Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 1, No. 1, Se-iee A

Homotopy Groups of Fibre Bundles.

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§ 1. Introduction. To determine the homotopy groups of a fibre bundle from the knowledge of those of the base space and of the fibre and the knowledge of the law of connection is an important problem in topology.

The present paper is devoted to the investigation of this problem.

Let Z be a fibre bundle over X with fiber Y and with a definite law of connection α. Then, combining the Hurewicz-Steenrod isomorphism[1]: $\pi_n(X) \approx \pi_n(Z, Y)$ with the well-known homotopy sequence[2] of the pair (Z, Y), we obtain the following sequence;

$$(1.1) \quad \to \pi_{n+1}(X) \xrightarrow{\partial_{n+1}} \pi_n(Y) \xrightarrow{i_n} \pi_n(Z) \xrightarrow{p_n} \pi_n(X) \xrightarrow{\partial_n} \pi_{n-1}(Y) \longrightarrow \cdots$$

which is exact in the sense, that the kernel of each homomorphism is identical with the image of the preceedingone. We denote the kernel-images in $\pi_n(Y)$, $\pi_n(Z)$, $\pi_n(X)$ by $\lambda_n(Y)$, $\mu_n(Z)$, $\nu_n(X)$, respectively.

As for the law of connection α, we may assume that it is represented by a continuous function f of the base space X into the space $\mathfrak{Y}$ (see, § 2). The law of connection α induces for each $n\,(n \geq 1)$ a homomorphism $\alpha_n : \pi_n(X) \longrightarrow \pi_{n-1}(\mathfrak{A}(Y))$, where $\mathfrak{A}(Y)$ denotes the group of automorphisms of Y (§ 4). If we let this homomorphism α_n be followed by the natural homomorphism $\kappa_{n-1} : \pi_{n-1}(\mathfrak{A}(Y)) \to \pi_{n-1}(Y)$, we obtain a homomorphism $\kappa_{n-1}\alpha_n$. In § 5 it will be shown that this homomorphism is identical with the boundary homomorphism ∂_n in (1.1). This shows that the groups λ_n, ν_n and hence the group $\mu_n \approx \pi_n(Y)/\lambda_n$ are calculable from the system of quantities $\mathfrak{S}_n = \{\pi_n(X), \pi_{n+1}(X); \pi_n(Y), \pi_{n-1}(Y); \alpha_n, \alpha_{n+1}\}$. Thus we have

Theorem 1. 1. *$\pi_n(Z)$ is an extension of a group by another, both of which are calculable if we know the system of quantities $\mathfrak{S}_n$.*

There are many special cases, where we are able to determine $\pi_n(X)$ without further observations. For example if $n \geq 2$ and $\pi_n(X)$ is free abelian, $\pi_n(Z) \approx \mu_n + \nu_n$. In §§ 6-7 we shall determine this extension in the case where Y undergoes some restrictions. In the

Appendix we shall give a generalization of Hurewicz-Steenrod isomorphism and a similar isomorphism, which may be applied to fibre bundles with a slicing map in the large.

§ 2. Let E^∞ be the Hilbert fundamental cube and Y a subset of E^∞. We define $\Theta(Y, E^\infty)$ ($\mathfrak{A}(Y)$) to be the totality of homeomorphic mappings of Y into E (onto Y). $\mathfrak{A}(Y)$ is a subset of $\Theta(Y, E^\infty)$.

A function $g: X \leftrightarrow \Theta(Y, E^\infty)$, where X is an arbitrary space, is called *continuous* if the 1-1 correspondence $Y \times X \ni (y, x) \leftrightarrow (g(x)y, x) \in E^\infty \times X$ is a homeomorphism. Further the totality of the subsets of E^∞ homeomorphic with Y will be denoted by $\mathfrak{Y}$. Every element σ of Θ determines an element of $\mathfrak{Y}$, which is the image of Y under σ and is called the projection of σ (notation: $|\sigma| = \sigma(Y)$). A function f of X into $\mathfrak{Y}$ is called *trivial* if it is representable as a projection of some continuous function g of X into Θ, i.e., $f(x) = |g(x)|$ for every $x \in X$. f is called *continuous* if it is locally trivial, i.e., if there exists for $x \in X$ an neighbourhood of x on which f is trivial. In the sequel we shall omit the adjective "continuous" for such functions as f and g. The space X will always be assumed to be compact metrisable.

Given a function f of X into $\mathfrak{Y}$ we denote by Z_f the graph of f over X, i.e., the subspace of $E^\infty \times X$ consisting of points of the form (ω, x) with $\omega \in f(x)$. It is casy to see that Z_f is a fibre bundle over X with fibre Y.

In [1] the following theorems have been proved.

