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Complex of the standard paths and n-ad homotopy groups

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Introduction

In a previous note [10] the author has defined a class of CW-complexes such that if K is such a complex, then there is a CW-complex $\omega(K)$ which is a subspace of the loops $\Omega(K)$ in K, and such that the injection map induces an isomorphism of the homotopy groups of $\omega(K)$ and $\Omega(K)$.

In this paper, we consider firstly CW-complexes which have free monoid structure. Secondly if L is such a complex we construct from L a new CW-complex K such that $L=\omega(K)$, and such that K is obtained by an identification $d: L\times I\to K$. Each point x of $\omega(K)$ defines a standard loop $d: (x)\times I\to K$, and $\omega(K)$ is regarded as the subset of $\Omega(K)$. As a standard path in K, we mean a linear part of a standard loop in K. We define a complex $\omega(K, K_0)$ of standard paths which start in a subcomplex K_0 of K and end at the base point e_0.

Our fundamental theorem is stated as follows (§4):

THEOREM. $\pi_i(\omega(K, K_0))\approx\pi_{i+1}(K, K_0)$ *for all* i.

The application of our theory to the homotopy theory is based on the fact that for any simply connected space X there exists a complex K on which we may define (K) and there exists a map $f: K\to X$ such that f induces isomorphisms of homo-opy groups.

One purpose of the paper is to prove the following connectedness theorem for $(n+1)$-ad homotopy groups (§6). Let X be a CW-complex and let $Y, Y_1, \cdots, Y_n$ be subcomplexes of X such that $Y_i\cap Y_j=Y$ for $i\neq j$ and $Y_1\cup\cdots\cup Y_n=X$. Set $X_i=X-(Y_i-Y)$. Let $\mathcal{C}$ be a class of abelian groups which satisfies the condition (I), ($\mathrm{II_B}$) and (III) of [9].

THEOREM. *If* Y *is simply connected,* (Y_i, Y) *are 2-connected and* $H_p(Y_i, Y)\in\mathcal{C}$ *for* $p<q_i+1$. *Then* $\pi_p(X; X_1, \cdots, X_n)\in\mathcal{C}$ *for* $p\leq Q=\sum q_i$ *and* $\pi_{Q+1}(X; X_1, \cdots, X_n)$ *is* $\mathcal{C}$*-isomorphic to the direct sum of* $(n-1)!$ *copies of* $H_{q_1+1}(Y_1, Y)\otimes\cdots\otimes H_{q_n+1}(Y_n, Y)$.

1. Preliminaries.

Denote by I^n the unit n-cube and by $\dot{I}^n$ its boundary:

$$I^n=\{(x_1, \cdots, x_n)\,|\,0\leq x_i\leq 1\}, \qquad \dot{I}^n=\{(x_1, \cdots, x_n)\in I^n\,|\,\Pi\, x_i(1-x_i)=0\}.$$

According to J. H. C. Whitehead [12] K is a CW-complex; if K is a closure finite cell complex, i. e., K is a Hausdorff space which is the union of disjoint open cells e_α^n with characteristic maps $\psi_\alpha: I^n \to \bar{e}_\alpha^n$ such that $\psi_\alpha | (I^n - \dot{I}^n)$ is a homeomorphism onto e_α^n and $\partial e_\alpha^n = \psi_\alpha(\dot{I}^n)$ is contained in the union of a finite number of cells whose dimensionalities do not exceed $n-1$; and if K has the weak topology, i.e., a subset $X \subset K$ is closed provided $X \cap \bar{e}$ is closed for each cell $e \subset K$. A subcomplex of a CW-complex is also a CW-complex. We list here some properties of CW-complex from [12]:

(1.1) A map $f: K \to X$ is continuous provided $f | K \cap \bar{e}$ is continuous for each cell $e \subset K$.

(1.2) If $X \subset K$ is compact, then X meets only a finite number of cells.

(1.3) Let $f: K \to L$ be a map of a CW-complex K onto a closure finite complex L which has the identification topology determined by f and if $f(\bar{e})$ meets only a finite number of cells of L for each $e \subset K$, then L is a CW-complex.

(1.4) If K is a CW-complex and L is a locally finite complex, then the topological product $K \times L$ is a CW-complex by the natural cell-decomposition.

(1.5) Let K and L be CW-complexes. Then a map $f: K \to L$ is a homotopy equivalence if and only if f induces isomorphisms of the homotopy groups.

Hereafter we consider that to each CW-complex characteristic maps of the cells are given and fixed.

Let K and L be CW-complexes. Consider the topological product $K \times L$ which is a closure finite complex, a cell of $K \times L$ is the product $e_\alpha^n \times e_\beta^m$ of cells $e_\alpha^n \subset K$ and $e_\beta^m \subset L$ and the characteristic map of $e_\alpha^n \times e_\beta^m$ is given by $\psi_{\alpha,\beta}(x, y) = (\psi_\alpha(x), \psi_\beta(y))$ for the characteristic maps ψ_α and ψ_β of e_α^n and e_β^m. We do not know whether the complex $K \times L$ has the weak topology or not. Hence we change the topology of $K \times L$ to the weak topology and let $K \times_w L$ be the resulting CW-complex. The natural map $K \times_w L \to K \times L$ is a homeomorphism on finite subcomplexes.

Let K and K_0 be a CW-complex and a subcomplex. Let $H_*(K, K_0) = \sum H_p(K, K_0)$ be the (cubical) singular homology groups, then $H_n(K^n, K^{n-1})$ is a free module generated by the classes of the characteristic maps $\psi_\alpha: (I^n, \dot{I}^n) \to (\bar{e}_\alpha^n, \partial e_\alpha^n) \subset (K^n, K^{n-1})$. We denote by the same symbol e_α^n the class of ψ_α. Set $H_n(K^n, K^{n-1}) = C_n(K)$ and $\sum C_n(K) = C(K)$, then $C(K)$ is a chain group with the boundary homomorphisms $\partial_n: C_n(K) \to C_{n-1}(K)$ defined by the composition $j_* \circ \partial: H_n(K^n, K^{n-1}) \to H_{n-1}(K^{n-1}) \to H_{n-1}(K^{n-1}, K^{n-2})$. As is well known

(1.6) $H_n(K) \approx H_n(C(K)) = \text{Kernel } \partial_n / \text{Image } \partial_{n+1}$ and $H_n(K, K_0) \approx H_n(C(K) - C(K_0))$.

A cellular map $f: (K, K_0) \to (L, L_0)$ induces chain homomorphisms $f_\#: C(K, K_0) = C(K) - C(K_0) \to C(L, L_0) = C(L) - C(L_0)$ and $f_\#$ induces the homomorphism $f_*: H_*(K, K_0) \to H_*(L, L_0)$.

Consider the natural map $f: K \times_w L \to K \times L$. Since a singular chain covers only a finite number of cells in $K\times L$ $(K\times_w L)$, f induces isomorphisms $f_*: H_*(K\times_w L)\approx H_*(K\times L)$ and $f_\#: C(K\times_w L)\approx C(K\times L)$. The generators of $C(K\times_w L)$ are chains $e^n_\alpha\times e^m_\beta$, the classes of $\psi_{\alpha,\beta}$, and the correspondence $e^n_\alpha\otimes e^m_\beta\to e^n_\alpha\times e^m_\beta$ induces an isomorphism: $C(K)\otimes C(L)\to C(K\times_w L)$. We have that $\partial(e^n_\alpha\times e^m_\beta)=\partial e^n_\alpha\times e^m_\beta+(-1)^n e^n_\alpha\times\partial e^m_\beta$. The chain group $C(K)\otimes C(L)$ is refered to as the tensor product of $C(K)$ and $C(L)$. Then we have the formula of Künneth [4], [2]: $H_p(C(K)\otimes C(L))\approx\sum_{i+j=p}H_i(C(K))\otimes H_j(C(L))+\sum_{i+j=p-1}H_i(C(K))*H_j(C(L))$.

Let $\mathcal{C}$ be a class of abelian group which satisfies the axioms (I) and ($\mathrm{II_B}$) of [9], then we have that

(1.6) Let $C_{(1)},\cdots,C_{(r)}$ be chain groups such that $H_p(C_{(i)})\in\mathcal{C}$ for $p<q_i$, $i=1,\cdots,r$, and let $Q=\sum q_i$. Then $H_p(C_{(1)}\otimes\cdots\otimes C_{(r)})\in\mathcal{C}$ for $p<Q$ and $H_Q(C_{(1)}\otimes\cdots\otimes C_{(r)})$ is $\mathcal{C}$-isomorphic to $H_{q_1}(C_{(1)})\otimes\cdots\otimes H_{q_r}(C_{(r)})$.

Let E^n_α be disjoint n-cubes, let $\sum^n$ be the union of E^n_α and let $\dot{\sum}^n$ be its boundary. Then the cross-products induces isomorphisms $H_p(X)\otimes H_n(\sum^n,\dot{\sum}^n)\approx H_{p+n}(X\times\sum^n, X\times\dot{\sum}^n)$ and the diagram

$$\text{(1.7)}\qquad\begin{array}{ccc} H_p(X)\otimes H_n(\sum^n,\dot{\sum}^n) & \approx & H_{p+n}(X\times\sum^n, X\times\dot{\sum}^n)\\ \downarrow f_*\otimes i_* & & \downarrow (f\times i)_*\\ H_p(Y)\otimes H_n(\sum^n,\dot{\sum}^n) & \approx & H_{p+n}(Y\times\sum^n, Y\times\dot{\sum}^n)\end{array}$$

is commutative, where $f: X\to Y$ is a map and i is the identity on $\sum^n$. This is a special case of the Künneth's formula and a simple proof was given in a remark of [11, p. 213].

Next we recall the following Hurewicz theorem:

(1.8) Let X and Y be a space and a subspace. If $H_p(X, Y)=0$ for $p<n$ and if (X, Y) is n-simple, then $\pi_n(X, Y)\approx H_n(X, Y)$.

For the proof see [5]. If X is arcwise connected, then $\pi_1(X)\to H_1(X)$ is onto. Therefore we have that

(1.8)′ if $\pi_p(X)=0$ for $0\leq p<n$, then $H_p(X)=0$ for $0<p<n$.

Let $\mathcal{C}$ be a class of abelian groups which satisfies (I), ($\mathrm{II_B}$) and (III), then from [9],

(1.9) If X and Y are simply connected, (X, Y) is 2-connected and if $H_p(X, Y)\in\mathcal{C}$ for $p<n$, then $\pi_p(X, Y)\in\mathcal{C}$ for $p<n$ and $\pi_n(X, Y)$ is $\mathcal{C}$-isomorphic to $H_n(X, Y)$.

