Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 6, No. 2, Series A

On uniform topology of functional spaces (II)

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(Received September 30, 1955)

In the previous paper[1] we defined a new uniform topology of functional space and studied analogous theories about uniform spaces as about topological spaces. However, the uniform topology was defined with the order of real numbers and was unnatural. In this paper we shall define a more natural uniform topology and shall simplify the theories.

In this paper we denote by R a uniform space and by $\{\mathfrak{U}_\alpha \mid \alpha \in A\}$, $\mathfrak{U}_\alpha = \{U_\alpha(x) \mid x \in R\}$ the u. nbd (=uniform neighbourhood) system of R.[2]

DEFINITION. We denote by $F(R)$ a family of functions defined on a subset of R and having values in a uniform space R' with the u. nbd system $\{\mathfrak{V}_{\alpha'} \mid \alpha' \in A'\}$, $\mathfrak{V}_{\alpha'} = \{V_{\alpha'}(x') \mid x' \in R'\}$. Moreover, we denote by D_f the domain of definition of a function $f \in F(R)$. For $f \in F(R)$ we define a subset of $F(R)$ by $U_{\alpha\alpha'}(f) = \{g \mid \forall x \in D_f \ \exists y \in U_\alpha(x) : g(y) \in V_{\alpha'}(f(x))\,;\ \forall x \in D_g \ \exists y \in U_\alpha(x) : f(y) \in V_{\alpha'}(g(x))\}$.

THEOREM 1. $\{\mathfrak{U}_{\alpha\alpha'} \mid \alpha \in A,\ \alpha' \in A'\}$ $(\mathfrak{U}_{\alpha\alpha'} = \{U_{\alpha\alpha'}(f) \mid f \in F(R)\})$ *satisfies the condition of u.nbd system in $F(R)$ consisting of continuous functions defined on a closed set of R.*

Proof. We denote by f, g, h elements of $F(R)$. Let $f \neq g$, then $f(x) \neq g(x)$ for some $x \in D_f \cap D_g$, or there exists $x \in (D_f \cap D_g^c) \cup (D_g \cap D_f^c)$. If the former is the case, then there exist $\alpha \in A$, $\alpha' \in A'$ such that $y \in U_\alpha(x)$ implies $g(y) \notin V_{\alpha'}(f(x))$. Hence $g \notin U_{\alpha\alpha'}(f)$. If the latter is the case, then $x \in D_f \cap D_f^c$ implies $U_\alpha(x) \cap D_g = \phi$ for some $\alpha \in A$. Hence $g \notin U_{\alpha\alpha'}(f)$ $(\alpha' \in A')$.

Since $\mathfrak{U}_\gamma < \mathfrak{U}_\alpha$[3], $\mathfrak{V}_{\gamma'} < \mathfrak{U}_{\alpha'}$ imply obviously $\mathfrak{U}_{\gamma\gamma'} < \mathfrak{U}_{\alpha\alpha'}$, for every $\mathfrak{U}_{\alpha\alpha'}$, $\mathfrak{U}_{\beta\beta'}$ there exists $\mathfrak{U}_{\gamma\gamma'}$ such that $U_{\gamma\gamma'}(x) \subseteq U_{\alpha\alpha'}(x) \cap U_{\beta\beta'}(x)$ $(x \in R)$.

Lastly, we prove that $\mathfrak{U}_\beta^* < \mathfrak{U}_\alpha$[4] and $\mathfrak{V}_{\beta'}^* < \mathfrak{V}_{\alpha'}$ imply $\mathfrak{U}_{\beta\beta'}^* < \mathfrak{U}_{\alpha\alpha'}$. If $g, h \in U_{\beta\beta'}(f)$, then for every $x \in D_g$ there exsists $y \in U_\beta(x)$ such that $f(y) \in V_{\beta'}(g(x))$ and $z \in U_\beta(y)$ such that $h(z) \in V_{\beta'}(f(y))$. Hence $z \in U_\alpha(x)$, $h(z) \in V_{\alpha'}(g(x))$. Similarly, for $x \in D_h$ we get $z \in U_\alpha(x)$ such that $g(z) \in V_{\alpha'}(h(x))$. Hence $h \in U_{\alpha\alpha'}(g)$.