Theorem 2. 1. (*The Classification Theorem*). *The totality of the fibre bundles over X with fibre Y is in one-to-one correspondence with the totality of homotopy classes of functions of X into $\mathfrak{Y}$.*

Theorem 2. 2. *If X_0 is a deformation retract of X and f a function of X into $\mathfrak{Y}$ which takes on the valve Y in X_0, there exists a function g of X into Θ such that $|g| = f$ and $g(x) = 1 =$ [the identity of $\mathfrak{A}(Y)$], for $x \in X_0$.*

§ 3. Let A be a separable metrizable space and a_0 fixed point of A. We denote by $\Pi_n(A)$ the mapping space $(A, a_0)^{(E^n, T^{n-1})}$, where E^n is the unit cube $0 \leq u_i \leq 1$, $i = 0, 1, \dots, n-1$, and T^{n-1} a subset of the boundary $\dot{E}$ of E^n defined by the equality $(1-u_{n-1})\Pi_{i=0}^{n-2} u_i (1-u_i) = 0$. We introduce in $\Pi_n(A)$ two operations as follows:

For $a_1(u^n)$, $a_2(u^n)$, $a(u^n)\in \Pi_n(A)$

$$a_1a_2(u^n) = a_1(2u_0, u_1^n) \qquad (0\leq u_0\leq 1/2),$$
$$= a_2(2u_0-1, u_1^n) \qquad (1/2\leq u_0\leq 1);$$
$$a^{-1}(u^n) = a(1-u_0, u_1^n) \qquad (0\leq u_0\leq 1).$$ [4]

Let $\Pi_n^0(A)$ be the mapping space $(A, a_0)^{(E^n, \dot{E}^n)}$, then it is a subset of $\Pi_n(A)$ which is closed with respect to the above two operations. We may define a natural homomorphism ρ_n of $\Pi_{n+1}(A)$ into $\Pi_n^0(A)$ by $\rho_n a(u) = a(u^n, 0)$ for $a(u^{n+1})\in \Pi_{n+1}(A)$. Let the image of $\Pi_{n+1}(A)$ under ρ_n be denoted by $\Pi_n^*(A)$, then the following propositions may easily been proved.

Proposition 3. 1. *Let a_1, a_2 be two elements of $\Pi_n^0(A)$. Then $a_1^{-1}a_2$ $\Pi_n^*(A)$ if and only if there exists a mapping $a(u^{n+1})$ of E^{n+1} into A such that for each u_n $a(u^n, u_n) = a_0$ for every $u^n\in E$, and $a(u^n, 0) = a_1(u^n)$. $a(u^n, 1) = a_2(u^n)$ for every $u^n\in E^n$.*

Proposition 3. 2. $\Pi_n^0(A)/\Pi_n^*(A) = \pi_n(A)$. [5]

§ 4. Let f be a function of X into $\mathfrak{Y}$ such that $f(x_0) = Y$, and let $x(u^n)$ be an element of $\Pi_n(X)$. Then $fx(u^n)$ is a function of E^n into $\mathfrak{Y}$ such that $fx(u^n) = Y$ in T^{n-1}. Since T^{n-1} is a deformation retract of E^n, applying theorem 2. 2 for these fx, E^n, T^{n-1}, we may find a function $\sigma = \sigma_x(u^n)$ of E^n into Θ such that $|\sigma_x(u^n)| = fx(u^n)$ for $u^n\in E^n$ and $\sigma_x(u^n) = 1$ for $u^n\in T^{n-1}$, In particular, if $x(u^n)\in \Pi_n^0$, $\sigma_x(u^n)\in \mathfrak{A}(Y)$ for $u^{n-1}=0$, or, to speak in another way, $\rho_{n-1}\sigma_x(u^{n-1})\in \Pi_{n-1}^0(\mathfrak{A}(Y))$. In order to define the operations α_n announced in § 1. we need several lemmas.

Lemma 4. 1. *Let $\sigma_x'(u^n)$ be any other function of E^n into Θ with the same properties as $\sigma_x(u^n)$, then $\rho_{n-1}\sigma_x(u^{n-1}) \equiv \rho_{n-1}\sigma_x'(u^{n-1})$ mod $\Pi_{n-1}^*(\mathfrak{A}(Y))$.*

Proof: $(\rho\sigma_x)^{-1}(\rho\sigma_x') = \rho(\sigma_x^{-1}\sigma_x')$, since ρ is a homomorphism. But, since $|\sigma_x(u^n)| = |\sigma_x'(u^n)|$ for $u^n\in E^n$ and $\sigma_x(u^n) = \sigma_x'(u^n) = 1$ for $u^n\in T^{n-1}$, $\sigma_x^{-1}\sigma_x'\in \Pi_n(\mathfrak{A}(Y))$. Hence $\rho\sigma_x \equiv \rho\sigma_x'$ mod $\Pi_{n-1}^*(\mathfrak{A}(Y))$.

