

On the Uniform Topology of Bicompatifications.

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In this note we shall characterise the uniform space, which is a uniform subspace of its bicompatification $\beta(R)$ or $w(R)$ ¹⁾, by introducing the notion of u -normality. Then we shall study some properties of u -normal spaces. We denote by R a uniform space having uniformity $\{\mathfrak{M}_x \mid \mathfrak{X}\}$.

Let A and B be subsets of R and $A \cap B = \phi$. When there exists $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$ such that $S(A, \mathfrak{M}_x) \cap B = \phi$, we say that A and B are u -separated. It is easy to see that A and B are u -separated, when and only when there exists a uniformly continuous function φ such that

$$\begin{aligned} \varphi(a) &= 0 \quad (a \in A), \\ \varphi(a) &= 1 \quad (a \in B), \end{aligned} \qquad 0 \leq \varphi(a) \leq 1.$$

A uniform space R is called a \check{C} ech u -normal space, when any disjoint completely closed sets²⁾ of R are u -separated, and is called a u -normal space, when any disjoint closed sets of R are u -separated.

Lemma 1. *In order that R is \check{C} ech u -normal, it is necessary and sufficient that every bounded continuous functions of R are uniformly continuous.*

Proof. Since for any disjoint completely closed sets F and G , there exists a bounded continuous function φ such that

$$\begin{aligned} \varphi(a) &= 0 \quad (a \in F), \\ \varphi(a) &= 1 \quad (a \in G), \end{aligned} \qquad 0 \leq \varphi(a) \leq 1.$$

the sufficiency of the condition is obvious.

Conversely let R be \check{C} ech u -normal, then any finite open covering $\mathfrak{N} = \{N_i \mid i = 1, \dots, n\}$ is an element of $\{\mathfrak{M}_x\}$, if every N_i are completely closed.³⁾

For put $N_i^c = F_i$, then $\bigcap_{i=1}^n F_i = \phi$ or $F_1 \cap (F_2 \cap \dots \cap F_n) = \phi$, where F_1 and $F_2 \cap \dots \cap F_n$ are completely closed. Hence there exists

$\mathfrak{M}_i \in \{\mathfrak{M}_x\}$ such that

$$S(F_1, \mathfrak{M}_1) \cap S(F_2 \cap \dots \cap F_n, \mathfrak{M}_1) = \phi.$$

Hence there exists a completely closed set H_1 such that

$$S(F_1, \mathfrak{M}_1) \subset S^c(F_2 \cap \dots \cap F_n, \mathfrak{M}_1) \subset H_1 \subset (F_2 \cap \dots \cap F_n)^c.$$

This implies that $H_1 \cap F_2 \cap \dots \cap F_n = \phi$.

Assume that we get completely closed sets H_1, \dots, H_i and elements $\mathfrak{M}_1, \dots, \mathfrak{M}_i$ of $\{\mathfrak{M}_x\}$ such that

$$H_1 \cap \dots \cap H_i \cap F_{i+1} \cap \dots \cap F_n = \phi, H_i \supset S(F_i, \mathfrak{M}_i).$$

Since F_{i+1} and $H_1 \cap \dots \cap H_i \cap F_{i+2} \cap \dots \cap F_n$ are completely closed, there exists $\mathfrak{M}_{i+1} \in \{\mathfrak{M}_x\}$ such that

$$S(F_{i+1}, \mathfrak{M}_{i+1}) \cap S(H_1 \cap \dots \cap H_i \cap F_{i+2} \cap \dots \cap F_n, \mathfrak{M}_{i+1}) = \phi.$$

Hence there exists a completely closed set H_{i+1} such that

$$S(F_{i+1}, \mathfrak{M}_{i+1}) \subset S^c(H_1 \cap \dots \cap H_i \cap F_{i+2} \cap \dots \cap F_n, \mathfrak{M}_{i+1}) \subset H_{i+1} \\ \subset (H_1 \cap \dots \cap H_i \cap F_{i+2} \cap \dots \cap F_n)^c$$

This implies that $H_1 \cap \dots \cap H_i \cap H_{i+1} \cap F_{i+2} \cap \dots \cap F_n = \phi$.

Therefore by induction there exist H_1, \dots, H_n such that $H_1 \cap \dots \cap H_n = \phi, H_i \supset S(F_i, \mathfrak{M}_i)$.

Hence $\bigcap_{i=1}^n S(F_i, \mathfrak{M}_i) = \phi \dots \dots (S)$.

Put $\mathfrak{M}_1 \wedge \mathfrak{M}_2 \wedge \dots \wedge \mathfrak{M}_n = \mathfrak{M} \in \{\mathfrak{M}_x\}$.

