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MULTIPLICATIVE IDEAL THEORY IN HEREDITARY ORDERS

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Let R be a Dedekind domain and K its quotient field, and Σ a central simple K-algebra with finite rank over K. We have already defined a hereditary order Λ over R in Σ in [5], [6] and [7], namely Λ is a hereditary ring in Σ such that Λ is a finitely generated R-module and $\Lambda K=\Sigma$. In §§3-6 in [5] and in [6] we have studied properties of hereditary orders (briefly h-order) over local ring in Σ, and in [7] we have extended those properties to the global case in the generalized quaternions.

In this paper we shall try to extend those results to the global case in any central simple K-algebra.

In §1 we shall generalize the results in [5], [7] (Theorem 1.5) and obtain a decomposition theorem of two-sided ideal in an h-order Λ (Theorem 1.4). Furthemore, we obtain that every left (right) order of a one-sided ideal of Λ is an h-order (Theorem 1.1). Hence, if we consider one-sided ideal A of Λ, we know that it is a left (right) ideal of the left (right) order of A which is hereditary.

We shall consider, in §3, a decomposition of one-sided ideal of h-order by the characteristic product of normal and maximal one-sided ideals. In order to cosider it in global case, in §2 we first study it in the local case.

We obtain that for two orders Λ^i, Λ^j which are of the same typs all left Λ^i- and right Λ^j-ideal A^{ij} except finite number is expressed as above (Theorem 3.2) and those decompositions are unique up to left (right) quasi-equivalence (Theorem 3.4). If A^{ij} is not contained in any regular and maximal one-sided ideals then A^{ij} is locally principal and Λ^i, Λ^j are locally isomorphic (Theorem 3.5).

In §4 we define the ideal class and isomorphic class of h-orders and we obtain that those numbers are finite if R is the ring of integers (Theorem 4.1).

Finally in §5 we consider the different (discriminant) theorem in h-orders which is a slight generalization of the theorem in maximal orders (Theorems 5.1 and 5.2).

1. Normal ideals and inversible ideals.

Let R be a Dedekind domain and K its quotient field, and Σ a central simple K-algebra. By order we mean an order Λ over R in Σ.

DEFINITION 1. *Let Λ be an order in Σ and A a left Λ-module in Σ. If $AK=\Sigma$, then we call A is a left ideal of Λ. Furthermore, if A is contained in Λ, then A is called "integral".*

DEFINITION 2. *Let A be a finitely generated R-module in Σ such that $AK=\Sigma$. $\Lambda^i=\{x \mid \in\Sigma, xA\subseteqq A\}$, $\Lambda^j=\{x\mid\in\Sigma, Ax\subseteq A\}$ are orders and called the left order of A and right order of A respectively. We denote Λ^i, Λ^j and A by $\Lambda^l(A)$, $\Lambda^r(A)$ and A^{ij}.*

DEFINITION 3. *Let A^{ij} be an ideal of Λ^i and Λ^j. For $A=A^{ij}$ $A^{-1}=\{x\in\Sigma, Ax\subseteqq\Lambda^i\}=\{x\mid\in\Sigma, xA\subseteqq\Lambda^j\}=\{x\mid\in\Sigma, AxA\subseteqq A\}$. If $\Lambda^i=A^{ij}(A^{ij})^{-1}$ and $\Lambda^j=(A^{ij})^{-1}A^{ij}$ then we call A^{ij} is "inversible".*

If we say that a left ideal A of Λ is "*normal*", then we mean that $\Lambda=\Lambda^l(A)$.

Let A be a left Λ-module. By trace ideal of A we mean the two-sided ideal in Λ which is generated by $f(a)$, where f runs through all elements of $\mathrm{Hom}^l_\Lambda(A,\Lambda)$ and $a\in A$. We shall denote it by $\tau^l_\Lambda(A)$.

From the definitions we have

LEMMA 1.1. *For orders Λ^i, Λ^j and the ideal A^{ij}, we have $A^{ij}(A^{ij})^{-1}=\tau^l_{\Lambda^i}(A)$, and $(A^{ij})^{-1}A^{ij}=\tau^r_{\Lambda^j}(A^{ij})$.*

COROLLARY 1.1. *An ideal A^{12} is inversible if and only if $\tau^l_{\Lambda^1}(A^{12})=\Lambda^1$, $\tau^r_{\Lambda^2}(A^{12})=\Lambda^2$.*

COROLLARY 1.2. *Let Λ be an h-order and A a left ideal of Λ. A is inversible if and only if $\tau^l_\Lambda(A)=\Lambda$.*

Proof. If A is inversible then $\tau^l_\Lambda(A)=AA^{-1}=\Lambda$. Conversely if $\tau^l_\Lambda(A)=\Lambda$, $\tau^r_{\Lambda^r(A)}(A)=\Lambda^r(A)$ by [3], Theorem A. 1.

It is clear from this Corollary and [3], Theorem A. 2 that the category of inversible ideals in an h-order coincides with that of normal ideals.

COROLLARY 1.3. *Let Λ be an order and A an integral inversible left ideal in Λ. Then A is an integral right $\Lambda^r(A)$-ideal.*

Proof. Since $\Lambda\subseteq A^{-1}$, $\Lambda^r(A)=A^{-1}A\supseteq\Lambda A=A$.

LEMMA 1.2. *Let S be a ring and E a finitely generated left projective*

S-module, and $T=\mathrm{Hom}_S^l(E, E)$. If $\tau_S^l(E)=S$ and E is a finitely generated T-module, then r.gl. dim $S\geqslant r$.gl. dim T, (l.gl. dim $T\geqq l$.gl. dim S).

Proof. Let F be a right T-module and $\to P_n\to P_{n-1}\cdots\to P_0\to F\otimes_T \mathrm{Hom}_T^r(E, T)\to 0$ a projective resolution of $F\otimes_T \mathrm{Hom}_T^r(E, T)$ as a right S-module by setting $(g\otimes f(e))s=g\otimes f(se)$ for $f\in \mathrm{Hom}_T^r(E, T)$, $e\in E$ and $g\in F$. Since E is S and T-projective, by [3], Theorem A. 2 $\to P_n\otimes_S E\to\cdots\to P_0\otimes_S E\to F\otimes_S \mathrm{Hom}_T^r(E, T)\otimes_S E\to 0$ is a projective resolution as a right T-module. However, $\mathrm{Hom}_T^r(E, T)\otimes_S E\approx T$ by [3], Theorem A. 4. Hence $\to P_n\otimes E\to\cdots\to P_0\otimes E\to F\to 0$ is a projective resolution of F, which proves the lemma. Since $\tau_T^r(E)=T$ and $S=\mathrm{Hom}_T(E, E)$, we obtain l.gl. dim $T\geqslant l$.gl. dim S by exchanging S and T.

THEOREM 1. 1. *Let Λ be an h-order and A a left ideal of Λ. Then $\Lambda^l(A)$ and $\Lambda^r(A)$ are h-orders. Hence A is inversible in $\Lambda^l(A)$ $(\Lambda^r(A))$.*

Proof. Let $\Lambda'=\Lambda^l(A)$, then $\Lambda'\supset\Lambda$. Hence Λ' is an h-order by [5], Coro. 1. 4. It is clear that $\mathrm{Hom}_{\Lambda'}^l(A, A)=\Lambda^l(A)$. Since A is a projective Λ'-module, $A=\tau_\Lambda^l(A)A$ by [3] Proposition A. 3. Hence, $\tau_{\Lambda'}^l(A)=AA^{-1}=\tau_{\Lambda'}^l(A)AA^{-1}=(\tau_{\Lambda'}^l(A))^2$. However, $\tau_{\Lambda'}^l(A)$ is a two-sided ideal in Λ', and hence $A=\tau_\Lambda^l(A)A$ implies $\Lambda'\supseteqq \mathrm{Hom}_{\Lambda'}^r(\tau_{\Lambda'}^l(A), \tau_{\Lambda'}^l(A))$. Therefore, $\tau_{\Lambda'}^l(A)=\Lambda'$ by [5], Theorem 1. 7. Thus A is a projective right $\Lambda^r(A)$-module. Hence $\Lambda^r(A)$ is hereditary by Lemma 1. 2.

Let $\Gamma\supseteqq\Lambda$ be h-orders. We recall the definition of left (right) conductors. Let $C_\Lambda(\Gamma)=\{x\mid\in\Sigma, \Gamma x\subseteq\Lambda\}$, and $D_\Lambda(\Gamma)=\{x\mid\in\Sigma, x\Gamma\subseteq\Lambda\}$. We call $C_\Lambda(\Gamma)$, $D_\Lambda(\Gamma)$ left and right conductor of Γ with respect to Λ. It is clear that $C_\Lambda(\Gamma)$ $((D_\Lambda)\Gamma))$ is a unique maximal left Λ- and right Γ- (right Γ- and left Λ-) module in Λ. If Λ is an h-order, then $\Gamma=\Lambda^l(C_\Lambda(\Gamma))$ by [5], Theorem 1. 7. Hence, $C_\Lambda(\Gamma)$ is regular ideal of Γ.