Lemma 4. 2. *If $x_1(u^n)\equiv x_2(u^n)$ mod $\Pi_{n-1}^*(x)$, then*

$$\rho_{n-1}\sigma_{x_1}(u^{n-1}) \equiv \rho_{n-1}\sigma_{x_2}(u^{n-1}) \ mod\ \Pi_{n-1}^*(\mathfrak{A}(Y)).$$

Proof: By assumption $x_1^{-1}x_2(u^n) = \rho x(u^n)$ for some $x(u^{n+1}\in \Pi_{n+1}(X)$. Since the set $M = \{u^{n+1} \mid (1-u_n)(1-u_{n-1})\Pi_{i=0}^{n-2}u_i(1-u_i)=0\} \cup \{u^{n+1}\mid u_n = 0, u_0 = 1/2\}$ is a deformation retract of E^{n+1} and $x(u^n) = x_0(u^{n+1}\in M)$, applying theorem 2. 2, we may find a function $\sigma(u^{n+1})$ of E^{n+1} into $\mathfrak{Y}$

with $|\sigma(u^{n+1})| = fx(u^{n+1})$ for $u^{n+1} \in E^{n+1}$ and $\sigma(u^{n+1}) = 1$ for $u^{n+1} \in M$. Put $\sigma_{x_1}(u^n) = \rho_n \sigma\left(\frac{1-u_0}{2}, u_1^n\right)$, $\sigma_{x_2}(u^n) = \rho_n \sigma\left(\frac{1+u_0}{2}, u_1^n\right)$ $(0 \leq u_0 \leq 1)$. It is obvious that $\sigma_{x_i}(u^n)$ $(i = 1, 2)$ have the property characteristic for their notations, and that $\rho_n \sigma(u^n) \sigma_{x_1}^{-1} \sigma_{x_2}(u^n)$. If we denote by ρ the operation which restricts the range of u^n to $\{u^n \mid u_{n+1} = 0\}$, from the last equality we have $\rho\, \rho_n \sigma(u^{n-1}) = \rho(\sigma_{x_1}^{-1} \sigma_{x_2}(u^{n-1})) = (\rho\, \sigma_{x_1})^{-1} (\rho\, \sigma_{x_2})(u^{n-1})$. But, since $\rho\, \rho_n \sigma(u^{n-1}) = \rho_n \{\rho(u^{n-1})\}$ and $\rho\, \sigma(u^{n-1}, u_n) \in \Pi_n(\mathfrak{A}(Y))$, the assertion of the lemma follows.

Lemma 4. 3. *Let f' be another f and homotopic to f: for some function $F(x, u_n)$ of $X \times E_n^{n+1}$ into $\mathfrak{Y}$, $f(x) = F(x, 0)$, $f'(x) = F(x, 1)$ (the condition $F(x_0, u_n) = Y$ is not necessarily satisfied for $0 < u_n < 1$. and let σ_x' be analogously defined for f'. Then $\rho_{n-1} \sigma_x(u^{n-1}) \equiv \rho_{n-1} \sigma_x'(u^{n-1})$ mod $\Pi_{n-1}^*(\mathfrak{A}(Y))$.*

Proof: Since $F(x(u^n), u_n)$ is continuous in E^{n+1} and $F(x(u^n), u_n) = \text{const.}$ in T^{n-1} for fixed u_n, we may find a function $\sigma(u^{n+1})$ of E^{n+1} into Θ such that $|\sigma(u^{n+1})| = F(x(u^n), u_n)$ for $u^{n+1} \in E^{n+1}$ and $\sigma(u^{n+1}) = \text{const.}$ in T^{n-1} for fixed u_n. Put $\tau(u^{n+1}) = \sigma(u^{n+1})[\sigma(0, u_n)]^{-1}$, where $[\sigma(0, u_n)]^{-1}$ denotes the homeomorphic mapping inverse to $\sigma(0, u_n)$. Then for each u_n, $\tau(u^{n-1}, 0, u_n) \in \Pi_{n-1}^0(\mathfrak{A}(Y))$ and $\tau(u^{n-1}, 0, 0) = \sigma(u^{n-1}, 0, 0) = \rho_{n-1} \sigma_x(u^{n-1})$, $\tau(u^{n-1}, 0, 1) = \rho_{n-1} \sigma_x'(u^{n-1})$, which, comparing with proposition 3. 1, assures the validity of the assertion.