Let M be an arbitrary element of \mathfrak{M} , then for every i there exists $M_i \in \mathfrak{M}_i$ such that $M_i \supset M$. Therefore it must be $M \cap F_i = \phi$ for

some i , because $M \cap F_i \neq \phi$ (for all i) contradicts the condition (S).

This implies $\mathfrak{M} \subset \mathfrak{N}$, i. e. $\mathfrak{N} \in \{\mathfrak{M}_x\}$.

Now let φ be an arbitrary bounded continuous function of R such that $0 < \varphi < 1$. We write

$$U_n = \{U_k \mid k = 0, 1, \dots, 2^n - 2\},$$

$$U_k = \{a \mid k/2^n < \varphi(a) < (k+2)/2^n\}.$$

Then, since U_n is a finite open covering of R and every U_k are completely closed, we get $U_n \in \{\mathfrak{M}_x\}$.

Since $a \in S(b, U_n)$ implies $|\varphi(a) - \varphi(b)| < 1/2^{n-1}$, φ is uniformly continuous.

Theorem 1. *In order that the uniform topology of $\beta(R)$ can be reduced to that of R , it is necessary and sufficient that R is Čech u -normal and totally bounded. (This shows a necessary and sufficient condition for that $\beta(R)$ and the completion R^* of R is identical.)*

Proof. Necessity: Since $\beta(R)$ is totally bounded, R is also totally

bounded.

Let F and G be complete closed sets of R and $F \cap G = \phi$, then there exists a continuous function φ such that

$$\begin{aligned}\varphi(a) &= 0 \quad (a \in F), \\ 0 &\leq \varphi(a) \leq 1. \\ \varphi(a) &= 1 \quad (a \in G),\end{aligned}$$

When we inflate φ to the continuous function φ^* of $\beta(R)$, φ^* is uniformly continuous from the bicomactness of $\beta(R)$. This implies the uniform continuity of φ .

Hence F and G are u -separated, and R is Čech u -normal.

Sufficiency: Let R be homeomorphic to the subspace $R_0 = f(R)$ of its Tychonoff's parallelotope $P\{I' \mid \Gamma\}$, (where Γ is the set of all continuous functions φ of R such that $0 \leq \varphi \leq 1$.)

We shall prove that R and R_0 with the weak uniform topology by means of bounded continuous functions are uniformly homeomorphic. When this fact is proved, the sufficiency of this condition is obvious.

We denote by $\mathfrak{N}(\varphi, \alpha)$ the uniform covering $\{N(\varphi(a), \alpha) \mid a \in R\}$ of R_0 , where $N(\varphi(a), \alpha) = \{f(x) \mid |\varphi(a) - \varphi(x)| < \alpha\}$, (f is the homeomorphism between R and R_0).

Now we consider an arbitrary uniform covering $\mathfrak{N} = \mathfrak{N}(\varphi_1, \alpha_1) \wedge \dots \wedge \mathfrak{N}(\varphi_k, \alpha_k)$ of R_0 . Since by Lemma 1, φ_i is uniformly continuous, there exists $\mathfrak{M}_i \in \{\mathfrak{M}_x\}$ such that $b \in S(a, \mathfrak{M}_i)$ implies $|\varphi_i(a) - \varphi_i(b)| < \alpha_i$. Putting $\mathfrak{M} = \mathfrak{M}_1 \wedge \dots \wedge \mathfrak{M}_k$, we get $\mathfrak{M} \in \{\mathfrak{M}_x\}$ and $f(\mathfrak{M}) < \mathfrak{N}$.

Conversely consider an arbitrary element \mathfrak{M} of $\{\mathfrak{M}_x\}$, and let $\mathfrak{M}_1^\wedge < \mathfrak{M}$, $\mathfrak{M}_2^* < \mathfrak{M}_1$, and $\mathfrak{M}_1, \mathfrak{M}_2 \in \{\mathfrak{M}_x\}$. Since R is totally bounded, there exists a finite subcovering $\{\mathfrak{M}_{2i} \mid i=1, \dots, k\}$ of \mathfrak{M}_2 . Let us assume that $S(M_{2i}, \mathfrak{M}_{2i}) \subset M_{1i} \in \mathfrak{M}_1$ ($i=1, \dots, k$), then we can construct continuous functions φ_i of R such that

$$\begin{aligned}\varphi_i(a) &= 0 \quad (a \in M_{2i}), \\ 0 &\leq \varphi_i \leq 1. \\ \varphi_i(a) &= 1 \quad (a \in M_{1i}^c),\end{aligned}$$

Let $a, b \in R$, and $b \notin S(a, \mathfrak{M})$, then $a \in M_{2i}$ implies $b \notin S(M_{2i}, \mathfrak{M}_{2i})$, whence $\varphi_i(a) = 0$, $\varphi_i(b) = 1$. Hence putting $\mathfrak{N} = \mathfrak{N}(\varphi_1, \frac{1}{2}) \wedge \dots \wedge \mathfrak{N}(\varphi_k, \frac{1}{2})$, we get $f^{-1}(\mathfrak{N}) < \mathfrak{M}_1^\wedge < \mathfrak{M}$. Therefore R and R_0 are uniformly homeomorphic. Thus the proof is complete.