The last two propositions are valid for an arbitrary family of functions defined

1) On uniform topology of functional spaces, this journal, Vol. 5, No. 2, 1954.
2) We assume without loss of generality that $y \in U_\alpha(x)$ implies $x \in U_\alpha(y)$ for every u. nbd.
3) $\mathfrak{U}_\gamma < \mathfrak{U}_\alpha$ means $U_\gamma(x) \subseteq U_\alpha(x)$ for every $x \in R$.
4) $\mathfrak{U}_\beta^* = \{\cup\{U_\beta(y) \mid y \in U_\beta(x)\} \mid x \in R\}$; hence $\mathfrak{U}_\beta^* < \mathfrak{U}_\alpha$, if and only if $y, z \in U_\beta(x)$ implies $y \in U_\alpha(z)$ for every $x \in R$.

on a subset of R and having values in R'.

From now on we concern ourselves only with real valued functions; hence this new uniform topology is defined by $\{\mathfrak{U}_{\alpha,\varepsilon}|\alpha\in A,\ \varepsilon>0\}$, $\mathfrak{U}_{\alpha\varepsilon}=\{U_{\alpha\varepsilon}(f)|f\in F(R)\}$, $U_{\alpha\varepsilon}(f)=\{g|\forall x\in D_f\ \exists y\in U_\alpha(x): |f(x)-g(y)|<\varepsilon;\ \forall x\in D_g\ \exists y\in U_\alpha(x): |g(x)-f(y)|<\varepsilon\}$. We denote by $F_u(R)$ ($F(R)$) the uniform space with this uniform topology consisting of all the uniformly continuous (continuous) functions defined on a closed set of R and having values between 0 and 1.

DEFINITION. We define $M(x)$ to mean the mapping which maps $x\in R$ to the function $x(f|f\in D_x): x(f)=f(x)$, $D_x=\{f|x\in D_f\}\subseteq F_u(R)$.

LEMMA 1. *$M(x)\in F(F_u(R))$ for every $x\in R$.*

Proof. If $f\notin D_x$, then $x\notin D_f$, and hence there exists α such that $U_\alpha(x)\cap D_f=\phi$. Therefore $g\in U_{\alpha\varepsilon}(f)$ implies $x\notin D_g$, i.e. $g\notin D_x$; hence D_x is closed. If $f\in D_x$, then for every $\varepsilon>0$ there exists $\alpha\in A$ such that $|f(U_\alpha(x))-f(x)|<\frac{\varepsilon}{2}$[5]. Since $g\in U_{\alpha\frac{\varepsilon}{2}}(f)$ implies $|g(x)-f(x)|<\varepsilon$, $x(f)$ is continuous.

THEOREM 2. *$M(x)$ is a uniformly homeomorphic mapping from R onto $M(R)\subseteq F(F_u(R))$.*

Proof. It is obvious that $M(x)$ is one-to-one. $M(x)$ is uniformly continuous. For given $\varepsilon>0$ and $\alpha\in A$ we take $\beta\in A$ such that $\mathfrak{U}_\beta{}^*<\mathfrak{U}_\alpha$. If $y\in U_\beta(x)$, then for $f\in D_x$, we take $\gamma\in A$ such that $|f(U_\gamma(x))-f(x)|<\varepsilon$, $|f(U_\gamma(y))-a|<\varepsilon$ for some a. Defining $g(y)=f(x)$, $g(x)=a(x\notin D_g$ for $U_\gamma(y)\cap D_f=\phi)$, $g(z)=f(z)(z\in U_\gamma{}^c(x)\cap U_\gamma{}^c(y))$, we obtain $g\in U_{\alpha\varepsilon}(f)$ such that $x(f)=y(g)$. Since $|y(f)-x(g)|<\varepsilon$ for $f\in D_y$, we have $M(y)\in U_{\alpha\varepsilon\varepsilon}(x)$[6]. Thus $M(x)$ is uniformly continuous.