COROLLARY 1. 4. *Let Λ be an h-order and Γ an order. Then $\Lambda^l(C_\Lambda(\Gamma))$ is an h-order which contains Γ and is contained in h-orders containing Λ and Γ. Furthermore, $C_\Lambda(\Gamma)$ is a two-sided Λ-module, if and only if $\Lambda^l(C_\Lambda(\Gamma))\supseteqq\Lambda$.*

Proof. Since $C_\Lambda(\Gamma)$ is a left Γ-module, $\Lambda^l(C_\Lambda(\Gamma))\supseteqq\Gamma$. By the theorem $\Lambda^l(C_\Lambda(\Gamma))$ is an h-order. We assume that the ring Ω generated by Λ and Γ is an order. Let $C=C_\Lambda(\Omega)$, then $C\subseteqq C_\Lambda(\Gamma)$. Let $x\in\Lambda^l(C_\Lambda(\Lambda))$, then $xC\subseteqq xC_\Lambda(\Gamma)\subseteqq C_\Lambda(\Gamma)\subseteqq\Lambda$. Since C is idempotent, $xC\subseteqq C$, and hence $x\in\Lambda^l(C)=\Omega$. If $\Lambda^l(C_\Lambda(\Gamma))\supseteqq\Lambda$, then $\Omega=\Lambda^l(C_\Lambda(\Gamma))$. Hence, $C_\Lambda(\Gamma)\subseteqq C\subseteqq C_\Lambda(\Gamma)$.

For two h-orders $\Lambda \subseteq \Gamma$, there exists a finite set $\Phi(\Lambda, \Gamma)$ of prime ideal p in R such that $\Lambda_p \subsetneqq \Gamma_p$ for $p \in \Phi$ and $\Lambda_q = \Gamma_q$ for $q \notin \Phi$. If Γ is maximal, then $\Phi(\Lambda, \Gamma)$ does not depend on Γ, and we call that Λ belongs to Φ-block (cf. [5], Section 7).

By simple argument (cf. [5], Lemma 7. 5), we have

LEMMA 1. 3. *Let Λ be an order over R and A a one-sided ideal in Λ_p. Then there exists a unique ideal A' in Λ such that $A'_p = A$, and $A'_q = \Lambda_q$ for $p \neq q$.*

Let Λ be an h-order, $\{M_{i,j}\}$ be the set of maximal two-sided ideals in Λ_{p_i}, and $N(\Lambda_{p_i})$ the radical of Λ_{p_i}. Then there exist, by Lemma 1. 3, maximal ideals P_{ij} and ideal $Q_\Lambda(p_i)$ in Λ such that $(P_{ij})_{p_i} = M_{ij}$, $Q_\Lambda(p_i)_{p_i} = N(\Lambda_{p_i})$ and $(P_{ij})_q = Q_\Lambda(p_i)_q = \Lambda_q$ for $q \neq p_i$.

We have obtained in [5], Theorem 7. 6

THEOREM 1. 2. *Let Λ be an h-order in Σ. Then the set of normal two-sided ideals in Λ is an abelian group which is generated by $\{Q_\Lambda(p), P_i\}$.*

Let Λ be an order and Ω a maximal order containing Λ. A unique maximal two-sided Ω-ideal in Λ is called the *two-sided conductor* (of Ω with respect to Λ) and we shall denote it by $F_\Lambda(\Omega)$.

THEOREM 1. 3. *Let Λ be an h-order in $\Phi = \{p_i\}_{i=1}^m$-block. Then $F_\Lambda(\Omega) = \prod_{p_i \in \Phi} Q_\Omega(p_i)$. Conversely, we assume that R/P is finite field, and $\Omega = (\Omega_0)_n$ where Ω_0 is a maximal order in the associated division ring Δ of Σ. If $\Gamma \supseteq \Omega_0$ is an order such that $F_\Gamma(\Omega) = \prod_{p_i \in \Phi} Q_\Omega(p_i)$ then there exists an h-order Γ' containing Γ such that $(\prod Q_{\Gamma'}(p_i)) \cap \Gamma = \prod Q_\Gamma(p_i)$ and the rank of Γ'_{p_i} is equal to or larger than the number of two-sided simple compoments of $\Gamma_{p_i}/Q_\Gamma(p_i)_{p_i} = \Gamma_{p_i}/N(\Gamma_{p_i})$. Furthermore, Γ' is a minimal one among h-orders containig Γ.*

Proof. Let p be a prime in Φ. We assume that Λ, Ω be orders over $R_p(=R)$. If Λ is an h-order, then $\Lambda \supset N(\Lambda) \supset N(\Omega)$, and hence $F_\Lambda(\Omega) = N(\Omega)$. Thus, we have obtained the first part of the theorem. Next, we assume that $\Gamma \supset N(\Omega)$, $\Gamma \supset \Omega_0$ and R/p is a finite field. Since $N(\Omega) \supset p\Omega \supset p\Gamma$ and $p\Omega = N(\Omega)^t$ for some t, $N(\Omega)^{t+1} = pN(\Omega) \subset p\Gamma$. Hence $N(\Omega) \subseteq N(\Gamma)$. We denote $N(\Omega)$, $N(\Gamma)$ by N, N' and $\bar{\Omega} = \Omega/N$ and so on. We may asume that R is complete by [6], Proposition 1. 1. Then by assumption $\bar{\Omega}_0$ is a field and $\bar{\Omega} = (\bar{\Omega}_0)_n \supset \bar{\Lambda} \supset \bar{\Omega}_0$. Let M be a simple left ideal in $\bar{\Omega}$, then $\bar{\Omega} = \mathrm{Hom}^r_{\bar{\Omega}_0}(M, M)$. Let $M = M_0 \supset M_1 \supset \cdots \supset M_r = (0)$ be a composition series of M as a left Γ-module. Since $\Gamma\supset\Omega_0$, the M_i's are Ω_0-modules. Let $\Gamma'=\{x|\in\Omega,\ xM_i\supseteq M_i$ for all $i\}$. Then $\Gamma'\supseteq N$ and $\Gamma'/N\cong\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$. Since $N'(M_i/M_{i+1})=0$, $N'\subseteq N(\Gamma')$. Now we shall consider a natural ring homomorphism φ of Γ/N' to $\Gamma'/N(\Gamma')$. Let $\varphi^{-1}(0)=\Gamma/N'\bar{e}$, $\bar{e}$ is idempotent in Γ, then $e(M_i/M_{i+1})=0$ for all i. Since M is a faithful $\bar{\Gamma}$-mocule, $\bar{e}=0$. Hence, φ is monomorphic. Since M is a faithful Γ-module, all left simple components appear in $\{M_{i-1}/M_i\}_{i=1}^{r}$. Hence r is equal to or larger than the number of simple component in Γ/N' as two-sided ideals. By [6], Theorem 6.2, Γ is an r th h-order in Ω. If Γ'' is an h-order such that $\Gamma'\supset\Gamma''\supset\Gamma$, then $M_0\supset M_1\supset\cdots$ is also a composition series as left Γ''-module. Hence the number of simple components in $\Gamma''/N(\Gamma'')$ as a two-sided does not exceed that of Γ'/N. Therefore $\Gamma'=\Gamma''$.

From the above proof we have

COROLLARY 1.5. *Let $\Omega=(\Omega_0)_n$; Ω_0 is a unique maximal order in a commutative field over K. If $F_\Lambda(\Omega)=\Pi Q_\Omega(p_i)$ for an order Λ in Ω, then there exists an h-order containing Λ as in the thorem.*

DEFINITION 4. *Let P be a two-sided ideal in Λ. P is called prime if $AB\subseteq P$ for two-sided ideals A, B in Λ implies $A\subseteq P$ or $B\subseteq P$.*

LEMMA 1.4. *Let Λ be an order over R, then every prime ideal is a maximal two-sided ideal.*

We have a special case of [8], Satz 18

PROPOSITION 1.2. *Let Λ be an h-order which belongs to $\Phi=\{p_i\}$-block. Then for a prime ideal P in Λ P is regular if and only if $(P, F_\Lambda(\Omega))=\Lambda$.*

Proof. If $(P, F)=\Lambda$, $\Lambda_p=(P_p, F_p)=(P_p, N(\Omega_p))=(P_p, N(\Lambda_p))$ for $p\in\Phi$. Since P_p is maximal, $P_p=\Lambda_p$. If $p\notin\Phi$, P_p is equal to Λ_p or $N(\Lambda_p)$ which is regular. Hence P is regular. Conversely if P_p is regular then so is P_p for all p. Hence, since P_p is maximal, $P_p=\Lambda_p$ for $p\in\Phi$. Therefore, $(P, F_\Lambda(\Omega))=\Lambda$.

