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On homotopy classification and extension

By Katuhiko MIZUNO

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It is the purpose of this paper to discuss the rôle of the groups $H(\Pi, n, \Pi', q, \boldsymbol{k})$ in the study of the obstruction and classification theorems for mappings of a geometric complex K into a topological space Y such that

$$\pi_i(Y)=0 \quad \text{for} \quad 0\leqq i<n,\ n<i<q, \quad \text{and} \quad q<i<r<2q-1,$$

along the line of Eilenberg-MacLane [3].

As is well-known, the space Y has the invariants $\boldsymbol{k}_n^{q+1}\in H^{q+1}(\pi_n, n;\pi_q)$ and $\boldsymbol{k}_q^{r+1}\in H^{r+1}(\pi_q, q;\pi_r)$.[1] In addition to these, as is shown in §6, there is an invariant $\{\boldsymbol{k}_{n\cdot q}^{r+1}\}$ which is a coset of $H^{r+1}(\pi_n, n, \pi_q, q, \boldsymbol{k}_n^{q+1};\pi_r)$.

Let K be a geometric complex with subcomplex L and $f: K^n\cup L\rightarrow Y$ be a mapping extensible to a map $K^{q+1}\cup L\rightarrow Y$. The third obstruction to the extension of f is then a coset of $H^{r+1}(K, L;\pi_r)$. This obstruction was treated by N. Shimada and H. Uehara in some special cases [1].

Our main purpose is the expression of this coset in the general cases, and by an application we shall explain the allied extension and classification theorems in terms of our new operators $\boldsymbol{y}_\gamma$ and $\boldsymbol{y}_\tau$ which are introduced is §4. Throughout, we omit the case $n=1$.

§1. The maps $T(x_n, x_q)$.

For any (discrete) abelian groups Π, Π', any integers n, q $(1<n<q)$, and any cocycle k of $Z^{q+1}(\Pi, n;\Pi')$ we shall introduce an R-complex $K(\Pi, n, \Pi', q, k)$ which is a k-prolongation of $K(\Pi, n)$ in a sence.[2]

A p-cell of $K(\Pi, n, \Pi', q, k)$ is a pair (ϕ, ψ), where ϕ is a p-cell of $K(\Pi, n)$, and ψ is an element of $F_p(\Pi', q)$ subject to the condition;

$$\textstyle\sum_{i=0}^{q+1}(-1)^i\psi(\gamma_{q+1}^i)+k(\phi\gamma)=0 \quad \text{for any map } \gamma\in K_{q+1}(p).$$

The internal product of two such p-simplices (ϕ, ψ), (ϕ', ψ') is $(\phi\circ\phi', \psi\circ\psi')$ where

$$(\phi\circ\phi')(\alpha)=\phi(\alpha)+\phi'(\alpha), \quad (\psi\circ\psi')(\beta)=\psi(\beta)+\psi'(\beta)$$

for arbitrary appropriate dimensional maps α, β. And the p-simplex $(\iota_{p.n}, \iota_{p.q})$ which is a pair of the neutral elements determines the unit for this product.

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1) For the sake of brevity, we write in the following $\pi_n=\pi_n(Y)$, $\pi_q=\pi_q(Y)$, and $\pi_r=\pi_r(Y)$.
2) Refer. [2].

The zero subgroup $\{0\}$ of Π determines the subcomplex $K(0, n, \Pi', q, k)$ of $K(\Pi, n, \Pi', q, k)$ which consists of the simplices of the type (ι, ψ) satisfying

$$\sum_{i=0}^{q+1}(-1)^i\psi(\gamma\varepsilon_{q+1}^i)=0 \quad \text{for any map } \gamma\in K_{q+1}(p)\,,$$

namely, this subcomplex is isomorphic with $K(\Pi', q)$.

We wish to classify simplicial maps of a complete semi-simplicial (C.S.S.) complex K

$$T: K \longrightarrow K(\Pi, n, \Pi', q, k)\,. \tag{1.1}$$

Such a map determines a cocycle and a cochain

$$x_n=T^{\#}b_n\in Z^n(K;\Pi)\,,\quad x_q=T^{\#}b_q\in C^q(K;\Pi')$$

where b_n is the basic cocycle in $Z^n(\Pi, n, \Pi', q, k;\Pi)\cong Z^n(\Pi, n;\Pi)$ and b_q is the basic cochain in $C^q(\Pi, n, \Pi', q, k;\Pi')$ defined by

$$b_n(\phi, \psi)=\phi(\varepsilon_n),\quad b_q(\phi, \psi)=\psi(\varepsilon_q)\,.$$

Then, it is easily verified that

$$kT(x_n)+\delta x_q=0$$

where $T(x_n): K\to K(\Pi, n)$ is the simplicial map induced by x_n as follows:

$$T(x_n)\sigma=\phi$$

where σ is any p-cell of K, and ϕ is the corresponding p-cell of $K(\Pi, n)$ determined by $\phi(\alpha)=x_n(\sigma_\alpha)$ for any map $\alpha\in K_n(p)$.

LEMMA 1.1. *Given the complex $K(\Pi, n, \Pi', q, k)$ and the C.S.S. complex K, the rule $T\to(x_n, x_q)$ establishes a one-to-one correspondence between simlicial maps and pairs (x_n, x_q) satisfying the conditions;*

$$x_n\in Z^n(K;\Pi)\,,\quad x_q\in C^q(K;\Pi')\,,\quad kT(x_n)+\delta x_q=0\,. \tag{1.2}$$

The map T corresponding in this fashion to the pair (x_n, x_q) will be denoted by $T(x_n, x_q)$. Then $T(x_n, x_q)$ is characterized as a simplicial map for which

$$T(x_n, x_q)\sigma=(\phi, \psi)$$

if σ is an p-simplex of K, where

$$\begin{aligned}\phi(\alpha)&=x_n(\sigma_\alpha) \quad \text{for any map } \alpha\in K_n(p)\\ \psi(\beta)&=x_q(\sigma_\beta) \quad \text{for any map } \beta\in K_q(p)\,.\end{aligned}$$

The proof of this lemma is an immediate consequence of the following:

$$\begin{aligned}\sum_{i=0}^{n+1}(-1)^i\phi(\lambda\varepsilon_{n+1}^i)&=\sum_{i=0}^{n+1}(-1)^i x_n(\sigma_{\lambda\varepsilon^i})=\sum_{i=0}^{n+1}(-1)^i x_n(\sigma_\lambda)^{(i)}\\ &=x_n\partial(\sigma_\lambda)=\delta x_n(\sigma_\lambda)=0 \qquad \text{for any may } \lambda\in K_{n+1}(p).\end{aligned}$$

$$\sum_{i=0}^{q+1}(-1)^i \psi(\gamma\varepsilon_{q+1}^i)+k(\phi_\gamma)=\sum_{i=0}^{q+1}(-1)^i x_q(\sigma_{\gamma\varepsilon^i})+k(\phi_\gamma)$$
$$=\sum_{i=0}^{q+1}(-1)^i x_q(\sigma_\gamma)^{(i)}+k(\phi_\gamma)$$
$$=x_q(\partial\sigma_\gamma)+k[T(x_n)\sigma]_\gamma$$
$$=\delta x_q(\sigma_\gamma)+k[T(x_n)\sigma_\gamma]$$
$$=(\delta x_q+kT(x_n))\sigma_\gamma=0 \qquad \text{for any map } \gamma\in K_{q+1}(p).$$

