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HEREDITARY ORDERS WHICH ARE DUAL

Manabu HARADA

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Let R be a Dedekind domain and Σ a central simple K-algebra, where K is the quotient field of R.

M. Auslander and O. Goldman have considered maximal orders in Σ from point of view of homological method in [1] and introduced the notion of hereditary orders in Σ. Recently, A. Brumer, H. Hijikata, S. Williamson and the author have studies structures and applications of hereditary orders in [2], [2′], [8], [9], [4], [5], [6] and [7].

In this paper we shall give some relations between Brumer's work [2], [2′] and the author's [4], [5] and [6] by the method mentioned in [3].

We shall call briefly a hereditary order an h-order.

Let Δ be a division K-algebra and $\Sigma=\Delta_n$, $\Sigma'=\Delta_m$. Then there exists a left Σ and right Σ' module V such that $\Sigma=\mathrm{Hom}^r_{\Sigma'}(V, V)$ and $\Sigma'=\mathrm{Hom}^l_{\Sigma}(V, V)$. Let E be a sub R-module of V such that $EK=V$, then we call E an R-lattice (in V). If there exists an R-lattice E which is Λ-Λ' module for orders Λ, Λ' in Σ, Σ' respectively, we call Λ, Λ' are dual with respect to E.

We shall show that if h-orders Λ and Λ' are dual then there exists a unique one-to-one correspondence between orders Γ and Γ' containing Λ and Λ' respectivety, which are dual with respect to special lattice (Theorem 1). Next, we show a relation between R-lattices E which is Λ'-module and right ideals of Λ (Theorem 2), which gives a bridge between [2] and [5], and we give an isomorphism of normal two-sided ideals[1] of Λ to those of Λ' through E as a corollary. In Theorem 3, we shall show that Λ and Λ' are dual if and only if they belong to the same block[2]. Finally we give an isomorphism of the groupoid[3] of two-sided ideals of Λ to the groupoid of those of Λ' through E (Theorem 4).

Let Λ, Λ' be dual with Λ-Λ' module E which is an R-lattice. If

1) See [6], §1.
2) See [7], §3.
3) See [4], §6.

Λ is an h-order, then E is Λ-projective by [4], Lemma 3.6 and $\tau_{\Lambda'}(E)$[4]$=\Lambda'$ by [1], Proposition A.3. Furthermore, by [2'], Appendix Theorem 5 or [6], Theorem 1.1 we know that Λ' is an h-order in Σ'.

LEMMA 1. *Let Λ' be an order in Σ' and E a torsion-free finitely generated R-module and right Λ'-module such that $\tau_{\Lambda'}(E)=\Lambda'$. Let $\mathfrak{M}$ be a two-sided ideal in Λ' and Γ' be the right order of $\mathfrak{M}$ in Σ'. If $\Gamma'\mathfrak{M}=\Gamma'$, $\tau_{\Gamma'}(E\mathfrak{M})=\Gamma'$.*

Proof. By the assumption and [4], Lemma 1.2 we have an exact sequence $0\to \mathrm{Hom}^r_{\Lambda'}(E\mathfrak{M}, \Lambda') \xrightarrow{\psi} \mathrm{Hom}^r_{\Lambda'}(E\mathfrak{M}, \Gamma')=\mathrm{Hom}^r_{\Gamma'}(E\mathfrak{M}, \Gamma')$. Furthermore, from an exact sequence $0\to E\mathfrak{M}\to E\to E/E\mathfrak{M}\to 0$ we obtain the monomorphism φ: $\mathrm{Hom}_{\Lambda'}(E, \Lambda')\to \mathrm{Hom}_{\Lambda'}(E\mathfrak{M}, \Lambda')$. Since $\tau_{\Lambda'}(E)=\Lambda'$, $1=\Sigma f_i(e_i)$, $f_i\in \mathrm{Hom}^r_{\Lambda'}(E, \Lambda')$ and $e_i\in E$. Hence $m=\Sigma f_i(e_i m)$ for any $m\in\mathfrak{M}$. Since $\psi\varphi f_i\in \mathrm{Hom}^r_{\Gamma'}(E\mathfrak{M}, \Gamma')$ and $\Sigma\psi\varphi f_i(e_i m)=m$, $\tau_{\Gamma'}(E\mathfrak{M})\supseteq\mathfrak{M}$. Hence $\tau_{\Gamma'}(E\mathfrak{M})\supseteq\Gamma'\mathfrak{M}\Gamma'=\Gamma'$. Therefore, $\tau_{\Gamma'}(E\mathfrak{M})=\Gamma'$.