THEOREM 1.4. *Let Λ be an h-order and A a two-sided ideal in Λ. Then $A=P_1P_2\cdots P_rQ_1Q_2\cdots Q_sA_0$, where the P's are normal prime ideals, the Q's are maximal ones among normal ideals Q in Λ such that $Q\supseteq F_\Lambda(\Omega)$, A_0 is not contained in normal ideals, and $A_0\supseteq F^\rho$ for some ρ. The P's commute with P, Q and A_0 and the Q's commute with Q_i. Furthermore, this expression is unique, where $F=F_\Lambda(\Omega)$ for a maximal order Ω containing Λ (cf. [8], Satz 19).*

Proof. Let P be a maximal normal ideal containing A. Then $\Lambda \supseteqq P^{-1}A$. Repeating this argument, we obtain a set of maximal normal ideals P_i such that $A=P_1P_2\cdots P_rA'_0$, and A'_0 is not contained in maximal normal ideals since Λ is noetherian. We assume $A'_0 \neq \Lambda$. If $A'_0 \not\supseteq F^\rho$ for all $\rho \geqq 0$, then there exists, by Theorem 1.3, p such that $A'_{0p} \subsetneqq \Omega_p = \Lambda_p$, which contradicts the assumption of A'_0. Next if we repeat the above argument for maximal ideals of normal ideals $Q_i \supseteqq A'_0$ such that $Q_i \supseteqq F$. We have $A'_0=Q_1Q_2\cdots Q_sA_0$ and A_0 has the property as in the theorem. Since $(P_i)_q=N(\Omega_q)=N(\Lambda_q)$ for $q\notin\Phi$ (Λ belongs to Φ-block) and $Q_j=Q_\Lambda(p_j)$ for $p_j\in\Phi$, we know P_i, Q_j satisfy the conditions in the theorem. Let $A=P_1P_2\cdots P_rQ_1\cdots Q_sA_0=P'_1P'_2\cdots P'_{r'}Q'_1\cdots Q'_{s'}A'_0$. Since $(P_i, F)=\Lambda$, P_i contains neither Q'_j nor A'_0, and hence $\{P_1, P_2, \cdots, P_r\}\equiv\{P'_1, P'_2, \cdots, P'_r\}$. Thus we have obtained $Q_1Q_2\cdots Q_sA_0=Q'_1Q'_2\cdots Q'_{s'}A'_0$. We may assume $Q_1=Q_2\cdots =Q_t=Q'_1=\cdots=Q'_{t'}=Q(p)$ for $p\in\Phi$, and $Q_j\neq Q(p)$ $Q'_{j'}\neq Q(p)$ for $j>t$, $j'>t'$. If $t>t'$, $N(\Lambda_p)^{t'-t}\supseteqq(A_0)_p$, and hence A_0 is contained in Q_1, which is a contradiction.

We shall recall the definition of characteistic product, see [5], § 6.

DEFINITION 5. *Let A B be module in Σ. If $AB=A'B'$ for modules $A'\supseteqq A$ and $B'\supseteqq B$ implies $A'=A$, $B'=B$, then we call the product AB is characteristic.*

The substasntial parts of the following results are already proved in the case of maximal orders in [1], [5].

Let Λ^i, Λ^j be orders and by a left Λ^i- and right Λ^i-ideal A we mean that $\Lambda^i=\Lambda^l(A)$ and $\Lambda^j=\Lambda^r(A)$, and we shall say briefly Λ^i, Λ^j-ideal, and denote it by A^{ij}.

LEMMA 1.5. *The product $A^{ij}A^{kl}$ is characteristic if and only if $\Lambda^j=\Lambda^k$, (see [1], or [5]).*

We have a generalization of [5], Theorem 6.1.

THEOREM 1.5. *The set of h-orders in Σ and the set of one-sided ideals have a structure of a groupoid with respect to characteristic products. The set of h-orders consists of units in this groupoid. The set of two-sided ideals of a given h-order is an abelian group. Conversely, let $\boldsymbol{G}$ be a set of R-submodules in Σ containing elements in K. We assume*

1) *$\boldsymbol{G}$ is a groupoid. If AB is defined in $\boldsymbol{G}$ for $A, B\in\boldsymbol{G}$ then AB is a product as a R-module in Σ.*

2) *The units in $\boldsymbol{G}$ are all orders.*

3) *Let Λ be a unit in $\boldsymbol{G}$ and L, S left ideal and right ideal in Λ, respec-*

tively. Then $L, S \in \boldsymbol{G}$ and the left unit Γ of L (the right unit Γ' of S) is left Λ-projective (right Λ-projectives).
Then the set of units in $\boldsymbol{G}$ consists of h-orders and $\boldsymbol{G}$ consists of groupoid obtained from one-sided ideals with characteristic product.

Proof. The first part of the theorem is clear from Theorem 1.1 and [5], Theorem 6.1. Let Λ be a unit in $\boldsymbol{G}$ and L a left ideal in Λ. Then $LL^{-1}=\Lambda^1$, $L^{-1}L=\Lambda^2$. Let Q be a two-sided ideal in Λ such that $Q_p=N(\Lambda_p)$ and $Q_q=\Lambda_q$ for $p\neq q$. $Q_p(Q^{-1})_p=(\Lambda^1)_p$ and $(Q^{-1})_pQ_p=(\Lambda^2)_p$. Hence, Q_p is left Λ^1_p-projective. Since Λ^1_p is left Λ_p-projective, so is Q_p. Therefore, Λ_p is an h-order, by [5], Lemma 3.6, which implies that Λ is an h-order. It is clear that $\Lambda^l(L)\supset\Lambda^1$. Since $LL^{-1}=\Lambda^1$, $\Lambda^l(L)\Lambda^1=\Lambda^l(L)LL^{-1}=\Lambda^1$. Hence $\Lambda^1=\Lambda^l(L)$. Similarly, we obtain $\Lambda^2=\Lambda^r(L)$. It is clear that $\boldsymbol{G}$ consists of all one-sided ideals of Λ's. If AB is defined in $\boldsymbol{G}$, then $\Lambda^r(A)=\Lambda^l(B)$. Hence AB is characteristic.

REMARK 1. We have considered this theorem in [5], Theorem 6.1 for two-sided ideals of an h-order Λ. In this case if we omit the assumption "*projective*", then the converse is not true, namely there exists non h-order Λ such that every two-sided ideal A of Λ is inversible in $\Lambda^l(A)$ and $\Lambda^r(A)$. For example, let $\Lambda=\begin{pmatrix}R & p\\ p & R\end{pmatrix}$. Then we can easily check that evey two-sided ideal Λ in is isomorphic as a two-sided Λ-module to one of the following:

i) $\begin{pmatrix}R & R\\ R & R\end{pmatrix}$ ii) $\begin{pmatrix}R & R\\ R & p\end{pmatrix}, \begin{pmatrix}R & R\\ p & R\end{pmatrix}, \begin{pmatrix}p & R\\ R & R\end{pmatrix}, \begin{pmatrix}R & p\\ R & R\end{pmatrix}, \begin{pmatrix}R & p\\ p & p\end{pmatrix}, \begin{pmatrix}p & p\\ p & R\end{pmatrix}, \begin{pmatrix}p & R\\ p & p\end{pmatrix},$
$\begin{pmatrix}p & p\\ R & p\end{pmatrix},$ iii) $\begin{pmatrix}R & R\\ p & p\end{pmatrix}, \begin{pmatrix}p & p\\ R & R\end{pmatrix},$ iv) $\begin{pmatrix}R & p\\ p & R\end{pmatrix},$ v) $\begin{pmatrix}p & R\\ R & p\end{pmatrix},$ vi) $\begin{pmatrix}R & p\\ p & p^2\end{pmatrix}, \begin{pmatrix}p^2 & p\\ p & R\end{pmatrix},$
$\begin{pmatrix}p & R\\ p^2 & p\end{pmatrix}, \begin{pmatrix}p & p^2\\ R & p\end{pmatrix}.$

The left (right) orders of i) iii) and vi) are maximal and those of ii) and iv) are h-orders. Hence, those ideals A are inversible in $\Lambda^l(A)$ and $\Lambda^r(A)$. Let $A=\begin{pmatrix}p & R\\ R & p\end{pmatrix}$ then $\Lambda^l(A)=\begin{pmatrix}R & p\\ p & R\end{pmatrix}=\Lambda^r(A)$ and $AA^{-1}=\Lambda^l(A)=\Lambda^r(A)$. Hence, A is inversible in $\Lambda^r(A)=\Lambda^l(A)$. However Λ is not an h-order.

