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On $(k+1)$*-ad homotopy groups*

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A. L. Blakers and W. S. Massey[1] have defined triad homotopy groups and used it for the problem of absolute and relative homotopy groups. They also refered to $(k+1)$-*ad* homotopy groups. I want here to prove the exactness of sequence of $(k+1)$-*ad* homotopy groups, and apply to the simplest case. The author wishes to express his cordial thanks to Prof. A. Komatu, and Mr. J. Nagata, for their encouragement in this paper.

1. Notation and terminology.

Let X and Y be topological spaces, $A_1, \cdots, A_k$ subspaces of X, and $B_1, \cdots, B_k$ subspaces of Y.

The notation

(1.1) $$f\colon (X;\ A_1, \cdots, A_k) \to (Y;\ B_1, \cdots, B_k)$$

means that f is a continuous function defined on X with values in Y, satisfying the condition

$$f(A_i) \subset B_i, \quad (i=1, 2, \cdots\cdots k).$$

If the sets $A_1, \cdots, A_k$ have a non-vacuous intersection, $A_1 \cap A_2 \cap \cdots \cap A_k = C \neq 0$, we call the ordered collection of spaces $(X;\ A_1, \cdots, A_k)$ a $(k+1)$-*ad*. An n-cell E^n is the set of vectors $\mathfrak{x}=(x_1, \cdots, x_n)$, where $0 \leqq x_i \leqq 1$.

The symbol $F_n^k(X;\ A_1, \cdots, A_k, p_0)(p_0 \in A_1 \cap A_2 \cap \cdots \cap A_k)$ will denote the function space of all maps

$$f\colon (E^n \times E^k) \to (X)\,,$$

such that, for all vectors $\mathfrak{x}=(x_1, \cdots, x_n) \in E^n$ and $\mathfrak{y}=(y_1, \cdots, y_k) \in E^k$,

(1.2) $$f(x_1, \cdots, x_n, y_1, \cdots, y_k) = p_0 \quad (\text{if one of } x_i=0 \text{ or } 1, \text{ or one of } y_j=1),$$
$$f(x_1, \cdots, x_n, y_1, \cdots, y_k) \in A_i \quad (\text{if } y_i=0),$$

and we introduce the compact open topology in it.

Analogously, we use another symbol $\overline{F}_n^k\ (X;\ A_1, \cdots, A_k)$, which is the function space of all maps

$$f\colon (\dot{E}^{n+1} \times E^k) \to X\,,$$

such that

(1.3) $$f(x_1, \cdots, x_{n+1}, y_1, \cdots, y_k) = p_0 \quad (\text{if } \mathfrak{x}=(x_1, 0 \cdots 0) \text{ or } x_i=0 \text{ or } x_i=1 \text{ or one of } y_i=1),$$

$$f(x_1, \cdots, x_{n+1}, y_1, \cdots, y_k) \in A_j \quad (\text{if } y_j=0)\,.$$

We can introduce in both of them an operation of addition as follows:

Namely, if $f, g \in F_n^k(X; A_1, \cdots, A_k)$ or $\bar{f}, \bar{g} \in \overline{F}_n^k(X; A_1, \cdots, A_k)$, we define $h=f+g$, $\bar{h}=\bar{f}+\bar{g}$ by

$$h(x_1, \cdots, x_n, y_1, \cdots, y_k)= \begin{matrix} f(2x_1, x_2, \cdots, x_n, y_1, \cdots, y_k) & (0\leqq x_1\leqq \tfrac{1}{2})\,, \\ g(2x_1-1, x_2, \cdots, x_n, y_1, \cdots, y_k) & (\tfrac{1}{2}\leqq x_1\leqq 1)\,, \end{matrix}$$

$$\bar{h}(x_1, \cdots, x_{n+1}, y_1, \cdots, y_k)= \begin{matrix} \bar{f}(2x_1, x_2, \cdots, x_{n+1}, y_1, \cdots, y_k) & (0\leqq x_1\leqq \tfrac{1}{2})\,, \\ \bar{g}(2x_1-1, x_2, \cdots, x_{n+1}, y_1, \cdots, y_k) & (\tfrac{1}{2}\leqq x_1\leqq 1)\,. \end{matrix}$$

We can readily prove that the homotopy classes of F_n^k or $\overline{F}_n^k$ make a group.