Corollary 1. *A completely regular space R has one and only one Čech u -normal and totally bounded uniformity agreeing with its topology.*

Corollary 2. *In order that a uniform space R is uniformly homeomorphic to a subspace of its Tychonoff's parallelotope P , it is necessary and sufficient that R is Čech u -normal and totally bounded, where P has the weak uniform topology by means of bounded continuous functions of R .*

Corollary 3. *In order that the uniform topology of $w(R)$ can be reduced to that of R , it is necessary and sufficient that R is u -normal and totally bounded.*

Proof. The validity of this corollary is obvious from the fact that $\beta(R)$ and $w(R)$ are identical, when and only when R is normal.

We can prove Corollary 3 as follows, too.

Lemma 2. *In order that two Cauchy filters $\mathfrak{F} = \{F_\alpha \mid A\}$ and $\mathfrak{G} = \{G_\beta \mid B\}$ of R are equivalent, it is necessary and sufficient that $S(F_\alpha, \mathfrak{M}_x) \frown G_\beta \neq \phi$ for all $F_\alpha \in \mathfrak{F}$, $G_\beta \in \mathfrak{G}$ and $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$.⁴⁾*

Proof. Necessity: Let \mathfrak{F} and \mathfrak{G} be equivalent, i. e. $\mathfrak{F} \smile \mathfrak{G}$ be a Cauchy filter, then for every $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$, there exist two sets $F \in \mathfrak{F}$ and $G \in \mathfrak{G}$ such that $a, b \in F \smile G$ implies $b \in S(a, \mathfrak{M}_x)$.

Take arbitrary $F_\alpha \in \mathfrak{F}$ and $G_\beta \in \mathfrak{G}$, then $F_\alpha \frown F \neq \phi$, $G_\beta \frown G \neq \phi$. Therefore, when we take two points $a \in F_\alpha \frown F$ and $b \in G_\beta \frown G$, we see that

$$S(F_\alpha, \mathfrak{M}_x) \frown G_\beta \neq \phi$$

Sufficiency: For an arbitrary $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$, we consider $\mathfrak{M}_y \in \{\mathfrak{M}_x\}$ such that $\mathfrak{M}_y^* \leq \mathfrak{M}_x$, and assume that $a, b \in F_\alpha$ or $a, b \in G_\beta$ implies $b \in S(a, \mathfrak{M}_y)$.

If $a \in F_\alpha$, $b \in G_\beta$, then by the assumption there exists $M \in \mathfrak{M}_y$ such that $M \frown F_\alpha \neq \phi$, $M \frown G_\beta \neq \phi$, and accordingly $L, N \in \mathfrak{M}_y$ such that $L \frown M \neq \phi$, $N \frown M \neq \phi$, $a \in L$, $b \in N$. Hence there exists $P \in \mathfrak{M}_x$ such that $a, b \in L \smile M \smile N \subset P \in \mathfrak{M}_x$, whence we get $b \in S(a, \mathfrak{M}_x)$. Therefore $\mathfrak{F} \smile \mathfrak{G}$ is a Cauchy filter, i. e. \mathfrak{F} and \mathfrak{G} are equivalent.

Let R be a uniform space with the uniform topology $\{\mathfrak{M}_x \mid \mathfrak{X}\}$. We classify all maximum Cauchy closed filters of R by equivalence, and denote by R^* the set of the classes $\{\mathfrak{F}\}$. We introduce a topology

in R^* by means of the closed basis: $\{\{\mathfrak{F}\} \mid \mathcal{A} \mathfrak{F}_0 \in \{\mathfrak{F}\} : F \in \mathfrak{F}_0\} (\subset R^*)$, where F is an arbitrary closed set of R , then it is obvious that R^* becomes a T_1 -space.

