$, the notation $\mathcal{L}[f]$ denotes the Laplace transform of $f$ and the inverse Laplace transform of $f$ is denoted by $\mathcal{L}^{-1}[f]$. With the help of (2.2), Hamana [5] showed that

$$\mathcal{L}[L_r](\lambda) = \frac{S_{d-1} r^{d-1}}{\sqrt{2\lambda^3}} \frac{K_{d/2}(r\sqrt{2\lambda})}{K_{d/2-1}(r\sqrt{2\lambda})} \tag{2.3}$$

for $\lambda > 0$.

The remainder of this section is devoted to giving some properties of modified Bessel functions. In general, $K_\nu$ is the function defined on $\mathbb{C} \setminus \{0\}$ for each complex number $\nu$. In this paper, however, it is sufficient to consider the case that $\nu$ is a half integer since we treat odd dimensional cases. In these cases, it is well-known that $K_{n+1/2}$ has the following explicit form for each integer $n \geqq 0$:

$$K_{n+1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[ \sum_{m=0}^{n} \frac{\langle n, m \rangle}{(2z)^m} \right], \tag{2.4}$$

where the branch of $\sqrt{z}$ is principal and

$$\langle n, m \rangle = \begin{cases} \dfrac{(n+m)!}{m!\,(n-m)!} & \text{if } n \geqq m, \\ 0 & \text{if } n < m. \end{cases}$$

In addition, the number of zeros of $K_{n+1/2}$ is $n$ and each zero lies in the half plain $\{z \in \mathbb{C};\, \mathrm{Re}\, z < 0\}$, denoted by $\mathbb{C}^-$. These are all described in [17]. Recall that $K_\nu$ is one of solutions of the modified Bessel differential equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 + \nu^2) w = 0.$$

Thus we obtain that all zeros of $K_\nu$ are of multiplicity one by the uniqueness of the solution of ordinary differential equations. This immediately implies that $K_{n+1/2}$ has exactly $n$ zeros with multiplicity one in $\mathbb{C}^-$ for $n \geqq 1$.

## 3. Formula for the mean volume of the Wiener sausage

We consider the case that $d$ is odd and not less than five. Our goal in this section is to give a proof of Theorem 1.1. Recall that $N = (d-3)/2$ and then $N$ is a positive integer. It follows from (2.3) that

$$\mathcal{L}[L_r](\lambda) = \frac{S_{d-1} r^{d-1}}{\sqrt{2\lambda^3}} \frac{K_{N+3/2}(r\sqrt{2\lambda})}{K_{N+1/2}(r\sqrt{2\lambda})}. \tag{3.1}$$

For $t > 0$ let $T_r(t) = L_r(2r^2 t)$. Since $\mathcal{L}[T_r](\lambda) = (1/2r^2)\mathcal{L}[L_r](\lambda/2r^2)$, then

$$\mathcal{L}[T_r](\lambda) = \frac{S_{d-1} r^d}{\sqrt{\lambda^3}} \frac{K_{N+3/2}(\sqrt{\lambda})}{K_{N+1/2}(\sqrt{\lambda})}. \tag{3.2}$$

We first intend to represent the right hand side of (3.2) as the sum of several rational functions of $\sqrt{\lambda}$. For $n \geqq 1$ and $z \in \mathbb{C}$ let

$$H_n(z) = \sum_{k=0}^{n} \frac{\langle n, n-k \rangle}{2^{n-k}} z^k. \tag{3.3}$$

Note that $H_n$ is a monic polynomial of degree $n$. It follows from (2.4) and (3.3) that

$$K_{n+1/2}(z) = \sqrt{\frac{\pi}{2z}} \frac{e^{-z}}{z^n} H_n(z) \tag{3.4}$$

for $n \geqq 1$. This yields that the ratio of modified Bessel functions can be represented as that of polynomials. Namely we obtain that

$$\frac{K_{n+3/2}(z)}{K_{n+1/2}(z)} = \frac{H_{n+1}(z)}{z H_n(z)}. \tag{3.5}$$

Therefore it is sufficient to consider the partial fraction decomposition of the right hand side of (3.5) in the case that $n = N$.