Let Λ^i and Λ^j be h-orders. If there exists Λ^i, Λ^j-ideal A^{ij} we call Λ^i and Λ^j are of the *same type*. It is clear that we may assume $A^{ij}\subset\Lambda^i\cap\Lambda^j$.

LEMMA 1. 6. *Let* Λ^i, Λ^j *be h-orders over R. If* Λ^i_p *and* Λ^j_p *are of the same rank*[1] *for all* p *then there exists a finite set* Φ *of prime ideals* p_i *in R such that* $\Lambda^i_q = \Lambda^i_q$ *for* $q \notin \Phi$.

Proof. Let $c = \{x \mid \in R, \Lambda^i x \subseteq \Lambda^j\}$. Then c is a non-zero ideal in R and let Φ be the set of prime ideals in R which divides c. If $q \notin \Phi$, then $c_q = R_q$ and hence $\Lambda^i_q \subseteq \Lambda^j_q$. However since Λ^i_q and Λ^j_q are of the same rank, we obtain $\Lambda^i_q = \Lambda^j_q$.

LEMMA 1. 7. *Let* Λ^i, Λ^j *be h-orders. If* Λ^i_p *and* Λ^j_p *are of the same rank for a prime* p *in R, then we can find an h-order* Λ^0 *such that* $\Lambda^0_p = \Lambda^j_p$, $\Lambda^0_q = \Lambda^i_q$ *for* $q \neq p$.

Proof. There exists a Λ^i_p, Λ^j_p-ideal $A^{ij}(p)$ by assumption and [5], Theorem 4. 4. Then we obtain, by Lemm 1. 3, a Λ^i, Λ^j-ideal A such that $A_p = A^{ij}(p)$, $A_q = \Lambda^i_q$ for $q \neq p$. Hence, $\Lambda^r(A)$ is a desired order.

THEOREM 1. 6. Λ^i *and* Λ^j *are of the same type if and only if* Λ^i_p *and* Λ^j_p *are of the same rank for all* p.

Proof. If there exists an ideal A^{ij}, then A^{ij}_p, is a Λ^i_p, Λ^j_p-ideal for any p. Hence Λ^i_p and Λ^j_p are of the same rank by [5], Propo. 4. 4. Conversely, we assume that Λ^i_p and Λ^j_p are of the same rank for all p, and hence there exists a Λ^i_p, Λ^j_p-ideal $A^{ij}(p)$ by [6], Theorem 4. 4. First we assume the set of prime ideals as in Lemma 1. 6 consists of a single element, say p. Then Λ^i and Λ^j are of the same type by the proof of Lemma 1. 7. We assume that the theorem is true for Φ' which consists of $n-1$ elements and $\Phi = \{p_1, p_2, \cdots, p_n\}$. Then there exists an h-order Λ^0 such that $\Lambda^0_{p_1} = \Lambda^i_{p_1}$, $\Lambda^0_{p_i} = \Lambda^j_{p_i}$ for $i = 2, \cdots, n$. Hence Λ^i, Λ^0 are of the same type and Λ^0, Λ^j are of the same type by induction hypothesis. Therfore, Λ^i, Λ^j are of the same type.

2. Decomposition of one-sided ideals in a local case.

By virtue of Theorem 1. 4 we are interested ourselves to study a decomposition of A_0. Thus we shall study, in this section a decomposition of a normal one-sided ideal A as a characteristic product of maximal normal one-sided ideals. If we say "*product*" then we mean a characteristic product, and ideals are always normal.

First, we consider h-orders over R_p, and all orders and ideas are considered over R_p and we denote them by Ω, Λ and A.

1) See the definition in [6].

PROPOSITION 2.1. *Let Λ be an h-order over R_p and A a maximal left ideal in Λ. Then A is a maximal right $\Lambda^r(A)$-ideal.*

Proof. Let N be the radical of Λ and $\Lambda^r(A)=\Gamma$. Since $\Lambda=\Lambda^l(A)$, $A/N=\Delta_{m_1}\oplus\cdots\oplus l_i\oplus\cdots\oplus\Delta_{m_r}$, and $\Gamma/N(\Gamma)\approx\mathrm{Hom}^l_{\Lambda/N}(A/NA, A/NA)$ by [5], Propo. 4.4. We may assume that R_p is complete by the standard argument. Then there exists an idement element e in A such that $A=\Lambda e+N$. Furthermore $A/NA=(\Lambda e+NA)/NA\oplus N/NA$, and N/NA is a simple left Λ-module, since if $N\supset B\supset NA$ for a left Λ-module B then we have $\Lambda\supseteq N^{-1}B\supset A$. It is clear from [6], Lemma 3.2, that $\Gamma\supset A\supset N(\Gamma)$. Since $(N/NA)A=(0)$, $A/N(\Gamma)\approx\mathrm{Hom}^l_\Lambda((\Lambda e+NA)/NA, (\Lambda e+NA)/NA)$ is a maximal right ideal in $\Gamma/N(\Gamma)$.

PROPOSITION 2.2. *Let Λ^i, Λ^j be r^{th} h-orders over R_p. Then Λ^i, Λ^j-integral ideal is linearly ordered with respect to inclusion.*

Proof. $(A^{ij})^{-1}B^{ij}$ is a two-sided normal ideal of Λ^j, and hence $(A^{ij})^{-1}B^{ij}=N(\Lambda^i)^t$ for some t. If $t\geqslant 0$, $B^{ij}=A^{ij}N(\Lambda^i)^t\subseteq A^{ij}$. If $t<0$, then $A^{ij}=B^{ij}N(\Lambda^j)^t\subseteq B^{ij}$.

By virtue of Proposition 2.2 we have

DEFINITION 6. *Let Λ^α, Λ^β be r^{th} h-orders. A unique maximal integral left Λ^α, Λ^β-ideal is called the distance ideal of Λ^α to Λ^β, (notion $D^{\alpha\beta}$).*

It is clear that $D^{\alpha\beta}\not\subset N(\Lambda^\alpha)$, $N(\Lambda^\beta)$.

PROPOSITION 2.3. *Let Λ^i, Λ^j be r^{th} h-orders. Then $(\Lambda^j\Lambda^i)^{-1}$ is the distance ideal if and only if there exists an integral ideal A^{ij} such that $A^{ij}\Lambda^i$ (Λ^jA^{ij}) is a two-sided regular ideal of Λ^i (Λ^j). In this case $A^{ij}\Lambda^i=\Lambda^i$, $A^{ij}=(\Lambda^j\Lambda^i)^{-1}$.*

Proof. Let A^{ij} be an integral ideal such that Λ^jA^{ij} is a regular ideal. Let $B=\Lambda^j\Lambda^i$. It is clear that B is an ideal. $BA^{ij}=\Lambda^j\Lambda^iA^{ij}=\Lambda^jA^{ij}$. Hence $\Lambda^l(B)=\Lambda^j$. Therefore, $\Lambda^r(B)=\Lambda^i$, since $\Lambda^r(B)\supseteq\Lambda^i$. Since $B\supset\Lambda^i$, $B^{ij}=(B)^{-1}\subset\Lambda^i$. We obtain for an integral ideal C^{ij} that $C^{ij}B=A^{ij}=A^{ij}\Lambda^j\Lambda^i\subseteq\Lambda^i$. Hence $C^{ij}\subset B^{ij}$ which implies $B^{ij}=D^{ij}$. Conversely, we assume $(\Lambda^j\Lambda^i)^{-1}=D^{ij}$. Then $\Lambda^i=(\Lambda^j\Lambda^i)^{-1}(\Lambda^j\Lambda^i)=D^{ij}\Lambda^i$. Let $A^{ij}\Lambda^i=\Lambda^i$. $A^{ij}=D^{ij}N^t$ for some $t\geqslant 0$. Hence $\Lambda^i=A^{ij}\Lambda^i=D^{ij}N^t\Lambda^i\subseteqq N^t$. Therefore $A^{ij}=D^{ij}$.

COROLLARY 2.1. $D^{ij}=(\Lambda^j\Lambda^i)^{-1}$ *if and only if* $D^{ij}+N(\Lambda^i)/N(\Lambda^i)$ *does contain all left simple components in* $\Lambda^i/N(\Lambda^i)$.