Lemma 3. $\{\{\mathfrak{F}\} \mid \mathcal{A} \mathfrak{F}_0 \in \{\mathfrak{F}\} : F \in \mathfrak{F}_0\} = \{\{\mathfrak{F}\} \mid (\forall \alpha) F_\alpha \in \mathfrak{F} \in \{\mathfrak{F}\} \rightarrow S(F_\alpha, \mathfrak{M}_x) \cap F \neq \phi (\forall \mathfrak{M}_x \in \{\mathfrak{M}_x\})\}$

Proof. Assume that $F \in \mathfrak{F}_0 \in \{\mathfrak{F}\}$, then $\mathfrak{F} \in \{\mathfrak{F}\}$ implies $\mathfrak{F} \sim \mathfrak{F}_0$. Therefore by Lemma 2 we get $S(F_\alpha, \mathfrak{M}_x) \cap F \neq \phi$ for all $F_\alpha \in \mathfrak{F}$, $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$.

Conversely assume that $F_\alpha \in \mathfrak{F} \in \{\mathfrak{F}\}$ implies $S(F_\alpha, \mathfrak{M}_x) \cap F \neq \phi$. Then $\mathfrak{G} = \{G \mid G \supset S(F_\alpha, \mathfrak{M}_x) \cap F, F_\alpha \in \mathfrak{F}, \mathfrak{M}_x \in \{\mathfrak{M}_x\}, G \text{ is closed.}\}$ is a closed filter containing F , where we consider a fixed \mathfrak{F} . Further we can see that \mathfrak{G} is a Cauchy filter.

For, for an arbitrary $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$, we take \mathfrak{M}_y such that $\mathfrak{M}_y^{**} \triangleleft \mathfrak{M}_x$, and assume that $a, b \in F_\alpha (\in \mathfrak{F})$ implies $a \in S(b, \mathfrak{M}_y)$. If $a, b \in \overline{S(F_\alpha, \mathfrak{M}_y)} (\in \mathfrak{G})$, then there exist elements M and N of \mathfrak{M}_y such that $a \in M, b \in N; M \cap S(F_\alpha, \mathfrak{M}_y) \neq \phi, N \cap S(F_\alpha, \mathfrak{M}_y) \neq \phi$. Hence there exist $P, Q \in \mathfrak{M}_y$ such that

$$P \cap M \neq \phi, Q \cap N \neq \phi; P \cap F_\alpha \neq \phi, Q \cap F_\alpha \neq \phi,$$

and $L \in \mathfrak{M}_y$ such that $P \cap L \neq \phi, Q \cap L \neq \phi$. Therefore there exists K such that $M \cup P \cup L \cup Q \cup N \subset K \in \mathfrak{M}_x$. Thus from $a \in M, b \in N$, we conclude that $a \in S(b, \mathfrak{M}_x)$, i. e. \mathfrak{G} is a Cauchy filter.

Now we denote by \mathfrak{G}_0 a maximum closed filter containing \mathfrak{G} , then $G \in \mathfrak{G}_0$ and $F_\alpha \in \mathfrak{F}$ imply $S(F_\alpha, \mathfrak{M}_x) \cap G \neq \phi$, for all $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$. For assume the contrary, i. e. $S(F_\alpha, \mathfrak{M}_x) \cap G = \phi$, then for $\mathfrak{M}_y \in \{\mathfrak{M}_x\}$: $\mathfrak{M}_y^\wedge \triangleleft \mathfrak{M}_x$, we get $\overline{S(F_\alpha, \mathfrak{M}_y)} \cap G = \phi$ from the fact that $\overline{S(F_\alpha, \mathfrak{M}_y)} \triangleleft S(F_\alpha, \mathfrak{M}_x)$. But this contradicts $\overline{S(F_\alpha, \mathfrak{M}_y)} \in \mathfrak{G}_0, G \in \mathfrak{G}_0$. Thus by Lemma 2 we conclude that $\mathfrak{F} \sim \mathfrak{G}_0$. Therefore \mathfrak{G}_0 is an element of $\{\mathfrak{F}\}$ containing F .

Thus the proof of Lemma 3 is complete.

For each $\mathfrak{M}_x = \{M_\alpha \mid A\} \in \{\mathfrak{M}_x\}$ we put $\mathfrak{N}_x = \{\overline{M_\alpha}^\circ \mid A\}$, where $\overline{M_\alpha}^\circ$ means the closure of M_α in R^* . Then we can see easily from Lemma 3 that $\{\mathfrak{N}_x \mid \mathfrak{X}\}$ becomes a basis of a uniformity agreeing with the topology of R^* , and that $R^* = \overline{R}$ and R^* is complete.

If R is u -normal and totally bounded, from Lemma 2 and the method of introducing the topology into R^* we see easily that R^* is identical with $w(R)$. The converse is obvious.

Therefore we can reproduce Corollary 3 in such a way. Though we used the metod of Theorem 1 at first to study the relation between R and its Tychonoff's parallelotope, the last method has more generality, when we study uniform topologies of general bicompaifications.⁵⁾

From now forth we study properties of u -normal spaces.