It follows from (3.4) that zeros of $H_N$ coincide with those of $K_{N+1/2}$ including multiplicities. Recall that we have written $z_1^N, z_2^N, \ldots, z_N^N$ for the zeros of $K_{N+1/2}$, and these are, of course, zeros of $H_N$. Since each zero is of multiplicity one, $H_N$ has the following form:

$$H_N(z) = \prod_{j=1}^{N} (z - z_j^N).$$

In virtue of this factorization, we obtain that

$$\frac{H_{N+1}(z)}{z H_N(z)} = 1 + \frac{\sigma_0}{z} + \sum_{j=1}^{N} \frac{\sigma_j}{z - z_j^N} \tag{3.6}$$

for some suitable sequence $\{\sigma_j\}_{j=0}^N$ of complex numbers. The following lemma is quite useful for determining them.

**Lemma 3.1.** *If $w \in \mathbb{C}^-$ is a zero of $H_N$, then*

$$H_{N+1}(w) = -w H_N'(w). \tag{3.7}$$

Proof. Note that $H_1(z) = z + 1$ and $H_2(z) = z^2 + 3z + 3$. Hence (3.7) is obvious in the case that $N = 1$.

We now prove (3.7) if $N \geqq 2$. It is sufficient to establish that

$$H_{N+1}(z) - z^2 H_{N-1}(z) = (2N + 1)H_N(z), \tag{3.8}$$

$$H'_N(z) + z H_{N-1}(z) = H_N(z). \tag{3.9}$$

It follows from (3.3) that the left hand side of (3.8) is equal to

$$\sum_{k=2}^{N+1}\left[\frac{\langle N+1, N-k+1\rangle}{2^{N-k+1}} - \frac{\langle N-1, N-k+1\rangle}{2^{N-k+1}}\right]z^k + \frac{\langle N+1, N\rangle}{2^N}z + \frac{\langle N+1, N+1\rangle}{2^{N+1}}.$$

It is obvious that the coefficient of $z^{N+1}$ is 0. For $2 \leqq k \leqq N$ a simple calculation shows that the coefficient of $z^k$ is $(2N + 1)\langle N, N-k\rangle/2^{N-k}$, which coincides with the coefficient of corresponding term in $(2N + 1)H_N(z)$. Moreover it is easy to see

$$\frac{\langle N+1, N\rangle}{2^N} = (2N+1)\frac{\langle N, N-1\rangle}{2^{N-1}},$$
$$\frac{\langle N+1, N+1\rangle}{2^{N+1}} = (2N+1)\frac{\langle N, N\rangle}{2^N}.$$

Therefore we can conclude (3.8).

The proof of (3.9) is similar to that of (3.8). It follows from (3.3) that the left hand side of (3.9) is equal to

$$z^N + \sum_{k=1}^{N-1}\left[\frac{(k+1)\langle N, N-k-1\rangle}{2^{N-k-1}} + \frac{\langle N-1, N-k\rangle}{2^{N-k}}\right]z^k + \frac{\langle N, N-1\rangle}{2^{N-1}}.$$

It is easy to obtain that the coefficient of $z^k$ is $\langle N, N-k\rangle/2^{N-k}$ for $0 \leqq k \leqq N$. □

By reduction of the fractions in the right hand side of (3.6) to a common denominator, we have that

$$\frac{H_{N+1}(z)}{z H_N(z)} = 1 + \frac{\sigma_0 \prod_{j=1}^N (z - z_j^N) + z \sum_{j=1}^N \sigma_j \prod_{i\neq j}(z - z_i^N)}{z \prod_{j=1}^N (z - z_j^N)}.$$

Then we may determine $\sigma_0, \sigma_1, \ldots, \sigma_N$ satisfying that

$$H_{N+1}(z) = z H_N(z) + \sigma_0 H_N(z) + z \sum_{j=1}^N \sigma_j \prod_{i\neq j}(z - z_i^N). \tag{3.10}$$