Therefore $T(x_n, x_q)$ may be represented as follows;

(1.3) $$T(x_n, x_q)=\gamma[i_n\times i_q][T(x_n)\times T(x_q)]e,$$

here, the first map e is the diagonal map $K\to K\times K$, the second map is the cartesian product of $T(x_n)$ and $T(x_q): K\to F(\Pi', q)$ which is defined similarly as $T(x_n)$ for the cochain x_q. The third map is the cartesian product of the inclusion maps

$$i_n:\ K(\Pi, n)\longrightarrow F(\Pi, n, \Pi', q, k)$$
$$i_q:\ F(\Pi', q)\longrightarrow F(\Pi, n, \Pi', q, k)$$

defined by

$$i_n(\phi)=(\phi, \iota),\quad i_q(\psi)=(\iota, \psi),$$

where $F(\Pi, n, \Pi', q, k)$ is the family of the pairs (ϕ, ψ) of $\phi\in K(\Pi, n)$ and $\psi\in F(\Pi', q)$ (no restriction is settled). Finally, the map γ is given in terms of the internal product in $F(\Pi, n, \Pi', q, k)$ which is given similarly as in $K(\Pi, n, \Pi', q, k)$. If $U: K'\to K$ is any simplicial map, the above characterization of $T(x_n, x_q)$ shows at once that

$$T(x_nU, x_qU)=T(x_n, x_q)U$$

since $T(x_nU)=T(x_n)U$, and $T(x_qU)=T(x_q)U$.

§2. The maps $\tau_{n\cdot q}(x_n)$.

For our future convenience, we now derive an explicit formula for the automorphisms $\eta(\phi, \psi)=(\phi', \psi')$ such that

(2.1) $$\eta:\ K(\Pi, n, \Pi', q, k)\longrightarrow K(\Pi, n, \Pi', q, k)$$
$$\phi\equiv\phi' \text{ for any } (\phi, \psi) \text{ of } K(\Pi, n, \Pi', q, k).$$

According to the Lemma 1.1, such a map η is represented as $T(b_n, b_q')$ where b_n is the basic cocycle of $Z^n(\Pi, n, \Pi', q, k; \Pi)$, and $b_q'=\eta^{\#}b_q$ is a cochain of $C^q(\Pi, n, \Pi', q, k; \Pi')$. Generally, b_q' is not equal to b_q, and their difference induces a cocycle $h_q=b_q'-b_q$ since $\delta b_q=-kT(b_n)=\delta b_q'$.

LEMMA 2.1. *Given the complex $K(\Pi, n, \Pi', q, k)$, the rule $\eta\to h_q$ establishes a one-to-one correspondence between the chain homotopic class of η and cohomology class of h_q.*

The map η corresponding in this fashion to the cocycle h_q is characterized as a simplicial map for which

(2.2) $$\eta \equiv T(b_n, b_q) \circ i_q T(h_q)$$

and so

$$\eta(\phi, \psi) = (\phi, \psi) \circ (\iota, T(h_q)(\phi, \psi)) .$$

The proof of this lemma is an immediate consequence of the theorem 5.2 of [3]. Namely, $\eta_1 \simeq \eta_2$ implies the existence of a $(q-1)$-cochain h_{q-1} satisfying:

$$\delta h_{q-1} = \eta_1{}^{\#} b_q - \eta_2{}^{\#} b_q = b_q^1 - b_q^2$$
$$= (b_q^1 - b_q) - (b_q^2 - b_q) = h_q^1 - h_q^2 .$$

Conversely, assume that h_q^1 and h_q^2 are cohomologous, then the maps $T(h_q^1)$, $T(h_q^2)$: $K(\Pi, n, \Pi', q, k) \to K(\Pi', q)$ are chain homotopic and also this shows that $\eta_1 \simeq \eta_2$ since

$$\eta_1 \equiv T(b_n, b_q) \circ i_q T(h_q^1) , \quad \eta_2 \equiv T(b_n, b_q) \circ i_q T(h_q^2) .$$

In the following, the homotopy class of the automorphism η corresponding to the cohomology class $\boldsymbol{h}_q$ will be denoted by $\eta(\boldsymbol{h}_q)$.

We shall now consider the replacement of x_q by another x_q' on the map $T(x_n, x_q)$. We have a cocycle $d_q = x_q' - x_q \in Z^q(K; \Pi')$, and also a simplicial map:

$$T(d_q) : K \longrightarrow K(\Pi', q) .$$

Being the map $T(x_n, x_q')$ represented by

$$T(x_n, x_q') \equiv T(x_n, x_q) \circ i_q T(d_q) ,$$

we can identify $T(x_n, x_q')$ with $T(x_n, x_q)$ if we identify the complex $K(\Pi, n, \Pi', q, k)$ with its image of automorphism (2.2).

We shall define $\tau_{n \cdot q}(x_n)$ as the family of $T(x_n, x_q)$ where x_n is a fixed cocycle of $Z^n(K; \Pi)$ satisfying $kT(x_n) \sim 0$.

LEMMA 2.2. *The cocycles* $x_n^1, x_n^2 \in Z^n(K; \Pi)$ *are cohomologous if and only if the maps* $\tau_{n \cdot q}(x_n^1)$, $\tau_{n \cdot q}(x_n^2)$ *are chain homotopic (i.e.,* $\tau_{n \cdot q}(x_n^1)$ *and* $\tau_{n \cdot q}(x_n^2)$ *contain* $T(x_n^1, x_q^1)$, $T(x_n^2, x_q^2) : K \to K(\Pi, n, \Pi', q, k)$ *respectively and* $T(x_n^1, x_q^1) \simeq T(x_n^2, x_q^2)$*)*.

