Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 1, No. 1, Series A

A Characterization of the Uniform Topology of a Uniform Space by the Lattice of its Uniformity.

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We shall denote in this paper by R a uniform space, and by $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ its uniformity.[1] We denote by $\mathfrak{M}_x < \mathfrak{M}_y$ the fact that $\mathfrak{M}_x$ is a refinement of $\mathfrak{M}_y$ and by $\mathfrak{M}_x \triangle \mathfrak{M}_y$ the fact that $\mathfrak{M}_x^\triangle < \mathfrak{M}_y$. $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ is a lattice by the order $<$, and has also the relation $\triangle$.

We shall show in this paper that in general a lattice-isomorphism between uniformities of two uniform spaces preserving the relations $\triangle$ and $<$ implies a uniform homeomorphism between the uniform spaces, and especially that when R has no isolated point, the structure of the lattice $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ or of $\mathfrak{X}$ defines R up to a uniform homeomorphism.

An element of $\{\mathfrak{M}_x \mid \mathfrak{X}\}$, which is an open covering of R, is called simply a *u-covering* in this paper. German capitals are used for u-coverings but in 6 of the proof of Lemma 3.

Definition. Let $\mathfrak{M}$, $\mathfrak{N}$ be two u-coverings. We denote by $\mathfrak{M} \not\!\not< \mathfrak{N}$ the fact that for every $M \in \mathfrak{M}$ there exists some $M' \in \mathfrak{M}$ such that $M \subset M'$ and $M' \not\subset N$ for all $N \in \mathfrak{N}$.

We denote by $\overline{\not\!\not<}$ the negation of $\not\!\not<$.

Lemma 1. *In order that $\mathfrak{M} \not\!\not< \mathfrak{N}$ holds, it is necessary and sufficient that*

(1) *$\mathfrak{M}^\triangle$ contains no set consisting of one point,*

(2) *whenever $\mathfrak{M} \not< \mathfrak{P}$, $\mathfrak{M} \not< \mathfrak{P} \smile \mathfrak{N}$ holds.*[2]

Proof. Necessity: The condition (1) is obvious from the definition of $\not\!\not<$.

From $\mathfrak{M} \not< \mathfrak{P}$ we get $M \in \mathfrak{M}$ such that $M \not\subset P$ for all $P \in \mathfrak{P}$. Since $\mathfrak{M} \not\!\not< \mathfrak{N}$, there exists $M' \in \mathfrak{M}$ such that $M' \supset M$, $M' \not\subset N$ for all $N \in \mathfrak{N}$.

1) Cf. J. W. Tukey, Convergence and uniformity in topology, (1940).
2) $\not<$ denotes the negation of $<$.

Therefore $M' \not\subset Q$ for all $Q \in \mathfrak{P} \smile \mathfrak{N}$. Thus we get $\mathfrak{M} \not< \mathfrak{P} \smile \mathfrak{N}$, i. e. the condition (2) is necessary.

Sufficiency: Let $\mathfrak{M} \not\leqq \mathfrak{N}$ and $\mathfrak{M}^{\triangle}$ contains no set consisting of one point, then there exists $M \in \mathfrak{M}$ such that for all $M' \in \mathfrak{M}$: $M' \supset M$ and for some $N \in \mathfrak{N}$, $N \supset M'$ holds, and M containsat least two points. Hence there exists an open set U such that $U \cdot M \neq \phi$, $U \not\supset M$. Taking a point $a \in U \cdot M$, we construct a covering $\mathfrak{P}$ from U and from $\mathfrak{M}$ as follows. $\mathfrak{P}$ consists of

1) $\{M'_\alpha - \{a\} \mid \alpha \in A\}$, where $\{M'_\alpha \mid A\}$ denotes the set of all element M'_α of $\mathfrak{M}$ satisfying $M'_\alpha \supset M$,
2) the set $\{M''_\beta \mid B\}$ of all elements M''_β of $\mathfrak{M}$ such that $M''_\beta \not\supset M$,
3) U.

Then it is easy to see that $M \not\subset P$ for all $P \in \mathfrak{P}$, i. e. $\mathfrak{M} \not< \mathfrak{P}$, and that at the same time $\mathfrak{M} < \mathfrak{P} \smile \mathfrak{N}$ holds.

Thus the sufficiency is proved.