It is denoted by $\pi_n^k(X; A_1, \cdots, A_k)$. And we shall call it the *$(k+n)$-th or $(k+n)$-dimensional homotopy group of $(k+1)$-ad* $(X; A_1, \cdots, A_k)$.

We can define isomorphisms,

$$\varphi_m: \pi_m(F_n^k(X; A_1, \cdots, A_k), k_0) \approx \pi_{m+n}^k(X; A_1, \cdots, A_k)\,.$$

And the group $\pi_n^k(X; A_1, \cdots, A_k)$ is abelian if $n\geqq 2$ as is readily seen, where k_0 denotes the constant map $k_0(\mathfrak{x}, \mathfrak{y})=p_0$.

2. Exact sequence.

THEOREM 2.1. *The sequence*

$$\rightarrow \pi_n^k(X; A_1, \cdots, A_k) \xrightarrow{\beta} \pi_n^{k-1}(A_1; A_1\frown A_2, \cdots, A_1\frown A_k) \xrightarrow{i} \pi_n^{k-1}(X; A_2, \cdots, A_k)$$
$$\xrightarrow{j} \pi_{n-1}^k(X; A_1, \cdots, A_k) \rightarrow$$

is an exact sequence. (i, j are injections and β is a boundary operator)

Proof.

The proof breaks up into six parts.

(a) $j\circ i=0$.

Let $\alpha \in \pi_n^{k-1}(A_1; A_1\frown A_2, \cdots, A_1\frown A_k)$ be represented by a map $f: E^n\times E^{k-1}\rightarrow X$, where $f(x_1, \cdots, x_n, y_2, \cdots, y_k)\in A_1\frown A_i$ (if $y_i=0$ $(i=2, \cdots, k)$). If $i(f)=g\in \bar{\alpha} = i\alpha \in \pi_n^{k-1}(X; A_2, \cdots, A_k)$,

$$g(x_1, \cdots, x_n, y_2, \cdots, y_k)=f(x_1, \cdots, x_n, y_2, \cdots, y_k)\,.$$

Let $(j\circ i)(f)=j(g)=h\in \bar{\bar{\alpha}} \in \pi_{n-1}^k(X; A_1, \cdots, A_k)$ be the $(j-)$ image of g. Then

$$h: E^{n-1}\times E^k \rightarrow X\,,$$

and $h(x_1, \cdots, x_{n-1}, y_1, \cdots, y_k)\in A_i$(if $y_i=0$, $i=1, 2, \cdots, k$). Therefore, we may write h as a function of $(x_1, \cdots, x_{n-1}, y_1, y_2, \cdots, y_k)$ such that

$$h(x_1, \cdots, x_{n-1}, y_1, y_2,\cdots, y_k)=g(x_1, \cdots, x_{n-1}, x_n, y_2, \cdots, y_k) \quad (\text{if } y_1=x_n)\,.$$

Then $h(x_1, \cdots, x_{n-1}, 0, y_2, \cdots, y_k)=p_0$.

Let $\Psi(x_1, \cdots, x_{n-1}, y_1, y_2, \cdots, y_k, t)=h(x_1, \cdots, x_{n-1}, (1-t)y_1+t, y_2, \cdots, y_k)$.

Then $\Psi(\mathfrak{x}, \mathfrak{y}, 0)=h(\mathfrak{x}, \mathfrak{y})$, $\Psi(\mathfrak{x}, \mathfrak{y}, 1)=p_0$ Hence $h\simeq 0$.

(b) $\imath\circ\beta=0$

Let $\alpha\in\pi_n^k(X; A_1, \cdots, A_k)$ be represented by $f: E^n\times E^k\to X$, then $\beta(f)=g$ is a function such that

$$g: E^n\times E^{k-1}\to A_1$$

and $g(x_1, \cdots, x_n, y_2, \cdots, y_k)=f(x_1, \cdots, x_n, 0, y_2, \cdots, y_k)$.