LEMMA 2. *Let $\Lambda_2\supset\Lambda_1\supset\Lambda$ be a tower of h-orders in Σ. Then $C_\Lambda(\Lambda_2)=C_{\Lambda_1}(\Lambda_2)C_\Lambda(\Lambda_1)$, where $C_\Lambda(\Lambda_i)=\{x\mid\in\Sigma,\ \Lambda_i x\subseteq\Lambda\}$.*

Proof. It is clear that $C_\Lambda(\Lambda_2)\supseteq C_{\Lambda_1}(\Lambda_2)C_\Lambda(\Lambda_1)$. We denote $C_\Lambda(\Lambda_2)$, $C_{\Lambda_1}(\Lambda_2)$, $C_\Lambda(\Lambda_1)$ by $\mathfrak{C}$, $\mathfrak{C}_2$, $\mathfrak{C}_1$, respectively. Let $\mathfrak{D}=\mathfrak{C}_2\mathfrak{C}_1$. Then $\mathfrak{D}^2=\mathfrak{C}_2\mathfrak{C}_1\mathfrak{C}_2\mathfrak{C}_1=\mathfrak{C}_2\mathfrak{C}_1\Lambda_1\mathfrak{C}_2\mathfrak{C}_1=\mathfrak{C}_2\mathfrak{C}_1=\mathfrak{D}$ by [4], Propositions 1.6 and 3.1. It is clear that $\mathrm{Hom}^r_\Lambda(\mathfrak{D}, \mathfrak{D})\supseteq\mathrm{Hom}^r_\Lambda(\mathfrak{C}_2, \mathfrak{C}_2)=\Lambda_2$. Furthermore, $\mathfrak{D}\Lambda_1=\mathfrak{C}_2\mathfrak{C}_1\Lambda_1=\mathfrak{C}_2\Lambda_1=\mathfrak{C}_2$. Hence $\Lambda_2=\mathrm{Hom}^r_\Lambda(\mathfrak{C}_2, \mathfrak{C}_2)\supseteq\mathrm{Hom}^r_\Lambda(\mathfrak{D}, \mathfrak{D})$. Therefore, $\mathrm{Hom}^r_\Lambda(\mathfrak{D}, \mathfrak{D})=\mathrm{Hom}^r_\Lambda(\mathfrak{C}, \mathfrak{C})$, which means $\mathfrak{D}=\mathfrak{C}$ by [4] Theorem 1.7.

LEMMA 3. *Let Λ and Λ' be h-orders in Σ and Σ' which are dual with E. Let $\mathfrak{A}$ be a right Λ-ideal in Σ. Then $\mathfrak{A}=\mathrm{Hom}^r_{\Lambda'}(E, \mathfrak{A}E)$.*

Proof. Since $\mathfrak{A}$ is Λ-projective, $\mathrm{Hom}^r_{\Lambda'}(E, \mathfrak{A}E)=\mathrm{Hom}^r_{\Lambda'}(E, \mathfrak{A}\otimes_\Lambda E)$. Since E is Λ'-projective, $\mathrm{Hom}^r_{\Lambda'}(E, \mathfrak{A}\otimes_\Lambda E)=\mathfrak{A}\otimes_\Lambda\mathrm{Hom}^r_{\Lambda'}(E, E)=\mathfrak{A}$ by setting $(a\otimes f)\ e=a\otimes f(e)$, $a\in\mathfrak{A}$ and $f\in\mathrm{Hom}^r_{\Lambda'}(E, E)$.

THEOREM 1. *Let Λ and Λ' be h-orders in Σ and Σ' which are dual with an R-lattice E. Let Γ be an order containing Λ. Then there exists a unique order Γ' containing Λ' such that $\mathfrak{C}E=E\mathfrak{D}'$ and Γ and Γ' are dual with respect to $\mathfrak{C}E$, where $\mathfrak{C}=C_\Lambda(\Gamma)$ and $\mathfrak{D}=D_{\Lambda'}(\Gamma')=\{x\mid\in\Gamma',\ x\Gamma'\subseteq\Lambda'\}$.*

Proof. Let $\mathfrak{M}$ be a maximal two-sided ideal in Λ. Since E is a

4) See [1], Appendix.