We put $z = 0$ in (3.10) and have that

$$\frac{\langle N+1, N+1\rangle}{2^{N+1}} = \sigma_0 \frac{\langle N, N\rangle}{2^N},$$

which yields that $\sigma_0 = 2N + 1$. For $1 \leqq k \leqq N$ it follows from (3.10) that

$$H_{N+1}(z_k^N) = z_k^N \sum_{j=1}^{n} \sigma_j \prod_{i \neq j} (z_k^N - z_i^N). \tag{3.11}$$

It is easy to see that the right hand side of (3.11) is

$$z_k^N \sigma_k \prod_{i \neq k} (z_k^N - z_i^N).$$

Note that

$$\prod_{i \neq k} (z_k^N - z_i^N) = \lim_{z \to z_k^N} \frac{H_N(z)}{z - z_k^N} = H_N'(z_k^N).$$

Then Lemma 3.1 shows that

$$-z_k^N H_N'(z_k^N) = z_k^N \sigma_k H_N'(z_k^N).$$

Since all zeros of $H_N$ are of multiplicity one, $H_N'(z_j^N) \neq 0$ for $1 \leqq j \leqq N$. Thus we can conclude that $\sigma_k = -1$ for $1 \leqq k \leqq N$. This implies that we have finished showing the following lemma.

**Lemma 3.2.** *We have that*

$$\frac{K_{N+3/2}(z)}{K_{N+1/2}(z)} = 1 + \frac{2N+1}{z} - \sum_{j=1}^{N} \frac{1}{z - z_j^N}.$$

For an integer $n \geqq 1$ let

$$\zeta_n^{(d)} = \sum_{j=1}^{N} \frac{1}{(z_j^N)^n}.$$

We have remarked in Section 1 that, if $w$ is one of zeros of $K_{N+1/2}$, the complex conjugate of $w$ is also a zero of $K_{N+1/2}$. Hence $\zeta_n^{(d)}$ is real for each $n \geqq 1$. We are now ready to establish the following proposition.

**Proposition 3.3.** *We have that*

$$\mathcal{L}[T_r](\lambda) = S_{d-1}r^d\left[\frac{2N+1}{\lambda^2} + \frac{1}{(2N-1)\lambda} + \frac{\zeta_3^{(d)}}{\lambda^{1/2}} - \sum_{j=1}^{N}\frac{1}{(z_j^N)^3(\lambda^{1/2}-z_j^N)}\right]$$

*for* $\lambda > 0$.

Proof. For simplicity, let $k_d = S_{d-1}r^d$. By (3.2) and Lemma 3.2, we obtain that

$$\mathcal{L}[T_r](\lambda) = k_d\left[\frac{1}{\lambda^{3/2}} + \frac{2N+1}{\lambda^2} - \frac{1}{\lambda^{3/2}}\sum_{j=1}^{N}\frac{1}{\lambda^{1/2}-z_j^N}\right]. \tag{3.12}$$

The equality

$$\frac{1}{z^3(z-w)} = -\frac{1}{z^3w} - \frac{1}{z^2w^2} - \frac{1}{zw^3} + \frac{1}{(z-w)w^3}$$

yields that the right hand side of (3.12) is equal to

$$k_d\left[\left(1+\sum_{j=1}^{n}\frac{1}{z_j^N}\right)\frac{1}{\lambda^{3/2}} + \frac{2N+1}{\lambda^2} + \sum_{j=1}^{N}\frac{1}{(z_j^N)^2\lambda} + \sum_{j=1}^{N}\frac{1}{(z_j^N)^3\lambda^{1/2}} - \sum_{j=1}^{N}\frac{1}{(z_j^N)^3(\lambda^{1/2}-z_j^N)}\right].$$

Therefore $\mathcal{L}[T_r](\lambda)$ is expressed by

$$\frac{k_d(1+\zeta_1^{(d)})}{\lambda^{3/2}} + k_d\left[\frac{2N+1}{\lambda^2} + \frac{\zeta_2^{(d)}}{\lambda} + \frac{\zeta_3^{(d)}}{\lambda^{1/2}} - \sum_{j=1}^{N}\frac{1}{(z_j^N)^3(\lambda^{1/2}-z_j^N)}\right]. \tag{3.13}$$