Theorem 2. *Any topological space R admits at most one u -normal metric (or enumerable uniformity), which is the finest uniform topology (so called α -structure).*

Proof. Since the u -separation of a u -normal space is defined by means of its topology, and by virtue of the Lemma obtained by the author,⁶⁾ the uniform topology of a metric space is defined by the notion of u -separation, we get the uniqueness of such a uniform topology.

We denote by $\{\mathfrak{M}_n \mid n = 1, 2, \dots\}$ a basis of such a uniformity of R . If we assume that there exists an open covering \mathfrak{N} such that $\mathfrak{M}_n \triangleleft \mathfrak{N}$ for all n , then taking $\mathfrak{N} = \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \dots$ such that $\mathfrak{N}_1 > \mathfrak{N}_2^c, \mathfrak{N}_2 > \mathfrak{N}_3^c, \dots$, we see that the metric $\{\mathfrak{N}_n \wedge \mathfrak{M}_n \mid n = 1, 2, \dots\}$ is u -normal and is not equivalent with $\{\mathfrak{M}_n\}$. But the last result contradicts the above mentioned uniqueness of such a uniformity. Therefore $\{\mathfrak{M}_n\}$ must be the finest uniform topology of R .

Theorem 3. *In order that a metric space R is u -normal, it is necessary and sufficient that the set H of all points of accumulation of R is bicompaict, and for all $\varepsilon > 0$ there exists $\delta > 0$ such that $S_\varepsilon^c(H) \subset A_\delta$, where*

$$S_\varepsilon(H) = \{a \mid \rho(a, b) < \varepsilon, b \in H\},$$

$$A_\delta = \{a \mid \rho(a, b) > \delta (\forall b \neq a)\}.$$

Proof. Necessity: Assume that H is not bicompaict, and an open covering \mathfrak{M} of H has no finite subcovering. Since H is fully normal, there exists an open covering \mathfrak{N} such that $\mathfrak{N}^{**} \triangleleft \mathfrak{M}$.

If there exist $N_1, N_2, \dots, N_k \in \mathfrak{N}$ such that for every $N \in \mathfrak{N}$, $S(N, \mathfrak{N}) \cap S(N_i, \mathfrak{N}) \neq \phi$ holds for some i , then there exists M_i such that $S(N, \mathfrak{N}) \cup S(N_i, \mathfrak{N}) \subset M_i \in \mathfrak{M}$, and this shows that \mathfrak{M} has a finite subcovering $\{M_i \mid i = 1, \dots, k\}$, which contradicts the assumption. Therefore we can select N_1, N_2, \dots , such that

$$S(N_i, \mathfrak{N}) \cap S(N_j, \mathfrak{N}) = \phi \quad (i \neq j).$$

Let $a_i, b_i \in N_i$; $\rho(a_i, b_i) < \frac{1}{2}$, $\neq 0$ then from the method of selecting

N_i it must be $\{\overline{a_i}\} = \{a_i\}$, $\{\overline{b_i}\} = \{b_i\}$, whence they are disjoint, but not u -separated, which contradicts the u -normality of R . Hence H must be bicomact.

Next, assume that $S_\varepsilon^c(H) \not\subseteq A_\delta$ for all δ , then there exist $a_1 \in S_\varepsilon^c(H)$ and $b_1 \neq a_1$ such that $\rho(a_1, b_1) < \frac{\varepsilon}{2}$.

We denote by $\varepsilon(a) = \inf\{\rho(ax) \mid x \neq a\}$, then, as $a_1, b_1 \notin H$, it must be $\varepsilon(a_1) > 0$, $\varepsilon(b_1) > 0$. Hence we can select $a_2 \in S_\varepsilon^c(H)$ and $b_2 \neq a_2$ such that $\rho(a_2, b_2) < \inf(\varepsilon(a_1), \varepsilon(b_1), \varepsilon/3)$, then a_1, b_1, a_2 and b_2 are different from the others. We can select in the same way $a_3, b_3; a_4, b_4; \dots$ such that $\{a_i\} \cap \{b_i\} = \emptyset$, $\rho(a_i, b_i) < 1/i+1$ and $\{a_i\} \subseteq S_\varepsilon^c(H)$. Since every points of $S_\varepsilon^c(H)$ are isolated, $\{\overline{a_i}\} = \{a_i\}$. Since $\rho(b_i, x) > \varepsilon/2 (x \in H)$, $\{\overline{b_i}\} = \{b_i\}$. But $\{a_i\}$ and $\{b_i\}$ are not u -separated, which contradicts the u -normality of R .