Proof. Since b_n is a cocycle, $T(x_n^1, x_q^1) \simeq T(x_n^2, x_q^2)$ implies that $x_n^1 = T(x_n^1, x_q^1)^{\#} b_n$ and $x_n^2 = T(x_n^2, x_q^2)^{\#} b_n$ are cohomologous. Conversely, assume that x_n^1 and x_n^2 are cohomologous, there is a cocycle $u_n \in Z^n(IK; \Pi)$ such that $x_n^1 = u_n i_0$, $x_n^2 = u_n i_1$ where $i_0, i_1 : K \to IK$ are the fixed simplicial injections defined as $i_0(\sigma) = 0 \times \sigma$, $i_1(\sigma) = 1 \times \sigma$. Then, if we fix a cochain $u_q \in C^q(IK; \Pi)$ satisfying the relation $\delta u_q + kT(u_n) = 0$, we have two cochains $x_q^1 = u_q i_0$, $x_q^2 = u_q i_1$, and $T(x_n^1, x_q^1)$ and $T(x_n^2, x_q^2)$ are the desired maps, q.e.d.

According to this lemma, we shall denote the homotopy class of $\tau_{n \cdot q}(x_n)$ corresponding to the cohomology class $\boldsymbol{x}_n$ of x_n by $\tau_{n \cdot q}(\boldsymbol{x}_n)$ in the following.

§3. The maps $\gamma_{n\cdot q}(x_{n_1}, x_{q_1})$, $\gamma_{n\cdot q}(x_{q_1})$.

As a preliminary to the definition of the basic operations, we shall consider first a certain maps. Given two C.S.S. pairs (K, L_i) $(i=1, 2)$ and two cocycles

$$x_{n_1} \in Z^{n_1}(K, L_1; \Pi), \quad x_{q_1} \in Z^{q_1}(K, L_2; \Pi'),$$

and given a pair of integers (n, q) where $1 < n_1 \leqq n$, $1 < q_1 \leqq q$ and $n < q$, we shall define a chain transformation

$$\gamma_{n\cdot q}(x_{n_1}, x_{q_1}) : (K, L) \longrightarrow K(\Pi, n, \Pi', q, k)$$

of degree $s=(n-n_1)+(q-q_1)$, here $L=L_1 \cup L_2$. This degree is called the defect.

The map $\gamma_{n\cdot q}(x_{n_1}, x_{q_1})$ is defined as the composite of the maps displayed in the following main diagram

$$\begin{array}{ccc}
(K, L) & & \\
\downarrow e & & \\
(K, L_1) \times (K, L_2) & \xrightarrow{\quad f \quad} & (K, L_1) \otimes (K, L_2) \\
& & \downarrow R(x_{n_1}) \otimes R(x_{q_1}) \\
& & K(\Pi, n_1) \otimes K(\Pi', q_1) \\
& & \downarrow S^{n-n_1} \otimes S^{q-q_1} \\
& & K(\Pi, n) \otimes K(\Pi', q) \\
& & \downarrow i_n \otimes i_q \\
K(\Pi, n, \Pi', q, k) \times K(\Pi, n, \Pi', q, k) & \xleftarrow{\quad g \quad} & K(\Pi, n, \Pi', q, k) \otimes K(\Pi, n, \Pi', q, k) \\
\downarrow \gamma & & \\
K(\Pi, n\, \Pi', q, k). & &
\end{array}$$

Here the first map e is the diagonal map. The second map f is the standard map of the cartesian into the tensor product defined by

$$f(\sigma \times \tau) = \textstyle\sum_\beta \beta_1^* \sigma \otimes \beta_2^* \tau \qquad \text{if } \dim \sigma = \dim \tau = r \tag{3.2}$$

where β is going round the family of pairs (β_1, β_2) such that

$$\beta_i : [m_i] \longrightarrow [m_1+m_2] \qquad 0 \leqq m_i \leqq r,\ m_1+m_2=r$$
$$\beta_1(i)=i \quad \text{for} \quad 0 \leqq i \leqq m_1,\ \beta_2(j)=j+m_1 \quad \text{for} \quad 0 \leqq j \leqq m_2.$$

The third map is the tensor product of the *FD*-maps $R(x_{n_1})$, $R(x_{q_1})$ each of which is defined by

$$R(x) = T(x) - T(0),$$

while the fourth map is the tensor product of the suspensions.

The fifth map is the tensor product of the inclusion maps

$$i_n : K(\Pi, n) \longrightarrow K(\Pi, n, \Pi', q, k)$$
$$i_q : K(\Pi', q) \longrightarrow K(\Pi, n, \Pi', q, k)$$

defined by $i_n(\phi)=(\phi, \iota)$, $i_q(\psi)=(\iota, \psi)$, here it is easily verified that the map i_n is meaningless when dim $\phi>q$. That is to say, if dim $\phi>q$ the image (ϕ, ι) of i_n belongs to $F(\Pi, n, \Pi', q, k)$ but not to $K(\Pi, n, \Pi', q, k)$ in general.

The sixth map g is the standard map of the tensor into the cartesian product defined by

$$g(\sigma\otimes\tau)=\sum_\alpha \wp(\alpha)\alpha_1^*\sigma\times\alpha_2^*\tau \quad \text{if dim } \sigma=m_1, \text{ dim } \tau=m_2 \tag{3.3}$$

where α is going round the family of pairs (α_1, α_2) such that

$$\alpha_i: [m_1+m_2]\longrightarrow[m_i]$$
$$\{\alpha_1(p)+\alpha_2(p)\}-\{\alpha_1(p-1)+\alpha_2(p-1)\}=1 \qquad 1\leqq p\leqq m_1+m_2,$$

and
$$\wp(\alpha)=\text{Sgn.}\begin{pmatrix}1, & \cdots\cdots, & m_1, & m_1+1, & \cdots\cdots, & m_1+m_2\\ r_1, & \cdots\cdots, & r_{m_1}, & s_1, & \cdots\cdots\cdots, & s_{m_2}\end{pmatrix}$$

where $r_1<\cdots<r_{m_1}$, $s_1<\cdots<s_{m_2}$ and $\alpha_1(r_i)-\alpha_1(r_i-1)=1$ and $\alpha_2(s_j)-\alpha_2(s_j-1)=1$.

Finally, the map γ is given in terms of the internal product in $K(\Pi, n, \Pi', q, k)$.

The final definition may be written as

$$\gamma_{n\cdot q}(x_{n_1}, x_{q_1})=\gamma g[i_n\otimes i_q][S^{n-n_1}\otimes S^{q-q_1}][R(x_{n_1})\otimes R(x_{q_1})]fe. \tag{3.4}$$

According to the dimensional restriction which is occured by the map i_n, our maps $\gamma_{n\cdot q}(x_{n_1}, x_{q_1})$ are meaningless in the case when $\gamma_{n\cdot q}(x_{n_1}, x_{q_1})$ operate upon the cells whose dimension are large than $2q-s$.