It remains to evaluate $\zeta_1^{(d)}$ and $\zeta_2^{(d)}$. The fact that $-1$ is the unique zero of $H_1$ shows that $\zeta_n^{(5)} = (-1)^n$ for each $n \geqq 1$. This immediately implies the assertion of Proposition 3.3 for dimension five. Thus we may concentrate on considering the higher dimensional cases. Note that $N \geqq 2$ in these cases. The formula

$$\frac{H_N'(z)}{H_N(z)} = \sum_{j=1}^{N}\frac{1}{z-z_j^N} \tag{3.14}$$

shows that

$$\sum_{j=1}^{N}\frac{1}{z_j^N} = -\frac{H_N'(0)}{H_N(0)}.$$

Since $H_N(0) = \langle N, N\rangle/2^N$ and $H_N'(0) = \langle N, N-1\rangle/2^{N-1}$, we have $\zeta_1^{(d)} = -1$. This implies that the first term of (3.13) vanishes. Applying (3.14) again, we have that

$$\frac{H_N''(z)}{H_N(z)} - \left[\frac{H_N'(z)}{H_N(z)}\right]^2 = -\sum_{j=1}^{N} \frac{1}{(z - z_j^N)^2},$$

which yields that

$$\zeta_2^{(d)} = [\zeta_1^{(d)}]^2 - \frac{H_N''(0)}{H_N(0)} = \frac{1}{2N-1}.$$

This completes the proof of Proposition 3.3. □

It is easy to give an explicit form of $V_r(t)$. For $\lambda > 0$ and $w \in \mathbb{C}$ let

$$f(\lambda) = \frac{1}{\lambda^{1/2} - w}.$$

It is well-known that

$$\mathcal{L}^{-1}[f](t) = \frac{1}{2\sqrt{\pi t^3}} \int_0^\infty x \exp\left(-\frac{x^2}{4t} + wx\right) dx,$$

which can be found in [13] for example. It follows from Proposition 3.3 that

$$T_r(t) = S_{d-1} r^d \left[ \frac{(2N+1)t}{\Gamma(2)} + \frac{1}{(2N-1)\Gamma(1)} + \frac{\zeta_3^{(d)}}{\Gamma(1/2)t^{1/2}} - \frac{1}{2\sqrt{\pi t^3}} \sum_{j=1}^{N} \frac{1}{(z_j^N)^3} \int_0^\infty x \exp\left(-\frac{x^2}{4t} + z_j^N x\right) dx \right],$$

where $\Gamma$ is the gamma function. Recall that $T_r(t) = L_r(2r^2 t)$. In virtue of (2.1), we can immediately derive the assertion of Theorem 1.1.

## 4. Asymptotic expansion and power series expansion

In this section we again consider odd dimensional cases. Since the compact forms of $V_r(t)$ in one and three dimensions have been given in Section 2, it is sufficient to consider higher dimensional cases.

One of our purpose in this section is to establish the following theorem.

**Theorem 4.1.** *Let $r > 0$ be fixed and $M$ be a given integer which is not less than $(d-5)/2$. If $d$ is odd and not less than five, we have that*

$$V_r(t) = S_{d-1}r^{d-2}\left[\frac{(d-2)t}{2} + \frac{2(d-2)r^2}{d(d-4)} + \frac{2r^3}{\sqrt{2\pi t}}\sum_{n=(d-5)/2}^{M}\frac{\gamma_n\zeta_{2n+3}^{(d)}}{t^n}\right] + O\left[\frac{1}{t^{M+3/2}}\right]$$

*as $t \to \infty$, where $\gamma_0 = 1$ and $\gamma_n = (-1)^n r^{2n}(2n-1)!!$ for $n \geqq 1$.*

Before giving a proof of Theorem 4.1, we consider five and seven dimensional cases. If $d = 5$, we have shown that $\zeta_n^{(5)} = (-1)^n$ for $n \geqq 1$. Hence it follows from Theorem 4.1 that, if $d = 5$,

$$V_r(t) \sim S_4 r^3\left[\frac{3t}{2} + \frac{6r^2}{5} - \sqrt{\frac{2}{\pi}}\left(\frac{r^3}{t^{1/2}} - \frac{r^5}{t^{3/2}} + \frac{3r^7}{t^{5/2}} - \frac{15r^9}{t^{7/2}} - \cdots\right)\right].$$