PROPOSITION 2.4. *Let Λ^i, Λ^j be distinct h-orders containing the same h-order. If $D^{ij}=(\Lambda^j\Lambda^i)^{-1}$, then $D^{ji}=\Lambda^jN(\Lambda^i)$.*

Proof. Let $C=\Lambda^j N$ then $D^{ij}\Lambda^j N=D^{ij}\Lambda^i N=N$, where $N=N(\Lambda^i)$. Hence, $\Lambda^r(C)=\Lambda^i$, and $\Lambda^l(C)=\Lambda^j$. Since $C=\Lambda^j N\subseteq\Lambda^j N(\Gamma)\subseteq\Lambda^j$, C is integral, where $\Gamma=\Lambda\cap\Lambda^j$. Since $C=D^{ji}N^t$ for $t\geqq 0$, $\Lambda^i\subsetneqq\Lambda^j\Lambda^i=D^{ji}N^{t-1}$. Hence $t=0$.

DEFINITION 7. *Let Λ^α, Λ^β be r^{th} h-orders over R_p. If there exists a set of r^{th} h-orders Λ^i such that $\Lambda^\alpha=\Lambda^0, \Lambda^1, \cdots, \Lambda^n=\Lambda^\beta$, and $\Lambda^i=\Lambda^l(A^{i\,i+1})$. $\Lambda^{i+1}=\Lambda^r(A^{i\,i+1})$ and $A^{i\,i+1}$ is a maximal left Λ^i-ideal, then we call that there exists a path from Λ^α to Λ^β.*

Since $A^{i i+1}$ is also a maximal right Λ^{i+1}-ideal by Proposition 2. 1, we can obtain the same path of Λ^β to Λ^α by making use of right ideals.

$\prod_i A^{i\,i+1}$ is called the ideal of a path of Λ^α to Λ^β.

PROPOSITION 2. 5. *Let Λ^α, Λ^β be r^{th} h-orders. Then the ideals of path are linearly ordered by inclusion. Hence the number of r^{th} h-orders which appears in the shortest path is the same.*

Proof. Let $\prod A^{i\,i+1}$ be a path ideal. Then $\prod A^{i\,i+1}$ is a Λ^α, Λ^β-ideal by Proposition 2. 3. Since $A^{i\,i+1}$ is a maximal left ideal in Λ^i, $\Lambda^\alpha\supset A^{01}\supset A^{01}A^{12}\supset\prod A^{i\,i+1}$ is a composition series of $\Lambda^\alpha/\prod A^{i\,i+1}$. Therefore, the number of orders in a path is equal to the length of composition series of $\Lambda^\alpha/\prod A^{i\,i+1}$, which proves the proposition.

We shall denote the ideal of a shortest path of Λ^α to Λ^β by $P^{\alpha\beta}$.

We shall prove that there exists a path between non-minimal h-orders.

LEMMA 2. 1. *There exists a path between two maximal orders.*

Proof. Let $\Lambda^\alpha\supset A_1\supset A_2\supset\cdots\supset A_n=D^{\alpha\beta}$ be a composition series of $\Lambda^\alpha/D^{\alpha\beta}$. Let $\Lambda^i=\Lambda^l(A_i^{-1}A_{i+1})$ is a maximal left ideal in Λ^i. Therefore, $\Lambda^\alpha=\Lambda^0, \Lambda^1, \cdots, \Lambda^n=\Lambda^\beta$ is a path of Λ^α to Λ^β.

LEMMA 2. 2. *Let Λ^α, Λ^β be r^{th} h-orders containing an h-order. Then there exists a path of Λ^α to Λ^β.*[2]

Proof. Let Λ_0 be a minimal h-order contained in $\Lambda^\alpha\cap\Lambda^\beta$ and $C_{\Lambda_0}(\Lambda)=I(S_1, S_2, \cdots)$, where for the normal sequence[3] $\{M_i\}$ of maximal two-sided ideals in Λ_0 $S_1'=\{M_1, M_2, \cdots, M_{m_1}\}$, $S_2'=\{M_{t_2}, M_{t_2+1}, \cdots, M_{t_2+m_2-1}\}, \cdots$; $S_i'\neq\phi$, $S_i'\cap S_j'=\phi$, $t_i=\sum_{j=1}^{i-1}m_j$; $\bigcup S_j'=\{M_i\}$; $S_i=S_i'-M_{t_1+m_1-1}$, (see [6], §2). Let Λ be an $r+1^{th}$ h-order in Λ^α such that $C_{\Lambda_0}(\Lambda)=I(S_1-M_1, S_2, \cdots)$. Then $A^{\alpha\gamma}=D_\Lambda(\Lambda^\alpha)$ is a maximal left Λ^α-ideal, (see the

2) cf. the proof of Theorem 2. 3.
3) See the definition in [6].

proof [6], Theorem 5.1), then $\Lambda=\Lambda^{\alpha}\cap\Lambda^{\gamma}$, and $A^{\alpha\gamma}=C_{\Lambda_0}(\Lambda^{\alpha})\Lambda=I(S_1, S_2, \cdots)\Lambda$. Since $\Lambda^{\gamma}=\Lambda^{r}(A^{\alpha\gamma})$, $D_{\Lambda}(\Lambda^{\gamma})=A^{\alpha\gamma}$. We obtain, by [5], Theorems 2.3 and 5.1, that $C_{\Lambda}(\Lambda^{\gamma})=ND_{\Lambda}(\Lambda^{\gamma})N^{-1}=I(M_n, \{S_1-M_1\}, S_2, \cdots)\Lambda$, where $N=N(\Lambda)$. Since $C_{\Lambda}(\Lambda^{\gamma})=C_{\Lambda_0}(\Lambda^{\gamma})\Lambda=I\{(S_r+M_n\}, \{S_1-M_1\}, S_2, \cdots)$. Thus we have proved the following facts: $\Lambda^{\alpha}\leftrightarrow\{S'_1, S'_2, \cdots, S'_r\}$; $S'_1=\{M_1, \cdots, M_m\}, \cdots$ $S'_i=\{M_{t_1}, \cdots, M_{t_1+m_1-1}\}, \cdots S'_r=\{\cdots, M_n\}$; $\Lambda'\leftrightarrow\{S'_1, \cdots, S'_i+M_{t_i+m_i}, S'_{i+1}-M_{t_i+m_i}, S'_{i+2}\cdots, S'_r\}$, then there exists a maximal left ideal A in Λ such that $\Lambda=\Lambda^{l}(A)$, $\Lambda'=\Lambda^{r}(A)$. Let Λ' be an r^{th} h-order containing Λ_0, which corresponds to $S'_1=\{M_1, \cdots, M_{n-r+1}\}$, $S'_2=\{M_{n-r+2}\}\cdots, S'_r=\{M_n\}$. Then we can find a path of Λ' to any r^{th} h-order Λ containing Λ_0 by the above facts, and conversely a path of Λ to Λ'. Therefore, there exists a path between Λ^{α} and Λ^{β}.

LEMMA 2.3. *Let Λ and Λ' be $n-1^{th}$ h-order in a maximal order Ω, then there exists a path of Λ to Λ', where $\Omega/N(\Omega)=\Delta_n$.*

Proof. Let L_i be a left ideal in Λ which contains exactly $N=N(\Omega)$. Then from [6], Theorem 5.3 we obtain $\Lambda=\Omega\cap\Lambda^{r}(L_1)\cap\Lambda^{r}(L_2)\cap\cdots\cap\Lambda^{r}(L_{n-2})$, and $L_i\supset L_{i+1}$ for all i. We shall denote Λ by $\Lambda(L_1, L_2, \cdots, L_{n-2})$. If L_{n-2}/N is minimal, there exists a left ideal L' such that $L_i\supset L'\supset L_{i+1}$ for some i. Hence $\Lambda(L_1, \cdots, L_i, L', L_{i+1}, \cdots, L_{n-2})=\Lambda(L_1, \cdots, L_i, L_{i+1}, \cdots, L_{n-2})\cap\Lambda(L_1, L_2, \cdots, L_i, L', L_{i+1}, \cdots, L_{n-3})$. Therefore we can find a path of $\Lambda(L_1, \cdots, L_{n-2})$ to $\Lambda(L_1, \cdots, L_i, L', L_{i+1}, \cdots, L_{n-3})$. Thus we may assume L_{n-2}/N is not a simple Ω-module and hence L_i/L_{i+1} is simple for all i. Let $\Lambda=\Lambda(L_1, \cdots, L_{n-2})$ and $\Lambda'=\Lambda(L'_1, \cdots, L'_{n-2})$. If $L_1\neq L'_1$, $L_1/L_1\cap L'_1$, $L'_1/L_1\cap L'_1$ are simple. Let $\Lambda_1=\Lambda(L_1, L_1\cap L'_1, L''_3, \cdots, L''_{n-2})$, and $\Lambda_2=\Lambda(L'_1, L'_1\cap L_1, L''_3, \cdots, L''_{n-2})$ and $\Lambda'_1=\Lambda(L_1\cap L'_1, L''_3, \cdots, L''_{n-1})$. Then $\Lambda_1\cap\Lambda'_1=\Lambda(L_1, L_1\cap L'_1, L''_3, \cdots, L''_{n-1})$. Therefore, there exists a path of Λ_1 to Λ'_1. Similary we have a path of Λ'_1 to Λ_2. Hence there exists a path of Λ_1 to Λ_2. Hence by using induction on t such that $L_i=L'_i$ for $i<t$ we can prove the lemma.