Replacement of x_{n_1} or x_{q_1} by a cohomologous cocycle replaces $R(x_{n_1})$ or $R(x_{q_1})$ by a chain homotopic map, therefore the homotopy class of the map $\gamma_{n\cdot q}$ depends only on the cohomology classes $\boldsymbol{x}_{n_1}$, $\boldsymbol{x}_{q_1}$ of x_{n_1}, x_{q_1} respectively; this homotopy class will be denoted by $\gamma_{n\cdot q}(\boldsymbol{x}_{n_1}, \boldsymbol{x}_{q_1})$.

We shall introduce another maps which are k-prolongation of the maps $R_q(x)$ in a sence. Given a C.S.S. pair (K, L) and a cocycle

$$x_{q_1}\in Z^{q_1}(K, L; \Pi')$$

and given a pair of integers (n, q) where $1<n<q$, $q_1\leqq q$, we shall define a chain transformation

$$\gamma_{n\cdot q}(x_{q_1}): (K, L)\longrightarrow K(\Pi, n, \Pi', q, k)$$

of degree $q-q_1$. The map $\gamma_{n\cdot q}(x_{q_1})$ is defined by

$$\gamma_{n\cdot q}(x_{q_1})=\gamma g[i_n\otimes i_q][I\otimes S^{q-q_1}][T(0)\otimes R(x_{q_1})]fe \tag{3.5}$$

in the main diagram same as above.

The homotopy class of the map $\gamma_{n\cdot q}(x_{q_1})$ depends only on the cohomology class $\boldsymbol{x}_{q_1}$ of x_{q_1}; this homotopy class will be denoted by $\gamma_{n\cdot q}(\boldsymbol{x}_{q_1})$.

The diagram may also be simplified if the defect is zero $(q=q_1)$: No suspension is involved, $R(x_{q_1})$ is an *FD*-map, and f is natural with respect to such maps, and $gf\cong I$, we have

$$\gamma_{n\cdot q}(x_q)=\gamma gf[i_n\times i_q][T(0)\times R(x_q)]e$$
$$\cong\gamma[i_n\times i_q][T(0)\times R(x_q)]=i_q R(x_q)\,,$$

and so

(3.6) $$\gamma_{n\cdot q}(\boldsymbol{x}_q)=i_q R(\boldsymbol{x}_q)\,.$$

§ 4. Definition of the operators.

Take abelian groups Π, Π', and G, positive integers n, q, and r $(1<n<q<r<2q)$, and a cohomology class $\boldsymbol{y}\in H^r(\Pi, n, \Pi', q, k; G)$. The γ-operation $\boldsymbol{y}_\gamma$ is defined for cohomology classes $\boldsymbol{x}_{n_1}\in H^{n_1}(K, L_1; \Pi)$ and $\boldsymbol{x}_{q_1}\in H^{q_1}(K, L_2; \Pi')$ (where $1<n_1\leqq n$, $1<q_1\leqq q$) by the formula

$$\boldsymbol{y}_\gamma(\boldsymbol{x}_{n_1}, \boldsymbol{x}_{q_1})=\gamma_{n.q}(\boldsymbol{x}_{n_1}, \boldsymbol{x}_{q_1})^*\boldsymbol{y}\,;$$

it is an element of $H^{r-s}(K, L; G)$, where $s=(n-n_1)+(q-q_1)$ is the defect already introduced.

LEMMA 4.1. *If $U_i:(K', L_i')\to(K, L_i)$ $(i=1, 2)$ are simplicial maps which agree on K' and thus determine a simplicial map $U:(K', L')\to(K, L)$, then*

(4.1) $$\boldsymbol{y}_\gamma(U_1^*\boldsymbol{x}_{n_1}, U_2^*\boldsymbol{x}_{q_1})=U^*[\boldsymbol{y}_\gamma(\boldsymbol{x}_{n_1}, \boldsymbol{x}_{q_1})]$$

Proof. Denoting by e and e' the respective diagonal maps, we have

$$\begin{aligned}\gamma_{n\cdot q}(x_{n_1}U_1, x_{q_1}U_2)&=V[R(x_{n_1}U_1)\otimes R(x_{q_1}U_2)]fe'\\&=V[R(x_{n_1})U_1\otimes R(x_{q_1})U_2]fe'\\&=V[R(x_{n_1})\otimes R(x_{q_1})][U_1\otimes U_2]fe'\\&=V[R(x_{n_1})\otimes R(x_{q_1})]f[U_1\times U_2]e'\\&=V[R(x_{n_1})\otimes R(x_{q_1})]feU\end{aligned}$$

since $[U_1\times U_2]e'=eU$, where V is a chain transformation of degree s, and is independent of the x_{n_1}, x_{q_1}.

Under the same conditions above, the τ-operation $\boldsymbol{y}_\tau$ is defined for cohomology class $\boldsymbol{x}_n\in H^n(K; \Pi)$ such that $kT(x_n)$ is cohologous zero for any representative cocycle x_n of $\boldsymbol{x}_n$. The operation $\boldsymbol{y}_\tau$ is defined by the formula

$$\boldsymbol{y}_\tau(\boldsymbol{x}_n)=\tau_{n\cdot q}(\boldsymbol{x}_n)^*\boldsymbol{y}\,.$$

THEOREM 4.2. *$\boldsymbol{y}_\tau(\boldsymbol{x}_n)$ is an element of the factor group*

$$H^r(K; G)/\boldsymbol{y}_\gamma(\boldsymbol{x}_n, H^q(K; \Pi'))+i_q^*\boldsymbol{y}\vdash H^q(K; \Pi')$$

where $\boldsymbol{y}\cdot(\boldsymbol{x}_n, H^q(K; \Pi'))+i_q^*\boldsymbol{y}\vdash H^q(K; \Pi')$ denote the subgroup of $H^r(K; G)$, generated by the classes which can be represented by the formulas

$$\boldsymbol{y}_\gamma(\boldsymbol{x}_n, \boldsymbol{x}_q) + i_q^* \boldsymbol{y} \vdash \boldsymbol{x}_q{}^{3)},$$

here $\boldsymbol{x}_q$ is cohomology classes going round the $H^q(K; \Pi')$.