If $d = 7$, then $N = 2$. Since $H_2(z) = z^2 + 3z + 3$, both $1/z_1^2$ and $1/z_2^2$ are zeros of $3z^2 + 3z + 1$. Hence we have $\zeta_1^{(7)} = -1$ and $\zeta_2^{(7)} = 1/3$. The Newton formula yields that $\{\zeta_n^{(7)}\}_{n=1}^{\infty}$ is the solution of

$$\begin{cases} a_{n+2} + a_{n+1} + \dfrac{a_n}{3} = 0 \quad (n \geqq 1) \\ a_1 = -1, \quad a_2 = \dfrac{1}{3}. \end{cases}$$

In particular, $\zeta_3^{(7)} = \zeta_9^{(7)} = 0$. It follows from Theorem 4.1 that, if $d = 7$,

$$V_r(t) \sim S_6 r^5\left[\frac{5t}{2} + \frac{10r^2}{21} - \sqrt{\frac{2}{\pi}}\left(\frac{r^5}{9t^{3/2}} - \frac{r^7}{9t^{5/2}} + \frac{35r^{11}}{81t^{9/2}} - \frac{35r^{13}}{27t^{11/2}} + \cdots\right)\right].$$

We begin to prove Theorem 4.1. Recall that we have already derived the explicit form of $V_r(t)$. According to Theorem 1.1, it is sufficient to consider

$$\int_0^{\infty} x\exp\left(-\frac{r^2x^2}{2t} + zx\right)dx$$

for $z \in \mathbb{C}^-$, which is denoted by $G_r(t, z)$. If $d = 5$, we need to treat the case that $M = 0$. Since $|G_r(t, z)|$ is dominated by

$$\int_0^{\infty} xe^{-|\mathrm{Re}\, z|x}\,dx = \frac{1}{|\mathrm{Re}\, z|^2}$$

uniformly for $t > 0$, Theorem 4.1 can be immediately obtained by Theorem 1.1 if $d = 5$ and $M = 0$.

We concentrate on considering the case that $M$ is positive and not less than $(d-5)/2$. The Taylor theorem implies that

$$e^{-x} = \sum_{n=0}^{M-1} \frac{(-1)^n}{n!} x^n + R_M(x)$$

for $x > 0$, where $R_M(x) = x^M e^{-\theta x}/M!$ for some $\theta \in [0, 1]$. Then it follows that

$$G_r(z, t) = \sum_{n=0}^{M-1} \frac{(-1)^n r^{2n}}{n!\,(2t)^n} \int_0^\infty x^{2n+1} e^{zx}\, dx + \int_0^\infty R_M\left(\frac{r^2x^2}{2t}\right) x e^{zx}\, dx. \tag{4.1}$$

Since $\operatorname{Re} z < 0$, the integral in the first term of (4.1) can be evaluated and thus we obtain that the first term of (4.1) is equal to

$$\sum_{n=0}^{M-1} \frac{(-1)^n r^{2n}(2n+1)!}{n!\,(2t)^n z^{2n+2}} = \sum_{n=0}^{M-1} \frac{(-1)^n r^{2n}(2n+1)!!}{t^n z^{2n+2}}.$$

The estimate of the second term of (4.1) is easy. Indeed, we have that

$$\left| \int_0^\infty R_M\left(\frac{r^2x^2}{2t}\right) x e^{zx}\, dx \right| \leqq \frac{r^{2M}}{M!\,(2t)^M} \int_0^\infty x^{2M+1} e^{-|\operatorname{Re} z|x}\, dx$$

by the fact that $|R_M(x)| \leqq x^M/M!$ for $x \geqq 0$. Therefore it follows that

$$G_r(z, t) = \sum_{m=1}^{M} \frac{(-1)^{m-1} r^{2m-2}(2m-1)!!}{t^{m-1} z^{2m}} + O\left[\frac{1}{t^M}\right],$$

which yields that

$$\frac{r^5}{\sqrt{2\pi t^3}} \sum_{j=1}^{N} \frac{G_r(z_j^N, t)}{(z_j^N)^3} = \frac{r^3}{\sqrt{2\pi t}} \sum_{m=1}^{M} \frac{(-1)^{m-1} r^{2m}(2m-1)!!\, \zeta_{2m+3}^{(d)}}{t^m} + O\left[\frac{1}{t^{M+3/2}}\right].$$