THEOREM 2.1. *Let Λ^{α}, Λ^{β} be non-minimal r^{th} h-order over R_p. Then there exists a path of Λ^{α} to Λ^{β}.*

Proof. First we assume that Λ^{α} and Λ^{β} are $n-1^{th}$ h-orders. Let Ω and Ω' be a maximal orders containing Λ^{α} and Λ^{β}, respectively. There exists a path $\Omega^{\alpha}=\Omega^{1}, \cdots, \Omega^{n}=\Omega^{\beta}$ by Lemma 2.1. Let Λ^{i} be a $n-1^{th}$ h-order in $\Omega^{i}\cap\Omega^{i+1}$; $\Lambda^{0}=\Lambda^{\alpha}, \cdots, \Lambda^{n-1}=\Lambda^{\beta}$. Then we obtain a path of Λ^{i} to Λ^{i+1} by Lemma 2.3 and hence we have a path of Λ^{α} to Λ^{β}. Let Λ^{α}

and Λ^β be any r^{th} h-orders and Γ^α, Γ^β be $n-1^{th}$ h-orders in Λ^α, Λ^β. Let $\Gamma^\alpha=\Gamma^0, \cdots, \Gamma^t=\Gamma^\beta$ be a path of Γ^α to Γ^β. If we take an r^{th} h-order Λ^i containing Γ^i such that $\Lambda^0=\Lambda^\alpha, \cdots, \Lambda^t=\Lambda^\beta$, then $\Lambda^i\cap\Lambda^{i+1}\supset\Gamma^i$. Hence there exists a path of Λ^i to Λ^{i+1} by Lemma 2.2. Therefore, we have a path of Λ^α to Λ^β.

PROPOSITION 2.6. *Let Λ^1 be a non-minimal h-order. Then P^{11} is equal to $N(\Lambda)$. Hence the length of shortest path of Λ^1 to itself does not depend on Λ^1.*

Proof. Since Λ is not minimal, we can find a maximal left Λ-ideal A^{12} containing $N(\Lambda)$. Let $B^{21}=(A^{12})^{-1}N(\Lambda)$, then $B^{21}\supseteqq(A^{12})^{-1}N(\Lambda)A^{12}=N(\Lambda^2)$. If $\Lambda\neq\Lambda^2$, then B^{21} is not a two-sided ideal in Λ^2. Hence $B^{21}/N(\Lambda^2)=l_1\oplus\cdots\oplus l_r$; $l_i\neq\Delta_{m_i}$ for some i. Therefore, there exists a maximal left ideal A^{23} containing B^{21}. Repeating this process, we have $N(\Lambda)=A^{12}B^{23}\cdots K^{m1}$. Since A^{12}, B^{23} are maximal one-sided ideals, m is equal to the length of composition series of $\Lambda/N(\Lambda)$, which does not depend on Λ by [6], Coro. to Lemma 2.5.

THEOREM 2.2. *Let Λ^α, Λ^β be non-minimal r^{th} h-orber over R_p. If $P^{\alpha\beta}$ is not contained in $N(\Lambda^\alpha)$ $(N(\Lambda^\beta))$, then any Λ^α, Λ^β-ideal is expressed as a characteristic product of maximal and normal integral ideals. If $P^{\alpha\beta}$ is contained in $N(\Lambda^\alpha)$ $(N(\Lambda^\beta))$, then any integral ideal except finite number is expressed as above.*

Proof. Let $P^{\alpha\beta}$ be the path ideal of Λ^α to Λ^β. $P^{\alpha\beta}=D^{\alpha\beta}N(\Lambda^\beta)^t$ for some t by Proposition 2.2, Therefore any Λ^α, Λ^β-integral ideal A except $D^{\alpha\beta}B(\Lambda^\beta)^i$, $i=1, \cdots, t-1$ is written as $P^{\alpha\beta}N(\Lambda^\beta)^k$. Since $P^{\alpha\beta}$, $N(\Lambda^\beta)$ are expressed as product of maximal normal ideals by Proposition 2.6, so is A.

PROPOSITION 2.7. *Let Λ^α, Λ^β be r^{th} h-orders and $(\Lambda^\beta\Lambda^\alpha)^{-1}=D^{\alpha\beta}$. Then $D^{\alpha\beta}$ is equal to the shortest path ideal of Λ^α to Λ^β, and every Λ^α, Λ^β-ideal is products of maximal and normal left ideals.*

Proof. Let $D^{\alpha\beta}=(\Lambda^\beta\Lambda^\alpha)^{-1}$ and $\Lambda^\alpha\supset A_1\supset A_2\cdots\supset A_n=D^{\alpha\beta}$ be a composition series of $\Lambda^\alpha/D^{\alpha\beta}$. Since $A_i\Lambda^\alpha\supset D^{\alpha\beta}\Lambda^\alpha=\Lambda^\alpha$, $\tau(A_i)=\Lambda^i$. Hence $\Lambda^l(A_i)=\Lambda^i$. Furthermore, since$A_i^{-1}A_{i+1}=B^{i\,i+1}$ is a maximal Λ^i,Λ^{i+1}-ideal in Λ^i. Hence $D^{\alpha\beta}$ is an ideal of a path of Λ^α to Λ^β. Therefore, $D^{\alpha\beta}$ is equal to the shortest path of Λ^α to Λ^β.

THEOREM 2.3. *Let Λ^α, Λ^β be r^{th} h-order over R_p. If Λ^α, Λ^β contain the same h-order, then $P^{\alpha\beta}=D^{\alpha\beta}$, and hence any Λ^α, Λ^β-integral ideal is expressed as a characteristic product of a maximal one sided ideals.*

Proof. Let $\Lambda=\Lambda^{\alpha}\cap\Lambda^{\beta}$, then Λ is an h-order by assumption. Let $A^{\alpha 2}=C_{\Lambda}(\Lambda^{\alpha})$. Since $A^{\alpha 2}\Lambda=\Lambda^{\alpha}$, $D^{\alpha 2}=(\Lambda^{2}\Lambda^{\alpha})^{-1}=A^{\alpha 2}$ by Proposition 2.3, and hence $A^{\alpha 2}$ is expressed as a product of maximal left ideals by Proposition 2.7. Furthermore, since $D^{\alpha\beta}\subsetneqq\Lambda$, $A^{\alpha 2}\supseteqq D^{\alpha\beta}$. Let $D=(A^{\alpha 2})^{-1}D^{\alpha\beta}$, then D is a Λ^{2}, Λ^{β}-integral ideal. $A^{\alpha 2}$ is a two-sided Λ-module, and nence $\Lambda^{2}=\Lambda^{r}(A^{\alpha 2})\supseteqq\Lambda$. Therefore, $\Lambda^{1}=\Lambda^{2}\cap\Lambda^{\beta}\supset\Lambda$. Let $A^{23}=C_{\Lambda^{1}}(\Lambda^{2})$, then $D\subseteq A^{23}$. Hence $D_{1}=(A^{23})^{-1}D$ is integral $\Lambda^{\alpha}\Lambda^{\beta}$-ideal and $D^{\alpha\beta}=A^{\alpha 2}A^{23}D_{1}$. Repeating this argument, we have $D^{\alpha\beta}=A^{\alpha 2}A^{23}\cdots A^{t\beta}$, and $A^{i\,i+1}$ is expressed as a product of maximal left ideals, which implies $D^{\alpha\beta}=P^{\alpha\beta}$. Thus, we have proved the theorem.

In general there exists an ideal $A^{\alpha\beta}$ which is not expressed as a charecteristic product of maximal left ideal. Now, we shall consider those ideals. If ideals $A^{\alpha\beta}$ is contained in a normal and maximal left ideal, we may divide $A^{\alpha\beta}$ by it. Hence we may assume that $A^{\alpha\beta}$ is not contained in a maximal and normal left Λ^{α}-ideal. Then it is clear that $A^{\alpha\beta}+N^{\alpha\alpha}$ is a two-sided Λ^{α}-module, and $A^{\alpha\beta}+N^{\alpha\alpha}/N^{\alpha\alpha}=\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_t}\oplus\overbrace{\Delta\oplus\cdots\oplus\Delta}^{k}$, and $\Lambda^{\alpha}/N^{\alpha\alpha}=\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_t}\oplus\overbrace{\Delta\cdots\oplus\Delta}^{s}$, $s\geqslant k$.