Suppose that in the following homomorphisms between two exact sequences $\{G_n\}$ and $\{H_n\}$:

$$\begin{array}{ccccccccc} G_{n+2} & \longrightarrow & G_{n+1} & \longrightarrow & G_n & \longrightarrow & G_{n-1} & \longrightarrow & G_{n-2}\\ \downarrow f_{n+2} & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \downarrow f_{n-2}\\ H_{n+2} & \longrightarrow & H_{n+1} & \longrightarrow & H_n & \longrightarrow & H_{n-1} & \longrightarrow & H_{n-2}\end{array}$$

the commutativity holds. Then we have that (cf. [4])

(1.10) a) if f_{n+1} and f_{n-1} are onto and if $f_{n-2}^{-1}(0)=0$, then f_n is onto; b) if f_{n+2} is onto and if f_{n+1}, f_{n-1} and f_{n-2} are isomorphisms, then f_n is an isomorphism.

2. *FM*-complexes and complexes of standard paths.

i) *FM-complexes*: An *FM*-complex is a *CW*-complex L having a free monoid structure. More precisely, an *FM*-complex L has a multiplication (product) $(x, y) \to x\cdot y$ which satisfies the following conditions $(2.1)_1$–$(2.1)_5$.

$(2.1)_1$ $f(x, y)=x\cdot y$ defines a continuous map $f: L\times_w L\to L$.

$(2.1)_2$ The 0-section L^0 constitutes of a single point e_0 which acts as the unit element: $x\cdot e_0=e_0\cdot x=x$ for all $x\in L$.

$(2.1)_3$ $(x\cdot y)\cdot z=x\cdot(y\cdot z)$ (denoted by $x\cdot y\cdot z$).

We denote $A\cdot B=\{a\cdot b|a\in A, b\in B\}$ for two subsets A and B of L.

$(2.1)_4$ The product $e^n\cdot e^m$ of 2-cells e^n and e^m is also a cell whose characteristic map $\psi_3: I^{n+m}\to\overline{e^n\cdot e^m}$ is given by $\psi_3(t, u)=\psi_1(t)\cdot\psi_2(u)$, where ψ_1 and ψ_2 are characteristic maps of e^n and e^m.

$(2.1)_5$ There are no relations but $(2.1)_2$ and $(2.1)_3$ in the product.

By a *primitive* point x we mean a point which is not decomposable, i.e., $x=y\cdot z$ implies $x=y$ or $x=z$. If $x=y\cdot z$ for $x\in e^n$, $y\in e^m$ and $z\in e^r$, then from $(2.1)_4$ we have that $e^r=e^n\cdot e^m$, $r=m+n$ and that $f|e^n\times e^m$ is a homeomorphism onth $e^n\cdot e^m$. Therefore if a point of a cell σ is primitive, then the other points of σ are also primitive, and the cell σ is said to be *primitive*. Then any cell e^n of a positive dimention $n>0$ is the product of a finite number of primitive cells σ_i of prositive dimention n_i such that $n=\sum n_i$. By $(2.1)_5$ the expression $e^n=\sigma_1\cdots\sigma_r$ has to be unique.

Therefore L is a free monoid whose generators are the primitive points.

Let L_0 be an *FM*-complex. Consider maps $f_\alpha: \dot{I}^{n_\alpha}\to L_0^{n_\alpha-1}$ and a *CW*-complex $L'=L_0+\sum e^{n_\alpha}$ which is obtained from L_0 attaching the cells e^{n_α} by the maps f_α.

(2.2) There exists an *FM*-complex L which contains L_0 as a submonoid and L' as a subcomplex such that the primitive cells of L are the those of L_0 and e^{n_α}.

In fact, we consider a free monoid L generated by the primitive points of L_0 and the points of e^{n_α}. Then the products of the primitive cells of L_0 and e^{n_α} form a decomposition of L, and L becomes *FM*-complex if we give the topology on each closure of a cell by $(2.1)_4$ and next take the weak topology on the whole of L. This process is possible since L is a closure finite complex.

The product $f: L\times_w L\to L$ induces the chain homomorphism $f_\#: C(L)\otimes C(L)\to C(L)$. We write as $c\cdot c'\in f_\#(c\otimes c')$ for $c, c'\in C(L)$. By a *primitive chain* σ we mean the class of the characteristic map of a primitive cell σ.

Proposition (2.3) The chain group $C(L)$ of a *FM*-complex L is a graded free ring (Pontrjagin ring) generated by the primitive chains. The boundary operator is an anti-derivation, i.e. $\partial(c^n\cdot c^m)=(\partial c^n)\cdot c^m+(-1)^n c^n\cdot(\partial c^m)$ for $c^n\in C_n(L)$ and $c^m\in C_m(L)$.

The proof is immediate.

ii) *Complexes of standard loops.* Here we shall construct a *CW*-complex K such that a subspace of its loops space is an *FM*-complex L. We shall refer to L as the complex of the standard loops in K, and K as a complex which admits standard paths.

LEMMA (2.4). *Let L be an FM-complex, then there exists a real valued function ρ of L such that $\rho(x\cdot y)=\rho(x)+\rho(y)$, $x, y\ni L$ and such that $\rho(x)>0$ for $x\neq e_0$.*

We define ρ presisely as follows: Let $L(n)$ be a subcomplex whose cells are the products of primitive cells of dimension $\leq n$. Since $L(0)=e_0$, we set $\rho(e_0)=0$ and ρ is defined on $L(0)$. Suppose that ρ is defined on $L(n-1)$. Let σ_α be a primitive n-cell and let $\psi_\alpha: I^n\rightarrow\bar{\sigma}_\alpha$ be its characteristic map, then ρ is defined on $\partial\sigma_\alpha$ since $L^{n-1}\subset L(n-1)$. For a point x of $\dot{I}^n$, we denote by $[x, t]$ the point which divides x and the center $(\frac{1}{2},\cdots,\frac{1}{2})$ of I^n in the ratio t; $1-t$. We set $\rho(\psi_\alpha[x, t])=(1-t)\rho(\psi_\alpha(x))+tn$ for each α, then ρ is extended over $L(n)$ by the linearity $\rho(x\cdot y)=\rho(x)+\rho(y)$. From $(2.1)_1$-$(2.1)_5$ and (1.1) ρ is single valued and continuous. Then ρ is defined by induction on n.

This function ρ is defined uniquely since we fixed the characteristic maps for each *CW*-complex.

Let $K=B(L)$ be a space which is defined from the product complex $L\times I$ by an identification $d: L\times I\rightarrow B(L)=K$ such that

$$d(e_0, t)=d(e_0, 0)=e_0\in K,$$

$$d(x\cdot y, t)=\begin{cases} d\left(x, \dfrac{t}{\lambda}\right), & 0\leq t\leq\lambda, \\ d\left(y, \dfrac{t-\lambda}{1-\lambda}\right) & \lambda\leq t\leq 1, \end{cases} \tag{2.5}$$

where $x, y\in L$, $x\cdot y\neq e_0$ and $\lambda=\dfrac{\rho(x)}{\rho(x\cdot y)}$.

We see that $d(L\times\dot{I})=e_0$ and $d(e\cdot e'\times I)=d(e\times I)\cup d(e'\times I)$. Therefore $B(L)$ is the union of the disjoint sets $d(\sigma\times(I-\dot{I}))$ for the primitive cells σ. Since there is no relation on $\sigma^n\times(I-\dot{I})$ if dimension $n>0$, $d|\sigma^n\times(I-\dot{I})$ is a homeomorphism. Denote the image $d(\sigma^n\times(I-\dot{I}))$ by $E\sigma^n$ and define a characteristic map $\psi': I^{n+1}\rightarrow\overline{E\sigma^n}$ by $\psi'(x_1,\cdots,x_{n+1})=d(\psi(x_1,\cdots,x_n), x_{n+1})$ where ψ is the characteristic map of σ^n. Then $B(L)=e_0+\sum E\sigma$ becomes a closure finite cell complex. Since $\overline{d(\sigma_1\cdots\sigma_r)}\subset\bigcup\overline{E\sigma_i}$, the identification $d: L\times I\rightarrow B(L)$ satisfies the condition of (1.3), and hence $K=B(L)$ is a *CW*-complex.

We write $L=\omega(K)$; this means that L is an *FM*-complex such that $K=B(L)$.

We say that a CW-complex K *admits standard paths* if there exists an FM-complex L such that $K=B(L)$.

For each point x of $\omega(K)$, we define a standard loop l_x: $(I, \dot{I}) \rightarrow (K, e_0)$ by the formula $l_x(t)=d(x, t)$, $t \in I$. Then the correspondence $x \rightarrow l_x$ defines a 1-1 continuous map i: $\omega(K) \rightarrow \Omega(K)$, where $\Omega(K)$ denotes the space of loops in K based at e_0. Hence $\omega(K)$ is regarded as the subset of $\Omega(K)$ changing its topology from weak topology, and it is called *the complex of standard loops in K.*

Define a suspension homomorphism E: $C_n(\omega(K)) \rightarrow C_{n+1}(K)$ by setting $E(c) = d_{\#}(c \otimes i_1)$, where i_1 is the class of the identity of $(I, \dot{I})$ on itself. Then we have that

(2.6) E is a chain homomorphism. E maps $C(\omega(K))-\{e_0\}$ onto $C(K)-\{e_0\}$ and its kernel is generated by the decomposable elements.

In the case that the union of the primitive cells forms a subcomplex L_0 of L, $K=B(L)$ is a suspension of L_0 and d shrinks $L_0 \times I \cup e_0 \times \dot{I}$ to a single point e_0. Then L becomes the reduced product space of L_0 in the sense of [6].

iii) *Complexes of standard paths.* Let $L=\omega(K)$ be an FM-complex. Define a space $\omega(K, K)$ from $\omega(K) \times I$ be the identification $\bar{d}$: $\omega(K) \times I \rightarrow \omega(K, K)$ such that

$$\begin{aligned} &\bar{d}(e_0, t)=d(e_0, 0)=e_0 \in \omega(K, K)\ , \\ &\bar{d}(x \cdot y, t)=\bar{d}\left(y, \frac{t-\lambda}{1-\lambda}\right) \qquad \text{if} \quad \lambda \leq t \leq 1\ , \end{aligned} \tag{2.7}$$

where $x, y \in \omega(K)$, $x \cdot y \neq e_0$ and $\lambda=\dfrac{\rho(x)}{\rho(x \cdot y)}$.

Since $\bar{d}$ has no relations on $\omega(K) \times (0)$, $\bar{d}|\omega(K) \times (0)$ is a homeomorphism onto a subset of $\omega(K, K)$. We imbed $\omega(K)$ into $\omega(K, K)$ by identifying each $x \in \omega(K)$ to $\bar{d}(x, 0) \in \omega(K, K)$. The product in $\omega(K)$ is extended to the product

$$\omega(K, K) \times_w \omega(K) \longrightarrow \omega(K, K) \tag{2.8}$$

by setting $d(x, t) \cdot y=\bar{d}\left(x \cdot y, \dfrac{t\rho(x)}{\rho(x \cdot y)}\right)$. Denote $A \cdot B=\{x \cdot y \mid x \in A, y \in B\}$ for $A \subset \omega(K, K)$ and $B \subset \omega(K)$, then $A \cdot e_0=A$, $e_0 \cdot B=B$ and $A \cdot (B \cdot B')=(A \cdot B) \cdot B'$.