Next, we show the uniform continuity of the inverse mapping $M^{-1}(x(f))$. If for a given $\alpha\in A$, it holds $y\notin U_\alpha(x)$, then for an element f of $F_u(R): f(x)=0$, $f(y)=1$, $D_f=\{x,y\}$, and for every $g\in U_{\alpha\frac{1}{2}}(f)$ we get $g(y)>\frac{1}{2}$. Hence $y(g)-x(f)=g(y)-f(x)>\frac{1}{2}$, and hence $M(y)\notin U_{\alpha\frac{1}{2}\frac{1}{2}}(M(x))$. Thus $M^{-1}(x(f))$ is uniformly homeomorphic.

Now, we consider a diverging cauchy filter $\mathfrak{F}'=\{F_\gamma'|\gamma\in C\}$ and denote by $\mathfrak{F}=\{F_\delta|\delta\in D\}$ the cauchy filter $\{S(F_\gamma',\mathfrak{U}_\alpha)|\gamma\in C,\ \alpha\in A\}$[7] defined from $\mathfrak{F}'$. We define a function u on the subset $D_u=\{f|D_f\cap F_\delta\neq\phi(\delta\in D)\}$ of $F_u(f)$ by $u(f)=\lim f(F_\delta)$.

LEMMA 2. *$u(f)\in F(F_u(R))$.*

Proof. If $f\notin D_u$, then $D_f\cap F_\delta=\phi$ for some $\delta\in D$, and hence $S(F_{\delta'},\mathfrak{U}_\alpha)\cap D_f=\phi$ for some $\delta'\in D$ and $\alpha\in A$. Since $g\in U_{\alpha\varepsilon}(f)$ implies $F_{\delta'}\cap D_g=\phi$, it also implies

5) We use the notation $|f(U_\alpha(x))-f(x)|<\frac{\varepsilon}{2}$ to mean $|f(y)-f(x)|<\frac{\varepsilon}{2}$ for every $y\in U_\alpha(x)$.

6) $U_{\alpha\varepsilon\varepsilon}(a)$ denotes the u.nbd of $a\in F(Fu(R))$ defined by $\mathfrak{U}_{\alpha\varepsilon}$ and ε.

7) $S(A,\mathfrak{U}_\gamma)=\cup\{U_\gamma(x)|U_\gamma(x)\cap A\neq\phi\}$.

$g \notin D_u$. Hence D_u is closed. If $f \in D_u$, then for every $\varepsilon > 0$ there exist $\delta \in D$, $\alpha \in A$ such that $|f(S(F_\delta, \mathfrak{U}_\alpha)) - u(f)| < \frac{\varepsilon}{2}$. Since $g \in U_{\alpha\frac{\varepsilon}{2}}(f)$ implies $|g(F_\delta) - u(f)| < \varepsilon$ and accordingly implies $|u(g) - u(f)| \leqq \varepsilon$, $u(f)$ is continuous.

LEMMA 3. *Using above notations, every diverging cauchy filter* $\mathfrak{F}' = \{F_\gamma' | \gamma \in C\}$ *converges to* $u(f)$ *in* $F(F_u(R))$.