THEOREM 2.4. *Let R be a local ring and $A^{\alpha\beta}$ an integral ideal which is not contained in maximal and normal left-ideals, then $A^{\alpha\beta}$ is not contained in maximal and nomal right Λ^{β}-ideals. In this case Λ^{α} is isomorphic to Λ^{β}. Furthermore those ideals are principal left Λ^{α}-ideals and hence they are isomorphic as a left Λ^{α}-module.*

Poof. Let $\Lambda^{\alpha}/N^{\alpha\alpha}=\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_t}\oplus\overbrace{\Delta\oplus\cdots\oplus\Delta}^{s}$, $m_i>1$; Λ^{α} is an h-order of r^{th} $(=t+s)^{th}$, and $n=\sum_{i=1}^{t}m_i+s$ is an invariant in Σ by [5], Coro. to Lemma 2.5. By usual argument (cf [6], Propo. 1.1.) we may assume that R is complete. Then there exists an idempotent element e in $A^{\alpha\beta}$ such that $\Lambda^{\alpha}e+N^{\alpha\alpha}/N^{\alpha\alpha}=A^{\alpha\beta}+N^{\alpha\alpha}/N^{\alpha\alpha}$. By the assumption on $A^{\alpha\beta}$, we obtain $A^{\alpha\beta}+N^{\alpha\alpha}/N^{\alpha\alpha}=\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_t}\oplus\overbrace{\Delta\cdots\oplus\Delta}^{k}$, $k\leq s$. Furthermore, $A^{\alpha\beta}/N^{\alpha\alpha}A^{\alpha\beta}=(\Lambda^{\alpha}e+N^{\alpha\alpha}A^{\alpha\beta})/N^{\alpha\alpha}$ $A^{\alpha\beta}\oplus A^{\alpha\beta}\cap N^{\alpha\alpha}/N^{\alpha\alpha}A^{\alpha\beta}$ and $\Lambda^{\alpha}e+N^{\alpha\alpha}A^{\alpha\beta}$ $N^{\alpha\alpha}A^{\alpha\beta}\approx A^{\alpha\beta}+N^{\alpha\alpha}/N^{\alpha\alpha}$ as a left Λ^{α}-module. Since $\Lambda^{\beta}/N^{\beta\beta}\approx\mathrm{Hom}^{l}_{\Lambda^{\alpha}/N^{\alpha\alpha}}(A^{\alpha\beta}/N^{\alpha\alpha}A^{\alpha\beta}, A^{\alpha\beta}/N^{\alpha\alpha}A^{\alpha\beta})$ and Λ^{β} is an order of r^{th}, $A^{\alpha\beta}\cap N^{\alpha\alpha}/N^{\alpha\alpha}A^{\alpha\beta}$ contains at least $r-(t+k)=s-k$ distinct simple left Λ^{α}-components which are different from those in $\Lambda^{\alpha}e+N^{\alpha\alpha}A^{\alpha\beta}/N^{\alpha\alpha}A^{\alpha\beta}$. However, the number of simple components in $A^{\alpha\beta}\cap N^{\alpha\alpha}/N^{\alpha\alpha}A^{\alpha\beta}$ is equal to $n-(\sum_{i=1}^{t}m_i+k)=s-k$. Hence $A^{\alpha\beta}\cap N^{\alpha\alpha}/N^{\alpha\alpha}A^{\alpha\beta}$ is a directsum of all distinct simple components which are different from those in $\Lambda^{\alpha}e+N^{\alpha\alpha}A^{\alpha\beta}/N^{\alpha\alpha}A^{\alpha\beta}$. On the other hand, $(A^{\alpha\beta}\cap N^{\alpha\alpha}/N^{\alpha\alpha}A^{\alpha\beta})A^{\alpha\beta}=(0)$ and e is the identity mapping on $(\Lambda^{\alpha}e+N^{\alpha\alpha}A^{\alpha\beta})/N^{\alpha\alpha}A^{\alpha\beta}$, and hence $A^{\alpha\beta}+N^{\beta\beta}/N^{\beta\beta}\approx\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_t}\oplus\overbrace{\Delta\cdots\oplus\Delta}^{k}\approx A^{\alpha\beta}+N^{\alpha\alpha}/N^{\alpha\alpha}$ as a ring and $\Lambda^{\beta}/N^{\beta\beta}\approx\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_t}\oplus\overbrace{\Delta\cdots\oplus\Delta}^{s}$. Therefore, $A^{\alpha\beta}$ is not contained in maximal and narmal right Λ^{β}-ideals. From the above observation, we know that $A^{\alpha\beta}/N^{\alpha\alpha}A^{\alpha\beta}\approx\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_t}\oplus\Delta\oplus\cdots\oplus\Delta\approx\Lambda^{\alpha}/N^{\alpha\alpha}$ as a left Λ^{α}-module. Therefore, $A^{\alpha\beta}=\Lambda^{\alpha}a$, which implies $\Lambda^{\beta}=a^{-1}\Lambda^{\alpha}a$.

COROLLARY 2.2. *Let Λ^{α} be an r^{th} h-order such that $\Lambda^{\alpha}/N^{\alpha\alpha}=\Sigma\oplus\Delta_{m_i}$, $m>1$ for all i. Then every one-sided Λ^{α}-ideal is expressed as a product of maximal left (right) ideals. Especially if Λ^{α}, is principal (non-minimal), then Λ^{α} satisfies the condition.*

Proof. The above arguments are true if we exchange "*left*" to the "*right*".

REMARK 2. Even if $A^{\alpha\beta}$ is expressed as a product of maximal left ideals by taking suitable maximal left ideals, there exist, in general, no expressions of product of maximal left such that the first factor of it is an arbitray maximal and normal left ideal containig $A^{\alpha\beta}$. However, if Λ^{α} satisfies the above condition, then $A^{\alpha\beta}$ is expressed as a product of maximal left ideals by any choice of maximal left ideals in each step of expression.

COROLLARY 2.2. *Let Λ be a minimal h-order, then every one-sided ideal is principal.*

Proof. In a minimal h-order every maximal left ideal (as a ring) is two-sided, and hence it is not regular.

THEOREM 2.5. *Any ideal $A^{\alpha\beta}$ is expressed as a product of two-sided ideals such that $A^{\alpha\beta}=A^{\alpha\kappa}A^{\kappa\beta}$: $A^{\alpha\kappa}$ is expressed as a product of normal and maximal left ideals and $A^{\kappa\beta}$ is not contained in normal and maximal left Λ^{κ}-ideals. If we have two expressions $A^{\alpha\beta}=A^{\alpha\kappa}A^{\kappa\beta}=A^{\alpha\kappa'}A^{\kappa'\beta}$, then $A^{\alpha\kappa}\approx A^{\alpha\kappa'}$ as a left Λ^{α}-module, and $A^{\kappa'\beta}\approx A^{\kappa\beta}$ as a right Λ^{β}-module.*

Proof. From Theorem 2.4 we obtain an regular element a such that $A^{\kappa'\beta}=aA^{\kappa\beta}$. Hence $A^{\alpha\beta}=A^{\alpha\kappa}a^{-1}aA^{\kappa\beta}=A^{\alpha\kappa}a^{-1}A^{\kappa'\beta}=A^{\alpha\kappa'}A^{\alpha'\beta}$. Therefore, $A^{\alpha\kappa'}=A^{\alpha\kappa}a^{-1}\Lambda^{\kappa'}$. Since $\Lambda^{\kappa'}=a\Lambda^{\kappa}a^{-1}$, $A^{\alpha\kappa'}=A^{\alpha\kappa}a^{-1}$.