finitely generated Λ-projective, we have an exact sequence $0\to\varphi^{-1}(0)\to \mathrm{Hom}^l_\Lambda(E, E)\xrightarrow{\varphi}\mathrm{Hom}^l_{\Lambda/\mathfrak{M}}(E/\mathfrak{M}E, E/\mathfrak{M}E)\to 0$. Since $\mathrm{Hom}_{\Lambda/\mathfrak{M}}(E/\mathfrak{M}E, E/\mathfrak{M}E)$ is a simple ring, $\varphi^{-1}(0)$ is a maximal two-sided ideal in $\Lambda'=\mathrm{Hom}^l_\Lambda(E, E)$ and $E\varphi^{-1}(0)\subseteqq\mathfrak{M}E$. Similarly we obtain $E\varphi^{-1}(0)\subseteqq\mathfrak{P}E$ for some maximal two-sided ideal $\mathfrak{P}$ in Λ. If $\mathfrak{M}\neq\mathfrak{P}$, $\Lambda=\mathfrak{P}+\mathfrak{M}$. Hence $E=\mathfrak{M}E+\mathfrak{P}E=\mathfrak{M}E$, which implies $\mathfrak{M}=\Lambda$ by Lemma 3. Hence, $\mathfrak{M}E=E\varphi^{-1}(0)$. First we assume that Γ is an order containing Λ such that there are no orders between Λ and Γ, then $\Lambda_q=\Gamma_q$ for any prime ideal q expect one prime p by [4], §7. Let $\mathfrak{C}=C_\Lambda(\Gamma)$, then $\Gamma\mathfrak{C}\Gamma=\Gamma$ by [4], Proposition 3.1. Hence $\tau_\Gamma(\mathfrak{C}E)=\Gamma$ by Lemma 1. Therefore, $\mathrm{Hom}^l_\Gamma(\mathfrak{C}E, \mathfrak{C}E)=\Gamma'$ and Γ are dual with respect to $\mathfrak{C}E$. Furthermore, Γ' is an order containing Λ' such that there are no orders between Γ' and Λ' by the above remark. It is clear that $\mathfrak{C}$ is a maximal two-sided ideal by [5], Lemma 2.3. Hence, $\mathfrak{C}E=E\mathfrak{D}'$ for a maximal two-sided ideal $\mathfrak{D}'$ in Λ'. Since, $\Gamma'\supseteqq\mathrm{Hom}^l_{\Lambda'}(\mathfrak{D}', \mathfrak{D}')\supsetneqq\Lambda'$, $\Gamma'=\mathrm{Hom}^l_{\Lambda'}(\mathfrak{D}', \mathfrak{D}')$. $\mathfrak{D}'$ is uniquely determined by Lemma 3. We assume the theorem is true for orders contained in Γ. There exists $\Lambda_0(\supsetneq\Lambda)$ contained in Γ such that there are no orders between Γ and Λ_0 by [4], Theorem 7.2. By the assumption there exists an order such that Λ_0 and Λ_0' are dual with respect to $\mathfrak{C}_\Lambda(\Lambda_0)E=E\mathfrak{D}'_{\Lambda'}(\Lambda_0')$. Hence there exists an order $\Gamma'\supsetneq\Lambda_0'$ such that Γ and Γ' are dual with respect to $C_{\Gamma_0}(\Gamma)C_\Lambda(\Lambda_0)E=ED'_{\Lambda'}(\Lambda_0')D'_{\Lambda_0'}(\Gamma')$. Hence $C_\Lambda(\Gamma)E=ED'_{\Lambda'}(\Gamma')$ by Lemma 2, Γ' is uniquely determined by Lemma 3.

COROLLARY 1. *Let Λ, Λ' and E be as above. Then every chain of h-orders containing Λ corresponds uniquely to the chain of h-orders containing Λ' in Σ'.*

THEOREM 2. *Let Λ and Λ' be h-orders in Σ and Σ' which are dual with respect to E. Then the set of $\boldsymbol{E}$ of R-lattic in V which is a Λ'-module corresponds to the set $\boldsymbol{R}$ of right Λ-ideal in Σ as follows*: 1). *For $E'\in\boldsymbol{E}$, $\mathfrak{A}\in\boldsymbol{R}$, $E'=\mathfrak{A}E$, $\mathfrak{A}=\mathrm{Hom}^r_\Lambda(E, E')$. This correspondence preserves the inclusion relation.* 2) *The left order $\Lambda^l(\mathfrak{A})$ of $\mathfrak{A}=\mathrm{Hom}^r_{\Lambda'}(E', E')$.* 3) *The right order $\Lambda^r(\mathfrak{A})$ of $\mathfrak{A}=\Lambda$ ($\mathfrak{A}$ is a normal right ideal) if and only if $\tau_{\Lambda'}(E')=\Lambda'$.* 4) *The number of ideal classes of normal right Λ-ideals is equal to the number of Λ'-isomorphic classes of the R-lattice E' in $\boldsymbol{E}$ such that $\tau_{\Lambda'}(E')=\Lambda'$.*

Proof. 1) Let E' be in $\boldsymbol{E}$. Then $\mathfrak{A}=\mathrm{Hom}^r_{\Lambda'}(E, E')$ is a right Λ-ideal in Σ, and $\mathfrak{A}E=\mathfrak{A}\otimes_\Lambda E=\mathrm{Hom}^r_{\Lambda'}(E, E')\otimes_\Lambda E=\mathrm{Hom}^r_{\Lambda'}(\mathrm{Hom}^l_\Gamma(E, E), E')=E'$. Conversely, if $\mathfrak{A}\in R$, then $\mathfrak{A}E\in\boldsymbol{E}$, and $\mathrm{Hom}^r_\Lambda(E, \mathfrak{A}E)=\mathfrak{A}$ by Lemma