Define a projection

$$p: \omega(K, K) \longrightarrow K \tag{2.9}$$

by the formula $p(\bar{d}(x, t))=d(x, t)$, then $p(z \cdot x)=p(z)$ for $z \in \omega(K, K)$ and $x \in \omega(K)$. Hence if $y=d(x, t)$ for a primitive point $x \in \omega(K)$ then $p^{-1}(y)=\bar{d}(x, t) \cdot \omega(K)$. Let σ be a primitive cell and denote by $D\sigma$ the image $\bar{d}(\sigma \times (I-\dot{I}))$. Since the identification $\bar{d}$ has no retation on $\sigma \times (I-\dot{I})$, $\bar{d}$ maps $\sigma \times (I-\dot{I})$ homeomorphically onto $D\sigma$.

(2. 10) The product defines a homeomorphism of $D\sigma\times\omega(K)$ onto a subset $D\sigma\cdot\omega(K)$ of $\omega(K, K)$. $\omega(K, K)$ is the union of the disjoint subset $D\sigma\cdot\omega(K)$ over all primitive cells σ. $(De_0=e_0)$.

Proof. First we prove that the product in $\omega(K)$ defines a homeomorphism of $\sigma\times\omega(K)$ onto $\sigma\cdot\omega(K)$. Since $\omega(K)$ is free this correspondence is one to one. Then it is sufficient to prove that $\sigma\times\bar{e}$ and $\sigma\cdot\bar{e}$ are homeomorphic, and this follows from $(2.1)_4$ since $\sigma\times\bar{e}$ and $\sigma\cdot\bar{e}$ both have the identification topology given by their characteristic maps. Let $\sigma\neq e_0$ and let f: $(\sigma\cdot\omega(K))\times(I-\dot{I})\rightarrow\sigma\cdot\omega(K)\times I\subset\omega(K)\times I$ be a map defined by $f(x\cdot y, t)=\left(x\cdot y, \frac{t\rho(x)}{\rho(x\cdot y)}\right)$, then f is a homeomorphism onto a subset M of $\omega(K)\times I$. The map $\bar{d}$ is a homeomorphism of M onto $D\sigma\cdot\omega(K)$ since $\bar{d}$ is one to one and has no relation on M. Since $\bar{d}|\sigma\times(I-\dot{I})$ is a homeomorphism onto $D\sigma$, $D\sigma\times\omega(K)$ is homeomorphic to $(\sigma\cdot\omega(K))\times(I-\dot{I})$, to M and to $D\sigma\cdot\omega(K)$, and this homeomorphism is given by the product (2. 8). The second part of (2. 10) is easily verified.

By (2. 10), $\omega(K, K)$ is a closure finite cell complex consisting of the cells $D\sigma\cdot e$, $e\subset\omega(K)$; σ primitive. Since $\bar{d}(\overline{\sigma_1\cdot\cdots\cdot\sigma_k\times I})\subset\bigcup_{i=1}^{k}\overline{D\sigma_i\cdot\sigma_{i+1}\cdot\cdots\cdot\sigma_k}$, $\bar{d}$ satisfies the condition of (1. 3). Therefore we have that

(2. 11) $\omega(K, K)$ is a CW-complex.

Let K' be a subcomplex of K. Then $p^{-1}(K')$ is a subcomplex of $\omega(K, K)$ which consists of cells $D\sigma\cdot e$ where $e\subset\omega(K)$ and σ is a primitive cell such that $E\sigma\subset K'$. We denote this complex by $\omega(K, K')$. Obviously $\omega(K, K')\cdot\omega(K)\subset\omega(K, K')$.

To each point $\bar{d}(x, t)$ of $\omega(K, K')$, $x\in(K)$, $t\in I$, we associate *a standard path* $l_{x(t)}: I\rightarrow K$ which is defined by $l_{x(t)}(u)=d(x, t+u-tu)$, Let f: $\omega(K, K')\times I\rightarrow K$ be the map given by $f(\bar{d}(x, t), u)=d(x, t+u-tu)$, then f is continuous. Therefore the correspondence $\bar{d}(x, t)\rightarrow l_{x(t)}$ defines a continuous map

$$i: \omega(K, K') \longrightarrow \Omega(K, K'),$$

where $\Omega(K, K')$ denotes the space of paths in K which start in K' and end at the point e_0. The map i maps $\omega(K, K')$ one to one continuously onto a subset of $\Omega(K, K')$. We remark that the map i is homeomorphism on compact subsets of $\omega(K, K')$ but not always homeomorphic on the whole of $\omega(K, K')$.

The product of (2. 8) defines a chain homomorphism: $C(\omega(K, K'))\otimes C(\omega(K))\rightarrow C(\omega(K, K'))$, and we denote the image of $c\otimes c'$ by $c\cdot c'$. Next we define a homomorphism

$$D: C_n(\omega(K)) \longrightarrow C_{n+1}(\omega(K, K))$$

by $D(c^n)=\bar{d}_*(c^n\otimes i_1)$, then $p_\# D(c)=E(c)$ for the projection p of (2. 9). Immediate calculation shows that

(2.12) a) $\partial(c^n \cdot c^m) = (\partial c^n) \cdot c^m + (-1)^n c^n \cdot (\partial c^m)$,
b) $\partial(Dc^n) = D(\partial c^n) + (-1)^{n+1} c^n$.

Define homotopy $r_t: \omega(K, K) \to \omega(K, K)$ by $(r_t f)(u) = f(t + u(1-t))$, then we have that

(2.13) $\omega(K, K)$ is contractible to e_0.

3. Some lemmas

In this §, K and K_0 means always a CW-complex which admits standard paths and a subcomplex. We shall use the notations of the previous §.

Since $\omega(K)$ and $\Omega(K)$ are both H-spaces [8], $\omega(K)$ and $\Omega(K)$ are simple for all dimensions.

Let $^\vee: \Omega(K, K_0) \times \Omega(K) \to \Omega(K, K_0)$ be a map which is given by

$$(f^\vee g)(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2}, \\ g(2t-1) & \frac{1}{2} \le t \le 1, \end{cases}$$

where $f: (I, (0), (1)) \to (K, K_0, e_0)$, $g: (I, \dot{I}) \to (K, e_0)$ and $t \in I$. Define a path $P(f, g, s): I \to K$ for $f \in \omega(K, K_0)$, $g \in \omega(K)$ and $s \in I$ by

$$P(f, g, s)(t) = \begin{cases} f\left(\dfrac{t}{\lambda_s}\right) & 0 \le t \le \lambda_s, \\ g\left(\dfrac{t-\lambda_s}{1-\lambda_s}\right) & \lambda_s \le t \le 1, \end{cases}$$

where $\lambda_s = \dfrac{(2-s)\,\rho(f) + s\rho(g)}{2\rho(f \cdot g)}$. Then we have a continuous map

(3.1) $$P: \omega(K, K_0) \times_w \omega(K) \times I \longrightarrow \Omega(K, K_0)$$

such that $P(f, g, 0) = f \cdot g$, $P(f, g, 1) = f^\vee g$ and $P(e_0, e_0, s) = e_0$.

LEMMA (3.2) *Let $i: \omega(K, K_0) \to \Omega(K, K_0)$ be the natural map, then the following two conditions are equivalent*:

a) $i_*: H_n(\omega(K, K_0)) \approx H_n(\Omega(K, K_0))$ *for all n*,
b) $i_*: \pi_n(\omega(K, K_0)) \approx \pi_n(\Omega(K, K_0))$ *for all n*.

Proof. Let Ω' be the mapping cylinder of i. We represent the points of Ω' by (x, t), $x \in \omega(K, K_0)$, $t \in I$ and by $y \in \Omega(K, K_0)$ with the relation $x = (x, 1)$ and $i(x) = (x, 0)$. Since $K^1 = K_0^1 = e_0$, $\pi_1(K, K_0) = \pi_0(\Omega(K, K_0)) = 0$, i.e. $\Omega(K, K_0)$ is arcwise-connected. Since $\omega(K, K_0)$ has only a vertex e_0, $\omega(K, K_0)$ is arcwise connected. Then the conditions a) and b) are equivalent to the following conditions a′) and b′) respectively:

a′) $H_n(\Omega', \omega(K, K_0)) = 0$ for all $n > 0$,
b′) $\pi_n(\Omega', \omega(K, K_0)) = 0$ for all $n > 0$.

Now we shall prove the following two assertions:

(3.3) $\pi_1(\Omega', \omega(K, K_0))=0$,

(3.4) $\pi_1(\omega(K, K_0))$ operates trivially on $\pi_n(\Omega', \omega(K, K_0))$.

Then Lemma (3.2) is proved immediately from (1.8).

Proof of (3.3) Since $K^1=K_0^1=e_0$, $\pi_1(K_0)=0$ and $\pi_2(K)\to\pi_2(K, K_0)$ is onto. Then $\pi_2(K, K_0)$ is generated by the classes of characteristic maps of 2-cells in $K-K_0$, which are the suspension of the classes of characteristic maps of 1-cells in $\omega(K)$. Therefore $\pi_1(\omega(K))\to\pi_1(\Omega(K, K_0))=\pi_2(K, K_0)$ is onto, and $\pi_1(\omega(K, K_0))\to\pi_1(\Omega(K, K_0))$ is onto, that is, $\pi_1(\Omega', \omega(K, K_0))=0$.

Proof of (3.4) Since $\omega(K, K_0)-\omega(K)$ has at least 2-dimension, $\pi_1(\omega(K))\to\pi_1(\omega(K, K_0))$ is onto. Therefore it is saficient to prove that

(3.4)′ $\beta^\alpha=\beta$ for $\alpha\in\pi_1(\omega(K))$ and $\beta\in\pi_n(\Omega', \omega(K, K_0))$.