Proof. Let x_n be an representative cocycle of $\boldsymbol{x}_n$, let $T(x_n, x_q)$ and $T(x_n, x_q')$ be two representative maps of $\tau_{n\cdot q}(\boldsymbol{x}_n)$, and let $d_q = x_q - x_q'$ be the difference cocycle of $Z^q(K; \Pi')$. Then,

$$\begin{aligned} &T(x_n, x_q) - T(x_n, x_q) \\ &\quad = \gamma[i_n \times i_q][T(x_n) \times T(x_q)]e - \gamma[i_n \times i_q][T(x_n) \times T(x_q')]e \\ &\quad = \gamma[i_n \times i_q][T(x_n) \times \{T(x_q) - T(x_q')\}]e \\ &\quad \simeq \gamma g[i_n \otimes i_q][T(x_n) \otimes \{T(x_q) - T(x_q')\}]fe, \end{aligned}$$

here

$$\begin{aligned} &T(x_q) - T(x_q') \\ &\quad = T(x_q - x_q') + T(x_q - x_q' + x_q') - T(x_q - x_q') - T(x_q') \\ &\quad = R(d_q) + R(d_q + x_q') - R(d_q) - R(x_q') \\ &\quad = R(d_q) + R(d_q) \circ R(x_q) \end{aligned}$$

hence

$$\begin{aligned} &T(x_n, x_q) - T(x_n, x_q') \\ &\simeq \gamma g[i_n \otimes i_q][T(x_n) \otimes R(d_q)]fe + \gamma g[i_n \otimes i_q][T(x_n) \otimes R(d_q) \circ R(x_q')]fe. \end{aligned}$$

In this formula, the latter term is homotopic zero whenever it operates on the cell whose dimension is less than $2q$. Because; in such a case

$$\begin{aligned} R(d_q) \circ R(x_q') &= \gamma[i_q \times iq][R(d_q) \times R(x_q')]e \\ &\simeq \gamma g[i_q \otimes i_q][R(d_q) \otimes R(x_q')]fe, \end{aligned}$$

and the last term operates on the cell of K trivially, since at least one of the dimensions of $\beta_i^* \sigma$ $(i=1, 2)$ is less than q for all $\beta = (\beta_1, \beta_2)$ in the formula (3.2). Then $R(d_q) \circ R(x_q') : K \to F(\Pi', q)$ is homotopic zero.

Therefore, under the restriction of dimensions $(r < 2q)$, we have

$$\begin{aligned} (4.2) \qquad &T(x_n, x_q) - T(x_n, x_q') \\ &\simeq \gamma g[i_n \otimes i_q][\{R(x_n) + T(0)\} \otimes R(d_q)]fe \\ &= \gamma g[i_n \otimes i_q][R(x_n) \otimes R(d_q)]fe + \gamma g[i_n \otimes i_q][T(0) \otimes R(d_q)]fe \\ &= \gamma_{n\cdot q}(x_n, d_q) + \gamma_{n\cdot q}(d_q). \end{aligned}$$

This formula shows precisely the desired result, q.e.d.

§5. The comboundary formulas.

THEOREM 5.1. *Consider a C.S.S. pair (K, L) and cohomology classes $\boldsymbol{x}_{n_1} \in H^{n_1}(K; \Pi)$, $\boldsymbol{x}_{q_1} \in H^{q_1}(L; \Pi')$ and $\boldsymbol{y} \in H^r(\Pi, n, \Pi', q, k; G)$ where $1 < n_1 \leqq n$,*

3) $i_q{}^* y \vdash x_q$ is the internal operation. See [3].

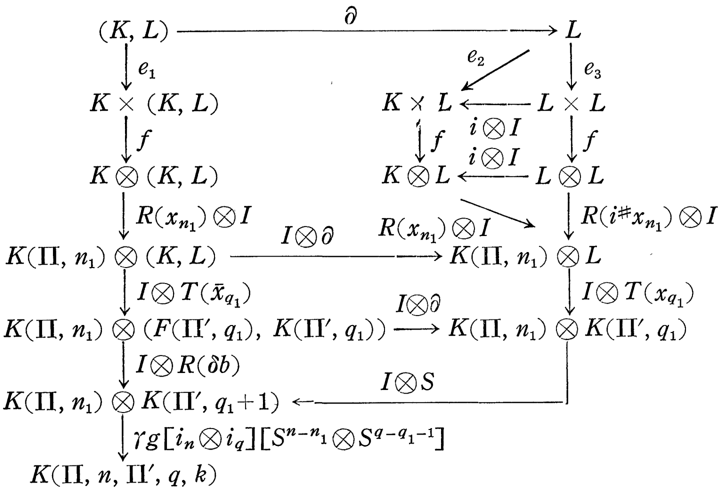
$1<q_1<q,\ n<q<r<2q,$ *then* $\delta \boldsymbol{x}_{q_1} \in H^{q_1+1}(K, L; \Pi')$. *In the case*

$$y_\gamma(\boldsymbol{x}_{n_1}, \delta\boldsymbol{x}_{q_1}) = \delta[y_\gamma(i^*\boldsymbol{x}_{n_1}, \boldsymbol{x}_{q_1})] \tag{5.1}$$

where $i: L \to K$ *is the inclusion map.*

This theorem remains valid if the pair (K, L) replaced by a triple, because the coboundary operation in the cohomology sequence of a triple is a composite of a map induced by inclusion and the coboundary operation of a pair.

The proof of this theorem is an immediate consequence of the commutativity in the following diagram.



Here e_1 and e_3 are the diagonal maps and e_2 is induced by e_1. The maps f, g, γ, i_n, i_q are same as before and the cocycles x_{n_1} and x_{q_1} are the representations of $\boldsymbol{x}_{n_1}$ and $\boldsymbol{x}_{q_1}$ respectively. The basic cocycle b of $K(\Pi', q_1)$ can be considered as an q_1-cochain on $F(\Pi', q_1)$, and $\delta b \in Z^{q_1+1}(F(\Pi', q_1), K(\Pi', q_1); \Pi')$. To the given cocycle $x_{q_1} \in Z^{q_1}(L; \Pi')$ we choose an extension $\bar{x}_{q_1} \in C^{q_1}(K; \Pi')$. Then it follows

$$\begin{aligned}
[I\otimes R(\delta x_{q_1})] &= [I\otimes R\{\delta\cdot T(\bar{x}_{q_1})^{\#}b\}] = [I\otimes R\{\delta b\cdot T(x_{q_1})\}] \\
&= [I\otimes R(\delta b)\cdot T(x_{q_1})] = [I\otimes R(\delta b)][I\otimes T(\bar{x}_{q_1})] \\
[I\otimes S][I\otimes R(x_{q_1})] &= [I\otimes S\cdot R\{T(\bar{x}_{q_1})^{\#}b\}] = [I\otimes S\cdot R\{b\cdot T(x_{q_1})\}] \\
&= [I\otimes SR(b)\cdot T(x_{q_1})] = [I\otimes S][I\otimes T(x_{q_1})]\,.
\end{aligned}$$

In the above diagram, the commutativity in the upper half is obvious, and the commutativity in the lowest triangle is due to [3].

Therefore the desired equality follows from the fact

$$y_\gamma(x_{n_1}, \delta x_{q_1}) = y\gamma g[i_n\otimes i_q][S^{n-n_1}\otimes S^{q-q_1-1}][I\otimes R(\delta x_{q_1})][R(x_{n_1})\otimes I]fe_1$$
$$y\ (i^{\#}x_{n_1}, x_{q_1}) = y\gamma g[i_n\otimes i_q][S^{n-n_1}\otimes S^{q-q_1}][I\otimes R(x_{q_1})][R(x_{n_1})\otimes I]fe_3$$

and $[S^{n-n_1}\otimes S^{q-q_1}] = [S^{n-n_1}\otimes S^{q-q_1-1}][I\otimes S]$, q.e.d.