3. 2) Let $\mathfrak{A}'=\mathrm{Hom}^r_{\Lambda'}(E', E)$. Then $\mathfrak{A}\mathfrak{A}'=\mathrm{Hom}^r_{\Lambda'}(E, E')\underset{\Lambda}{\otimes}\mathrm{Hom}^r_{\Lambda'}(E', E)=\mathrm{Hom}^r_{\Lambda'}(\mathrm{Hom}^l_{\Lambda}(\mathrm{Hom}^r_{\Lambda'}(E', E),E), E')=\mathrm{Hom}^r_{\Lambda'}(E'\underset{\Lambda'}{\otimes}\mathrm{Hom}^l_{\Lambda}(E, E), E')=\mathrm{Hom}^r_{\Lambda'}(E', E')$. Therefore, $\Lambda^l(\mathfrak{A})\subseteq\mathrm{Hom}^r_{\Lambda'}(E', E')$. However, it is clear that $\mathrm{Hom}^r_{\Lambda}(E', E')\subseteq\Lambda^l(\mathfrak{A})$. 3) If $\Lambda=\Lambda^r(\mathfrak{A})$, then $\mathfrak{A}$ is inversible by [4], §2. Hence $\mathfrak{A}Ex\subseteq\mathfrak{A}E$ implies $Ex\subseteq E$. Hence $\mathrm{Hom}^l_{\Lambda}(\mathfrak{A}E, \mathfrak{A}E)=\Lambda'$. Therefore, $\tau_{\Lambda'}(\mathfrak{A}E)=\Lambda'$. Conversely, if $\tau_{\Lambda'}(E')=\Lambda'$, then we can prove similarly as above that $\mathfrak{A}'\mathfrak{A}=\mathrm{Hom}^r_{\Lambda'}(E, E)=\Lambda$. Hence $\mathfrak{A}$ is an inversible Λ-ideal and $\Lambda^r(\mathfrak{A})=\Lambda$. 4) Let $\mathfrak{A}$, $\mathfrak{A}'$ be normal right Λ-ideals in Σ. If $\lambda\mathfrak{A}=\mathfrak{A}'$ for some λ in Σ, then $\mathfrak{A}E\approx\mathfrak{A}'E$ as a right Λ'-module. Conversely if $\mathfrak{A}E$ is isomorphic by f, then $f\in\mathrm{Hom}^r_{\Lambda'}(\mathfrak{A}E, \mathfrak{A}'E)=\mathfrak{F}$, $f^{-1}\in\mathrm{Hom}^r_{\Lambda}(\mathfrak{A}'E, \mathfrak{A}E)=\mathfrak{F}^{-1}$, and $\mathfrak{A}'E=f\mathfrak{A}E$ by 1). Since $f\Gamma=f\mathfrak{F}^{-1}\mathfrak{F}\supseteq ff^{-1}\mathfrak{F}=\mathfrak{F}\supseteq f\Gamma$, $f\Gamma=\mathfrak{F}$, where $\Gamma=\mathrm{Hom}^r_{\Lambda'}(\mathfrak{A}E, \mathfrak{A}E)$. Therefore, $\mathfrak{A}'E=f\mathfrak{A}E$. Hence $\mathfrak{A}'=f\mathfrak{A}$.

Corollary 2. (cf. [3], Theorem 4, [4], Theorem 7.6). *Let Λ and Λ' be h-orders as in Theorem 2. Then the group $\boldsymbol{G}$ of normal two-sided ideals $\mathfrak{A}$ of Λ and the group of $\boldsymbol{G}'$ of those $\mathfrak{A}'$ of Λ' are isomorphic by the correspondence $\mathfrak{A}E=E\mathfrak{A}'$. Hence they are abelian groups.*

It is clear that $\mathfrak{A}\mathfrak{B}E=\mathfrak{A}E\mathfrak{B}'=E\mathfrak{A}'\mathfrak{B}'=E\mathfrak{B}'\mathfrak{A}'$. Hence $\mathfrak{A}'\mathfrak{B}'=\mathfrak{B}'\mathfrak{A}'$.

Corollary 3. *Let R be local and Λ and Λ' be as above. Λ is principal*[5] *if and only if any two R-lattice E, E' in V which are Λ-Λ' module with $\tau_{\Lambda'}(E)=\tau_{\Lambda'}(E')=\Lambda'$ are isomorphic as a Λ'-module.*

It is clear from the theorem and [4], Corollary 4.5.

Let R be local and $\mathfrak{N}$, $\mathfrak{N}'$ the radicals of Λ and Λ'. We assume that $\Lambda/\mathfrak{N}=\Delta'_{m_1}\oplus\Delta'_{m_2}\oplus\cdots\oplus\Delta'_{m_r}$ and $\mathfrak{R}$ is a right ideal with $\tau_{\Lambda}(\mathfrak{R})=\Lambda$ in Λ which contains $\mathfrak{N}$. Let Λ and Λ' be dual with E. Then $\tau_{\Lambda'}(\mathfrak{R}E)=\Lambda'$ and $\mathfrak{R}E\supseteq E\mathfrak{N}'$. Let $F=\mathfrak{R}E$. Since E is Λ'-projective, $0\to\mathfrak{R}=\mathrm{Hom}^r_{\Lambda}(E, F)\to\Lambda=\mathrm{Hom}^r_{\Lambda}(E, E)\to\mathrm{Hom}^r_{\Lambda'}(E, E/F)=\mathrm{Hom}^r_{\Lambda'/\mathfrak{N}'}(E/E\mathfrak{N}, E/F)\to 0$ is exact. Similarly we know $\Lambda/\mathfrak{N}\approx\mathrm{Hom}^r_{\Lambda'/\mathfrak{N}'}(E/E\mathfrak{N}', E/E\mathfrak{N}')$. Hence $E/E\mathfrak{N}'\approx\mathfrak{r}_1'^{m_1}\oplus\mathfrak{r}_2'^{m_2}\oplus\cdots\oplus\mathfrak{r}_r'^{m_r}$, where the $\mathfrak{r}_i'$'s are the set of simple components of $\Lambda'/\mathfrak{N}'$. On the other hand $E/E\mathfrak{N}'=E/F\oplus F/E\mathfrak{N}'$ and we assume $E/F\approx\mathfrak{r}_1^{t_1}\oplus\mathfrak{r}_2^{t_2}\oplus\cdots\oplus\mathfrak{r}_r^{t_r}$. Then $\mathfrak{R}/\mathfrak{N}\approx\mathfrak{r}_r^{m_1-t_1}\oplus\cdots\oplus\mathfrak{r}_r^{m_r-t_r}$, where the $\mathfrak{r}_i$'s are the set of simple right ideals of $\Lambda/\mathfrak{N}$. Especially, if we take $\mathfrak{R}/\mathfrak{N}=\Delta'_{m_1}\oplus\cdots\oplus\Delta'_{m_{i-1}}\oplus\mathfrak{r}_i^{s_i}\oplus\Delta'_{m_{i+1}}\oplus\cdots\oplus\Delta'_{m_r}$, then $E/\mathfrak{R}E$ is a direct sum of one simple component of $\Lambda/\mathfrak{N}$.