Let $a: (I, \dot{I})\to(\omega(K), e_0)$ and $b: (I^n; I^{n-1}, J^{n-1})\to(\Omega', \omega(K, K_0), e_0)$ be representatives of α and β respectively, where $I^{n-1}=I^{n-1}\times(0)\subset I^n$ and $J^{n-1}=\dot{I}^n-\text{Int.}\, I^{n-1}$. Denote by $[x, t]$ a point of I^n which divides a point x of $\dot{I}^n$ and the center $(\frac{1}{2}, \cdots, \frac{1}{2})$ of I^n in the ratio $t: 1-t$. Define a homotopy $b_s: (I^n, I^{n-1}, J^{n-1})$ $(\Omega', \omega(K, K_0), e_0)$ by

$$b_s([x, t])=\begin{cases} r_{3t}(b([x, 0])) & 0\leq t\leq \frac{s}{3}, \\ r_s\left(b\left(\left[x, \frac{3t-s}{3-s}\right]\right)\right) & \frac{s}{3}\leq t\leq 1, \end{cases}$$

for $0\leq s\leq 1$ and by

$$b_s=([x, t])=\begin{cases} r_{3t}(b([x, 0])) & 0\leq t\leq \frac{1}{3}, \\ P_0(r_1(b([x, 0])), 2-3t) & \frac{1}{3}\leq t\leq \frac{s}{3}, \\ P_0\left(r_1\left(b\left(\left[x, \frac{3t-s}{3-s}\right]\right)\right), 2-s\right) & \frac{s}{3}\leq t\leq 1, \end{cases}$$

for $1\leq s\leq 2$, where $r_t; \Omega'\to\Omega'$ is a retraction of Ω' onto $\Omega(K, K_0)$ given by $r_t(x, u)=(x, (1-t)u)$, and $P_0(f, u): I\to K(f\in\Omega(K, K_0))$ is a path defined by

$$P_0(f, u)\ (t)=\begin{cases} f\left(\frac{2t}{1+u}\right) & 0\leq t\leq\frac{1+u}{2}, \\ e_0 & \frac{1+u}{2}\leq t\leq 1. \end{cases}$$

Then $b=b_0$ and b_2 represent the same element β. Next define a homotopy $h_s: (I^n, I^{n-1}, J^{n-1})\to(\Omega', \omega(K, K_0), a(s))$, $s\in I$, by

$$h_s([x, t])=\begin{cases} r_{3t}(b([x, 0])\cdot a(s)) & 0\leq t\leq\frac{1}{3}, \\ P(b([x, 0]), a(s), 3t-1) & \frac{1}{3}\leq t\leq\frac{2}{3}, \\ b\left(\left[x, \frac{3t-s}{3-s}\right]\right)\vee a(s) & \frac{2}{3}\leq t\leq 1. \end{cases}$$

Then we have that $h_0=h_1=b_2$. In the other words, the homotopy h_s shows that the class of h_0 is β^α. Therefore (3.3)′ is proved. Consequently Lemma (3.2) is established.

Let $\psi_\alpha: I^n \to \bar{\sigma}_\alpha^n$ be the characteristic map of a primitive n-cell σ_α^n of $\omega(K_0)$. Let $V^n=\{[x,t] \mid \frac{1}{2}\leq t\leq 1\}$, and denote by E_α^{n+1} the subset $\bar{d}(\psi_\alpha(V^n)\times V^1)$ of $D\sigma_\alpha^n$, and let Σ^{n+1} be the union of E_α^{n+1} for all $\sigma_\alpha^n \subset \omega(K_0)$. E_α^{n+1} are $(n+1)$-cubes disjoint from each other. Denote by $\dot{\Sigma}^{n+1}$ the boundary of Σ^{n+1}.

Define a homotopy r_s of $\overline{D\sigma_\alpha^n}$ on itself by

$$r_s(\bar{d}(\psi_\alpha([x,t]),u))=\begin{cases} \bar{d}\left(\psi_\alpha\left(\left[x, \dfrac{t-2u}{1-2u}\right]\right), 0\right) & u\leq \dfrac{t}{2},\ 0\leq u\leq \dfrac{1-s}{4}, \\ \bar{d}\left(\psi_\alpha([x,0]), \dfrac{2u-t}{2-2t}\right) & 0\leq t\leq \dfrac{1-s}{2},\ \dfrac{t}{2}\leq u\leq \dfrac{2-t}{2}, \\ \bar{d}\left(\psi_\alpha\left(\left[x, \dfrac{2t+s-1}{s+1}\right]\right), \dfrac{4u+s-1}{2s+2}\right) & \dfrac{1-s}{2}\leq t\leq 1, \\ & \dfrac{1-s}{4}\leq u\leq \dfrac{3+s}{4}, \\ \bar{d}\left(\psi_\alpha\left(\left[x, \dfrac{2u+t-2}{2u-1}\right]\right), 1\right) & \dfrac{2-t}{2}\leq u,\ \dfrac{3+s}{4}\leq u\leq 1, \end{cases}$$

$x, t, s\in I$, $x\in I^n$. Then r_1 is the identity and r_0 maps the interior of E_α^{n+1} onto $D\sigma_\alpha^n$. r_s fixes the points of $\partial\sigma_\alpha^n$.

Define homotopies

$$\phi_s : \Sigma^{n+1}\times\omega(K) \longrightarrow \omega(K, K_0^{n+1}),$$
$$\phi_s' : \Sigma^{n+1}\times\Omega(K) \longrightarrow \Omega(K, K_0^{n+1}),$$

by $\phi_s(x,y)=(r_s x)\cdot y$ and $\phi_s'(x,y')=(r_s x)\vee y'$ for $y\in\omega(K)$, $y'\in\Omega(K)$ and $x\in\Sigma^{n+1}$. Denote $\phi=\phi_0$ and $\phi'=\phi_0'$, then $\phi(\dot{\Sigma}^{n+1}\times\omega(K))\subset\omega(K,K_0^n)$ and $\phi'(\dot{\Sigma}^{n+1}\times\Omega(K))\subset\Omega(K,K_0^n)$.

PROPOSITION (3.5) *The maps ϕ and ϕ' induce isomorphisms of relative homology groups and the diagram*

$$\begin{array}{ccc} H_*(\Sigma^{n+1}\times\omega(K),\ \dot{\Sigma}^{n+1}\times\omega(K)) & \overset{\phi_*}{\approx} & H_*(\omega(K,K_0^{n+1}),\ \omega(K,K_0^n)) \\ \downarrow & & \downarrow \\ H_*(\Sigma^{n+1}\times\Omega(K),\ \dot{\Sigma}^{n+1}\times\Omega(K)) & \overset{\phi_*'}{\approx} & H_*(\Omega(K,K_0^{n+1}),\ \Omega(K,K_0^n)) \end{array}$$

is commutative, where the vertical homomorphisms are induced by the natural maps.

Proof. The commutativity follows from the homotopy (3.1). By (2.10), ϕ maps $(\Sigma^{n+1}-\dot{\Sigma}^{n+1})\times\omega(K)$ homeomorphically onto $\omega(K,K_0^{n+1})-\omega(K,K_0^n)$. Hence ϕ_* is an isomorphism.

Let $p: \Omega(K,K_0)\to K_0$ be the projection, then $\Omega(K,K_0^{n+1})=p^{-1}(K_0^{n+1})$ and $\Omega(K,K_0^n)=p^{-1}(K_0^n)$. p maps Σ^{n+1} homeomorphically onto a subset of $K_0^{n+1}-K_0^n$ and we denote this subset by Σ_0^{n+1} and its bundary by $\dot{\Sigma}_0^{n+1}$. Let X be the closure of $K_0^{n+1}-\Sigma_0^{n+1}$. Consider the diagram

$$\begin{array}{ccc}
H_*(\sum^{n+1}\times\Omega(K),\ \dot{\sum}^{n+1}\times\Omega(K)) & \xrightarrow{\phi_*'} & H_*(p^{-1}(K_0^{n+1}),\ p^{-1}(K_0^n)) \\
\downarrow \phi_{1*}' & & \downarrow j_*' \\
H_*(p^{-1}(\sum_0^{n+1}),\ p^{-1}(\dot{\sum}_0^{n+1})) & \xrightarrow{j_*} & H_*(p^{-1}(K_0^{n+1}),\ p^{-1}(X))
\end{array}$$

where j and j' are injections. From the homotopy ϕ_s', we have that the diagram is commutative. Now we shall prove that the homomorphisms j_*, j_*' and ϕ_{1*}' are isomorphisms, then ϕ_*' is an isomorphism.

As is easily seen K_0^n is a deformation retract of X, therefore $p^{-1}(K_0^n)$ is a deformation retract of $p^{-1}(X)$ by the covering homotopy theorem. Then $H_*(p^{-1}(X), p^{-1}(K_0^n))=0$ and this implies that j_*' is the isomorphism. Let $W^{n+1}=p(\phi_{\frac{1}{2}}'(\sum^{n+1}))$ and $W_0^{n+1}=W^{n+1}-(\sum_0^{n+1}-\dot{\sum}_0^{n+1})$. Then j_* is the composition of two injection homomorphisms $j_{1*}: H_*(p^{-1}(\sum_0^{n+1}),\ p^{-1}(\dot{\sum}_0^{n+1}))\to H_*(p^{-1}(W^{n+1}),\ p^{-1}(W_0^{n+1}))$ and $j_{2*}: H_*(p^{-1}(W^{n+1}), p^{-1}(W_0^{n+1}))\to H_*(p^{-1}(K_0^{n+1}), p^{-1}(X))$. The two pairs $(\sum_0^{n+1}, \dot{\sum}_0^{n+1})$ and (W^{n+1}, W_0^{n+1}) have the same homotopy type, then $(p^{-1}(\sum_0^{n+1}), p^{-1}(\dot{\sum}_0^{n+1}))$ and $(p^{-1}(W^{n+1}), p^{-1}(W_0^{n+1}))$ have the same homotopy type by the covering homotopy. Thus j_{1*} is the isomorphism. Since Int. $p^{-1}(X)\cup$ Int. $p^{-1}(W^{n+1})=p^{-1}(K_0^{n+1})$, j_{2*} is the excision isomorphism. Therefore j_* is an isomorphism.

The map ϕ_1' is a homotopy equivalence. In fact, define a map $\bar{\phi}$: $(p^{-1}(\sum_0^{n+1}), p(\dot{\sum}_0^{n+1}))\to(\sum^{n+1}\times\Omega(K),\ \dot{\sum}^{n+1}\times\Omega(K))$ and homotopies $Q_s: (\sum^{n+1}\times\Omega(K), \dot{\sum}^{n+1}\times\Omega(K))\to(\sum^{n+1}\times\Omega(K),\ \dot{\sum}^{n+1}\times\Omega(K))$ and R_s: $(p^{-1}(\sum_0^{n+1}),\ p^{-1}(\dot{\sum}_0^{n+1}))\to(p^{-1}(\sum_0^{n+1}), p^{-1}(\dot{\sum}_0^{n+1}))$ by

$$\bar{\phi}(f)=(p(f),\psi(f)),\quad \psi(f)(t)=\begin{cases} l(1-2t), & 0\leq t\leq\frac{1}{2}, \\ f(2t-1), & \frac{1}{2}\leq t\leq 1,\end{cases}$$

$$Q_s(x, g)=(x, Q_s'(g)),\quad Q_s'(g)(t)=\begin{cases} l'(1-2t), & 0\leq t\leq\frac{s}{2}, \\ l'(4t-3s+1), & \frac{s}{2}\leq t\leq\frac{3s}{4}, \\ g\left(\frac{4t-3s}{4-3s}\right), & \frac{3s}{4}\leq t\leq 1;\end{cases}$$

$$R_s(f)(t)=\begin{cases} l(1-2t), & 0\leq t\leq\frac{s}{2}, \\ l(4t-3s+1), & \frac{s}{2}\leq t\leq\frac{3s}{4}, \\ f\left(\frac{4t-3s}{4-3s}\right), & \frac{3s}{4}\leq t\leq 1;\end{cases}$$

where l and l' are standard paths in $\sum^{n+1}$ such that $p(l)=p(f)$ and $p(l')=x$. Then $Q_1=\phi\circ\phi_0'$, $R_1=\bar{\phi}_0'\circ\phi$ and Q_1 and R_1 are identities. Therefore ϕ_0' is a homotopy equivalence and ϕ_{0*}' is an isomorphism. Conseuuently Proposition (3.5) is proved.