Hence we can conclude that $V_r(t)$ is equal to

$$S_{d-1} r^{d-2} \left[ \frac{(d-2)t}{2} + \frac{2(d-2)r^2}{d(d-4)} + \frac{2r^3}{\sqrt{2\pi t}} \sum_{n=0}^{M} \frac{\gamma_n \zeta_{2n+3}^{(d)}}{t^n} \right] + O\left[\frac{1}{t^{M+3/2}}\right]. \tag{4.2}$$

It remains to see that the summation on $n$ in the right hand side of (4.2) begins from $(d-5)/2$. However it has been already proved by Le Gall [11] under the general situation. This completes the proof of Theorem 4.1.

REMARK 4.2. Recall that $d = 2N + 3$. The Newton formula implies that

$$\begin{cases} \zeta_n^{(d)} + a_1\zeta_{n-1}^{(d)} + a_2\zeta_{n-2}^{(d)} + \cdots + a_{n-1}\zeta_1^{(d)} = -na_n & \text{if } 1 \leqq n \leqq N, \\ \zeta_n^{(d)} + a_1\zeta_{n-1}^{(d)} + a_2\zeta_{n-2}^{(d)} + \cdots + a_N\zeta_{n-N}^{(d)} = 0 & \text{if } n \geqq N+1, \end{cases} \tag{4.3}$$

where $a_k = 2^k\langle N, N-k\rangle/\langle N, N\rangle$ for $1 \leqq k \leqq N$. It is easy to see by (4.3) that $\zeta_3^{(d)}$ is equal to $-1$ if $d = 5$ and 0 if $d \geqq 7$. In seven or more dimensions, we expect to show that $\zeta_{2n+1}^{(d)} = 0$ for $1 \leqq n \leqq N-1$ and that $\zeta_{2N+1}^{(d)} \neq 0$ by only (4.3) without using the result by Le Gall. However we could not succeed in showing them.

We now discuss the coefficients in the power series expansion of $V_r(t)$. In virtue of (2.1), we may concentrate on considering $L_r(t)$. If $d$ is odd and not less than five, Hamana [5] recently proved that

$$L_r(t) = \frac{S_{d-1}r^{d-1}}{\sqrt{2}} \sum_{n=1}^{\infty} \alpha_n^{(d)} t^{n/2} \tag{4.4}$$

for any $t > 0$ and that the right hand side of (4.4) converges absolutely, where $\{\alpha_n^{(d)}\}_{n=1}^{\infty}$ is the sequence of real numbers defined as

$$\alpha_n^{(d)} = \frac{\beta_{n-1}^{(d)}}{(\sqrt{2}r)^{n-1}\Gamma(n/2+1)} \tag{4.5}$$

and the sequence $\{\beta_n^{(d)}\}_{n=0}^{\infty}$ is determined by

$$\frac{1}{2^k}\left\langle \frac{d-1}{2}, k \right\rangle = \sum_{j=0}^{k} \frac{1}{2^{k-j}} \left\langle \frac{d-3}{2}, k-j \right\rangle \beta_j^{(d)}$$

for $k \geqq 0$.

Another purpose in this section is to represent $\alpha_n^{(d)}$ by zeros of a modified Bessel function. Recall that $z_1^N, z_2^N, \ldots, z_N^N$ are zeros of $K_{N+1/2}$. Let

$$\delta_N = \max_{1 \leqq j \leqq N} |z_j^N|. \tag{4.6}$$

Since $z_j^N \neq 0$ for any $1 \leqq j \leqq N$, we have that $\delta_N > 0$. Lemma 3.2 shows that

$$\begin{aligned} \frac{K_{N+3/2}(z)}{K_{N+1/2}(z)} &= 1 + \frac{2N+1}{z} - \frac{1}{z}\sum_{j=1}^{N}\sum_{n=0}^{\infty}\left(\frac{z_j^N}{z}\right)^n \\ &= 1 + \frac{N+1}{z} - \sum_{n=1}^{\infty}\frac{\xi_n^{(d)}}{z^{n+1}} \end{aligned}$$