§6. The invariants k_n^{q+1}, k_q^{r+1}, $\{k_{n\cdot q}^{r+1}\}$

Let Y be a topological space with base point y_0 such that

$$\pi_i(Y)=0 \quad \text{for} \quad i<n,\ n<i<q,\ q<i<r\ \ (r<2q-1).$$

Relative to the base point y_0, we choose as in [2] a minimal subcomplex M of the total singular complex $S(Y)$, and we denote by $S_m(Y)$ the subcomplex of $S(Y)$ which consists of all singular simplices whose faces in dimensions less than m reduce to y_0, and we denote $M\cap S_m(Y)$ by M_m. It is obvious that $M=M_n$ in our case.

As in [2], there are natural simplicial maps

$$\begin{aligned} \kappa: &\quad M \longrightarrow K(\pi_n, n) \\ \bar{\kappa}: &\quad K(\pi_n, n) \longrightarrow M \\ \kappa': &\quad M \longrightarrow K(\pi_n, n, \pi_q, q, k) \\ \bar{\kappa}': &\quad K(\pi_n, n, \pi_q, q, k) \longrightarrow M. \end{aligned}$$

Here κ is isomorphic in dimensions less than q, $\bar{\kappa}$ is defined in dimensions $\leqq q$ in such fashion that $\kappa\bar{\kappa}$ is the identity and the map $\bar{\kappa}$ presents an obstruction $k_n^{q+1}\in Z^{q+1}(\pi_n, n; \pi_q)$ whose cohomology class is the Eilenberg-MacLane invariant $\boldsymbol{k}_n^{q+1}$.

And, κ' is an k-prolongation of κ (where k is an representative cocycle of $\boldsymbol{k}_n^{q+1}$) and isomorphic in dimensions less than r, $\bar{\kappa}'$ is defined in dimensions $\leqq r$ in such fashion that $\kappa'\bar{\kappa}'$ is the identity and the map $\bar{\kappa}'$ presents an obstruction $k_{n\cdot q}^{r+1}\in Z^{r+1}(\pi_n, n, \pi_q, k; \pi_r)$. We described the cohomology class $\boldsymbol{k}_{n\cdot q}^{r+1}$ of this cocycle as a topological invariant if we pay no heed to the identification of the complex $K(\pi_n, n, \pi_q, q, k)$.[4] But it is not enough for our purpose, and we shall explain the topological invariant more precisely.

If we restrict the map κ' on the subcomplex M_q of M, we have a natural simplicial map

$$\kappa'|M_q: \quad M_q \longrightarrow K(0, n, \pi_q, q, k),$$

namely

$$\kappa_0: \quad M_q \longrightarrow K(\pi_q, q).$$

And, the restriction $\bar{\kappa}'|K(0, n, \pi_q, q, k)$ of $\bar{\kappa}'$ similarly gives a natural simplicial map

$$\bar{\kappa}_0: K(\pi_q, q) \longrightarrow M_q.$$

Here κ_0 and $\bar{\kappa}_0$ have the properties same as κ and $\bar{\kappa}$, and $\bar{\kappa}_0$ presents an obstruction $k_q^{r+1}\in Z^{r+1}(\pi_q, q; \pi_r)$ whose cohomology class is the secondary Eilenberg-MacLane invariant $\boldsymbol{k}_q^{r+1}$ of Y.

It is obvious from our definitions that

$$i_q^* \boldsymbol{k}_{n\cdot q}^{r+1} = \boldsymbol{k}_q^{r+1}.$$

In the identification of the complexes $K(\pi_n, n, \pi_q, q, k)$, the only essential part

4) See [2].

is the identification of the complex with its image of automorphism (2.2). Namely, we can recognize the invariant as the family $\{\eta(\boldsymbol{h}_q)^*\boldsymbol{k}_{n\cdot q}^{r+1}\}$ of the cohomology class of $H^{r+1}(\pi_n, n, \pi_q, q, k; \pi_r)$ for the fixed complex $K(\pi_n, n, \pi_q, q, k)$, where $\boldsymbol{h}_q$ is the cohomology class going round the $H^q(\pi_n, n, \pi_q, q, k; \pi_q) \cong H^q(Y; \pi_q)$. In other words, the invariant is an element $\boldsymbol{k}_{n\cdot q}^{r+1}{}_{\tau}(\boldsymbol{b}_n)$ of the factor group

$$H^{r+1}(\pi_n, n, \pi_q, q, k; \pi_r)/\boldsymbol{k}_{n\cdot q}^{r+1}{}_{l}(\boldsymbol{b}_n, H^q(\pi_n, \pi_q, \boldsymbol{k}; \pi_q)) + \boldsymbol{k}_q^{r+1} \vdash H^q(\pi_n, \pi_q, \boldsymbol{k}; \pi_q)$$

since $i_q^* \boldsymbol{k}_{n\cdot q}^{r+1} = \boldsymbol{k}_q^{r+1}$. In the following we shall denote this element simply as $\{\boldsymbol{k}_{n\cdot q}^{r+1}\}$.

§7. The obstruction theorem.

Let K be a geometric complex. We shall be interested in continuous maps $f: K \to Y$. Such a map induces a simplicial map $K \to S(Y)$ which is also denoted by f. Conversely, every simplicial map $K \to S(Y)$ arises in this fashion from a unique continuous map $K \to Y$. The map f is called *minimal* if it maps K into M.

In the theory of the minimal complex we have constructed for each map f a homotopy D_f which deforms f into a minimal map, and which has the following two important properties:

a) If f is already minimal on a subcomplex L of K, then D_f is stationary on L: i.e., $D_f(t, x) = f(x)$ for $x \in L$ and all t.

b) If the maps f and g coincide on a subcomplex L then D_f and D_g coincide on L.

In the following, without loss of generality we shall assume that the map $K \to Y$ are minimal. Then, a map $f: K^n \to Y$ determines a cochain $a_f^n = a^n(f) \in C^n(K; \pi_n)$ which assigns to each n-simplex σ of K the element of $\pi_n(Y)$ determined by the map $f|\sigma$. The cochain a_f^n is a cocycle if and only if the map f admits an extension $f_q: K^q \to Y$. This extension f_q presents an obstruction cocycle $c^{q+1}(f_q) \in Z^{q+1}(K; \pi_q)$ which is represented by

$$c^{q+1}(f_q) = k_n^{q+1} T(a_f^n) + \delta(l^q f_q)$$

where l^q is a cochain $C^q(M; \pi_q)$ determined by setting

$$l^q \sigma = d(\sigma, \bar{\kappa}\kappa\sigma) \quad \text{for any } q\text{-simplex } \sigma \text{ of } M.$$

This obstruction $c^{q+1}(f_q)$ is zero if and only if the map f_q admits an extension $f_r: K^r \to Y$. This extension f_r presents an obstruction cocycle $c^{r+1}(f_r) \in Z^{r+1}(K; \pi_r)$.