Lemma 2. *In order that $\mathfrak{M}^{\triangle}$ contains a set consisting one point, it is necessary and sufficient that there exists a covering $\mathfrak{N} > \mathfrak{M}^{\triangle}$ such that*

1) $\mathfrak{N} = \mathfrak{N}^{\triangle} \neq \mathfrak{R}$,
2) $\mathfrak{N} \lneqq \mathfrak{P}$ *implies* $\mathfrak{P}^{\triangle} = \mathfrak{R}$,

where we denote by $\mathfrak{R}$ the largest covering $\{R\}$.
(If we use the relation $\triangle$, then the relation $\mathfrak{N}^{\triangle} = \mathfrak{R}$ can be replaced by the proposition: $\mathfrak{N} \triangle \mathfrak{M}$ implies $\mathfrak{M} = \mathfrak{R}$.)

Proof. If $\mathfrak{M}^{\triangle}$ contains a set $\{a\}$ consisting of one point a, then the u-covering $\mathfrak{N} = \{\{a\}, R - \{a\}\}$ has the property of $\mathfrak{N}$ in the lemma.

Conversely, let $\mathfrak{N}$ be such a u-covering.

From $\mathfrak{N}^{\triangle} = \mathfrak{N}$ we see that $S(a, \mathfrak{N}) \cdot S(b, \mathfrak{N}) \neq \phi$ implies $S(a, \mathfrak{N}) = S(b, \mathfrak{N})$.

For let $a, c \in N \in \mathfrak{N}$, then there exists $N' \in \mathfrak{N}$ such that $S(a, \mathfrak{N}) \subset N'$, and hence $S(a, \mathfrak{N}) \subset N' \subset S(c, \mathfrak{N})$. In the same way $S(c, \mathfrak{N}) \subset S(a, \mathfrak{N})$ holds, whence $S(a, \mathfrak{N}) = S(c, \mathfrak{N})$. Therefore if $c \in S(a, \mathfrak{N}) \cdot S(b, \mathfrak{N}) \neq \phi$, we get $S(a, \mathfrak{N}) = S(c, \mathfrak{N}) = S(b, \mathfrak{N})$.

If more than two of elements $S(a, \mathfrak{N})$ of $\mathfrak{N}^{\triangle}$ are different from the others, i. e. $\mathfrak{N}^{\triangle} = \{S_1, S_2, S_3,\} \smile \{T_\alpha\}$ $(S_i \neq S_j \ (i \neq j))$, then the u-covering $\mathfrak{P} = \{S_1 + S_2, S_3\} \smile \{T_\alpha\}$ has the property: $\mathfrak{P} \gneqq \mathfrak{N}$. $\mathfrak{P}^{\triangle} \neq \mathfrak{R}$.

which contradicts the condition 2). Since $\mathfrak{N}^{\triangle} \neq \mathfrak{N}$, $\mathfrak{N}^{\triangle}$ contains just two different elements. If S_1 and S_2 contains at least two points, then there exists an open set U such that

$$U \not\supset S_1,\ U \not\supset S_2\ ;\ U \cdot S_1 \neq \phi,\ U \cdot S_2 \neq \phi.$$

Hence, putting $\mathfrak{P} = \{S_1,\ S_2,\ U\}$, we get $\mathfrak{P} \gneqq \mathfrak{N}$, $\mathfrak{P}^{\triangle} \neq \mathfrak{N}$, which contradicts the condition 2). Hence S_1 or S_2 consists of one point a, i. e. $\mathfrak{N} = \{\{a\},\ R - \{a\}\}$. Since $\mathfrak{M}^{\triangle} < \mathfrak{N}$, it must be $\{a\} \in \mathfrak{M}^{\triangle}$.

Thus the proof of Lemma 2 is complete.

We notice that Lemma 1 and Lemma 2 show that the relation $\lll$ can be replaced by the relations $<$ and $\triangle$, and that if R has no isolated point, $\triangle$ is needless.

Lemma 3. *Let R and S be two uniform spaces with the uniformities $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ and $\{\mathfrak{N}_y \mid \mathfrak{Y}\}$ respectively.*

In order that R and S are uniformly homeomorphic. it is necessary and sufficient that $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ and $\{\mathfrak{N}_y \mid \mathfrak{Y}\}$ are lattice-isomorphic by a correspondence preserving the relations $\lll$ and $<$.

Proof. We concern ourselves only with R at first.

Definition. We denote by $\mathfrak{M}_0$ the u-covering such that

1) $\mathfrak{M}_0 \neq \mathfrak{N}$,

2) $\mathfrak{N} \neq \mathfrak{N}$ implies $\mathfrak{N} < \mathfrak{M}_0$.

It is obvious that $\mathfrak{M}_0 = \{R - \{a\} \mid a \in R\}$.