Proof. For given $\varepsilon > 0$ and $\alpha \in A$, we take α', β' such that $\mathfrak{U}_{\beta'}^* < \mathfrak{U}_{\alpha'} < \mathfrak{U}_{\alpha'}^* < \mathfrak{U}_\alpha$ and $F_\delta \in \mathfrak{F}$ such that $F_\delta \subseteqq U_{\alpha'}(x_0)$ for some x_0. We shall prove $x \in U_{\alpha\varepsilon\varepsilon}(u)$ for every $x \in F_\delta$. Let $f \in D_u$, then there exists $F_{\delta'} \in \mathfrak{F}$ such that $V(x) \cup S(F_\delta, \mathfrak{U}_{\beta'}) \subseteqq U_{\alpha'}(x_0)$, $|f(V(x)) - a| < \varepsilon$, $|f(S(F_{\delta'}, \mathfrak{U}_{\beta'})) - u(f)| < \frac{\varepsilon}{2}$ for some nbd $V(x)$ of x and $\beta' \in A$. If we define $g \in F_u(R)$ by $g(x) = u(f)$, $g(F_{\delta'}) = a$ $(F_{\delta'} \frown D_g = \phi$ for $V(x) \frown D_f = \phi)$, $g(z) = f(z) (z \in V^c(t) \frown S^c(F_\delta, \mathfrak{U}_{\beta'}))$, then $g \in U_{\alpha\varepsilon}(f)$ and $u(f) = x(g)$. If $x \in D_f$, then $u(g) = a$ and $|x(f) - u(g)| < \varepsilon$. Hence $x \in U_{\alpha\varepsilon\varepsilon}(u)$. Therefore $\mathfrak{F}$ and $\mathfrak{F}'$ converge to $u(f)$. Thus this lemma is established.

From this lemma we get

THEOREM 3. $\overline{M(R)}$ *is complete, where* $\overline{M(R)}$ *denotes the closure of* $M(R)$ *in* $F(F_u(R))$.

It is easy to prove the well known uniqueness of the completion by our method.

COROLLARY. *If* $\bar{R} = R^*$ *and if* R^* *is complete, then* R^* *is uniformly homeomorphic with* $\overline{M(R)}$ *by a correspondence which maps* $x \in R$ *to* $M(x)$.

Proof. R^* and $M^*(R^*) \subseteqq F(F_u(R))$ are obviously uniformly homeomorphic by the mapping $M^*(x^*)$: $R^* \ni x^* \to x^*(f | f \in F_u(R))$ $(x^*(f) = \lim \{f(U(x^*) \frown R) | U(x^*)$ is a nbd of $x^*\})$. Since $\bar{R} = R^*$, $M^*(R^*) \subseteqq \overline{M(R)}$. Moreover, since R^* is complete, it must be $M^*(R^*) = \overline{M(R)}$.

REMARK. Generally, the topology deduced from this new uniform topology is weaker than the strong topology and stronger than the weak topology in any space consisting of continuous functions. Moreover, it is obvious that this uniform topology agrees with the strong topology in any space consisting of uniformly continuous functions.

DEFINITION. We use $L(R)$ to mean the uniform space[8] of all the bounded uniformly continuous functions of R with the natural lattice-order, where R is a complete uniform space.

DEFINITION. If a non-vacuous subset J of $L(R)$ satisfies the conditions,

i) $f \leqq g \in J$ implies $f \in J$,

ii) if there exists $\bigcap_{\gamma \in c} f_\gamma$ for $f_\gamma \notin J$, then $\bigcap_\gamma f_\gamma \notin J$,

then we call J an *i-set*.

8) The uniform topology is of course the above defined one.

We call a non-vacuous subset satisfying the dual conditions an *s–set.*

DDFINITION. We mean by an *i–deal* a subset I of $L(R)$ satisfying

1) $I=\frown\{J_\lambda|\lambda\in M\}$, where J_λ are *i–sets*, and for every λ, $\mu\in M$ there exists $\nu\in M$ such that $J_\nu\subseteqq J_\lambda\frown J_\mu$,

2) if $f_\gamma\in I(\gamma\in C)$ and if $\{f\ |\gamma\in C\}$ is upper bounded, then for every $\mathfrak{U}_{\alpha\varepsilon}$ there exist $f_\gamma'(\gamma\in C)$ such that $f_i'\geqq f_\gamma$ and $f\in U_{\alpha\varepsilon}(f_\gamma')$ for some $f\in I$.

3) I is a non-trivial ideal.

We call a subset satisfying the dual conditions an *s–ideal.*

LEMMA 4. *For any open set V, $\{f|\exists x\in V: f(x)<a\}=J_a(V)$ is an i–set. $\{f|\exists x\in V: f(x)>a\}=S_a(V)$ is an s–st.*

Proof. It is obvious.