From (1.7) and Proposition (3.5), we have the following lemma:

LEMMA (3.6) If i_*: $H_p(\omega(K))\approx H_p(\Omega(K))$ for $p\leq N$, then i_*: $H_p(\omega(K, K_0^m), \omega(K, K_0^{m-1}))\approx H_p(\Omega(K, K_0^m), \Omega(K, K_0^{m-1}))$ for $p\leq N+m$,

4. The fundamental theorem.

Let K be a complex which admits standard paths, and let K_0 be a subcomplex of K. Then the fundamental theorem of our theory is stated as follows:

THEOREM (4.1) *The natural map* $i_*: \omega(K, K_0) \to \Omega(K, K_0)$ *induces isomorphisms*

$$i_*: \pi_p(\omega(K, K_0)) \approx \pi_p(\Omega(K, K_0)), \qquad p \geq 0.$$

To a map $f: (I^p, \dot{I}^p) \to (\Omega(K, K_0), e_0)$, we associate a map $f: (I^{p+1}, I^p, J^p) \to (K, K_0, e_0)$ given by $f(x_1, \cdots, x_{p+1}) = (f(x_1, \cdots, x_p))(x_{p+1})$. Then we have the isomorphism $\Omega': \pi_p(\omega(K, K_0)) \approx \pi_{p+1}(K, K_0)$ which is induced by the correspondence $f \leftrightarrow \Omega f$. In the same way we have homomorphism $\Omega: \pi_p(\omega(K, K_0)) \to \pi_{p+1}(K, K_0)$, then combining the isomorphisms Ω' and i_* we have that

THEOREM (4.1)′ $\qquad \Omega: \pi_p(\omega(K, K_0)) \approx \pi_{p+1}(K, K_0)$.

By (3.2) the theorem (4.1) is equivalent to the following proposition:

PROPOSITION (4.2) $i_*: H_p(\omega(K, K_0)) \approx H_p(\Omega(K, K_0))$, $p \geq 0$.

First we shall prove (4.2) in the case $K_0 = e_0$, that is,

(4.3) $$i_*: H_p(\omega(K)) \approx H_p(\Omega(K)), \qquad p \geq 0.$$

Proof. Denote $H_p(\omega(K, K^m)) = H^m_p$, $H_p(\omega(K, K^m), \omega(K, K^{m-1})) = H^{m,m-1}_p$, $H_p(\Omega(K, K^m)) = {}'H^m_p$, and $H_p(\Omega(K, K^m), \Omega(K, K^{m-1})) = {}'H^{m,m-1}_p$. Since $\omega(K)$ and $\Omega(K)$ are both arcwise connected, (4.3) is true if $p=0$. Now suppose that (4.3) is true for $p < n$. Then from (3.6), $i_*: H^{m,m-1}_p \approx {}'H^{m,m-1}_p$ for $p < n+m$. Applying (1.10), a) and b) to the following diagram

$$\begin{array}{ccccccccccccc}
H^{m,m-1}_{n+2} & \longrightarrow & H^{m-1}_{n+1} & \longrightarrow & H^m_{n+1} & \longrightarrow & H^{m,m-1}_{n+1} & \longrightarrow & H^{m-1}_n & \longrightarrow & H^m_n & \longrightarrow & H^{m,m-1}_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
{}'H^{m,m-1}_{n+2} & \longrightarrow & {}'H^{m-1}_{n+1} & \longrightarrow & {}'H^m_{n+1} & \longrightarrow & {}'H^{m,m-1}_{n+1} & \longrightarrow & {}'H^{m-1}_n & \longrightarrow & {}'H^m_n & \longrightarrow & {}'H^{m,m-1}_n ,
\end{array}$$

we have that

$(4.4)_m$ if $i_*: H^m_{n+1} \to H^m_{n+1}$ is onto and if $i_*: H^m_n \to H^m_n$ is an isomorphism, that $i_*: H^{m-1}_{n+1} \to {}'H^{m-1}_{n+1}$ is onto for $m \geq 3$ and $i_*: H^{m-1}_n \to {}'H^{m-1}_n$ is an isomorphism for $m \geq 2$.

If $m > p$, then $\pi_{p+1}(K, K^m) = \pi_p(\Omega(K, K^m)) = 0$, and by (1.8)′ $H_p(\Omega(K, K^m)) = 0$ for $m > p > 0$. By (2.13), $H_p(\omega(K, K)) = 0$ for $p > 0$. Since the dimension of the cells of $\omega(K, K) - \omega(K, K^m)$ are at least $m+1$, $H_p(\omega(K, K), \omega(K, K^m)) = 0$ for $m \geq p$. Hence $H_p(\omega(K, K^m)) = 0$ for $m > p > 0$. Then the hypothesis of $(4.4)_m$ is true for $m > n+1$. Applying $(4.4)_m$ for $m = n+2, n+1, \cdots, 3, 2$, we have that $i_*: H^1_n \to {}'H^1_n$ is an isomorphism. Since $K^1 = e_0$, this means that (4.3) is true for $p = n$. Therefore (4.3) is proved by the induction on p.

Proof of (4.2) Denote $H_p(\omega(K, K^m_0)) = G^m_p$, $H_p(\omega(K, K^m_0), \omega(K, K^{m-1}_0)) = G^{m,m-1}_p$,

$H_p(\Omega(K, K^m)) = {}'G_p^m$, and $H_p(\Omega(K, K_0^m), \Omega(K, K_0^{m-1})) = {}'G_p^{m,m-1}$. We shall prove that

$$(4.5)_m \qquad i_*: G_p^m \approx {}'G_p^m \quad \text{for all } p.$$

By (4.3), $(4.5)_0$ is true. Then, by (3.6), $i_*: G_p^{m,m-1} \approx {}'G_p^{m,m-1}$ for all p and m. Now suppose that $(4.5)_{m-1}$ is true. Applying the five lemma (1.10), b) to the diagram

$$\begin{array}{ccccccccc} G_{p+1}^{m,m-1} & \longrightarrow & G_p^{m-1} & \longrightarrow & G_p^m & \longrightarrow & G_p^{m,m-1} & \longrightarrow & G_{p-1}^{m-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ {}'G_{p+1}^{m,m-1} & \longrightarrow & {}'G_p^{m-1} & \longrightarrow & {}'G_p^m & \longrightarrow & {}'G_p^{m,m-1} & \longrightarrow & {}'G_{p-1}^{m-1}, \end{array}$$

we have that $(4.5)_m$ is true. Therefore $(4.5)_m$ is true for all m.

We may consider that $H_p(\omega(K, K_0))$ and $H_p(\Omega(K, K_0))$ are limit groups of $\{G_p^m\}$ and $\{'G_p^m\}$ respectively since any compact subsets of $\Omega(K, K_0) = p^{-1}(K_0)$ and $\omega(K, K_0)$ are in $\Omega(K, K_0^m) = p^{-1}(K_0^m)$ and $\omega(K, K_0^m)$ respectively for sufficiently large m. Then (4.5) implies (4.2).

For the application of our theory to homotopy problems the following theorem is useful.

THEOREM (4.6) *Let X be a simply connected space. Then there is a CW-complex K which admits standard paths, and there is a map $f: K \to X$ such that $f_*: \pi_p(K) \approx \pi_p(X)$, $p \geq 0$.*

Proof. We shall construct a CW-complex $K(n)$ which admits standard paths and a map $f_n: K(n) \to X$ such that $K(n) \supset K(n-1)$, $f_n | K(n-1) = f_{n-1}$ and that $f_{n*}: \pi_p(K(n)) \to \pi_p(X)$ is onto for $r = n$ and isomorphic for $p < n$. Set $K(0) = e_0$ and take f_0 arbitary. Now suppose that $K(n)$ and f_n are constructed for $n \leq m$. Consider the generators ζ_α of the kernel of $f_{m*}: \pi_m(K(m)) \to \pi_m(X)$. By (4.1)′ there exist maps $g_\alpha: \dot{I}^m \to \omega(K(m))$ which represent $\Omega^{-1}\zeta_\alpha$. Let ξ_β be the generators of $\pi_{m+1}(X)$. Attaching cells e_α^m by the maps g_α and e_β^m by the trivial maps $\dot{I}^m \to e_0$, we have a CW-complex $\omega(K(m)) + \sum e_\alpha^m + \sum e_\beta^m$. According to (2.2) we construct an FM-complex L whose primitive cells are those of $\omega(K(m))$, e_α^m and e_β^m. Define $K(m+1) = B(L)$, then $K(m+1) = K(m) + \sum Ee_\alpha^m + \sum Ee_\beta^m$, and Ee_α^m and Ee_β^m are attached by representatives of ζ_α and the trivial maps respectively. Since $f_{m*}(\zeta_\alpha) = 0$, the map $f_m | \partial Ee_\alpha^m$ is extendable over $\overline{Ee_\alpha^m}$. Next extend f_m over $\overline{Ee_\beta^m}$, which is $(m+1)$-sphere, such that $\overline{Ee_\beta^m} \to X$ represents ξ_β. Then we obtain an extension $f_{m+1}: K(m+1) \to X$ of f_m. As is easily seen that $\pi_p(K(m)) \approx \pi_p(K(m+1))$ for $p < m$, hence $f_{m+1*}: \pi_p(K(m+1)) \approx \pi_p(X)$ for $p < m$. The injection homomorphism $\pi_m(K(m)) \to \pi_m(K(m+1))$ is onto and its kernel is generated by ζ_α. Therefore $f_{m+1*}: \pi_m(K(m+1)) \to \pi_m(X)$ is an isomorphism. Since $f_{m+1} | \overline{\sum e_\beta^m}$ represent the generators of $\pi_{m+1}(X)$, $f_{m+1*}: \pi_{m+1}(K(m+1)) \to \pi_{m+1}(X)$ is onto. By the induction on n, $K(n)$ and f_n are constructed. We set $K = \cup K(n)$ and define $f: K \to X$ by $f | K(n) = f_n$. Then f satisfies the condition of (4.6).