Definition. We mean by *base-element* a collection μ of u-coverings which satisfies the following four conditions.

i) $\mathfrak{M} \in \mu$, $\mathfrak{M} < \mathfrak{N}$ implies $\mathfrak{N} \in \mu$,

ii) for every u-coverings $\mathfrak{M}_x$ there exists $\mathfrak{N} \in \mu$ such that $\mathfrak{N} \lll \mathfrak{M}_x$,

iii) let $\{\mathfrak{N}_\alpha \mid A\}$ be a set of u-coverings $\mathfrak{N}_\alpha$, and each $\mathfrak{N}_\alpha \lll \mathfrak{M}_\alpha$ for some $\mathfrak{M}_\alpha \in \mu$, then $\underset{\alpha \in A}{\smile} \mathfrak{N}_\alpha \neq \mathfrak{M}_0$,

iv) μ is a minimum set satisfying the above conditions 1), 2), 3).

Definition. Let U be an open set of R. We denote by $\mathfrak{P}(U)$ the u-covering $\{U,\ R - \{a\} \mid a \in U\}$ of R.

1. We consider an arbitrary base-element μ.

Let $\mathfrak{M}_x$ be an arbitrary u-covering, then by the condition ii) of μ

there exists $\mathfrak{N}_x \in \mu$ such that $\mathfrak{N}_x \not\ll \mathfrak{M}_x$. Hence there exists $N_x \in \mathfrak{N}_x$ such that for all $N'_x \in \mathfrak{N}_x$: $N'_x \supset N_x$ there exists $M \in \mathfrak{M}_x$: $M \supset N'_x$.

For a definite point $a_x \in N_x$ we construct the u-covering $\mathfrak{P}(S(a_x, \mathfrak{M}_x))$ and denote it by $\mathfrak{P}_x$ for simplicity. Then $\mathfrak{N}_x < \mathfrak{P}_x$ holds. For if $N \supset N_x$, $N \in \mathfrak{N}_x$, then $N \subset S(a_x, \mathfrak{M}_x)$, and if $N \not\supset N_x$, $N \in \mathfrak{N}_x$, then for a point $b \in N_x - N \subset S(a_x, \mathfrak{M}_x)$, $N \subset R - \{b\}$, which shows $\mathfrak{N}_x < \mathfrak{P}_x$. Since $\mathfrak{N}_x \in \mu$, by the condition i) of μ we get $\mathfrak{P}_x \in \mu$.

2. Next we shall show that $\prod_{x > x_0} S(a_x, \mathfrak{M}_x) \neq \phi$ for some $x_0 \in \mathfrak{X}$.

Assume that the contrary holds, i. e. $\prod_{x > x_0} S(a_x, \mathfrak{M}_x) = \phi$ for all $x_0 \in \mathfrak{X}$.

When we take three points c_1, c_2, c_3 of R and take x_0 such that for each $b \in R$, $S(b, \mathfrak{M}_{x_0})$ contains at most one point of c_1, c_2, c_3, then for every $x > x_0$ there exist at least two points in R which are not contained in $S(a_x, \mathfrak{M}_x)$.

Let b be an arbitrary point of R, then from the assumption there exists $x \in \mathfrak{X}$ such that $b \notin S(a_x, \mathfrak{M}_x)$, $x > x_0$, and hence there exists a point c of R such that $c \neq b$, $c \notin S(a_x, \mathfrak{M}_x)$.

Putting $\mathfrak{Q}_b = \{R - \{b\}, R - \{c\}\}$, we see easily that $\mathfrak{Q}_b \not\ll \mathfrak{P}_x$, and $\bigvee_{b \in R} \mathfrak{Q}_b = \mathfrak{M}_0$, which contradicts the condition iii) of μ.

This contradiction shows the validity of $\prod_{x > x_0} S(a_x, \mathfrak{M}_x) \neq \phi$ for some $x_0 \in \mathfrak{X}$.

3. We notice that in general $U \subset V$ implies $\mathfrak{P}(U) < \mathfrak{P}(V)$.

Let $b \in \prod_{x > x_0} S(a_x, \mathfrak{M}_x)$, then $S(a_x, \mathfrak{M}_x)$ $(x > x_0)$ is a nbd-basis (nbd=neighbourhood) of b. Combining the last conclusion in 1, the above remark and the condition i) of μ we get $\mathfrak{P}(U(b)) \in \mu$ for all nbds $U(b)$ of b.

Putting $\mu(b) = \{\mathfrak{P} \mid \exists \mathfrak{P}(U(b))$ such that $\mathfrak{P}(U(b)) < \mathfrak{P}$, $U(b)$ is some nbd of $b\}$, we get $\mu(b) \subseteq \mu$. $\mu(b)$ satisfies obviously the conditions i), ii) of μ.