LEMMA 7.1. *If f_r is a map $K^r \to Y$, then*

$$c^{r+1}(f_r) = k_{n\cdot q}^{r+1} T(a_f^n, l^q f_q) + \delta(l^r f_r) \tag{7.1}$$

where l^r is a cochain in $C^r(M: \pi_r)$ determined by setting

$$l^r \sigma = d(\sigma, \bar{\kappa}'\kappa'\sigma) \quad \textit{for any } r\textit{-simplex } \sigma \textit{ of } M.$$

Proof. In the complex M a cocycle $j^n \in Z^n(M:\pi_n)$ is defined by assigning to each n-simplex of M the element of π_n which this simplex represents. It is then easy to see that

$$\kappa = T(j^n)\,,\ a_f^n = j^n f \ \text{ and } \ \kappa' = T(j^n, l^q)\,.$$

By the naturality properties of T, it follows that

$$\kappa' f = T(j^n, l^q) f = T(j^n f, l^q f) = T(a, l^q f)\,.$$

Now consider the map

$$g = \bar{\kappa}' \kappa' f = \bar{\kappa}' T(a_f^n, l^q f) : K^r \longrightarrow M\,.$$

Since $T(a_f^n, l^q f)$ is defined in the whole complex K it follows that

$$c^{r+1}(g) = k_{n\cdot q}^{r+1} T(a_f^n, l^q f_q)\,.$$

Because $\bar{\kappa}'\kappa'$ is the identity in dimensions less than r, the maps f and g must coincide on K^{r-1}; hence the difference cochain $d^r(f, g)$ is defined. For each r-simplex σ of K we have

$$d^r(f, g)\sigma = d^r(f\sigma, g\sigma) = d^r(f\sigma, \bar{\kappa}'\kappa' f\sigma) = l^r f(\sigma)$$

and hence

$$d^r(f, g) = l^r f\,.$$

Since $c^{r+1}(f) - c^{r+1}(g) = \delta d^r(f, g)$, this implies

$$c^{r+1}(f) = c^{r+1}(g) + \delta d^r(f, g) = k_{n\cdot q}^{r+1} T(a_f^n, l^q f_q) + \delta(l^r f_r)\,,$$

which is the desired conclusion.

Let L be a subcomplex of K and let $f: K^n \cup L \to Y$ be a map extensible to a map $f': K^r \cup L \to Y$. The cohomology class $\boldsymbol{z}^{r+1}(f')$ of the obstruction cocycle $c^{r+1}(f_r)$ depends on the choice of the extension $f' | K^q \cup L$ of f.

THEOREM 7.2. *Let $f_1, f_2: K^q \cup L \to Y$ be two extensions of the map $f: K^n \cup L \to Y$ and which are extensible to $K^{q+1} \cup L \to Y$. Then*

(7.2) $$\boldsymbol{z}^{r+1}(f_1) - \boldsymbol{z}^{r+1}(f_2) = \boldsymbol{k}_{n\cdot q}^{r+1} \gamma (\boldsymbol{a}^n(f_1, f_2)) + \boldsymbol{k}_q^{r+1} \vdash (\boldsymbol{a}^q(f_1, f_2))\,,$$

where $\boldsymbol{a}^q(f_1, f_2) \in H^q(K, L; \pi_q)$ is the cohomology class of the cocycle $l^q f_1 - l^q f_2$ and $\boldsymbol{a}^n(f) \in H^n(K; \pi_n)$ is the cohomology class of the cocycle $a^n(f)$.

THEOREM 7.3. *Let f: $K^n \to Y$ be a map extendible to a map $K^{q+1} \to Y$, then the third obstruction of f is determined as follows:*

$$\{\boldsymbol{z}^{r+1}(f)\} = \boldsymbol{k}_{n\cdot q}^{r+1}{}_{\tau}\, \boldsymbol{a}^n(f)\,.$$

Proof, Let $f_1', f_2': K^r \cup L \to Y$ be extensions of f_1 and f_2 respectively. By the Lemma 7.1. We have

$$c^{r+1}(f_1') - c^{r+1}(f_2') = k_{n\cdot q}^{r+1}[T(a_f^n, l^q f_1) - T(a_f^n, l^q f_2)] + \delta(l^r f_1' - l^r f_2')\,.$$

Since f_1' and f_2' coincide on L, it follows that $l^r f_1' - l^r f_2'$ is zero on L; this yields the cohomology

$$c^{r+1}(f_1') - c^{r+1}(f_2') \sim k_{n \cdot q}^{r+1}[T(a_\gamma^n, l^q f_1) - T(a_\gamma^n, l^q f_2)],$$

and, from (4.2)

$$k_{n \cdot q}^{r+1}[T(a_\gamma^n, l^q f_1) - T(a_\gamma^n, l^q f_2)] \sim k_{n \cdot q}^{r+1}[\gamma_{n \cdot q}(a_\gamma^n, l^q f_1 - l^q f_2) + \gamma_{n \cdot q}(l^q f_1 - l^q f_2)].$$

Then, we have the desired conclusion since

$$\gamma_{n \cdot q}(\boldsymbol{a}^q(f_1, f_2))^* \boldsymbol{k}_{n \cdot q}^{r+1} = R(\boldsymbol{a}^q(f_1, f_2))^* i_q^* \boldsymbol{k}_{n \cdot q}^{r+1} = R(\boldsymbol{a}^q(f_1, f_2))^* \boldsymbol{k}_q^{r+1}.$$

As an application we prove the following extension theorem.

THEOREM 7.4. *Let $f: K^n \cup L \to Y$ and let $g: K^r \cup L \to Y$ be an extension of f. Then the map f admits an extension $f': K^{r+1} \cup L \to Y$ if any only if there is an element*

$$\boldsymbol{e}^q \in H^q(K, L; \pi_q)$$

such that

$$\boldsymbol{z}^{r+1}(g) + \boldsymbol{k}_{n \cdot q}^{r+1} \gamma(\boldsymbol{a}^n(f), \boldsymbol{e}^q) + \boldsymbol{k}_q^{r+1} \vdash \boldsymbol{e}^q = 0.$$

Proof. Let $f': K^{r+1} \cup L \to Y$ be an extension of f. An application of the previous theorem then shows that the element $\boldsymbol{e}^q = \boldsymbol{a}^q(f', g)$ satisfies above relation since $\boldsymbol{z}^{r+1}(f') = 0$.