COROLLARY (4.7) *Any simply connected CW-complex is homotopy equivalent to a CW-complex which admits standard paths.* (*by* (1.5))

THEOREM (4.8) *Let* $X \supset X_0$ *be simply connected spaces. Then there exist a CW-complex K which admits standard paths, its subcomplex* K_0 *and a map* $f: (K, K_0) \to (X, X_0)$ *such that* $f_*: \pi_p(K) \approx \pi_p(X)$, $\pi_p(K_0) \approx \pi_p(X_0)$ *and* $\pi_p(K, K_0) \approx \pi_p(X, X_0)$.

Proof. First construct K_0 and $f_0(=f|K_0): K_0 \to X_0$ as (4.6). Next set $K(0)=K_0$ in the proof of (4.6), then we obtain K and $f: (K, K_0) \to (X, X_0)$ such that $f_*: \pi_p(K) \approx \pi_p(X)$. Then the proof of $\pi_p(K, K_0) \approx \pi_p(X, X_0)$ is a simple application of the five lemma.

5. A filtration.

The notations in §2 will be used in this §.

Denote by $C_{(r)}(\omega(K, K_0))$ the subgroup of $C(\omega(K, K_0))$ which is generated by the products $\sigma_1 \cdot \cdots \cdot \sigma_r$ and $D\sigma \cdot \sigma_1 \cdot \cdots \cdot \sigma_{r-1}$ for primitive elements $\sigma_1, \cdots, \sigma_r \in C(\omega(K))$ and $\sigma \in C(\omega(K_0))$ of positive dimensions. Note that $C_{(r)}(\omega(K)) = C_{(r)}(\omega(K, e_0))$ and $C_{(0)}(\omega(K, K_0)) = \{e_0\}$. Next define $C^{(r)}(\omega(K, K_0))$ by $C^{(r)}(\omega(K, K_0)) = \sum_{i \geq r} C_{(i)}(\omega(K, K_0))$, then $C^{(r)}$ gives a filtration of $C(\omega(K, K_0))$:

(5.1)
$$C^{(r)}(\omega(K, K_0)) \supset C^{(r+1)}(\omega(K, K_0)),$$
$$C^{(r)}(\omega(K, K_0)) \cdot C^{(s)}(\omega(K, K_0)) = C^{(r+s)}(\omega(K, K_0)),$$
and
$$\partial C^{(r)}(\omega(K, K_0)) \subset C^{(r)}(\omega(K, K_0)).$$

Proof. The first two formulas are obvious. Since each 1-cell σ of $\omega(K, K_0)$ is primitive and forms a circle S^1 with the vertex e_0, $\partial\sigma = 0$. Hence $\partial C^{(1)}(\omega(K, K_0)) \subset C(\omega(K, K_0)) - \{e_0\} = C^{(1)}(\omega(K, K_0))$. Now suppose that $\partial C^{(r-1)} \subset C^{(r-1)}$, then $\partial C^{(r)} = \partial(C^{(1)} \cdot C^{(r-1)}) = (\partial C^{(1)}) \cdot C^{(r-1)} + C^{(1)} \cdot (\partial C^{(r-1)}) \subset C^{(1)} \cdot C^{(r-1)} = C^{(r)}$. Therefore the last formula is proved by the induction on r.

Define the boundary operator on $C_{(r)}$ as that of the difference chain group $C^{(r)} - C^{(r-1)}$. Then from (2.3) and (2.12) we have a chain isomorphism $(r>0)$:

(5.2)
$$C_{(1)}(\omega(K, K_0)) \otimes \overbrace{C_{(1)}(\omega(K)) \otimes \cdots \otimes C_{(1)}(\omega(K))}^{(r-1)\text{-fold}} \approx C_{(r)}(\omega(K, K_0))$$

given by $c \otimes c_1 \otimes \cdots \otimes c_{r-1} \to c \cdot c_1 \cdot \cdots \cdot c_{r-1}$. By (2.6) we have a chain isomorphism (suspension) $E: C_{(1)}(\omega(K)) \approx C(K) - \{e_0\}$, and we have that

(5.3)
$$H_p(C_{(1)}(\omega(K)),\ C_{(1)}(\omega(K_0))) \approx H_{p+1}(K, K_0), \quad p \geq 0.$$

We see that $C_{(1)}(\omega(K, K_0)) = C_{(1)}(\omega(K_0, K_0)) + (C_{(1)}(\omega(K)) - C_{(1)}(\omega(K_0))$ and $C_{(1)}(\omega(K_0, K_0))$ is closed under the boundary operator of $C_{(1)}(\omega(K, K_0))$. The formula b) of (2.12) shows that $C_{(1)}(\omega(K_0, K_0))$ is chain equivalent to 0. Then we have easily that

(5.4) the injection $C_{(1)}(\omega(K)) - C_{(1)}(\omega(K_0)) \to C_{(1)}(K, K_0))$ is chain equivalence and the inverse is given by the projection of $C_{(1)}(\omega(K, K_0))$ onto its direct factor $C_{(1)}(\omega(K))-C_{(1)}(\omega(K_0))$.

As a corollary we have that

$$(5.4)' \qquad H_p(C_{(1)}(\omega(K, K_0)))\approx H_{p+1}(K, K_0) \qquad \text{for } p>0.$$

Let $M, M_1, \cdots, M_n$ be subcomplexes of K such that $M_i \cap M_j = M$ for $i \neq j$ and $K = M_1 \cup \cdots \cup M_n$. Let $K_1, \cdots, K_n$ be subcomplexes of K given by $K_i = K-(M_i - M)$, $i=1, \cdots, n$. Let $\mathcal{C}$ be a class of abelian groups which satisfies the conditions (I) and ($\mathrm{II_B}$) of [9].

LEMMA (5.6) *If $H_p(K, K_i)\in\mathcal{C}$ for $p<q_i+1$ and if $H_{q_i+1}(K, K_i)$ is $\mathcal{C}$-isomorphic to a group G_i. Then $H_p(C_{(r)}(\omega(K, K_n)), \sum_{i=1}^{n-1} C_{(r)}\omega(K_i, K_i \cap K_n)))=0$ for $r<n$, and $\in\mathcal{C}$ for $r\geq n$ and $p<Q+r-n$, where $Q=\sum q_i$. The group $H_Q(C_{(n)}(\omega(K, K_n)), \sum_{i=1}^{n-1} C_{(n)}(\omega(K_i, K_i\cap K_n)))$ is $\mathcal{C}$-isomorphic to the direct sum of $(n-1)!$ copies of $G_1\otimes\cdots\otimes G_n$.*

Proof. By (5.2), $C_{(r)}(\omega(K, K_n))-\sum_{i=1}^{n-1} C_{(r)}(\omega(K_i, K_i\cap K_n))$ is chain isomorphic to $C_{(1)}(\omega(K, K_n))\otimes[C_{(1)}(\omega(K))]^{r-1}-\sum_{i=1}^{n-1}(C_{(1)}(\omega(K_i, K_i\cap K_n))\otimes[C_{(1)}(\omega(K_i))]^{r-1})$, where $[A]^t$ indicates the t-fold tensor product $A\otimes\cdots\otimes A$. Since $M_n - M = K_i - (K_i\cap K_n) = K-K_n$, $i=1, \cdots, n-1$, the injections $C_{(1)}(M_n)-C_{(1)}(M)\to C_{(1)}(\omega(K_i, K_i\cap K_n))$ and $C_{(1)}(M_n)-C_{(1)}(M)\to C_{(1)}(\omega(K, K_n))$ are chain equivalences by (5.4), and their inverse are the projections to the factor $C_{(1)}(M_n)-C_{(1)}(M)$. Then we have that the injection of $(C_{(1)}(M_n)-C_{(1)}(M))\otimes([C_{(1)}(\omega(K))]^{r-1}-\sum_{i=1}^{n-1}[C_{(1)}(\omega(K_i))]^{r-1})$ into $(C_{(1)}(\omega(K, K_n))\otimes[C_{(1)}(\omega(K))]^{r-1})-\sum_{i=1}^{n-1}(C_{(1)}(\omega(K_i, K_i\cap K_n))\otimes[C_{(1)}(\omega(K_i))]^{r-1})$ is a chain equivalence.

For the simplicity we denote that $C_{(1)}(\omega(M))=B_0$, $C_{(1)}(\omega(M_i))-C_{(1)}(\omega(M))=B_i$ for $i=1, \cdots, n$, then $C_{(1)}(\omega(K))=\sum_{i\geq 0} B_i$, $C_{(1)}(\omega(K_i))=\sum_{i\neq j} B_j$, $\partial B_0\subset B_0$ and $\partial B_i\subset B_0+B_i$. Then the assertion of (5.6) is reworded to that

$$(5.6)' \qquad H_p(B_n\otimes([\sum_{j=0}^{n} B_j]^{r-1}-\sum_{i=1}^{n-1}[\sum_{j\neq i} B_j]^{r-1})$$

satisfies the assertion of (5.6).

Applying the Künnth's formula (1.6), (5.6)′ is rewritten as

(5.6)″ If $H_0(B_0)=0$, $H_p(B_i)\in\mathcal{C}$ for $1\leq i\leq n-1$ and $p<q_i$ and if $H_{q_i}(B_i)$ is $\mathcal{C}$-isomorphic to a group G_i. Then $H_p(([\sum_{j=0}^{n} B_j]^{r-1}-\sum_{i=1}^{n-1}[\sum_{j\neq i} B_j]^{r-1})=0$ for $r<n$, and $\in\mathcal{C}$ for $r\geq n$ and $p<Q'+r-m$, where $Q'=Q-q_n$. The group $H_{Q'}(\sum_{j=0}^{n}[B_j]^{n-1}-\sum_{i=1}^{n-1}[\sum_{j\neq i} B_j]^{n-1})$ is $\mathcal{C}$-isomorphic to the sum of $(n-1)!$ copies of $G_1\otimes\cdots\otimes G_{n-1}$.

Proof of (5.6)″ A factor $B_{i_1}\otimes\cdots\otimes B_{i_{r-1}}$ of $[\sum_{j=1}^{n} B_j]^{r-1}$ is in $\sum_{i=1}^{n-1}[\sum_{j\neq i} B_j]^{r-1}$

if and only if the set $\{i_1, \cdots, i_{r-1}\}$ of indices is contained in one of $\{0, 1, \cdots, n-1, n\} -\{i\}$ for $i=1, \cdots, n-1$. Hence a factor $B_{i_1}\otimes\cdots\otimes B_{i_{r-1}}$ is in $[\sum_{j=1}^{n} B_j]^{r-1} - \sum_{i=1}^{n-1}[\sum_{j\neq i} B_j]^{r-1}$ if and only if $\{i_1, \cdots, i_{r-1}\}$ contains $\{1, \cdots, n-1\}$.