Let $i(g)=h$, then

$$h: E^n\times E^{k-1}\to X,$$

and $h(x_1, \cdots, x_n, y_2, \cdots, y_k)=g(x_1, \cdots, x_n, y_2, \cdots, y_k)$. Therefore, $f(x_1, \cdots, x_n, 0, y_2, \cdots, y_k)=h(x_1, \cdots, x_n, y_2, \cdots, y_k)\in X$ and $f(x_1, \cdots, x_n, 1, y_2, \cdots, y_k)=p_0$. This means that $h\simeq 0$.

(c) $\beta\circ j=0$.

As above we take one mapping $f\in\alpha\in\pi_n^{k-1}(X; A_2, \cdots, A_k)$. And, let $j(f)=g$, $\beta(g)=h$. Then $g(x_1, \cdots, x_{n-1}, y_1, y_2, \cdots, y_k)=f(x_1, \cdots, x_{n-1}, x_n, y_2, \cdots, y_k)$ (if $y_1=y_n$) and $h(x_1, \cdots, x_{n-1}, y_2, \cdots, y_k)=g(x_1, \cdots, x_{n-1}, 0, y_2, \cdots, y_k)$, whence $h(x_1, \cdots, x_{n-1}, y_2, \cdots, y_k)=f(x_1, \cdots, x_{n-1}, 0, y_2, \cdots, y_k)=p_0$ and $h=0$.

(d) $(\beta)^{-1}(0)\subset$ image of j.

Let $f\in\alpha\in\pi_n^k(X; A_1, \cdots, A_k)$, and

$$\beta(f)=g\simeq 0.$$

$$f(x_1, \cdots, x_n, 0, y_2, \cdots, y_k)=g(x_1, \cdots, x_n, y_2, \cdots, y_k)\in A_1.$$

We may assume that $g(x_1, \cdots, x_n, y_2, \cdots, y_k)=p_0$. Then, let φ be a mapping such that

$$\varphi(x_1, \cdots, x_n, x_{n+1}, y_2, \cdots, y_k)=f(x_1, \cdots, x_n, y_1, \cdots, y_k) \quad (\text{if } x_{n+1}=y_1).$$

Obviously,

$$\varphi: E^{n+1}\times E^{k-1}\to X,$$

$$\varphi(x_1, \cdots, x_n, 0, y_2, \cdots, y_k)=f(x_1, \cdots, x_n, 0, y_2, \cdots, y_k)=p_0.$$

Hence $$\varphi\in\alpha\in\pi_{n+1}^{k-1}(X; A_2, \cdots, A_k)$$

and $j(\varphi)=f$.

(e) $(i)^{-1}\subset$ image of β.

Take an element f of $\alpha\in\pi_n^{k-1}(A_1; A_1\cap A_2, \cdots, A_1\cap A_k)$, and assume $i(f)=g\simeq 0$. Then

$$g(x_1, \cdots, x_{n-1}, x_n, y_2, \cdots, y_k)=f(x_1, \cdots, x_n, y_2, \cdots, y_k).$$

And there exists a mapping such that

$$\Psi: E^n\times E^{k-1}\times E^1 \to X, \quad \text{and}$$
$$\Psi(x_1, \cdots, x_n, y_2, \cdots, y_k, t)\in X,$$
$$\Psi(x_1, \cdots, x_n, y_2, \cdots, y_k, t)\in A(\text{if } y_i=0,\ i=2, \cdots, k),$$
$$\Psi(x_1, \cdots, x_n, y_2, \cdots, y_k, 0)=g(x_1, \cdots, x_n, x_2, \cdots, y_k),$$
$$\Psi(x_1, \cdots, x_n, y_2, \cdots, y_k, 1)=p_0.$$

New, let φ be such a mapping that satisfies

$$\varphi(x_1, \cdots, x_{n-1}, x_n, y_1, y_2, \cdots, y_n)=\Psi(x_1, \cdots, x_n, y_2, \cdots, y_k, t) \quad (\text{if } y_1=t).$$

Then

$$\varphi: E^n\times E^k\to X, \quad \text{and}$$
$$\varphi(x_1, \cdots, x_{n-1}, x_n, 0, y_2, \cdots, y_k)\in A_1.$$

Therefore

$$\beta\varphi=f.$$

(f) $(j)^{-1}(0)\subset$ image of i.