Sufficiency: Assume that closed sets F and G of R are disjoint but not u -separated, then there exist two sequences of points $\{a_i\}$ and $\{b_i\}$ such that $a_i \in F, b_i \in G$ and $\rho(a_i, b_i) < 1/i$. If $\{a_i\}$ is cofinal in H , there exists a cluster point a of $\{a_i\}$ from the bicomactness of H , and it becomes at the same time a cluster point of $\{b_i\}$, which means $a \in F \cap G$, and this is a contradiction.

If $\{a_i\}$ is cofinal in H^c , since $\rho(a_i, b_i) < 1/i$, $\{a_i\}$ is residual in every A_δ^c and accordingly in every $S_\varepsilon(H)$ by assumption. Hence we can assume that $a_i \in S_{\varepsilon_i}(H)$ and $\varepsilon_i \rightarrow 0 (i \rightarrow \infty)$, i. e. we can select $h_i \in H$ such that $\rho(a_i, h_i) < \varepsilon_i$. From the bicomactness of H , $\{h_i\}$ has a cluster point h , which is at the same time a cluster point of $\{a_i\}$ and $\{b_i\}$. Therefore $h \in F \cap G$, which contradicts the assumption. This contradiction completes the proof of Theorem 3.

Corollary 4. *A u -normal metric space R is bicomact, if every $A_\delta (\delta > 0)$ are finite sets, (for example when R has no isolated point)*

Corollary 5. *Any u -normal metric space is complete.*

Theorem 4. *In order that a metrizable space R is homeomorphic to a u -normal metric space, it is necessary and sufficient that the set H of all points of accumulation of R is bicomact.*

Proof. Necessity of the condition is obvious from Theorem 3.

Sufficiency: We shall show that the enumerable set $\mathfrak{S}_n = \{S_{1/n}(a), x \mid a \in H, x \in H^c\}$ of open coverings is a basis for the a -structure of

R , where $S_{1/n}(a) = \{x \mid \rho(xa) < 1/n\}$.

Let M be an arbitrary open covering of R and

$$H \subset \bigcup_{i=1}^k M_i \quad (M_i \in \mathfrak{M}).$$

Let us select k closed sets N_i such that $N_i \subset M_i$, $\bigcup_{i=1}^k \bar{N}_i = H$, then

$$\min_{i=1 \dots k} d(\bar{N}_i, M_i^c) > \frac{1}{n}.$$

For, if $d(\bar{N}_i, M_i^c) = 0$, then there exist $a_j \in N_i$ and $b_j \in M_i^c$ such that $\rho(a_j b_j) < 1/j$. Since \bar{N}_i is bicompat, $\{a_j\}$ has a cluster point a , which is a cluster point of $\{b_j\}$ at the same time. Therefore $a \in \bar{N}_i \cap M_i^c$, which contradicts the assumption, i. e. it must be $d(N_i, M_i^c) > 0$ and accordingly $\min_{i=1 \dots k} d(\bar{N}_i, M_i^c) > \frac{1}{n}$.

Then it is easy to see that $\mathfrak{S}_n \subset \mathfrak{M}$ for this n ; hence $\{\mathfrak{S}_n\}$ is a basis for the α -structure of R , and the metric by means of $\{\mathfrak{S}_n\}$ is u -normal.

Now let us consider the case of non-metric spaces. If R is u -normal, then it is obvious that any Cauchy sequence of points converges, but, when we concern ourselves with general directed sets of points, it is difficult to study relations between u -normality and completeness.

In the general case we get the following

Theorem 5. *If a fully normal space R is a u -normal, locally complete and uniformly connected space⁷⁾, then R is complete.*

Proof. Let R be a uniform space satisfying the assumption, and let $\{\mathfrak{M}_\alpha\}$ be a basis for its uniformity such that every \mathfrak{M}_α consist of connected sets only.

1. Since R is fully normal and locally complete, by the result of A. H. Stone⁸⁾ we get a neighbourhood-finite open covering $\mathfrak{U} = \{U_\alpha \mid A\}$ such that \bar{U}_α are complete and an open covering $\mathfrak{B} = \{V_\alpha\}$ such that $V_\alpha \subset U_\alpha$. For these coverings we construct continuous functions f_α such that

$$\begin{aligned} f_\alpha(a) &= 1 \quad (a \in V_\alpha), \\ 0 &\leq f_\alpha(a) \leq 1, \\ f_\alpha(a) &= 0 \quad (a \in U_\alpha^c), \end{aligned}$$

We denote by $\Delta = \{\delta\}$ the collection of finite sets consisting of α , and

by $n(\delta)$ the number of δ . Put

$$N_k = \{a \mid \exists \gamma : f_\alpha(a) > \gamma (\alpha \in \delta), f_\alpha(a) < \gamma (\alpha \notin \delta)\}$$

for some δ such that $n(\delta) = k$, then $\{N_k \mid k=1, 2, \dots\}$ is an open covering of R . Put

$$M_\delta = \{a \mid \exists \gamma : f_\alpha(a) > \gamma (\alpha \in \delta), f_\alpha(a) < \gamma (\alpha \notin \delta)\},$$

then M_δ is an open set, and $\bigcup \{M_\delta^{(k)} \mid \delta \in \Delta, n(\delta) = k\} = N_k$, where $M_\delta^{(k)} = M_\delta \cap N_k$.