Thus, we have from [5], Theorem 5.3 the following corollary, which is a generalization of [2].

Corollary 4. *Let R be local and h-orders Λ, Λ' be dual with E.*

5) See [5], §2.

Then every h-order contained in Λ *is uniquely written as* $\bigcap_{i,j=1}^{r,s(i)} \Lambda_{ij}$, *where* Λ_{ij}, Λ' *are dual with* F_{ij} *which satisfies the conditions for all* i: 1) $F_{i0} = E \supset F_{i1} \supset \cdots \supset F_{is(i)} \supset E\mathfrak{N}'$ *and* 2) $E/F_{is(i)}$ *is a direct sum of one simple component of* $\Lambda'/\mathfrak{N}'$.

Furthermore, from [6], Theorem 2.5 we have

COROLLARY 5. *Let* R, Λ *and* Λ' *be as above and* E' *be a sub R-lattice in* E *such that* $\tau_{\Lambda'}(E)=\Lambda'$. *Then there exists a sub R-lattice* E^* *between* E *and* E' *such that* $E^* \approx E'$ *as a right* Λ'*-module and there exists a composition series* $E \supset E_1 \supset E_2 \supset \cdots \supset E_m = E^*$ *and* $\tau_{\Lambda'}(E_i)=\Lambda'$ *for all* i.

REMARK. Brumer considers in [2] the (Λ, Λ_0')-chain where Λ_0' is a maximal order in $\Delta=\Sigma'$ and Λ is an h-order in Δ_n. From the above observation we know that it is nothing but studying the inclusion relations of distance ideals[6] between maximal orders containing Λ. Since Λ_0' is maximal, $r=1$ in Corollary 4, and $\{F_{1j}\}_{j=1}^{s(1)}$ is the set of distance ideals of $\mathrm{Hom}^r_{\Lambda_0'}(E, E)=\Omega$ to a maximal order containing Λ. Hence the set of R-lattice in V which is Λ-Λ_0' module is linearly ordered with period $s(1)$, (cf. [2]).

Thus, we shall insert here one proposition related to distance ideals.

LEMMA 4. *Let* R *be local and* $\Omega \supset \Lambda$ *be h-orders of rank* r_1 *and* r_2[7]. *Let* $\Omega_1=\Omega, \Omega_2, \cdots, \Omega_t, \Omega_{t+1}=\Omega_1$ *be a sequence of h-orders containing* Λ *such that* $\Omega_{i+1}=\Lambda^r(\mathfrak{C}_i)$, $\mathfrak{C}_i=C_\Lambda(\Omega_i)$. *Then* $\mathfrak{C}_1\cdots\mathfrak{C}_t=\mathfrak{N}^k$ *for* $k \geqslant t-r_1$, *where* $\mathfrak{N}$ *is the radical of* Ω. *Furthermore,* $k=r_2-r_1$ *if and only if* $t=r_2$.

Proof. Let $\{\mathfrak{M}_1, \mathfrak{M}_2, \cdots, \mathfrak{M}_{r_2}\}$ be the normal sequence[8] of Λ and $\Lambda/\mathfrak{M}_i=\Delta'_{m_i}$. Then we obtain $\mathfrak{C}_{i+1}=\mathfrak{N}\mathfrak{C}_i\mathfrak{N}^{-1}$ by [5], Theorem 5.1. We shall use the same notations as in [5], §2. $S_i=\{\mathfrak{M}_{t_i}, \mathfrak{M}_{t_i+1}, \cdots, \mathfrak{M}_{t_i+\rho_i-1}\}$ and $t_1=1$, $t_i+\rho_i+1=t_{i+1}$, and $t_{r_1}+\rho_{r_1}+1=t_1$. We shall assume $\mathfrak{C}_1=I(S_1, S_2, \cdots, S_{r_1})$. Then we know the composition length $l(\Omega_1/\mathfrak{C}_1)$ of $\Omega_1/\mathfrak{C}_1$ is equal to $\sum_{i=1}^{r_1}\sum_{k=0}^{\rho_i-1} m_{t_i+k}=n-\sum_{i=1}^{r_1} m_{t_i+\rho_i}$ by [5], Theorem 2.3 and the same argument as the remark before Corollary 4, where $n=\sum_i m_i$. Since $\mathfrak{C}_i=\mathfrak{N}\mathfrak{C}_{i-1}\mathfrak{N}^{-1}$, we obtain similarly $l(\Omega/\mathfrak{C}_i)=n-\sum_{j=1}^{r_1} m_{t_j+\rho_j-i+1}$. Therefore, $l(\Omega/\mathfrak{C}_1\cdots\mathfrak{C}_t) \geqslant (t-r_1)n$. Since $\mathfrak{C}_1\mathfrak{C}_2\cdots\mathfrak{C}_t$ is a normal two-sided ideal in Ω_1 by [4], §6, $\mathfrak{C}_1\cdots\mathfrak{C}_t=\mathfrak{N}^k$. Furthermore, since $l(\Omega_1/\mathfrak{N})=n$ by [6], Proposition