We shall show that $\mu(b)$ satisfies iii) too.

Assume that the assertion is false, i. e. $\bigvee_{\alpha \in A} \mathfrak{N}_\alpha = \mathfrak{M}_0$, $\mathfrak{N}_\alpha \not\ll \mathfrak{P}_\alpha$, $\mathfrak{P}_\alpha > \mathfrak{P}(U_\alpha(b))$ for some $\{\mathfrak{N}_\alpha \mid A\}$, then $\mathfrak{N}_\alpha \not\ll \mathfrak{P}(U_\alpha(b))$ for each α.

Since $\underset{\alpha}{\smile} \mathfrak{N}_\alpha = \mathfrak{M}_0$, there must be $\mathfrak{N}_\alpha$ such that $R-\{b\} \in \mathfrak{N}_\alpha$.
Since $R-\{b\} \in \mathfrak{P}(U_\alpha(b))$ for every $U_\alpha(b)$, remarking that $R \notin \mathfrak{N}_\alpha$, we get $\mathfrak{N}_\alpha \overline{\nless} \mathfrak{P}(U_\alpha(b))$ for every $U_\alpha(b)$, which contradicts the assumption. Hence $\underset{\alpha}{\smile} \mathfrak{N}_\alpha$ cannot be $\mathfrak{M}_0$, and $\mu(b)$ satisfies the condition iii).

From the condition iv) of μ and the above obtained relation $\mu(b) \subseteqq \mu$ we get $\mu = \mu(b)$.

4. As we saw above, an arbitrary base-element has the form $\mu(b)$ and an arbitrary $\mu(b)$ satisfies the conditions i), ii), iii) of μ. Now we shall show that $\mu(b)$ satisfies the condition iv) of μ.

Let $\mu(b) \supseteqq \mu$ and let μ satisfy the conditions i), ii), iii) of μ, then by the same consideration as above there exists some $\mu(a)$ such that $\mu(b) \supseteqq \mu \supseteqq \mu(a)$.

If $a \neq b$, then there exist a point c of R and a nbd $U(a)$ of a such that $b \neq c$; $b, c \notin U(a)$. Since for each nbd $V(b)$ of b, $R-\{b\} \in \mathfrak{P}(V(b))$ and $R-\{b\} \not\subset P$ for all $P \in \mathfrak{P}(U(a))$, we get $\mathfrak{P}(V(b)) \nless \mathfrak{P}(U(a))$ for every nbds $V(b)$.

Hence $\mathfrak{P}(U(a)) \notin \mu(b)$, but $\mathfrak{P}(U(a)) \in \mu(a)$, which is a contradiction. Therefore it must be $a = b$, and accordingly $\mu(b) = \mu(a) = \mu$; hence we conclude that $\mu(b)$ satisfies the condition iv) and is a base-element.

Thus we have obtained a one-to-one correspondence between R and the set $B(R)$ of all base-elements of R. We shall denote this correspondence by B.

5. We introduce a topology in $B(R)$ as follows.

Let $B(A)$ be a subset of $B(R)$.

We say that $\nu \in \overline{B(A)}$, when and only when

(1) $\nu \in B(A)$,

or (2) for every $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$ there exist $\mathfrak{N}$, $\mathfrak{M}$ and μ such that $\mathfrak{N} \in \nu$, $\mathfrak{M} \in \mu \in B(A)$; $\mathfrak{M}_x \nless \mathfrak{M} \smile \mathfrak{N}$.

Then the topological space $B(R)$ with this closure-operation is homeomorphic with R.

To see this we shall show that $a \in \bar{A}$ and $\mu(a) \in \overline{B(A)}$ are equivalent.

If $a \notin \bar{A}$, then $\mu(a) \notin \overline{B(A)}$ is obvious.
When we consider the u-covering $\mathfrak{M}_x = \{R-\{a\}, (\bar{A})^c\}$,[3] for every $\mathfrak{N} \in \mu(a)$ and $\mathfrak{M} \in \mu(b) \in B(A)$ we get $\mathfrak{M}_x < \mathfrak{M} \smile \mathfrak{N}$, because $R-\{a\} \in \mathfrak{N}$

and $(\bar{A})^c \subset R-\{b\} \in \mathfrak{M}$ from $b \in A$.

Conversely let $a \in \bar{A}$. If $a \in A$, then $\mu(a) \in B(A)$ is obvious. If $a \notin A$, then for every $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$ there exists $M \in \mathfrak{M}_x$ such that $a, b \in M$; $b \in A$, $a \neq b$. Hence we may construct nbds $U(a)$ of a and $V(b)$ of b such that $b \notin U(a) \subset M$, $a \notin V(b) \subset M$.