Let α be a homotopy class of functions of X into $\mathfrak{Y}$, and ξ an element of $\pi_n(X)$. Choose representatives f and x arbitrarily from α and ξ respectively and define $\alpha_n(\xi)$ as the element of $\pi_{n-1}(\mathfrak{A}(Y))$ represented by $\rho_{n-1} \sigma_x(u^{n-1}) \in \Pi_{n-1}^0(\mathfrak{A}(Y))$. Then $\alpha_n(\xi)$ is uniquely determined by π and ξ, and the correspondence $\alpha_n : \pi_n(X) \to \pi_{n-1}(\mathfrak{A}(Y))$ is easily seen to be a homomorphism. Thus

Theorem 4. 4. *To each fibre bundle correspond a series of invariants α_n $(n = 1, 2, \ldots)$, each being a homomorphism of $\pi_n(X)$ into $\pi_{n-1}(\mathfrak{A})$.*

§ 5. Proof of the identity $\partial_n = \kappa_{n-1} \pi_n$. Let f be as before, and Z_f the corresponding fibre bundle. Let σ_x be chosen for each $x \in \Pi_n(x)$ as in § 4. Let the natural homomorphism $\kappa_{n-1} : \pi_n(\mathfrak{A}(Y)) \to \pi_{n-1}(Y)$ be the one induced from the correspondence $\Pi_{n-1}^0(\mathfrak{A}(Y)) \ni \sigma(u^{n-1}) \to \sigma(u^{n-1}) y_0 \in \Pi_{n-1}^0(Y)$. Then, whatever the representative $x(u^n) \in \Pi_n^0(X)$ of $\xi \in \pi_n(X)$ may be, the element $y(u^{n-1}) = \rho_{n-1} \sigma_x(u^{n-1}) y_0 \in \Pi_{n-1}^0(Y)$

represents the element $\kappa_{n-1}\alpha_n(\xi)$.

On the other hand, putting $\omega(u^n)=\sigma_x(u^n)y_0$, we get an element $z(u^n)=(\omega(u^n), x(u^n))$ of $\Pi_n(Z_f)$ over $x(u^n)$. But, since $\rho_{n-1}z(u^{n-1})=(\rho_{n-1}\omega(u^{n-1}), \rho_{n-1}x(u^{n-1}))=(y(u^{n-1}), x_0)$ determines no other than the boundary $\partial_n\xi$ (by definition!), as desired.

Theorem 5. 1. *The n-th boundary operator ∂_n in the sequence (1.1) depends only on the n-th invariant α_n of the fibre bundle.*

Corollary 5. 3. *Kernel images $\lambda_n(Y)$, $\nu_n(X)$, $\mu_n(Z)$ in (1.1) are calculable as soon as we know the quantities $\mathfrak{S}_n=\{\pi_n(X), \pi_{n+1}(X); \pi_{n-1}(Y); \pi_n(Y): \alpha_n, \alpha_{n+1}\}$.*

6. The function $C(\xi_1, \xi_2)$ and the automorphism $D(\xi)$. In the sequel of this paper, we assume there have been done the following arbitrary but never reviced choices: for each element of $\pi_{n-1}(\mathfrak{A}(Y))$, the choice of its representative $\sigma(u^{n-1})\in\Pi^0_{n-1}(\mathfrak{A}(Y))$, for each element $y(u^{n-1})\in\Pi^*_{n-1}(Y)$, the choice of $\tilde{y}(u^n)\in\Pi_n(Y)$ with $\rho_{n-1}\tilde{y}(u^{n-1})=y(u^{n-1})$.

For $\xi_i\in\nu_n(X)$ let $\sigma_i(u^{n-1})\in\Pi^0_{n-1}(\mathfrak{A}(Y))$ be the representative of $\alpha_n(\xi_i)$. Then $y_i(u^{n-1})y_0\in\Pi^*_{n-1}(Y)$; hence $\tilde{y}_i(u^n)\in\Pi_n(Y)$ with $\rho_{n-1}\tilde{y}_i(u^{n-1})=y_i(u^{n-1})a$ is determined.

Let $\xi_3=\xi_1\xi_2$. Then $\sigma_3(u^{n-1})\equiv\sigma_1\sigma_2(u^{n-1})$ mod $\Pi^*_{n-1}(\mathfrak{A}(Y))$. But, since $\sigma_1\sigma_2(u^{n-1})\equiv\sigma_1(u^{n-1})\sigma_2(u^{n-1})\equiv\sigma_2(u^{n-1})\sigma_1(u^{n-1})$ mod $\Pi^*_{n-1}(\mathfrak{A}(Y))$. There exists a function $\lambda(u^n)\in[\mathfrak{A}(Y)]^{E^n}$ with $\lambda(u^{n-1}, u_{n-1})\in\Pi^0_{n-1}(\mathfrak{A}(Y))$ for each fixed u_{n-1}, $\lambda(u^{n-1}, 0)=\sigma_2(u^{n-1})\sigma_1(u^{n-1})$, $\lambda(u^{n-1}, 1)=\sigma_3(u^{n-1})$. Define $C(\xi_1, \xi_2)\in\pi_n(Y)$ to be the element represented by

$$
\begin{aligned}
c(u^n) &= y_0 && (0\le u_0\le 1/6),\\
&=(\sigma_1(u^n))^{-1}\tilde{y}_1(u_1^n, 6u_0-1) && (1/6\le u_0\le 2/6),\\
&=(\sigma_1(u_1^n))^{-1}y_0 && (2/6\le u_0\le 3/6),\\
&=(\sigma_1(u_1^n))^{-1}(\sigma_2(u_1^n))\tilde{y}_2(u_1^n, 6u_0-3) && (3/6\le u_0\le 4/6),\\
&=[\lambda(u_1^n, 6u_0-4)]^{-1}\tilde{y}_3(u_1^n, 5/6), && (4/6\le u_0\le 5/6),\\
&= y_0 && (5/6\le u_0\le 1).
\end{aligned}
$$