In the case $r<n$, we have obviously $[\sum_{j=1}^{n} B_j]^{r-1}-\sum_{i=1}^{n-1}[\sum_{j\neq i} B_j]^{r-1}=0$. Therefore (5.6)″ is proved for $r<n$.

Let $r\geq n$. Since $H_p(B_i)\in\mathcal{C}$ for $p<1$, $i=0, 1, \cdots, n-1$, we have easily from (1.6) that

(5.7) a) if $\{i_1, \cdots, i_{r-1}\}$ contains $\{1, \cdots, n-1\}$, then $H_p(B_{i_1}\otimes\cdots\otimes B_{i_{r-1}})\in\mathcal{C}$ for $p<Q'+r-n$; b) if $\{i_1, \cdots, i_{n-1}\}=\{1, \cdots, n-1\}$, then $H_p(B_{i_1}\otimes\cdots\otimes B_{i_{r-1}})$ is $\mathcal{C}$-isomorphic to $G_1\otimes\cdots\otimes G_{n-1}$.

Now we can arrange the factors $B_{i_1}\otimes\cdots\otimes B_{i_{r-1}}$ in an order such that if $\{D_k;\ k=1, 2, \cdots\}$ is such an ordered set of $\{B_{i_1}\otimes\cdots\otimes B_{i_{r-1}}\}$ then $\partial D_k\subset\sum_{i\leq k} D_i$. Denote $E_k=\sum_{i\leq k} D_i$, then E_k are chain subgroups, and $E_k=[\sum_{j=1}^{n} B_j]^{r-1}-\sum_{i=1}^{n-1}[\sum_{j\neq i} B_j]^{r-1}$ for sufficiently large k. Consider an exact sequence $H_p(E_k)\rightarrow H_p(E_{k+1})\rightarrow H_p(E_{k+1}, E_k)=H_p(D_k)$. By (5.7), a) $H_p(D_k)\in\mathcal{C}$ for $p<Q'+r-n$. Hence $H_p(E_k)\in\mathcal{C}$ implies $H_p(E_{k+1})\in\mathcal{C}$ for $p<Q'+r-n$. By induction on k, we have that $H_p(E_k)\in\mathcal{C}$ for $p<Q'+r-n$ and for all k, and (5.6)″ is proved for the case $r\geq n$ and $p<Q'+r-n$. In the case $r=n$, $D_k=B_{i_1}\otimes\cdots\otimes B_{i_{n-1}}$ for some $\{i_1, \cdots, i_{n-1}\}=\{1, \cdots, n-1\}$. Then $\partial D_k\subset D_k$. Therefore $H_{Q'}(E_{(n-1)!})=\sum_k H_{Q'}(D_k)$ and it is $\mathcal{C}$-isomorphic to the sum of $(n-1)!$ copies of $G_1\otimes\cdots\otimes G_{n-1}$, by (5.7), b).

Consequently (5.6)″ and hence (5.6) is proved.

LEMMA (5.8) *From the hypothesis of* (5.6) *we have that* $H_p(\omega(K, K_n), \bigcup_{j=1}^{n-1}\omega(K_i, K_i\cap K_n))\in\mathcal{C}$ *for* $p<Q$ *and* $H_Q(\omega(K, K_n), \bigcup_{n=1}^{n-1}\omega(K_i, K_i\cap K_n))$ *is* $\mathcal{C}$-*isomorphic to the direct sum of* $(n-1)!$ *copies of* $G_1\otimes\cdots\otimes G_n$.

Proof. Denote $C^{(r)}=C^{(r)}(\omega(K, K_n))-\sum_{i=1}^{n-1}C^{(r)}(\omega(K_i, K_i\cap K_n))$ and $C_{(r)}=C_{(r)}(\omega(K, K_n))-\sum_{i=1}^{n-1}C_{(r)}(\omega(K_i, K_i\cap K_n))$, then $C_{(r)}=C^{(r)}-C^{(r+1)}$. Since the chains of $C(\omega(K, K_n))-C^{(r)}(\omega(K, K_n))$ have at least dimension r, $H_p(\omega(K, K_n), \sum_{i=1}^{n-1}\omega(K_i, K_i\cap K_n))=H_p(C^{(0)})\approx H_p(C^{(0)}, C^{(p+2)})$. Consider the exact sequence: $H_p(C_{(r)})\approx H_p(C^{(r)}, C^{(r+1)})\rightarrow H_p(C^{(0)}, C^{(r+1)})\rightarrow H_p(C^{(0)}. C^{(r)})\rightarrow H_{p-1}(C^{(r)}, C^{(r+1)})\approx H_{p-1}(C_{(r)})$. If $r<n$, $H_p(C_{(r)})=0$ by (5.6) and then $H_p(C^{(0)}, C^{(r+1)})\approx H_p(C^{(0)}, C^{(r)})$. Hence $H_p(C^{(0)}, C^{(n)})\approx H_p(C^{(0)}, C^{(0)})=0$. Therefore $H_p(C_{(n)})\approx H_p(C^{(0)}, C^{(n+1)})$. If $r\geq n$, by (5.6) $H_p(C_{(r)})\in\mathcal{C}$ for $p<Q+1$, then $H_p(C^{(0)}, C^{(r+1)})$ and $H_p(C^{(0)}, C^{(r)})$ are $\mathcal{C}$-isomorphic for $p\leq Q$. Hence $H_p(C^{(0)})\approx H_p(C^{(0)}, C^{(p+2)})$ is $\mathcal{C}$-isomorphic to $H_p(C^{(0)}, C^{(n+1)})\approx H_p(C_{(n)})$ for $p\leq Q$. Then (5.8) follows from (5.6).

6. Connectedness theorem for $(n+1)$-ad homotopy groups.

Let $(X; X_1, \cdots, X_n, x_0)$ be $(n+1)$-ad and let $\pi_p(X; X_1, \cdots, X_n)$ be the homotopy group of the $(n+1)$-ad [1].

We consider the group $\pi_p(X; X_1, \cdots, X_n)$ as the set of the homotopy classes of maps $f: (I; I_1^{p-1}, \cdots, I_n^{p-1}, J_n^{p-1}) \to (X; X_1, \cdots, X_n, x_0)$, where $I_i^{p-1} = \{(x_1, \cdots, x_p) \in I_p \mid x_i = 0\}$ and $J_n^{p-1} = \dot{I}^p - \text{Int.}\ (\bigcup_{i=1}^{n} I_i^{p-1})$.

For a map $g: (I^p; I_1^{p-1}, \cdots, I_{n-1}^{p-1}, J_{n-1}^{p-1}) \to (\Omega(X, X_n); \Omega(X_1, X_1 \cap X_n), \cdots, \Omega(X_{n-1}, X_{n-1} \cap X_n), f_0)$, define a map $\Omega g: (I^{p+1}; I_1^p, \cdots, I_n^p, J_n^p) \to (X; X_1, \cdots, X_n, x_0)$ by $\Omega g(x_1, \cdots, x_{p+1}) = g(x_1, \cdots, x_{n-1}, x_{n+1}, \cdots, x_{p+1})(x_n)$, where $f_0(I) = x_0$. Then the correspondance $g \leftrightarrow \Omega g$ defined the isomorphism

$$\Omega: \pi_p(\Omega(X, X_n); \Omega(X_1, X_1 \cap X_n), \cdots, \Omega(X_{n-1}, X_{n-1} \cap X_n)) \approx \pi_{p+1}(X; X_1, \cdots, X_n). \tag{6.1}$$

We introduce here some elementary properties of the homotopy groups of $(n+1)$-ad (cf. [1, I]).

(6.2) $\pi_p(X; X_1, \cdots, X_n) \approx \pi_p(X; X_{\sigma(1)}, \cdots, X_{\sigma(n)})$ for a permutation σ of $\{1, \cdots, n\}$.

$$\pi_p(X; X_1, \cdots, X_n) \approx \pi_p(X; X_1, \cdots, X_{n-1}) \quad \text{if } X_{n-1} \supset X_n. \tag{6.3}$$

(6.4) The following sequence of homomorphisms is exact:

$$\cdots \longrightarrow \pi_{p+1}(X; X_1, \cdots, X_n) \longrightarrow \pi_p(X_1; X_1 \cap X_2, \cdots, X_1 \cap X_n) \longrightarrow \pi_p(X; X_2, \cdots, X_n) \longrightarrow \pi_p(X; X_1, \cdots, X_n) \longrightarrow \cdots.$$

A map $f: (X; X_1, \cdots, X_n) \to (Y; Y_1, \cdots, Y_n)$ defines the induced homomorphism $f_*: \pi_p(X; X_1, \cdots, X_n) \to \pi_p(Y; Y_1, \cdots, Y_n)$.

(6.5) The induced homomorphisms commute with the exact sequences (6.4) of $(X; X_1, \cdots, X_n)$ and $(Y; Y_1, \cdots, Y_n)$.

Let K be a CW-complex and let $K_1, \cdots, K_n$ be subcomplexes such that $K_1 \cap \cdots \cap K_n \ni e_0$ a vertex. Denote by $I(n)$ the set of indices $\{1, \cdots, n\}$. For each subset J of $I(n)$, we associate the subcomplex $K_J = K_{j_1} \cap \cdots \cap K_{j_r}$ where $\{j_1, \cdots, j_r\} = J$. Denote $\partial K_J = \bigcup_{J' \subsetneqq J} K_{J'}$, $M = K_{I(n)}$ and $M_i - K_{I(n)-\{i\}}$.

Then the connectedness theorem for $(n+1)$-ad homotopy groups is stated as follows:

THEOREM $(6.6)_n$ *Assume that $K = M_1 \cup \cdots \cup M_n$, $\pi_0(M) = \pi_1(M) = \pi_0(M_i) = \pi_1(M_i) = 0$, $\pi_2(M_j, M) = 0$ and $H_p(M_i, M) \in \mathcal{C}$ for $p \leq q_i$, $i = 1, \cdots, n$. Let $Q = \sum q_i$. Then $\pi_p(K; K_1, \cdots, K_n) \in \mathcal{C}$ for $p \leq Q$ and $\pi_{Q+1}(K; K_1, \cdots, K_n)$ is $\mathcal{C}$-isomorphic to the direct sum of $(n-1)!$ copies of $H_{q_1+1}(M_1, M) \otimes \cdots \otimes H_{q_n+1}(M, M)$.*

Here $\mathcal{C}$ indicates a class of abelian groups which satisfies the conditions (I), (II_B) and (III) of [9]. For a general combinatorial $(n+1)$-ad, we have the following:

THEOREM $(6.7)_n$ *Assume that* $\pi_0(K_J)=\pi_1(K_J)=\pi_2(K_J, \partial K_J)$ *and* $H_p(K_J, \partial K_J) \in \mathcal{C}$ *for* $p \leq q_J$, $J \subset I(n)$. *Let* Q *be the ninimum of the sums* $q_{J_1}+\cdots+q_{J_s}$ *such that* $J_1 \cap \cdots \cap J_s=\phi$ *(empty set). Then* $\pi_p(K; K_1, \cdots, K_n) \in \mathcal{C}$ *for* $p \leqq Q$.