Let $f\in\alpha\in\pi_n^{k-1}(X;\ A_2, \cdots, A_k)$, and

$$j(f)=g\simeq 0.$$

Then, $f: E^n\times E^{k-1}\to X$,

$$g: E^{n-1}\times E^k\to X,$$
$$g(x_1, \cdots, x_{n-1}, y_1, y_2, \cdots, y_k)=f(x_1, \cdots, x_{n-1}, x_n, y_2, \cdots, y_k) \quad (\text{if } y_1=x_n).$$

There exists a mapping Ψ such that

$$\Psi: E^{n-1}\times E^k\times E^1\to X,$$
$$\Psi(x_1, \cdots, x_{n-1}, y_1, \cdots, y_k, 0)=g(x_1, \cdots, x_{n-1}, y_1, \cdots, y_k),$$
$$\Psi(x_1, \cdots, x_{n-1}, y_1, \cdots, y_k, 1)=p_0,$$

and $\quad \Psi(x_1, \cdots, x_{n-1}, 0, y_2, \cdots, y_k, t)\in A_1.$

Now, let h be such a mapping that satisfies following relations,

$$h: E^n\times E^{k-1}\to A_1,$$
$$h(x_1, \cdots, x_{n-1}, x_n, y_2, \cdots, y_k)=\Psi(x_1, \cdots, x_{n-1}, 0, y_2, \cdots, y_k, t) \quad (\text{if } x_n=t).$$

Then, obviously,

$$j(f)=g\simeq 0, \qquad i(h)\simeq f.$$

3. Applications.

Let $X=S^{n_1}\vee S^{n_2}\vee S^{n_3}\vee\cdots\vee S^{n_k}$, ($S^{n_i}$ is an n_i-dimensional sphere, and they have only one point in common.) and assume that

$$n_1\leqq n_2\leqq\cdots\leqq n_k.$$

And consider the $(k+1)$-$ad(X; S^{n_1}, S^{n_2}, \cdots, S^{n_k})$, if $Y \supset X$,

$$\rightarrow \pi_q^{k+1}(Y; X, S^{n_1}, S^{n_2}, \cdots, S^{n_k}) \rightarrow \pi_q^k(X; S^{n_1}, S^{n_2}, \cdots, S^{n_k}) \rightarrow \pi_q^k(Y; S^{n_1}, S^{n_2}, \cdots, S^{n_k}) \rightarrow \cdots$$ is exact.

And $\pi_q^{k+1}(Y; X, S^{n_1}, \cdots, S^{n_k}) = \pi_{q+k+1}(Y, X)$ from the following theorem.

THEOREM 3.1. *If* $(Y; X, A_1, \cdots, A_k)$ *is* $a(k+2)$-*ad and* $X \supset A_i (i=1 \cdots k)$, *then*

$$\pi_q^{k+1}(Y; X, A_1, \cdots, A_k) = \pi_{q+k+1}(Y, X).$$

Proof.

Let
$$f \in \alpha \in \pi_q^{k+1}(Y; X, A_1, \cdots, A_k).$$

Then
$$f(\mathfrak{x}, y_1, \cdots, y_{k+1}) \in Y \quad (x \in E^q),$$
$$f(\mathfrak{x}, 0, y_2, \cdots, y_{k+1}) \in X,$$
$$f(\mathfrak{x}, y_1, \cdots, y_i, 0, y_{i+2}, \cdots, y_{k+1}) \in A_i.$$

Let φ be such a mapping that

$$\varphi: E^q \times E^{k+1} \times E^1 \rightarrow E^q \times E^{k+1}$$
$$\varphi(\mathfrak{x}, \mathfrak{y}, 0) = (\mathfrak{x}, \mathfrak{y}), \ \varphi(\mathfrak{x}, \mathfrak{y}, t) \in (E^q \times E^{k+1})^{\bullet}$$

for $(\mathfrak{x}, \mathfrak{y}) \in (E^q \times E^{k+1})^{\bullet}$,

$$\varphi(\mathfrak{x}, \mathfrak{y}, 1) = p_0 \quad \text{for} \quad \mathfrak{x} \in E^q, \ y_1 \neq 0.$$

Then $f(\mathfrak{x}, \mathfrak{y}) = f\varphi(\mathfrak{x}, \mathfrak{y}, 0) \simeq f\varphi(\mathfrak{x}, \mathfrak{y}, 1) \in \pi_{q+k+1}(Y, X)$.