For $n=1$, define also $D(\xi_1)\in\mathfrak{A}(\pi_1(Y))$ as follows: let $y(u^n)$ represent an element η of $\pi_1(Y)$. Then the curve

$$
\begin{aligned}
Y(u_0) &= y_0 && (0\le u_0\le 1/5),\\
&=\sigma_i^{-1}\tilde{y}_i(5u_0-1) && (1/5\le u_0\le 2/5),\\
&=\sigma_i^{-1}y(5u_0-2) && (2/5\le u_0\le 3/5),\\
&=\sigma_i^{-1}\tilde{y}_i(4-5u_0) && (3/5\le u_0\le 4/5).\\
&= y_0 && (4/5\le u_0\le 1),
\end{aligned}
$$

determines an element of $\pi_1(Y)$. This element does not depend on the choice of $y(u^n)$ but only on η. We denote it by $D(\xi)\eta$. The correspondence $\eta \to D(\xi)\eta$ is easily seen to be an automorphism of $\pi_1(Y)$.

§ 7. **Theorem 7. 1.** *If Y satisfies the condition: $y(u^n) = \sigma(u^n) y_0$ for some $\sigma(u^n) \in \Pi_n^0(\mathfrak{A}(Y))$ implies $y(u^n) \in \Pi_n^*(Y)$, then $\pi_n(Z_f)$ is calculable from the system of quantities $\mathfrak{S}_n = \{\pi_n(X), \pi_{n+1}(X); \pi_{n-1}(Y), \pi_n(Y); \alpha_n, \alpha_{n+1}\}$: more precisely if we introduce in the totality $\mathfrak{G}$ of symbols $(\eta; \xi)$, where $\eta \in \pi_n(Y)$, $\xi \in \nu_n(X)$, a law of multiplication by*

$$(7.2) \qquad (\eta_1; \xi_1)(\eta_2; \xi_2) = (\eta_1 D(\xi_1) \eta_2 C(\xi_1, \xi_2); \xi_1 \xi_2),$$

$\mathfrak{G}$ becomes a group isomorphic with $\pi_n(Z_f)$.

Proof: We fix arbitrarily a function $f: X \to \mathfrak{Y}$ with the given invariants α_n, α_{n+1}. For each $\xi_i \in \nu_n(X)$ choose a representative $x_i(u^n) \in \Pi_n^1(X)$ and an element $\tau_i(u^n) \in \Pi_n(\Theta)$ with $|\tau_i(u^n)| = fx(u^n)$, $u^n \in E^n$. We may hereby arrange that $\rho_{n-1}\tau_i(u^{n-1}) = \sigma_i(u^{n-1})$.

$$\text{Put } z_i(u^n) = (\tau_i(u_1^n, 1-2u_0) y_0, x_i(u_1^n, 1-2u_0)) \qquad (0 \leq u_0 \leq 1/2),$$
$$= (\check{y}_i(u_1^n, 2u_0-1), x_0) \qquad (1/2 \leq u_0 \leq 1).$$

Then $z_i(u^n)$ determines an element $\bar{\xi}_i \in \pi_n(Z_f)$ with projection $\xi_i: p_n(\bar{\xi}_i) = \xi_i$.

According to Schreier's theory of group extensions, what are to be proved are then the following relations:

$$(7.3) \qquad \bar{\xi}_1 \bar{\xi}_2 \overline{\xi_1 \xi_2}^{-1} = i_n(C(\xi_1, \xi_2)) \qquad \text{for } n \geq 1,$$

$$(7.4) \qquad \bar{\xi}_i i_1(\eta) \bar{\xi}_i^{-1} = i_1(D(\xi_i)\eta) \qquad \text{for } n = 1.$$

Lemma 7. 5. *Let $X^*(u^{n+1}) \in \Pi_{n+1}(X)$ and $\Sigma^*(u^{n+1}) \in \Pi_{n+1}(\Theta)$ satisfy $|\Sigma^*(u^{n+1})| = f X^*(u^{n+1})$, $u^{n+1} \in E^{n+1}$. Let $z(u^n) = (\Omega(u^n), \rho_n X^*(u^n)) \in \Pi_n^0(Z_f)$. Then*

$$z(u^n) \equiv ((\Sigma^*(u^n, 0))^{-1} \Omega(u^n), x_0) \text{ mod } \Pi_n^*(Z_f).$$

Proof: $\bar{z}(u^{n+1}) = (\Sigma^*(u^{n+1})(\Sigma^*(u^n, 0))^{-1}\Omega(u^n), X^*(u^{n+1}))$ satisfies $\bar{z}(u^n, 0) = z(u^n)$, $\bar{z}(u^n, 1) = ((\Sigma^*(u^n, 0))^{-1}\Omega(u^n), x_0)$, $\bar{z}(u^{n+1}) = (y_0, x_0)$ for $\Pi_{i=0}^{n-1} u_i(1-u_i) = 0$. Comparing with the proposition 3.1, we obtain the proof of the lemma.