Conversely, assume that $\boldsymbol{e}^q$ satisfies this equation. By changing the map g on the interiors of q-simplices of $K - L$ we can construct a map $\tilde{f}: K^q \cup L \to Y$ which agrees with g on $K^{q-1} \cup L$ and has the representative cocycle $d^q(\tilde{f}, g)$ of $\boldsymbol{a}^q(\tilde{f}, g) = \boldsymbol{e}^q$. As $c^{q+1}(\tilde{f}) = c^{q+1}(g) + \delta d^q(\tilde{f}, g) = 0$ $\tilde{f}$ admits an extension $K^r \cup L \to Y$, and, an application of the preceding theorem then shows that $\boldsymbol{z}^{r+1}(\tilde{f}) = 0$. Therefore f admits an extension $f': K^{r+1} \cup L \to Y$, as desired.

§8. The homotopy classification theorem.

THEOREM 8.1. *Let L be a subcomplex of K such that dim $(K-L) \leqq r$, let $f_0, f_1: K \to Y$ be two maps which agree on $K^{r-1} \cup L$ and let $d^r(f_0, f_1)$ be their difference cocycle. Then $f_0 \simeq f_1$ rel L if and only if there exists a cohomology class*

$$\boldsymbol{e}^{q-1} \in H^{q-1}(K, L; \pi_q)$$

such that

$$\boldsymbol{d}^r(f_0, f_1) + \boldsymbol{k}_{n \cdot q}^{r+1} \gamma(\boldsymbol{a}^n(f_0), \boldsymbol{e}^{q-1}) + \boldsymbol{k}_q^{r+1} \vdash \boldsymbol{e}^{q-1} = 0.$$

Proof. This theorem will be reduced to the extension theorem of the previous section by the usual technique. We introduce the maps

$$l: (K, L) \longrightarrow (IL \cup 0K \cup 1K, IL \cup 1K),$$
$$l_1: \quad K \longrightarrow IL \cup 0K \cup 1K,$$

defined by $lx = l_1 x = (0, x)$. Since

$$IL \cup 0K \cup 1K = (IL \cup 1K) \cup 0K, \quad 0L = (IL \cup 1K) \cap 0K$$

it follows by excision that the map l induces isomorphisms

$$l^*: H^i(IL \cup 0K \cup 1K,\ IL \cup 1K) \cong H^i(K, L)$$

of the cohomology groups for any coefficient group. By making use of the exact sequence of the triple $(IK,\ IL \cup 0K \cup 1K,\ IL \cup 1K)$ we also have the isomorphism

$$\delta: H^i(IL \cup 0K \cup 1K,\ IL \cup 1K) \cong H^{i+1}(IK,\ IL \cup 0K \cup 1K)\,.$$

Now define a map F: $(IK)^n \cup IL \cup 0K \cup 1K \to Y$ by setting

$$F(t, x) = f_0(x) \quad \text{for } x \in K^{n-1} \cup L\,,$$
$$F(i, x) = f_i(x) \quad \text{for } x \in K\,, \quad i = 0,\ 1.$$

And we define an extension

$$F': (IK)^r \cup IL \cup 0K \cup 1K \longrightarrow Y$$

of F, by setting $F'(t, x) = f_0 x$ for all x in $(IK)^r$ and not in $IL \cup 0K \cup 1K$; this extension is continuous since f_0 and f_1 agree on $K^{r-1} \cup L$.

It now follows from the Theorem 7.4 applied to the pair $(IK,\ IL \cup 0K \cup 1K)$, that the desired homotopy $D: IK \to Y (f_0 \simeq f_1)$ exists if and only if there is an element

$$\boldsymbol{e}^q \in H^q(IK,\ IL \cup 0K \cup 1K;\ \pi_q)$$

satisfying

$$\boldsymbol{z}^{r+1}(F') + \boldsymbol{k}_{n\cdot q}^{r+1}\ \gamma(\boldsymbol{a}^n(F), \boldsymbol{e}^q) + \boldsymbol{k}_q^{r+1} \vdash \boldsymbol{e}^q = 0\,. \tag{8.1}$$

We shall show that this condition is equivalent to the one stated in the theorem.

First observe that $F(0, x) = f_0(x)$ and therefore that

$$a_{f_0}^n = a^n(F)\, il_1\,,$$

where i: $IL \cup 0K \cup 1K \to IK$ is the inclusion map. Next we define G: $IK \to Y$ by setting $G(t, x) = f_1(x)$. Then,

$$c^{r+1}(F') = c^{r+1}(F') - c^{r+1}(G) = \delta d^r(F', G)\,,$$
$$d^r(f_0, f_1) = d^r(F', G)l\,.$$

Finally we write the element $\boldsymbol{e}^q$ in the form $\delta\bar{\boldsymbol{e}}^{q-1}$, where

$$\bar{\boldsymbol{e}}^{q-1} \in H^{q-1}(IL \cup 0K \cup 1K,\ IL \cup 1K)\,.$$

Equation (8.1) now becomes

$$\delta \boldsymbol{d}^r(F', G) + \boldsymbol{k}_{n\cdot q}^{r+1}\ \gamma(\boldsymbol{a}^n(F),\ \delta\bar{\boldsymbol{e}}^{q-1}) + \boldsymbol{k}_q^{r+1} \vdash (\delta\bar{\boldsymbol{e}}^{q-1}) = 0\,.$$

In view of the cobundary formula (5.1) for the operations, this may be rewritten as

$$\delta \boldsymbol{d}^r(F', G) + \delta[\boldsymbol{k}_{n\cdot q}^{r+1}\ \gamma(i^*\boldsymbol{a}^n(F),\ \bar{\boldsymbol{e}}^{q-1})] + \delta[\boldsymbol{k}_q^{r+1} \vdash \bar{\boldsymbol{e}}^{q-1}] = 0\,.$$

Since δ, as is noted above, is an isomorphism, this equation is equivalent to

$$\boldsymbol{d}^r(F', G)+\boldsymbol{k}_{n\cdot q}^{r+1}\,\gamma(i^*\boldsymbol{a}^n(F),\ \bar{\boldsymbol{e}}^{q-1})+\boldsymbol{k}_q^{r+1}\vdash\bar{\boldsymbol{e}}^{q-1}=0\,.$$

An application of the isomorphism l^* yields

$$\boldsymbol{d}^r(f_0, f_1)+\boldsymbol{k}_{n\cdot q}^{r+1}\,\gamma(a_{f_0}^n,\ l^*\bar{\boldsymbol{e}}^{q-1})+\boldsymbol{k}_q^{r+1}\vdash l^*\bar{\boldsymbol{e}}^{q-1}=0\,.$$

This is precisely the desired equation, with $\boldsymbol{e}^{q-1}=l^*\bar{\boldsymbol{e}}^{q-1}$.

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