First we show that

LEMMA (6.8) $(6.6)_n$ *and* $(6.7)_r$ *for* $r<n$ *imply* $(6.7)_n$.

Proof. Let $\{J_k; k=1, \cdots, 2^n\}$ be an ordered set of the indices $J \subset I(n)$ such that $J_1=I(n)$, $J_{i+1}=I(n)-\{i\}$ for $i=1, \cdots, n$ and that $J_k \subset J_{k'}$ implies $k \geq k'$. Set $K(k)=\sum_{j \leq k} K_{J_j}$, then $K(k)$ is a subcomplex of K and $K(k)-K(k-1)=K_{J_k}-\partial K_{J_k}$. $(6.6)_n$ means that $(6.7)_n$ is true if $K=K(n+1)$, or that $(6.7)_n$ is true for $(K(n+1); K_1 \cap K(n+1), \cdots, K_n \cap K(n+1))$. Now suppose that $(6.7)_n$ is true for an $(n+1)$–ad $(K(k-1); K_1 \cap K(k-1), \cdots, K_n \cap K(k-1))$, $k \geq n+2$. Let $J=\{j_1, \cdots, j_r\}$ be a subset of $I(n)$ such that $K(k)-K(k-1)=K_J-\partial K_J$. By (6.4) we have the exact sequence: $\pi_p(K(k-1); K_1 \cap K(k-1), \cdots, K_n \cap K(k-1)) \rightarrow \pi_p(K(k); K_1 \cap K(k), \cdots, K_n \cap K(k)) \rightarrow \pi_p(K(k); K(k-1), K_1 \cap K(k), \cdots, K_n \cap K(k))$. Since $K(k) \cap K_i=K(k-1)$ for $i \in I(n)-J$, we have from (6.2) and (6.3) that $\pi_p(K(k); K(k-1), K_1 \cap K(k), \cdots, K_n \cap K(k)) \approx \pi_p(K(k); K(k-1), K_{j_1} \cap K(k), \cdots, K_{j_r} \cap K(k))$. Since $k \geq n+1$, $r \leq n-2$ and $r+1<n$, we can apply $(6.7)_{r+1}$ to the group $\pi_p(K(k); K(k-1), K_{j_1} \cap K(k), \cdots, K_{j_r} \cap K(k))$, and we shall prove that

$$(6.9) \qquad \pi_p(K(k); K(k-1), K_{j_1} \cap K(k), \cdots, K_{j_r} \cap K(k)) \in \mathcal{C} \qquad \text{for } p \leq Q.$$

Then $\pi_p(K(k-1); K_1 \cap K(k-1), \cdots, K_n \cap K(k-1)) \in \mathcal{C}$ implies $\pi_p(K(k); K_1 \cap K(k), \cdots, K_n \cap K(k)) \in \mathcal{C}$ for $p \leq Q$. By induction on $k \geq n+2$, $(6.7)_n$ is verified and (6.8) is proved.

Proof of (6.9) Set $K(k)=L$, $K(k-1)=L_1$ and $K_{j_i} \cap K(k)=L_{i+1}$ for $i=1, \cdots, r$. The conditions $\pi_0(L_A)=\pi_1(L_A)=\pi_2(L_A, \partial L_A)=0$, $A \subset I(r+1)$, are easily verified. Let p_A be an integer such that $H_p(L_A, \partial L_A) \in \mathcal{C}$ for $p \leq p_A$. If $A=I(r+1)-\{1\}$, then $L_A-\partial_, L_A=K_J-\partial K_J$ and hence $p_A=q_J$. If $A \subsetneqq I(r+1)-\{1\}$, then $L_A-\partial L_A=\phi$ and $p_A=\infty$. Consider subsets $A_1, \cdots, A_s$ of $I(r+1)$ such that $A_1 \cap \cdots \cap A_s=\phi$, then there is at least one A_i which does not contain 1. If $A_i \subsetneqq I(r+1)-\{1\}$, then $p_{A_1}+\cdots+p_{A_s}=\infty$. Now we suppose that $A_i \ni 1$ for $1 \leq i \leq t$ and $Ai=I(r+1)-\{1\}$ for $t<i \geq s$, $(t<s)$. Denote by B_i a subset $\{j_b | b+1 \in (I(r+1)-A_i)\}$ of $I(n)$, $i \leq t$, then $L_{A_i}-\partial L_{A_i}$ is the union of $K_{J_k}-\partial K_{J_k}$ such that $J_k \cap B_i=\phi$ and $r>k$. Therefore $p_{A_i} \geq$ Min. $(q_{J_k}: J_k \cap B_i=\phi)$ and $p_{A_1}+\cdots+p_{A_s} \geq$ Min. $(q_{J_1'}+\cdots+q_{J_t'}+(s-t)q_J: J_i' \cap B_i=\phi)$. Since $A_1 \cap \cdots \cap A_t=\{1\}$, $B_1 \cup \cdots \cup B_t=\{j_1, \cdots, j_r\}=J$ and $J_1' \cup \cdots \cup J_t' \cap J=\phi$ if $J_i' \cap B_i=\phi$. From the hypothesis of $(6.7)_n$, $p_{A_1}+\cdots+p_{A_s} \geq Q$, and we have (6.9) from $(6.7)_{r+1}$.

Proof of $(6.6)_n$ By (6.8), it is sufficient to prove that $(6.7)_r$, $r<n$ implies $(6.6)_n$. According to (4.8), we construct CW–complex M', M_1', $\cdots$, M_n' which admit standard paths and maps $f_i: M_i' \rightarrow M_i$ such that $M_i' \cap M_j'=M'$ and $f_i | M'=f_j | M'$

for $i \neq j$ and that $f_{i*}: \pi_p(M_i') \approx \pi_p(M_i)$ and $\pi_p(M') \approx \pi_p(M)$. By (2.2) we see that the union $K' = \cup M_i'$ admits standard paths. Define a map $f: K' \to K$ by $f|M_i' = f_i$ and set $K_i' = K' - (M_i' - M')$. Since the complexes are simply connected, the isomorphisms of homotopy groups provide isomorphisms of homology groups $f_*: H_*(M') \approx H_*(M)$ and $H_*(M_i') \approx H_*(M_i)$, and hence $f_*: H_*(K_J') \approx H_*(K_J)$. Then f induces isomorphisms $f_*: \pi_p(K'_J) \approx \pi_p(K_J)$. Applying (6.5) and the five lemma, we have that f induces isomorphisms $f_*: \pi_p(K'; K_1', \cdots, K_n') \approx \pi_p(K; K_1, \cdots, K_n)$. Therefore we may assume that $K' = K$, i.e., K admits standard paths.

By (4.2), $i_*: H_*(\omega(K, K_n)) \approx H_*(\Omega(K, K_n))$ and $H_*(\omega(K_i, K_i \cap K_n)) \approx H_*(\Omega(K_i, K_i \cap K_n))$. As is easily seen that $\Omega(K, K_n)$ and $\Omega(K_i, K_i \cap K_n)$ are simply connected. Repeating the above discussion on the map $f: K' \to K$ for the injection $i: \omega(K, K_n) \to \Omega(K, K_n)$, we have isomorphisms $\pi_p(\omega(K, K_n); \omega(K_1, K_1 \cap K_n, \cdots, \omega(K_{n-1}, K_{n-1} \cap K_n)) \approx \pi_p(\Omega(K, K_n); \Omega(K_1, K_1 \cap K_n), \cdots, \Omega(K_{n-1}, K_{n-1} \cap K_n))$. Combining (6.1) to this isomorphisms, we have isomorphisms $\pi_{p+1}(K; K_1, \cdots, K_n) \approx \pi_p(\omega(K, K_n); \omega(K_1, K_1 \cap K_n), \cdots, \omega(K_{n-1}, K_{n-1} \cap K_n))$. Set $L = \omega(K, K_n)$ and $L_i = \omega(K_i, K_i \cap K_n)$ for $i = 1, \cdots, n-1$. We apply $(6.7)_{n-1}$ to an n-ad $(\partial L; L_1, \cdots, L_{n-1})$. The simply connectedness of L_J and L is easily verified. By (5.8), $H_p(L_J, \partial L_J) = 0$ for $p < 4$ and $\pi_2(L_J, \partial L_J) = 0$, this is a special case of $(6.6)_2$. Applying (5.8) to $L_J = \omega(K_J, K_J \cap K_n)$, we have that $H_p(L_J, \partial L_J) \in \mathcal{C}$ for $p \leq (\sum_{i \in I(n-1)-J} q_i) + q_n - 1$. If $J_1 \cap \cdots \cap J_s = \emptyset\ (s > 1)$, $J_k \subset I(n-1)$, $k = 1, \cdots, s$, then $\sum_{k=1}^{s} ((\sum_{i \in I(n-1)-J_k} q_k) + q_n - 1) \geq (\sum_{k=1}^{n-1} q_k) + s(q_n - 1) \geq Q$. Therefore we have from $(6.7)_{n-1}$ that $\pi_p(\partial L; L_1, \cdots, L_{n-1}) \in \mathcal{C}$ for $p < Q+1$. From the exact sequence (6.4) for an $(n+1)$-ad $(L; \partial L, L_1, \cdots, L_{n-1})$, we have that $\pi_p(L; L_1, \cdots, L_{n-1})$ is $\mathcal{C}$-isomorphic to $\pi_p(L; \partial L, L_1 \cdots, L_{n-1})$ for $p < Q+1$. Since $L_i \subset \partial L$, $i = 1, \cdots, n-1$, we have from (6.2) and (6.3) that $\pi_p(L; \partial L, L_1, \cdots, L_{n-1}) \approx \pi_p(L; \partial L)$. Consequently $\pi_{p+1}(K; K_1, \cdots, K_n) \approx \pi_p(L; L_1, \cdots, L_{n-1})$ is $\mathcal{C}$-isomorphic to $\pi_p(L; \partial L) = \pi_p(\omega(K, K_n), \bigcup_{i=1}^{n-1} \omega(K_i, K_i \cap K_n))$ for $p < Q+1$. By (1.9) and (5.8), $\pi_p(L, \partial L)$ is $\mathcal{C}$-isomorphic to $H_p(L, \partial L)$ for $p < Q+1$. Then $(6.6)_n$ follows from (5.8).

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