We notice that $M_\delta^{(k)} \cap M_{\delta'}^{(k)} = \phi$ ($n(\delta) = n(\delta') = k, \delta \neq \delta'$), and $M_\delta^{(k)} \cap V_\alpha = \phi$ ($\alpha \notin \delta$), i. e., $M_\delta^{(k)}$ meets a finite number of V_α only. Since $\mathfrak{N} = \{N_k\}$ is enumerable, by K. Morita's theorem it has a star-finite refinement $\mathfrak{B}' = \{P_{i'} \mid i=1, 2, \dots\}$.

Let $\mathfrak{B} = \{P_i \mid i=1, 2, \dots\}$, $\mathfrak{Q} = \{Q_i \mid i=1, 2, \dots\}$ be open coverings such that $\bar{P}_i \subset P_{i'}, \bar{Q}_i \subset P_{i'}$.

If $\bar{P}_i \subset N_k$, then $\mathfrak{B}_i = \{M_\delta^{(k)} \cap V_\alpha \cap \bar{P}_i \mid \delta \in \Delta, n(\delta) = k, \alpha \in A\}$ is a star-finite refinement of \mathfrak{B} in \bar{P}_i .

2. When we construct $S_i^\infty = \bigcup_{n=1}^\infty S^n(Q_i, \mathfrak{Q})$, then S_i^∞ and S_j^∞ are disjoint or identical: hence every elements of any $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$ are contained in some S_i^∞ , because they are connected.

Let $\varphi(t \mid \mathfrak{X})$ be an arbitrary Cauchy directed set of points, then φ is residual in some S_i^∞ by the above fact.

We shall prove that there exists $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$ such that for some m , every elements M of \mathfrak{M}_x , in which φ is residual, is contained in $\bigcup_{n=1}^m S^n(Q_i, \mathfrak{Q}) = S^m$.

To see this, assume that the contrary holds. Let \mathfrak{M}_x be an arbitrary element of $\{\mathfrak{M}_x\}$, and let $\mathfrak{M}_y^* \subset \mathfrak{M}_x, \mathfrak{M}_y \in \{\mathfrak{M}_x\}$. If φ is residual in $M_y \in \mathfrak{M}_y$, then every elements of \mathfrak{M}_y , in which φ is residual, meet M_y . Hence when $S(M_y, \mathfrak{M}_y) \subset M_x \in \mathfrak{M}_x$, from the assumption we get $M_x \not\subset S^m (= \bigcup_{n=1}^m S^n(Q_i, \mathfrak{Q}))$ for all m .

Since M_x is connected, $M_x \cap (S^n - S^m) \neq \phi$ ($n \geq m$), when $M_x \cap S^m \neq \phi$. Therefore two closed sets $F = \bigcup_{n=1}^\infty (S^{2n-1} - S^{2n-2})$ and $G = \bigcup_{n=1}^\infty (S^{2n} - S^{2n-1})$ are disjoint but not u-separated, which contradicts the u-normality of R .

3. We denote by \mathfrak{M}_x an element of $\{\mathfrak{M}_x\}$ such that every elements of \mathfrak{M}_x , in which φ is residual, are contained in S^m for a definite m .

If $S^m = \bigcup_{i=1}^h Q_i$, then $\mathfrak{P}_0 = \{\overline{S^m} \cap P_i \mid i = 1 \dots h\}$ is a finite open covering of $\overline{S^m}$. Hence from the u-normality of $\overline{S^m}$, we get a refinement $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$ of \mathfrak{P}_0 in $\overline{S^m}$.

Let $\mathfrak{M}_z^* \subset \mathfrak{M}_x \wedge \mathfrak{M}_y$, $\mathfrak{M}_z \in \{\mathfrak{M}_x\}$, and let $\varphi(t \mid \mathfrak{X})$ be residual in a definite element M_0 of \mathfrak{M}_z . Let M be an arbitrary element of \mathfrak{M}_z , in which $\varphi(t \mid \mathfrak{X})$ is residual, then $M \subset S(M_0, \mathfrak{M}_z) \subset M_x \in \mathfrak{M}_x$; $M \subset S(M_0, \mathfrak{M}_z) \subset M_y \in \mathfrak{M}_y$, for some M_x and M_y . Since $\varphi(t \mid \mathfrak{X})$ is residual in M_x , we get $M_x \subset \overline{S^m}$.