6) See [6], §2.
7) See [4], §3.
8) See [5], §2.

2.6, $nk \geqslant (t-r_1)n$. Hence $k \geqslant t-r_1$. It is clear that $\sum_{i=1}^{t} \sum_{j=1}^{r_1} m_{tj+pj-i+1} = r_1 n$ if and only if $t=r_2$.

PROPOSITION 1[9]. *Let R be local and Λ an h-order of rank r. We assume that Ω is either a maximal order or a minimal order*[10] *containing Λ. Then the sequence in Lemma 4 gives the complete set of maximal or minimal orders containing Λ. $\Omega \supset \mathfrak{N}^{2-r}\mathfrak{C}_1 \cdots \mathfrak{C}_{r-1} \supset \mathfrak{N}^{3-r}\mathfrak{C}_1 \cdots \mathfrak{C}_{r-2} \supset \cdots \supset \mathfrak{N}^{-1}\mathfrak{C}_1\mathfrak{C}_2 \supset \mathfrak{C}_1$ is the set of distance ideals of Ω to Ω_i if Ω is maximal, and $\Omega \supset \mathfrak{C}_1 \supset \mathfrak{C}_1\mathfrak{C}_2 \supset \cdots \supset \mathfrak{C}_1\mathfrak{C}_2 \cdots \mathfrak{C}_{r-1}$ is the set of distance ideals of Ω to Ω_i if Ω is minimal over Λ, where $\mathfrak{N}$ is the radical of Ω.*

Proof. The first part is clear from [5], Theorem 5.1 and [4], Theorem 1.7. If Ω is minimal, then $\mathfrak{C}_1 \cdots \mathfrak{C}_r = \mathfrak{N}$ by Lemma 4. $\mathfrak{C}_1 \cdots \mathfrak{C}_{t-1}$ is a normal Ω-Ω_0 ideal which is not contained in $\mathfrak{N}$. Hence $\mathfrak{C}_1 \cdots \mathfrak{C}_{t-1}$ is the distance ideal of Ω to Ω_t. If Ω is maximal then $l(\Omega_i/\mathfrak{C}_i) = n - m_{r-(i-2)}$. Hence $l(\Omega/\mathfrak{C}_1 \cdots \mathfrak{C}_t) = tn - \sum_{i=1}^{t} m_{r-(i-2)} = (t-1)n + \sum_{i=2}^{r-(t-1)} m_i$. Let $\mathfrak{L}_{t-1}$ be the distance ideal of Ω to Ω_{t-1}. Then $\mathfrak{C}_1 \cdots \mathfrak{C}_t = \mathfrak{N}^s \mathfrak{L}_{t-1}$ and $l(\Omega/\mathfrak{N}^s\mathfrak{L}_{i-1}) = ns + \alpha = (t-1)n + \sum_{i=2}^{r-(t-1)} m_i$ where $\alpha = l(\Omega - \mathfrak{L}_{t+1})$. Hence, $|n(t-1-s)| = |\sum_{i=2}^{r-(t-1)} m_i - \alpha| < n$, because $\alpha < n$ since $\mathfrak{L}_{t+1} \supsetneqq \mathfrak{N}$. Therefore, $s = t-1$ and $\mathfrak{L}_{t+1} = \mathfrak{N}^{-(t-1)}\mathfrak{C}_1 \cdots \mathfrak{C}_t$. We have proved the proposition from Remark.

COROLLARY 6. *Let R be local and Λ, Λ' be h-orders in Σ and Σ'. We asssume that the rank r of Λ is larger than that of Λ' by one, then the set of R-lattices E in V such that E is a Λ-Λ' module with $\tau_{\Lambda'}(E) = \Lambda'$ is linearly ordered with period r.*

LEMMA 5. *Let Λ and Λ' be h-orders in Σ and Σ' which are dual with E. Orders Γ and Γ' are dual then Γ_p and Γ'_p are of the same rank for all p. Conversely, if furthermore, the rank of Γ_p and Γ'_p is equal to or less than rank of Λ_p and Λ'_p for all p, then Γ and Γ' are dual.*

Proof. If Γ and Γ' are dual with respect to E, then so are Γ_p and Γ'_p with E_p for any p. Hence Γ_p and Γ'_p are of the same rank by [1], Theorem A.5. Conversely, we assume that Γ_p and Γ'_p are of the same rank. We obtain an order Γ'' containing Λ' such that Γ and Γ'' are dual with E' by Theorem 1. Hence Γ''_p and Γ'_p are of the same rank for any p. Therefore, there exists by [5], Theorem 6.1 a normal Γ''-Γ' ideal $\mathfrak{C}'$. It is clear that Γ and Γ' are dual with $E'\mathfrak{C}'$.