Obviously $\mathfrak{P}(U(a)) \in \mu(a)$, $\mathfrak{P}(V(b)) \in \mu(b) \in B(A)$, and on the other hand $\mathfrak{M}_x \not< \mathfrak{P}(U(a)) \smile \mathfrak{P}(V(b))$ holds, because $M \not\subset P$ for all $P \in \mathfrak{P}(U(a)) \smile \mathfrak{P}(V(b))$.

Therefore $\mu(a) \in \overline{B(A)}$ according to the definition.

Thus B is a homeomorphism between R and $B(R)$.

6. Next we introduce a uniform topology in $B(R)$ as follows.

Let $B(\mathfrak{U}) = \{B(U_\alpha) \mid A\}$ be an open covering of $B(R)$.

We say that $B(\mathfrak{U})$ is a u-covering of $B(R)$, when and only when there exists some $\mathfrak{M}_x$ such that

$\mathfrak{M}_x < \underset{\alpha \in A}{\smile} \mathfrak{N}_\alpha$, whenever $\mathfrak{N}_\alpha$ and μ_α are selected so that $\mathfrak{N}_\alpha \in \mu_\alpha \in B(U_\alpha)^c$.

We shall show that R and $B(R)$ are uniformly homeomorphic.

When $\mathfrak{U}$ is a u-covering of R, $\mathfrak{U}$ itself satisfies the condition of $\mathfrak{M}_x$ in the above definition.

For if $\mathfrak{N}_\alpha \in \mu(a_\alpha) \in B(U_\alpha)^c$, then $a_\alpha \in U_\alpha^c$, and accordingly $U_\alpha \subset R-\{a_\alpha\} \in \mathfrak{N}_\alpha$ holds, whence $\mathfrak{U} = \{U_\alpha\} < \underset{\alpha \in A}{\smile} \mathfrak{N}_\alpha$ Hence $B(\mathfrak{U})$ is a u-covering of $B(R)$ according to the definition.

Conversely let $\mathfrak{U}$ be no u-covering of R. Then for each u-covering $\mathfrak{M}_x$ of R there exists an element M of $\mathfrak{M}_x$ such that $M \not\subset U_\alpha$ for all $U_\alpha \in \mathfrak{U}$, i.e. $M \cdot U_\alpha^c \neq \phi$ for all $U_\alpha \in \mathfrak{U}$.

Since $\mathfrak{U}$ is a covering, M contains at most two points. Hence taking $a_\alpha \in M \cdot U_\alpha^c$ for each $\alpha \in A$, we can construct a nbd $U(a_\alpha)$ of a_α such that $U(a_\alpha) \subsetneqq M$.

Obviously $\mathfrak{P}(U(a_\alpha)) \in \mu(a_\alpha) \in B(U_\alpha)^c$ holds, and on the other hand $\mathfrak{M}_x \not< \underset{\alpha \in A}{\smile} \mathfrak{P}(U(a_\alpha))$ holds, because $M \not\subset P$ for all $P \in \underset{\alpha \in A}{\smile} \mathfrak{P}(U(a_\alpha))$. Hence $B(\mathfrak{U})$ is not a u-covering of $B(R)$. Thus we have shown that B is a uniform homeomorphism between R and $B(R)$.

7. Now it is easy to prove Lemma 3.

Since we can construct the uniform space $B(R)$ from $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ by

3) M^c means the complement of M.

using only the relations $<$ and $\not\!\not<$, a lattice isomorphism between two uniformities $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ and $\{\mathfrak{N}_y \mid \mathfrak{Y}\}$ preserving the relation $\not\!\not<$ generates a uniform homeomorphism between $B(R)$ and $B(S)$, and this in turn generates a uniform homeomorphism between R and S.

Since the necessity of the condition is obvious, the proof of Lemma 3 is complete.

Combinining Lemma 1, Lemma 2 and Lemma 3, we get the following principal result.

Theorem. *Let R and S be two uniform spaces with uniformities $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ and $\{\mathfrak{N}_y \mid \mathfrak{Y}\}$ respectively.*

In order that R and S are uniformly homeomorphic, it is necessary and sufficient that $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ and $\{\mathfrak{N}_y \mid \mathfrak{Y}\}$ are lattice-isomorphic by a correspondence preserving the relations $\triangle$ and $<$.

Especially if R and S have no isolated point, then an ordinary lattice-isomorphism between $\{\mathfrak{M}_x \mid \mathfrak{X}\}$ and $\{\mathfrak{N}_y \mid \mathfrak{Y}\}$ generates a uniform homeomorphism between R and S.

(Received Nov. 1, 1949)