for $|z| > \delta_N$, where

$$\xi_n^{(d)} = \sum_{j=1}^{N} (z_j^N)^n.$$

This implies that

$$\mathcal{L}[T_r](\lambda) = S_{d-1} r^d \left[ \frac{1}{\lambda^{3/2}} + \frac{d-1}{2\lambda^2} - \sum_{n=1}^{\infty} \frac{\xi_n^{(d)}}{\lambda^{(n+4)/2}} \right]$$

for $\lambda > \delta_N^2$. Note that $\xi_n^{(d)}$ is real for each $n \geqq 1$. Since $\mathcal{L}[L_r](\lambda) = 2r^2 \mathcal{L}[T_r](2r^2\lambda)$, we have that

$$\mathcal{L}[L_r](\lambda) = \frac{S_{d-1} r^{d-1}}{\sqrt{2}} \left[ \frac{1}{\lambda^{3/2}} + \frac{d-1}{4r^2\lambda^2} - \sum_{n=3}^{\infty} \frac{\xi_{n-2}^{(d)}}{(\sqrt{2}r)^{n-1} \lambda^{(n+2)/2}} \right] \tag{4.7}$$

for $\lambda > \delta_N^2/2r^2$.

On the other hand, Hamana [5] proved that there is a constant $\kappa > 0$ such that

$$\mathcal{L}[L_r](\lambda) = \frac{S_{d-1} r^{d-1}}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{\alpha_n^{(d)} \Gamma(n/2+1)}{\lambda^{n/2+1}} \tag{4.8}$$

for $\lambda > \kappa$. By the comparison of corresponding coefficient in (4.7) and (4.8), we obtain the following theorem.

**Theorem 4.3.** *Let $r > 0$ be fixed. If $d$ is odd and not less than five, we have that*

$$V_r(t) = S_{d-1} r^d \sum_{n=0}^{\infty} \eta_n^{(d)} t^{n/2} \tag{4.9}$$

*for any $t > 0$ and that the right hand side of* (4.9) *converges absolutely, where*

$$\eta_0^{(d)} = \frac{1}{d}, \quad \eta_1^{(d)} = \frac{1}{r}\sqrt{\frac{2}{\pi}}, \quad \eta_2^{(d)} = \frac{d-1}{4r^2}$$

*and*

$$\eta_n^{(d)} = -\frac{\xi_{n-2}^{(d)}}{(\sqrt{2}\, r)^n \Gamma(n/2+1)}$$

*for $n \geqq 3$.*

We remark that $\beta_n^{(d)} = -\xi_{n-1}^{(d)}$ for $n \geqq 2$, which can be derived by (4.5) and Theorem 4.3.

## 5. Expected surface area of the Wiener sausage

Let $T > 0$ be fixed. The notation $S_r(T)$ is used to denote the expected surface area of $C_r(T)$. Rataj, Schmidt and Spodarev [14] proved that

$$S_r(T) = \frac{\partial}{\partial r} V_r(T) \tag{5.1}$$

for any $r > 0$ in two and three dimensions. In four or more dimensional cases, the formula (5.1) holds at least for almost all $r > 0$. In particular, if $d = 3$, they remarked that $S_r(T) = 2\pi T + 8r\sqrt{2\pi T} + 4\pi r^2$. This can be also obtained by the formula for $V_r(T)$ given in [16].

**Theorem 5.1.** *If $d$ is odd and more than or equal to five, we have that*

$$\begin{aligned} S_r(T) = S_{d-1} r^{d-3} \Bigg[ & \frac{(d-2)^2 T}{2} + \frac{2(d-2)r^2}{d-4} + \frac{2(d+1)r^3}{\sqrt{2\pi T}} \sum_{j=1}^{N} \frac{1}{(z_j^N)^3} \\ & - \frac{2(d+3)r^5}{\sqrt{2\pi T^3}} \sum_{j=1}^{N} \frac{1}{(z_j^N)^3} \int_0^\infty x \exp\left( -\frac{r^2 x^2}{2T} + z_j^N x \right) dx \\ & + \frac{2r^7}{\sqrt{2\pi T^5}} \sum_{j=1}^{N} \frac{1}{(z_j^N)^3} \int_0^\infty x^3 \exp\left( -\frac{r^2 x^2}{2T} + z_j^N x \right) dx \Bigg] \end{aligned}$$