LEMMA 5. $\{f|f(x)\leqq k\}=I_k(x)$ *is an i–ideal.*

$\{f|f(x)\geqq k\}=S_k(x)$ *is an s–ideal.*

Proof. $I_k(x)=\frown\{J_a(V)|a>k,\ V$ is an open nbd of $x\}$ is obvious. Since Condition 2) is obviously valid for an isolated point x, we prove 2) for an accumulating point x. Let $f_\gamma\in I_k(x)$ $(\gamma\in C)$ and let $f_\gamma\leqq q$ for a real number $q(>k)$, then for a given $\mathfrak{U}_{\alpha\varepsilon}$ we take $\beta\in A$ and a natural number n such that $\mathfrak{U}_\beta{}^*<\mathfrak{U}_\alpha$, $\frac{q-k}{n}<\varepsilon$. Since there exist different n points $x_1, x_2,, \cdots, x_n$ in $U_\beta(x)$, we can define $f\in L(R)$ such that $f(x)=k$, $f(x_i)=k+\frac{q-k}{n}i(\ i=1\cdots n)$, $f(U_\beta{}^c(x))=q$, $k\leqq f\leqq q$.

Taking $x_i{}^\gamma\in U_\beta(x)$ $(i=1\cdots n)$ such that $x_i{}^\gamma\neq x_j{}^\gamma$ $(i\neq j)$, $f\ (x_i{}^\gamma)\leqq k+\frac{q-k}{n}i$, we can define f_γ' such that $f_\gamma'(x_i{}^\gamma)=k+\frac{q-k}{n}i$, $f_\gamma'\geqq f_\gamma$, $f_\gamma'(U_\beta{}^c(x))=q$, $k\leqq f_\gamma'\leqq q$. Since $f\in I_k(x)$ and $f\in U_{\alpha\varepsilon}(f_\gamma')$ are obvious, $I_k(x)$ satisfies 2).

LEMMA 6. *For every i–set J there exist a real number a and an open set P such that $f(x)\leqq a$ for some $x\in P$ implies $f\in J$.*[9]

Proof. From now forth we omit the same proofs as those of the previous paper.[10] See (I).

LEMMA 7. *If $I=\frown\{J_\lambda|\lambda\in M\}$ is an i–ideal and if sup $\{a|$ there exists P such that $x\in P$, $f(x)\leqq a$ imply $f\in J_\lambda\}-\varepsilon=a_\lambda$ $(\varepsilon>0)$, then* $\inf\{a_\lambda|\lambda\in M\}\neq-\infty$.

Proof. See (I).

Let us put $\inf\{a_\lambda|\lambda\in M\}=a$ $(\neq-\infty)$, then for every J_λ there exists some open set P such that $f(x)\leqq a$ and $x\in P$ imply $f\in J_\lambda$.

Hence we can give the following

DEFINITION. $\smile\{P|f(x)\leqq a$ and $x\in P$ (open) imply $f\in J_\lambda\}=P_\lambda$.

9) Lemmas 6–15 admit the dual propositions.

10) loc. cit. We call this paper (I) for brevity.

LEMMA 8. *$J_\lambda \subseteq J_\mu (\lambda, \mu \in M)$ implies $P_\lambda \subseteq P_\mu$.*

Proof. See (I).

LEMMA 9. *$\{P_\lambda | \lambda \in M\}$ is a cauchy filter.*

Proof. See (I).

Since R is complete, $\{P_\lambda\}$ converges to a point x of R. Then $\{f | f(x) \leqq a - \varepsilon\} \subseteq J_{a-\varepsilon}(x) \subseteq I$ $(\varepsilon > 0)$ is obvious.

LEMMA 10. *$\{f | f(x) < c\} \subseteq I$ for $c = sup\{k | J_k(x) \subseteq I\}$.*

Proof. It is obvious.

LEMMA 11. *$f(x) > c$ impllies $f \notin I$ for $c = sup\{k | J_k(q) \subseteq I\}$.*

Proof. See (I).