Now let us prove (7.4). $\bar{\xi}_i i_1(\eta) \bar{\xi}_i^{-1}$ may be represented by $(\Omega(u_0), X(u_0))$ defined by

$$X(u_0) = x_i(1-5u_0) \qquad \Omega(u_0) = \tau_i(1-5u_0) y_0 \qquad (0 \leq u_0 \leq 1/5),$$
$$= x_0 \qquad = \check{y}_i(5u_0-1) \qquad (1/5 \leq u_0 \leq 2/5),$$
$$= x_0 \qquad = y(5u_0-2) \qquad (2/5 \leq u_0 \leq 3/5),$$
$$= x_0 \qquad = \check{y}_i(4-5u_0) \qquad (3/5 \leq u_0 \leq 4/5),$$
$$= x_i(5u_0-4) \qquad = \tau_i(5u_0-4) y_0 \qquad (4/5 \leq u_0 \leq 1).$$

Define $\Sigma(u^n)$ by

$$\Sigma(u^n) = \tau_t(1-5u_0),\ \sigma_t,\ \sigma_t,\ \sigma_t,\ \tau_t(5u_0-4).$$

It is easily seen that, for suitably chosen X^*, Σ^* with $\rho_1\Sigma^* = \Sigma$, $\rho_1 X^* = X$, and for the above Ω, the assumption of the lemma 7.5 is satisfied.

Thus we have

$$(\Omega(u_0),\ X(u_0)) \equiv ((\Sigma(u_0))^{-1}\Omega(u_0),\ x_0) = (Y(u_0),\ x_0) \mod \Pi_n^*(Z),$$

where $Y(u_0)$ is defined in § 6, which proves (7.4).

It remains to prove (7.3). $\xi_1\xi_2\overline{\xi_1\xi_2}^{-1}$ may be represented by $\Omega'(u^n)$, $X'(u^n))$ defined by

$$\begin{aligned}
X'(u^n) &= x_1(u_1^n,\ 1-6u_0) & \Omega'(u^n) &= \tau_1(u_1^n,\ 1-6u_0)\,y_0 & &(0\leq u_0\leq 1/6),\\
&= x_0 & &= y_1(u_1^n,\ 6u_0-1) & &(1/6\leq u_0\leq 2/6),\\
&= x_2(u_1^n,\ 3-6u_0) & &= \tau_2(u_1^n,\ 3-6u_0)\,y_0 & &(2/6\leq u_0\leq 3/6),\\
&= x_0 & &= y_2(u_1^n,\ 6u_0-3) & &(3/6\leq u_0\leq 4/6),\\
&= x_0 & &= y_2(u_1^n,\ 5-6u_0) & &(4/6\leq u_0\leq 5/6),\\
&= x_3(u_1^n,\ 6u_0-5) & &= \tau_3(u_1^n,\ 6u_0-5)\,y_0 & &(5/6\leq u_0\leq 1).
\end{aligned}$$

Define $\Sigma'(u^n)$ by

$$\Sigma'(u_0) = \tau_1(u_1^n,\ 1-6u_0),\ \sigma_1(u_1^n),\ \tau_2(u_1^n,\ 3-6u_0)\,\sigma_1(u_1^n),\ \sigma_2(u_1^n)\,\sigma_1(u_1^n),$$
$$\lambda(u_1^n,\ 6u_0-4),\ \tau_2(u_1^n,\ 6u_0-5).$$

Then $[\Sigma'(u^n)]^{-1}\Omega'(u^n) = c(u^n)$ of § 6, that is to say, $([\Sigma'(u^n)]^{-1}\Omega'(u^n),\ x_0)$ belongs to the class $i_n C(\xi_1,\ \xi_2)$.

To see that $(\Omega'(u^n),\ X'(u^n)) \equiv ([\Sigma'(u^n)]^{-1}\Omega'(u^n),\ x_0)$ and hence $\in i_n C(\xi_1,\ \xi_2)$, we have only to remark that, under the assumption of the theorem, the change of $\lambda(u^n)$ does not affect the value mod $\Pi_n^*(Y)$ of $[\Sigma'(u^n)]^{-1}\Omega'(u^n)$, and that, choosing $\lambda(u^n)$ suitably, we may find X^*, Σ^*, with $\rho X^* = X'$, $\rho\Sigma^* = \Sigma'$, so that the lemma 7.5 is applicable.