On the other hand from $M_y \in \mathfrak{M}_y$ we get $M_y \cap S^m \subset P_i \cap S^m$, whence

$$M \subset M_x \cap M_y = M_x \cap \overline{S^m} \cap M_y \subset P_i \cap \overline{S^m} \subset P_i$$

for a definite P_i and for every $M \in \mathfrak{M}_z$, in which $\varphi(t \mid \mathfrak{X})$ is residual.

4. Since from 3 for every $\mathfrak{M}_u \subset \mathfrak{M}_z$, $\mathfrak{M}_u \in \{\mathfrak{M}_x\}$ and for every $M \in \mathfrak{M}_u$, in which $\varphi(t \mid \mathfrak{X})$ is residual, we get $M \subset \overline{P_i}$, it must be for a definite $M_{\delta, \alpha}$, $M \subset \bigcup_{n=1}^{\infty} S^n(M_{\delta, \alpha}, \mathfrak{B}_i)$ for such M , where

$$M_{\delta, \alpha} = M_{\delta}^{(k)} \cap V_{\alpha} \cap \overline{P_i} \in \mathfrak{B}_i, \overline{P_i} \subset N_k.$$

(See the last part of 1.)

Since $\overline{P_i}$ is u-normal, by the same method as in 2 we can prove that there exists $M \in \mathfrak{M}_u \subset \mathfrak{M}_z$ such that $\varphi(t \mid \mathfrak{X})$ is residual in M , and $M \subset \bigcup_{n=1}^l S^n(M_{\delta, \alpha}, \mathfrak{B}_i)$ for some l . Since \mathfrak{B}_i is star-finite, $\bigcup_{n=1}^l S^n(M_{\delta, \alpha}, \mathfrak{B}_i)$ is a union of some finite number of $M_{\delta, \alpha}$.

Since $\varphi(t \mid \mathfrak{X})$ is residual in $\bigcup_{n=1}^l S^n(M_{\delta, \alpha}, \mathfrak{B}_i)$, it must be residual in a definite element $M_{\delta, \alpha}$ of \mathfrak{B}_i . From the fact that $\overline{M_{\delta, \alpha}} \subset \overline{V_{\alpha}}$, we get the completeness of $\overline{M_{\delta, \alpha}}$, whence $\varphi(t \mid \mathfrak{X})$ must converge. Thus the proof of Theorem 5 is complete.

Notes.

- 1) We denote by $\beta(R)$ Čech's bicompaification of R and by $w(R)$ Wallman's one.
- 2) A closed set F of R is called a completely closed set, when there exists a continuous function φ of R such that

$$F = \{a \mid \varphi(a) = 0\}.$$

- 3) We denote by N^c the complement of N .
- 4) In this note we consider filters consisting of closed sets only.
- 5) See N. A. Shanin, On Special Extensions of Topological Spaces, and his other papers, C. R. URSS, 38 (1943) No. 1, No. 4, No. 5-6.
- 6) On the Lattices of Functions on Topological Spaces and of Functions on Uniform Spaces, Osaka Math. Journal, 1 (1949) No. 2, Lemma 2.

7) We mean by a uniformly connected space a uniform space, which has a basis $\{\mathfrak{M}_\alpha\}$ for its uniformity such that every \mathfrak{M}_α consist of connected sets only.

8) A. H. Stone, On Paracompactness and Product Space, Bull. of Amer. Math. 54 (1948) No. 10.

An open covering \mathfrak{U} of R is called neighbourhood-finite, when each point x of R has a neighbourhood $V(x)$, which meets only finite number of elements of \mathfrak{U} .

\mathfrak{U} is called star-finite, when each element of \mathfrak{U} meets only finite number of elements of \mathfrak{U} .

9) K. Morita, Star-Finite Coverings and the Star-Finite Property, Math. Japonicae, Vol. 1, No. 2.

To see this in this case, for example consider an open covering $\mathfrak{M} = \{\overline{M}_k\}$, $M_k \subset N_k$, and construct continuous functions f_k such that

$$f_k(a) = k \quad (a \in \overline{M}_k), \quad 0 \leq f_k(a) \leq k.$$

$$f_k(a) = 0 \quad (a \notin N_k),$$

Putting $f = \sup f_k$, $L_n = \{a \mid n-1 < f(a) < n+1\}$, $\mathfrak{L} = \{L_n \mid n = 0, 1, 2, \dots\}$, we get a star-finite refinement $\mathfrak{L} \wedge \mathfrak{M}$ of \mathfrak{M} .

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