DEFINITION. We denote by $I(x, c)$ an i-ideal I satisfying Lemmas 10, 11.

LEMMA 12. *Every i-ideal is represented uniquely by the form $I(x, c)$.*

Proof. It is obvious.

DEFINITION. For two i-ideals I_1, I_2 we define $I_1 \sim I_2$ to mean that there exists some s-ideal S such that $S \frown I_1 = \phi$, $S \frown I_2 = \phi$.

LEMMA 13. *$I(x, c) \sim I(y, d)$, if and only if $x = y$.*

Proof. It is obvious.

DEFINITION. For an i-ideal I and an s-ideal S, we define $S \sim I$ to mean that there exist some i-ideal I_1 and s-ideal S_1 such that $S \sim S_1$, $I \sim I_1$; $S_1 \frown I_1 = \phi$.

LEMMA 14. *$I(x, c) \sim S(y, d)$, if and only if $x = y$.*

Proof. It is obvious.

Hence we can classify all the i-deals and all the s-ideals by $\sim$. We denote by $\mathfrak{L}(R)$ the totality of such classes and by $\mathfrak{L}(x)$ the one-to-one mapping from R onto $\mathfrak{L}(R)$, which maps x to the classes consisting of $I(x, a)$ and $S(x, b)$.

DEFINITION. If for a family $\{I(x, a(x)) | x \in A\}$ of i-ideals there exists $f \in \bigcap_{x \in A} I(x, a(x))$, then we call this family *lower bounded*. "*Upper bounded*" is defined as the dual.

LEMMA 15. *$\{I(x, a(x)) | x \in A\}$ is lower bounded, if and only if $inf\{a(x) | x \in A\} \neq -\infty$.*

Proof. It is obvious.

DEFINITION. $\mathfrak{L}(U)$ and $\mathfrak{L}(A)$ are called *u-disjoint*, if and only if for every lower bounded $\{I(x, a(x)) | x \in U\}$ and upper bounded $\{S(x, b(x)) | x \in A\}$ there exists $f \in \bigcap_{x \in U} I(x, a(x)) \bigcap_{x \in A} S(x, b(x))$.

LEMMA 16. *$\mathfrak{L}(U)$ and $\mathfrak{L}(A)$ are u-disjoint, if and only if there exists a uniformly continuous function f such that $f(U) = 0$, $f(A) = 1$, $0 \leqq f \leqq 1$.*

Proof. It is obvious.

DEFINITION. A family $\{\mathfrak{L}(V(x))\,|\,x\in R\}$ of non-void subsets of $\mathfrak{L}(R)$ is called a *uniform nbd* of $\mathfrak{L}(R)$, if and only if

(1) $\mathfrak{L}(y)\in L(V(x))$ implies $\mathfrak{L}(x)\in\mathfrak{L}(V(y))$, and there exists $\{\mathfrak{L}(U(x))\,|\,x\in R\}$ such that

(2) $\mathfrak{L}(U(x))$ and $\mathfrak{L}(V^c(x))$ are u-disjoint,

(3) there exist $a(y)$, $b(x)$, α, ε such that $\{I(y,\ a(y))\,|\,y\in U(x)\}$ is lower bounded; $f_x\in\frown\{I(y,\ a(y))\,|\,y\in U(x)\}$ and $g_x\in S(x,\ b(x))$ imply $g_x\notin U_{\alpha\varepsilon}(f_x)$ for every $x\in R$.

LEMMA 17. *$\{\mathfrak{L}(V(x))\,|\,x\in R\}$ is a uniform nbd of $\mathfrak{L}(R)$, if and only if $\{V(x)\,|\,x\in R\}$ is a uniform nbd of R.*

Proof. If $\{V(x)\,|\,x\in R\}$ is a u. nbd of R, then there exists $\alpha\in A$ such that $U_\alpha(x)$ and $V^c(x)$ are u-disjoint for every $x\in R$. Let $a(y)=0$, $b(x)=1$, $\varepsilon=1$, then $f_x\in\frown\{I(y,\ 0)\,|\,y\in U_\alpha(x)\}$ and $g_x\in S(x,1)$ mean $f_x(U_\alpha(x))\leqq 0$ and $g_x(x)\geqq 1$ respectively, and hence $g_x\notin U_{\alpha 1}(f_x)$ $(x\in R)$.