8. Remarks on the general cases. Theorem 7.1 contains an undesirable restriction on Y, but, as is seen from the proof of the theorem, it is of use only in the appropriate determination of $\lambda(u^n)$. Consequently, we may do without this restriction as soon as we find the way of determining $\lambda(u^n)$. Indeed we might give a theorem for the most general cases, but it is far from taking such an elegant from as theorem 7.1, and it is preferable to follow the arguments of the above proof in each case, than to give such a theorem.

Appendix

A) A generalization of Hurewicz-Steenrod isomorphism,

Let X_0 be a subset of X containing x_0, and let f be a function of

X into $\mathfrak{Y}$. Put $f_0 = f \mid X_0$. Then $\pi_n(Z_f Z_{f_0}) \approx \pi_n(X, X_0)$, $n = 1, 2, \ldots$

Proof: Let $z(u^n) = (\omega(u^n), x(u^n)) \in \Pi_n(Z_f)$ with $\rho_{n-1} z(u^{n-1}) \in \Pi_{n-1}^0(z_{f0})$. Then $x(u^n) \in \Pi_n(X)$ and $\rho_{n-1} x(u^{n-1}) \in \Pi_{n-1}^0(X_0)$. Obviously the correspondence $z(u^n) \to x(u^n)$ induces a homomorphism $\gamma: \pi_n(Z_f, Z_{f_0})$. We shall show that γ is an isomorphism on. For any $x(u^n) \in \Pi_n(X)$ with $\rho_{n-1} x(u^{n-1}) \in \Pi_{n-1}^0(X_0)$, choose a function $\sigma(u^n) \in \Pi_n(\Theta)$ with $|\sigma(u^n)| = f x(u^n)$, and put $\omega(u^n) = \sigma(u^n) y_0$. Then $z(u^n) = (\omega(u^n), x(u^n))$ determines an element of $\pi_n(Z_f, Z_{f_0})$ which is mapped under γ into the element of $\pi_n(X, X_0)$ determined by $x(u^n)$. Thus γ is on.

To see that γ is an isomorphism, let $z(u^n) = (\omega(u^n), x(u^n))$, $\{z(u^n)\} = 1$. Then there exists a function $x(u^{n+1}) \in X^{E^{n+1}}$, with $\bar{x}(u^n, 0) = x(u^n)$, $\bar{x}(u^{n+1}) = x_0$ in $M = \{u^{n+1} \mid (1-u_{n-1})(1-u_n)\Pi_{i=2}^{n-2} u_i(1-u_i) = 0\}$, and $\bar{x}(u^{n+1}) \in X_0$ for $u_{n-1} = 0$. Choose $\sigma(u^{n+1}) \in \Theta^{E^{n-1}}$ with $\sigma(u^{n+1}) = 1$ in M, $|\sigma(u^{n+1})| = fx(u^{n+1})$, and put $\bar{\omega}(u^{n+1}) = \sigma(u^{n+1})[\sigma(u^n, 0)]^{-1}\omega(u^n)$, $\bar{z}(u^{n+1}) = (\bar{\omega}(u^{n+1}), \bar{z}(u^{n+1}))$. Then $\bar{z}(u^{n+1}) \in Z_f$ with $\bar{z}(u^n, 0) = z(u^n)$, $\bar{z}(u^{n+1}) = (y_0, x_0)$ for $u^n \in T^{n-1}$, $\bar{z}(u^{n+1}) \in Z_f$ for $u_{n-1} = 0$, $\bar{z}(u^{n+1}) \in Y$ for $u_n = 1$. From this we may easily conclude that γ is an isomorphism.

B) Fibre bundles with a slicing map in the large.

Let $Y' \subset Y$ and let Θ', $\mathfrak{Y}'$ be defined for Y' in the same way as Θ, $\mathfrak{Y}$ were defined for Y. Lei f' be a function of X into $\mathfrak{Y}'$ such that $f'(x) \subset f(x)$ for every $x \in X$. Let $Z_{f'}$ be the fibre budle correspodig to f'. It is obvious that $Z_{f'} \subset Z_f$. Then we have

(B, 1) $\quad \pi_n(Z_f, Z_{f'}) \approx \pi_n(Y, Y')$, $n = 1, 2, \ldots\ldots$.