Hence $\varphi^*(\pi_q^{k+1}(Y; X, A_1, \cdots, A_k)) = \pi_{q+k+1}(Y, X)$. Now, let $i^*: \pi_{q+k+1}(Y, X) \rightarrow \pi_q^{k+1}(Y; X, A_1, \cdots, A_k)$ be the homomorphism induced by injection,

then
$$\varphi^* \circ i^* = \text{identity}, \quad i^* \circ \varphi^* = \text{identity}.$$

Therefore
$$\pi_q^{k+1}(Y; X, A_1, \cdots, A_k) = \pi_{q+k+1}(Y, X).$$

Now, let $Y = S^{n_1} \times \cdots \times S^{n_k}$.

If
$$f \in \alpha \in \pi_q^k(Y, S^{n_1}, \cdots, S^{n_k}),$$

then $f(x_1, \cdots, x_q, y_1, \cdots, y_k) = f_1(\mathfrak{x}, \mathfrak{y}) \times f_2(\mathfrak{x}, \mathfrak{y}) \times \cdots \times f_k(\mathfrak{x}, \mathfrak{y})$, where $f_i(\mathfrak{x}, \mathfrak{y})$ is a projection of $f(\mathfrak{x}, \mathfrak{y})$ into S^{n_i}.

Let $\Phi_i(\mathfrak{x}, \mathfrak{y}, t)$ be such a mapping that

$$\Phi_i: E^q \times E^k \times E^1 \rightarrow E^q \times E^k$$
$$\Phi_i(\mathfrak{x}, y_1, \cdots, y_k, 0) = (\mathfrak{x}, y_1, \cdots, y_k)$$
$$\Phi_i(\mathfrak{x}, y_1, \cdots, y_{i-1}, y_i, y_{i+1}, \cdots, y_k, t) = (\mathfrak{x}, y_1, \cdots, y_{i-1}, (1-t)y_i + t, y_{i+1}, \cdots, y_k)$$
$$\Phi_i(\mathfrak{x}, y_1, \cdots, y_{i-1}. y_i, y_{i+1}, \cdots, y_k, 1) = (\mathfrak{x}', \mathfrak{y}') \text{ where } \mathfrak{y}' = (y_1', \cdots, y_k') \text{ and } y_i' = 1.$$

Such a mapping obviously exists.

Then, we consider following mappings,

$$f_i(\Phi_i(\mathfrak{x}, \mathfrak{y}, t)) = \varphi_i(\mathfrak{x}, \mathfrak{y}, t),$$
$$\varphi_i(\mathfrak{x}, \mathfrak{y}, 1) \in S^{n_1} \vee \cdots \vee S^{n_{i-1}} \vee S^{n_{i+1}} \vee \cdots \vee S^{n_k},$$

and also $\varphi_i(\mathfrak{x}, \mathfrak{y}, 1) \in S^{n_i}$. Therefore $\varphi_i(\mathfrak{x}, \mathfrak{y}, t) = p_0$ (the common point of S^{n_i}),

$$\underset{\substack{(i)\\ \uparrow \\ \text{direct product}}}{\Pi} \varphi_i(\mathfrak{x}, \mathfrak{y}, 0) = f(\mathfrak{x}, \mathfrak{y}), \qquad \underset{\substack{(i)\\ \uparrow \\ \text{direct product}}}{\Pi} \varphi_i(\mathfrak{x}, \mathfrak{y}, 1) = p_0 .$$

Therefore $$f(\mathfrak{x}, \mathfrak{y}) \simeq 0.$$

That is to say $$\pi_q^k(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}; S^{n_1}, \cdots, S^{n_k}) = 0$$