Conversely, if $\{V(x)\,|\,x\in R\}$ is no u. nbd of R and if (1) is valid for $\{\mathfrak{L}(V(x))\,|\,x\in R\}$, then we can show that Condition (3) is not valid for any $\{U(x)\}$ such that $U(x)$ and $V^c(x)$ are u-disjoint. Take any $a(y)$, $b(x)$, $\alpha\in A$, $\varepsilon>0$ such that $\{I(y,\ a(y))\,|\,y\in U(x)\}$ is lower bounded, and put $c(x)=\inf\{a(y)\,|\,y\in U(x)\}$. Let $\mathfrak{U}_\beta^*<\mathfrak{U}_\alpha$ and let $U_\beta(x)\subseteqq\!\!\!\!/\ V(x)$, then there exists $y\in U_\beta(x)-V(x)$. If x, y are isolated points, then defining $f(y)=b(x)+1$, $f(z)=c(x)$ $(z\neq y)$; $g(x)=b(x)+1$, $g(z)=c(x)$ $(z\neq x)$, we obtain f, g such that $f\in\frown\{I(y,a(y))\,|\,y\in U(x)\}$, $g\in S(x,b(x))$ and $g\in U_{\alpha\varepsilon}(f)$.

If y is an accumulating point, then take a natural number n and a nbd $W(y)$ of y such that $\dfrac{b(x)+1-c(x)}{n}<\varepsilon$; $W(y)\frown U(x)=\phi$, $W(y)\subseteqq U_\beta(x)$. Since there exist different n points y_i $(i=1\cdots n)$ in $W(y)$, we can define f, $g\in L(R)$ such that $f(y_i)=\dfrac{b(x)+1-c(x)}{n}i+c(x)$, $f(W^c(y))=c(x)$, $c(x)\leqq f\leqq b(x)+1$; $g(x)=b(x)+1$, $g(y_i)=\dfrac{b(x)+1-c(x)}{n}i+c(x)$, $g(U_\beta{}^c(x))=c(x)$, $c(x)\leqq g\leqq b(x)+1$. Then $f\in\frown\{I(y,a(y))\,|\,y\in U(x)\}$, $g\in S(x,b(x))$ and $g\in U_{\beta\varepsilon}(f)$.

If x is an accumulating point, then since $x\in U_\beta(y)-V(y)$, we can prove the existence of such an element g of $L(R)$ in the same way. Hence $\{\mathfrak{L}(V(x))\,|\,x\in R\}$ is no u. nbd of $\mathfrak{L}(R)$.

Thus we get

THEOREM 4. *In order that two complete uniform spaces R_1 and R_2 are uniformly homeomorphic, it is necessary and sufficient that $L(R_1)$ and $L(R_2)$ are uniformly isomorphic.*[11]

11) A uniform isomorphism means a uniform homeomorphism preserving the lattice-order or the ring-operation.

COROLLARY. *In order that two complete uniform spaces R_1 and R_2 are uniformly homeomorphic, it is necessary and sufficient that $C(R_1)$ and $C(R_2)$ are uniformly isomorphic*[11], *where $C(R_i)$ is the ring of all the bounded uniformly continuous functions on R_i and having our uniform topology.*

If we concern ourselves with the uniformly topological lattice consisting of functions having values between 0 and 1, then the discussions are simpler.

THEOREM 5. *In order that two complete uniform spaces R_1 and R_2 are uniformly homeomorphic it is necessary and sufficient that $L'(R_1)$ and $L'(R_2)$ are uniformly isomorphic, where $L'(R_i)$ is the lattice of all the uniformly continuous functions on R_i having values between* 0 *and* 1 *and has our uniform topology.*

Proof. Since the proof is analogous to that of Theorem 4 and is simpler than it, we omit this proof.