9) This proposition was pointed out to the author by Mr. Takeuchi.
10) See [4], §3.

THEOREM 3. *Let Σ and Σ' be the central simple K-algebras and let Λ and Λ' be h-orders in Σ and Σ'. Λ and Λ' are dual if and only if Λ_p and Λ'_p are of the same rank for all p.*

Proof. We assume $\Sigma=\Delta_n$ and $\Sigma'=\Delta_m$, $n\geqslant m$. Let Λ' be an h-order in Σ' which belongs to $\Phi=\{p_1,\cdots,p_r\}$-block. Then there exists an h-order Λ'_0 in Δ such that the Λ'_{0p_i} is a minimal h-order over R_{p_i} in Δ for $p_i\in\Phi$, and Λ'_q is a maximal order over R_q for $q\notin\Phi$ by [5], Theorem 1.2. Furthermore, we can find the two-sided ideal $\mathfrak{A}'_0$ in Λ'_0 such that $\mathfrak{A}'_{0p_i}$ is the radical of Λ'_{0p_i} for $p_i\in\Phi$ and $\mathfrak{A}'_{0q}=\Lambda'_{0q}$ for $q\notin\Phi$, cf. [5], Lemma 1.3. Let

$$\Lambda_0^*=\left.\begin{pmatrix}\Lambda'_0 & \mathfrak{A}'_0 & \mathfrak{A}'_0 & \cdots & \cdots & \mathfrak{A}'_0\\ \Lambda'_0 & \Lambda'_0 & \mathfrak{A}'_0 & \mathfrak{A}'_0 & \cdots & \mathfrak{A}'_0\\ \Lambda'_0 & \Lambda'_0 & \Lambda'_0 & \cdots & \cdots & \vdots\\ \vdots & & & & & \vdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \Lambda'_0\end{pmatrix}\right\}m.$$

Then Λ_0^* is an h-order in Σ', such that $\Lambda^*_{0p_i}$ is a minimal order for $p_i\in\Phi$ and Λ^*_{0q} is maximal for $q\notin\Phi$ by [5], Lemma 1.2. Let $E=I_1\oplus I_2\cdots\oplus I_m$, where

$$I_i=\left.\begin{pmatrix}\left.\begin{matrix}\mathfrak{A}'_0\\ \mathfrak{A}'_0\end{matrix}\right\}i-1\\ \mathfrak{A}'_0\\ \Lambda'_0\\ \vdots\end{pmatrix}\right\}n.$$

We can define naturally the operation of elements of Λ_0^* from the right side, namely first we consider the I_j as a right Λ'_0-module and $(x_{ij})e_{1m}=(x_{ij})\in I_m$ if $i=1$ and $=0$ if $i\neq1$, where the e'_{ij}'s are matrix units in Σ'. Let

$$f\begin{pmatrix}a_{i1}\\ a_{i2}\\ a_{ii}\\ b_{i\,i+1}\\ b_{in}\end{pmatrix}=\begin{pmatrix}\vdots & \overset{i}{a_{i1}} & \vdots\\ \vdots & a_{i\,i-1} & \vdots\\ 0 & a_{ii} & 0\\ \vdots & b_{i\,i+1} & \vdots\\ \vdots & b_{im} & \vdots\end{pmatrix}\in\Lambda_0^*\ \text{for}\ \begin{pmatrix}a_{ij}\\ b_{ik}\end{pmatrix}\in I_i,$$

where $a_{ij}\in\mathfrak{A}'_0$ $b_{ij}\in\Lambda'_0$. Then it is clear that $f\in \mathrm{Hom}^r_{\Lambda_0^*}(E,\Lambda_0^*)$ and $f(E)=\Lambda_0^*$. Therefore $\tau_{\Lambda_0^*}(E)=\Lambda_0^*$. Since E is an R-lattice in V, $\mathrm{Hom}^r_{\Lambda_0^*}(E,E)=\Lambda_0(\subset\Sigma)$ and Λ_0^* are dual. Hence Λ'_{0p} and Λ^*_{0p} are of the same rank for all p. Let Γ, Γ' be h-orders in Σ and Σ' such that Γ_p and Γ'_p are of the same rank for all p. We assume Γ and Γ' belong to $\Phi=\{p_1,\cdots,p_r\}$. Since $\Lambda^*_{0p_i}$ is minimal, the rank of Γ'_{p_i} is equal to or larger than that of Λ^*_{0p} for all i. Hence, Γ and Γ' are dual by Lemma 5.

COROLLARY 7. *Let $\Sigma=\Delta_n$ and $\Sigma'=\Delta_m$. If $n\geqslant m$, for every h-order Γ' in Σ' there exists an h-order Γ in Σ such that Γ and Γ' are dual.*

COROLLARY 8. *The relation of duality is an equivalent relation.*

We shall generalize Corollary 2 by using the method of [1], Theorem A. 5.