Proof: Let $z(u^n) = (\omega(u^n), x(u^n)) \in \Pi_n(Z_f)$ with $\rho_{n-1} z(u^{n-1}) \in \Pi_{n-1}^0(Z_{f'})$. We choose $\tau(u^n) \in \Pi_n(\Theta')$ with $|\tau(u^n)| = f'x(u^n)$, and put

$$\bar{\omega}(u^n) = \omega(u^{n-1}, 2u_{n-1}-1),\ \bar{x}(u^n) = x(u^{n-1}, 2u_{n-1}-1),\ (1/2 \leq u_{n-1} \leq 1),$$
$$= \tau(u^{n-1}, 1-2u_{n-1})[\tau(u^{n-1}, 0)]^{-1}\omega(u^{n-1}, 0),$$
$$x(u^n) = x(u^{n-1}, 1-2u_{n-1}) \quad (0 \leq u_{n-1} \leq 1/2).$$

$\bar{z}(u^n)$ obviously determines the same element as $z(u^n)$ of $\pi_n(Z_f, Z_{f'})$; moreover $\bar{x}(u^n) \in \Pi_n^*(X)$, i.e. $\bar{x}(u^n) = x'(u^n, 0)$ for some $x'(u^{n+1}) \in \Pi_{n+1}(X)$. Let $\sigma(u^{n+1}) \in \Pi_{n+1}(\Theta)$ with $|\sigma(u^{n+1})| = f x'(u^{n+1})$, and define $z'(u^{n+1})$ by $z'(u^{n+1}) = (\sigma(u^{n+1})[\sigma(u^n, 0)]^{-1}\bar{\omega}(u^n), x'(u^{n+1}))$. Then $z'(u^n, 0) = (\bar{\omega}(u^n), \bar{x}(u^n)) = z(u^n)$, $z'(u^n, 1) = ([\sigma(u^n, 0)]^{-1}\bar{\omega}(u^n), x_0)$, $z'(u^n, u_n) = (y_0, x_0)$ for $u^n \in T^{n-1}$, $z'(u^{n-1}, 0, u_n) \in Z_{f'}$. From this we may conclude that any element $\zeta \in \pi_n(Z_f, Z_{f'})$ contains a representative of the form $(y(u^n), x_0)$ with $y(u^{n-1}, 0) \in Y'$. Let $\delta\zeta$ be the element of $\pi_n(Y,$

Y') represented by $y(u^n)$. The correspondence $\zeta \rightarrow \delta\zeta$ is easily seen to be an isomorphism of $\pi_n(Z_f, Z_f')$ onto $\pi_n(Y, Y')$. The arguments in the proof are similar to those of the above.

Combining (B. 1) with the homotopy sequence of the pair (Z_f, Z_f') we obtain *a new exact sequence*:

(B. 2) $\ldots \rightarrow \pi_{n+1}(Y, Y') \rightarrow \pi_n(Z_f') \rightarrow \pi_n(Z_f) \rightarrow \pi_n(Y, Y') \rightarrow \ldots$.

In particular, if Y' reduces to a point, it becomes

(B. 3) $\rightarrow \pi_{n+1}(Y) \rightarrow \pi_n(X) \rightarrow \pi_n(Z_f) \rightarrow \pi_n(Y) \rightarrow \ldots$.

(B. 3) is in some sense inverse to the sequence (1.1), and they together assure that, if Z_f has a slicing map in the large $\pi_n(Z_f)$ is a spalt extension of μ_n by ν_n.

LITERATURES

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2) Kelley, J. L. and Pitcher, E.: Exact homomorphism sequences in homology theory, Ann. of Math., vol. 48, 1947.

3) Whitehead, G. W. : Homotopy properties of the real orthogonal groups, Ann. of Math., v. 43, 1942.

4) Fox, R. H.: Homotopy groups and torus homotopy groups, Ann. of Math., v. 49, 1948, 471-510.

(Received Nov. 1, 1949)

(*) Prof. A. Komatu took up this problem in 1945, when the concept of exact homotopy sequence was not yet familiar to us, and found the formula: $\pi_n(Z)/\mu_n \approx \nu_n$. This paper continues and developes his unpublished researches by the aid of the results in [1]. I offer here my sincere thanks to Prof. A. Komatu who gave me many valuable suggestions during the preparation of this paper.

(1) Hurewicz, W. and Steenrod, N. E.: Homotopy Relations in Fibre Spaces, Proc. Nat. Acad. Sci. U. S. A., 27 (1941). They proved this isomorphism when Z is a *fibre space*, but, since it may be proved by making use of the covering homotopy theorem only, it holds as well in our case. See also, the appendix of the present paper.

(2) The concept of exact sequence is now familiar to us. It was Kelley and Pitcher who treated this sequence systematically at the first time [2]. The proof of the exactness of the homotopy sequence was given by G. W. Whitehead [3].

(3) cf. [1].

(4) Following Fox [4] we denote by u_h^k the set of real variable $u_h, u_{h+1}, \ldots, u_{k-1}$. $u^k = u_0^k$, $(0 \leqq u_i \leqq 1)$. E_h^k=[the $(k$-$h)$-dimensional element generated by u_h^k].

(5) Although $\Pi_n^0(A)$, $\Pi_n^*(A)$ do not form groups, the meaning of the left hand side is clear.