*at least for almost all $r > 0$.*

Proof. Theorem 1.1 and (5.1) imply that it is sufficient to consider the derivative of $G_r(z, T)$ with respect to $r$. We can easily show this theorem with the help of the formula

$$\frac{\partial}{\partial r} G_r(z, t) = -\frac{r}{T} \int_0^\infty x^3 \exp\left( -\frac{r^2 x^2}{2T} + zx \right) dx \tag{5.2}$$

for $z \in \mathbb{C}^-$ and $r > 0$. This formula can be obtained by a standard argument for justification of the interchange of differentiation and integration. Indeed, we have that

$$\left| \frac{\partial}{\partial r} x \exp\left( -\frac{r^2 x^2}{2T} + zx \right) \right| \leqq \frac{R x^3}{T} e^{-|\mathrm{Re}\, z| x}$$

for $0 < r < R$ with a given $R > 0$. The right hand side is an integrable function which is independent of $r$. This yields that (5.2) is valid for $0 < r < R$. Since $R > 0$ is arbitrary, the formula (5.2) holds for any $r > 0$. □

In addition, we can represent $S_r(T)$ as the power series of $T^{1/2}$.

**Theorem 5.2.** *If $d$ is odd and not less than five, we have that*

$$S_r(T) = S_{d-1} r^{d-1} \sum_{n=0}^{\infty} (d-n) \eta_n^{(d)} T^{n/2} \tag{5.3}$$

*at least for almost all $r > 0$ and that the right hand side of* (5.3) *converges absolutely.*

Proof. For $n \geqq 0$ let $\eta_n(r) = \eta_n^{(d)} r^d$. Then it follows from (4.9) that

$$\frac{\partial}{\partial r} V_r(T) = S_{d-1} \frac{\partial}{\partial r} \sum_{n=0}^{\infty} \eta_n(r) T^{n/2}.$$

We need to justify the interchange of differentiation and summation. The definition of $\eta_n(r)$ immediately shows that

$$\eta_0'(r) = r^{d-1}, \quad \eta_1'(r) = \sqrt{\frac{2}{\pi}} (d-1) r^{d-2}, \quad \eta_2'(r) = \frac{(d-1)(d-2)}{4} r^{d-3}$$

and

$$\eta_n'(r) = -\frac{(d-n)\xi_{n-2}^{(d)}}{(\sqrt{2}\, r)^n \Gamma(n/2+1)} r^{d-1}$$

for $n \geqq 3$. This yields that $\eta_n'(r) = (d-n)\eta_n^{(d)} r^{d-1}$ for each $n \geqq 0$. Moreover it follows from (4.6) that

$$|\eta_n'(r)| \leqq \frac{N(n-d)}{\Gamma(n/2+1)} \left( \frac{\delta_N}{\sqrt{2}} \right)^n \frac{1}{r^{n-d+1}}$$

for $n \geqq d$. Hence the estimate

$$|\eta_n'(r)| T^{n/2} \leqq \frac{2N r_0^{d-1}}{\Gamma(n/2)} \left( \frac{\delta_N^2 T}{2 r_0^2} \right)^{n/2}$$

is valid for $n \geqq d$ and $r > r_0$ with a given $r_0 > 0$. The sum of the right hand side on $n$ over $[d, \infty)$ converges, which yields that

$$\begin{aligned} \frac{\partial}{\partial r} \sum_{n=0}^{\infty} \eta_n(r) T^{n/2} &= \sum_{n=0}^{\infty} \eta_n'(r) T^{n/2} \\ &= r^{d-1} \sum_{n=0}^{\infty} (d-n) \eta_n^{(d)} T^{n/2} \end{aligned} \tag{5.4}$$

for $r > r_0$. Since $r_0 > 0$ is arbitrary, the formula (5.4) holds for all $r > 0$. This means that we finished proving (5.3). □

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