THEOREM 4. *Let Λ and Λ' be h-orders in Σ and Σ' which are dual with respect to E. Then the groupoid G of two-sided Λ-ideals is isomorphic to the groupoid G' of two-sided Λ'-ideals and the units Γ of G correspond to the units* $\mathrm{Hom}^l_\Lambda(\mathfrak{D}E, \mathfrak{D}E)$ *of G', where $\mathfrak{D}'=D_\Lambda(\Gamma)$.*

Proof. Let $\mathfrak{A}'$ be a two-sided ideal of Λ'. Then

$$F(\mathfrak{A}') = E\underset{\Lambda'}{\otimes}\mathfrak{A}'\underset{\Lambda'}{\otimes}\mathrm{Hom}^l_\Lambda(E,\Lambda)$$

is a two-sided ideal of Λ since $r\mathfrak{A}'\subseteq\Lambda'$ for $r\in R$ and $0\to rF(\mathfrak{A}')\to E\underset{\Lambda'}{\otimes}\Lambda'\underset{\Lambda'}{\otimes}\mathrm{Hom}^l_\Lambda(E,\Lambda)=E\underset{\Lambda'}{\otimes}\mathrm{Hom}^l_\Lambda(E,\Lambda)\approx\tau_\Lambda(E)=\Lambda$. $F(\mathfrak{A}')F(\mathfrak{B}')=F(\mathfrak{A}')\underset{\Lambda}{\otimes}F(\mathfrak{B}')=E\underset{\Lambda'}{\otimes}\mathfrak{A}'\underset{\Lambda'}{\otimes}\mathrm{Hom}_\Lambda(E,\Lambda)\underset{\Lambda}{\otimes}E\underset{\Lambda'}{\otimes}\mathfrak{B}'\underset{\Lambda'}{\otimes}\mathrm{Hom}^l_\Lambda(E,\Lambda)=E\underset{\Lambda'}{\otimes}\mathfrak{A}'\underset{\Lambda'}{\otimes}\mathfrak{B}'\underset{\Lambda'}{\otimes}\mathrm{Hom}^l_\Lambda(E,\Lambda)=F(\mathfrak{A}'\mathfrak{B}')$ by [1], Proposition A. 1. Let $\mathfrak{A}$ be a two-sided ideal of Λ. Then

$$G(\mathfrak{A}) = \mathrm{Hom}^l_\Lambda(E,\mathfrak{A}\underset{\Lambda}{\otimes}E)$$

is a two-sided ideal of Λ' since if $r\mathfrak{A}\subseteq\Lambda$, $G(r\mathfrak{A})\subseteq\mathrm{Hom}^l_\Lambda(E,E)=\Lambda'$. Furthermore, $FG(\mathfrak{A})=E\underset{\Lambda'}{\otimes}\mathrm{Hom}_\Lambda(E,\mathfrak{A}\underset{\Lambda}{\otimes}E)\underset{\Lambda}{\otimes}\mathrm{Hom}_\Lambda(E,\Lambda)=\mathrm{Hom}^l_\Lambda(\mathrm{Hom}^r_{\Lambda'}(E,E),\mathfrak{A}\underset{\Lambda}{\otimes}E)\underset{\Lambda'}{\otimes}\mathrm{Hom}^l_\Lambda(E,\Lambda)=\mathfrak{A}\underset{\Lambda}{\otimes}E\underset{\Lambda'}{\otimes}\mathrm{Hom}^l_\Lambda(E,\Lambda)=\mathfrak{A}$. $GF(\mathfrak{A}')=\mathrm{Hom}^l_\Lambda(E,E\otimes\mathfrak{A}')\underset{\Lambda'}{\otimes}\mathrm{Hom}_\Lambda(E,\Lambda)\underset{\Lambda}{\otimes}E)=\mathrm{Hom}_\Lambda(E,E\otimes\mathfrak{A}'\underset{\Lambda'}{\otimes}\mathrm{Hom}_\Lambda(E,E))=\mathrm{Hom}_\Lambda(E,E\underset{\Lambda'}{\otimes}\mathfrak{A}')=\mathrm{Hom}(E,E)\otimes\mathfrak{A}'=\mathfrak{A}'$ by Lemma 3. Finally, let Γ be an order containing Λ, and $\mathfrak{D}=\mathfrak{D}_\Lambda(\Gamma)$. Then $\Gamma=\mathrm{Hom}^l_\Lambda(\mathfrak{D},\mathfrak{D})$. $G(\Gamma)=\mathrm{Hom}_\Lambda(E,\Gamma\underset{\Lambda}{\otimes}E)=\mathrm{Hom}_\Lambda(E,\mathrm{Hom}^l_\Lambda(\mathfrak{D},\mathfrak{D})\underset{\Lambda}{\otimes}E)=\mathrm{Hom}_\Lambda(E,\mathrm{Hom}^l_\Lambda(\mathfrak{D},\mathfrak{D}\underset{\Lambda}{\otimes}E))=\mathrm{Hom}^l_\Lambda(\mathfrak{D}\underset{\Lambda}{\otimes}E,\mathfrak{D}\underset{\Lambda}{\otimes}E)=\Gamma'$.

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