Therefore $$\pi_q^k(X; S^{n_1}, S^{n_2}, \cdots, S^{n_k}) = \pi_{q+k}(S^{n_1} \times \cdots \times S^{n_k}, X) .$$

If $n_1 = n_2 = \cdots = n_k < n_{k+1}$,

then
$$\begin{aligned}
&\pi_q^k(S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k}; S^{n_1}, \cdots, S^{n_k}) \\
&= \pi_{q+k}(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}, S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k}) \\
&= H_{q+k}(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}, S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k}) \quad \text{if } q+k = 2n_1 . \\
&= \sum_{\substack{i+j=2n_1 \\ i<j}} H_{n_i}(S^{n_i}) \otimes H_j(S^{n_j})
\end{aligned}$$

Therefore $\pi_{2n_1}(S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k})$
$$= \pi_{2n_1}(S^{n_1}) + \pi_{2n_1}(S^{n_2}) + \cdots + \pi_{2n_1}(S^{n_k}) + \sum_{\substack{i+j=2n_1 \\ i<j}} H_i(S^{n_i}) \otimes H_j(S^{n_j}).$$

Theorem 3.2.

We consider a $(k+1)$-ad $(X; A_1, \cdots, A_k)$, and let X^{**} be a fibre space over X with fibre F in the sense of Serre.[4] *Analogously let $A_i^{**}(i \geqq 2)$ be a fibre space over $A_i(i \geqq 2)$ with the same fibre.*

Now, also, we consider a fibre space over A_1 with fibre $F_1 \subset F$, and denote this by A_1^*, and denote by $A_{[1,\alpha]}^*$ the fibre space over $A_{[1,\alpha]}$ with fibre F_1, where $[\alpha]$ is a subset of $(2, \cdots, k)$. Then, using these notations,

$$\pi_q^k(X; A_1, \cdots, A_k) = \pi_q^k(X^{**}; A_1^*, A_2^{**}, \cdots, A_k^{**}) .$$

Proof. We consider the diagram

$$\begin{array}{ccccc}
\to \pi_q^k(X^{**}; A_1^*, A_2^{**}, \cdots, A_k^{**}) & \to & \pi_q^{k-1}(A_1^*, A_{12}^*, \cdots, A_{1k}^*) & \to & \pi_q^{k-1}(X^{**}; A_2^{**}, \cdots, A_k^{**}) \\
\downarrow p_1 & & \downarrow p_2 & & \downarrow p_1 \\
\to \pi_q^k(X; A_1, A_2, \cdots, A_k) & \longrightarrow & \pi_q^{k-1}(A_1; A_{12}, \cdots, A_{1k}) & \to & \pi_q^{k-1}(X; A_2, \cdots, A_k) ,
\end{array}$$

where $A_i^*(i \geqq 2)$ denotes the fibre space over A_i and with fibre F_1, and $A_{1,i} = A_{[1,i]}$. Moreover, p_1, p_2 are projections and

$$p_1 | A_1^* = p_2 ,$$

Obviously p_1, p_2 are isomorphisms onto by the induction hypothesis. Therefore by the "five" lemma $\pi_q^k(X^{**}, A_1^*, A_2^{**}, \cdots, A_k^{**}) = \pi_q^k(X; A_1, \cdots, A_k)$

THEOREM 3.3.

If $(k+1)$-ad $(X; A_1, \cdots, A_k)$ is given, then

$$\pi_q^k(X;\ A_1,\ \cdots,\ A_k)=\pi_q^{k-1}(A_1^*;\ A_{12}^*,\ \cdots,\ A_{1k}^*)\ ,$$

where

$$A_1^*\ =\{\omega(t)\,|\,0\leqq t\leqq 1,\quad \omega(0)=p_0\ \omega(t)\in A_1,\quad \omega(1)\in A_1\}\ ,$$
$$A_{1\alpha}^*=\{\omega(t)\,|\,\omega(0)=p_0,\quad \omega(t)\in A_{[1]},\quad \omega(1)\in A_{[1\alpha]})\ .\qquad (\alpha\geqq 2)$$

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