PROPOSITION 2.8. *Let Λ be an h-order with radical N. For any given two-sided Λ-module A in Λ containing N there exists a left Λ-ideal A_0 such* *that* $A_0+N=A$. *Hence if one of the simple components of* Λ/N *is a division ring, then there exists a left* Λ*-ideal which is not expressed as a product of maximal and normal left ideals.*

Proof. We may assume that R is compete and Λ is written as a subring of matrix ring over a unique maximal order O in *a* division ring by [6], Theorem 6.2. Let $\Lambda/N=\Delta_{m_1}\oplus\Delta_{m_2}\oplus\cdots\oplus\Delta_{m_r}$, and $A/N=\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_t}$. Furthermore, we may assume

$$\Lambda=\begin{array}{c}m_1\{\\ m_2\{\\ \\ m_i\{\\ \\ \end{array}\left(\begin{array}{c|c|c|c|c}\overbrace{\cdots O\cdots}^{m_1} & \overbrace{\cdots\pi\cdots}^{m_2} & \cdots & \overbrace{\cdots\pi\cdots}^{m_i} & \cdots\\ \hline \cdots O\cdots & \cdots O\cdots & & \cdots\pi\cdots & \\ \hline \vdots & & \ddots & & \\ \hline \cdots O\cdots & \cdots O\cdots & & \cdots O\cdots & \cdots\pi\cdots\\ \hline \vdots & \vdots & & \vdots & \vdots\end{array}\right)\quad\text{and}\quad A=\begin{array}{c}\\ \\ m_{t+1}\{\\ \\ \end{array}\left(\begin{array}{c|c|c|c}\cdots O\cdots & \cdots\pi\cdots & \cdots & \overbrace{\cdots\pi\cdots}^{m_{t+1}}\\ \hline & \begin{matrix}O & \\ & \ddots \\ & & O\end{matrix} & & \\ \hline \cdots O\cdots & & \cdots\pi\cdots & \cdots\pi\cdots\\ \hline \cdots O\cdots & & & \cdots\pi\cdots\end{array}\right)$$

Let

$$\alpha=\begin{array}{c}\\ \\ \\ m_{t+1}\{\\ \\ \end{array}\left(\begin{array}{c|c|c|c|c}\begin{matrix}1 & & 0\\ & 1 & \\ 0 & & 1\end{matrix} & & & & \\ \hline 0 & \begin{matrix}1 & & 0\\ & 1 & \\ 0 & & 1\end{matrix} & & 0 & \\ \hline 0 & 0 & \ddots & & \\ \hline & & & \overbrace{\begin{matrix}\pi & & 0\\ & \ddots & \\ 0 & & \pi\end{matrix}}^{m_{t+1}} & \\ \hline & & & & \begin{matrix}\pi & & 0\\ & \ddots & \\ 0 & & \pi\end{matrix}\end{array}\right)$$

Then α is a unit element in Σ and hence $\Lambda\alpha$ is a normal left Λ-ideal in Λ. It is clear that $\Lambda\alpha+N=A$. If $m_s=m_{s+1}=\cdots=m_r=1$ and $A/N=\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_{s-1}}$, then every maximal left ideal containing $\Lambda\alpha$ is a two-sided Λ-ideal. Therefore, $\Lambda\alpha$ is not expressed as a product of maximal and normal left ideals

REMARK 3. In general $\Lambda^i\Lambda^j$ is not a Λ^i,Λ^j-ideal. Hence $D^{ji}\neq(\Lambda^i\Lambda^j)^{-1}$ Furthermore, ever if $D^{ij}=(\Lambda^j\Lambda^i)^{-1}$, but $D^{ji}\neq(\Lambda^i\Lambda^j)^{-1}$.

For example, let

$$\Lambda^i=\begin{pmatrix}R & R & p\\ R & R & p\\ R & R & R\end{pmatrix},\quad \Lambda^j=\begin{pmatrix}R & R & R\\ p & R & p\\ R & R & R\end{pmatrix}.\quad\text{Then}\quad \Lambda^i\Lambda^j=\begin{pmatrix}R & R & R\\ R & R & R\\ R & R & R\end{pmatrix}$$

$\Lambda^j\Lambda^i=\begin{pmatrix} R & R & R \\ R & R & p \\ R & R & R \end{pmatrix}$ is a $\Lambda^j\Lambda^i$-ideal. Hence $D^{ij}=(\Lambda^j\Lambda^i)^{-1}$, but $D^{ji}\neq(\Lambda^i\Lambda^j)^{-1}$. We note that $\Lambda^i\cap\Lambda^j$ is an h-order.

REMARK 4. The second ideals in Proposition 2. 8 are distance ideals which are not equal to the shortest path ideal.

LEMMA 2. 4. *Let Λ^i,Λ^k, Λ^l, and Λ^j be of the same rank and A^{ij} an ideal. If $(A^{ij})^{-1}D^{ik}A^{ij}D^{lj}\not\subseteq N^{jj}$, then there exist integral ideals C^{ik}, C^{lj} for any ideal $B^{kl}\supseteqq A^{ij}$ such that $A^{ij}=C^{ik}B^{kl}C^{lj}$. If $(A^{ij})^{-1}D^{ik}A^{ij}D^{lj}\subseteq N^{jj}$, then there exists an ideal $B^{kl}\supsetneqq A^{ij}$ which is not expressed as above.*

Proof. Let $N=N^{jj}$. $A^{ij}=D^{ij}N^{t_1}$, $D^{ik}B^{kl}D^{lj}=D^{ij}N^{t_2}$. Then $B^{kl}=(D^{ik})^{-1}A^{ij}N^{t_2-t_1}(D^{lj})^{-1}\supseteqq A^{ij}\leftrightarrow N^{t_2-t_1}\supseteqq(A^{ij})^{-1}D^{ik}A^{ij}D^{lj}=L$. Hence if $L\not\subseteq N$, $t_2<t_1$. Therefore, $A^{ij}=D^{ij}N^{t_1-t_2}N^{t_2}=D^{ik}B^{kl}D^{lj}N^{t_1-t_2}$. If $L\subseteq N$, then $B^{kl}=(D^{ik})^{-1}A^{ij}N(D^{lj})^{-1}\supseteqq A^{ij}$. If $A^{ij}=C^{ik}B^{kl}C^{lj}$, then $A^{il}=D^{ik}B^{kl}D^{lk}N^{t}$ for some $t\geqslant 0$. Hence, $A^{ij}=A^{ij}N^{t+1}$, which is a contradiction,

PROPOSITION 2. 9. *If either $D^{ik}=(\Lambda^k\Lambda^i)^{-1}$ or $D^{lj}=(\Lambda^j\Lambda^l)^{-1}$, then $A^{ij}\subseteq B^{kl}$ if and only if there exists integral ideals C^{ik}, C^{lj} such that $A^{ij}=C^{ik}B^{kl}C^{lj}$. Especially if $i=k$, then $A^{ij}=B^{il}C^{lj}$. If $D^{ik}\neq(\Lambda^k\Lambda^i)^{-1}$, then for any ideal A^{ij} there exists ideal $B^{kl}\supsetneqq A^{ij}$ which is not related as above.*

Proof. The first part is clear from Lemma 2. 4. We assume that $D^{ik}\neq(\Lambda^k\Lambda^i)^{-1}$ which means that $D^{ik}\Lambda^i+N^{ii}$ is a proper two-sided Λ^i-module. Hence $D^{ik}\Lambda^i+N^{ii}=\bigcap_{1\leqslant i<r}M_i$. Let A^{ij} be any ideal, then $(A^{ij})^{-1}D^{ik}\Lambda^iA^{ij}+(A^{ij})^{-1}N^{ii}A^{ij}=\bigcap_{1\leqslant i<r}(A^{ij})^{-1}M_iA^{ij}$. Since $\{(A^{ij})^{-1}M_iA^{ij}\}$ is the normal sequence in Λ^j, there exists a two-sided Λ^j-module $C\supsetneqq N^{jj}$ such that $(C/N^{jj})(\bigcap_{1<i\leqslant r}(A^{ij})^{-1}M_iA^{ij}/N^{jj})=(0)$. By Proposition 2. 8 we can find a right Λ^j-ideal D such that $D+N^{jj}=C$. Let $\Lambda^l=\Lambda^l(D)$. Then there exists an ideal B^{kl} as in the proposition by Lemma 2. 4.

3. Decomposition of one-sided ideals over Dedekind rings.

In this section we shall generalize the results in §2 to the global case. Let R be a Dedekind domain and K its quotient field. Σ is the central simple K-algebra.

Let Λ^i be order and L^{ij} an integral ideal. Let $C(L^{ij})=\{x|\in R, x\Lambda^i\subset L^{ij}\}$. It is clear that $C(L^{ij})$ is an ideal in R.

LEMMA 3.1. *L^{ij} is maximal if and only if $C(L^{ij})=p^{\rho}$ and L_p^{ij} is maximal, where p is prime in R.*

Proof. It is clear by Lemma 1.3.

PROPOSITION 3.1. *If L^{ij} is a maximal left Λ^i-ideal then L^{ij} is a maximal right Λ^j-ideal.*

Proof. Let $C(L^{ij})=p^{\rho}$. Then L_p^{ij} is a maximal right Λ_p^j-ideal by Proposition 2.1. Hence L^{ij} is maximal in Λ^i.

PROPOSITION 3.2. *Let Λ^i, Λ^j be of the same type. Then there exists a unique maximal integral ideal D^{ij}.*

Proof. There exists an ideal D^{ij} in Λ^i sucn that $D_p^{ii}=$ the distance ideal of Λ_p^i to Λ_p^j for all p by Lemma 1.3. It is clear that D^{ij} is a unique maximal integral ideal in $\Lambda^i \cap \Lambda^j$.

We shall call D^{ij} the *distance ideal* of Λ^i to Λ^j.

COROLLARY 3.1. *Let Λ^i, Λ^j be of the same type. For any integral ideal A^{ij}, we have $A^{ij}=D^{ij}B^{jj}$, where B^{jj} is a normal two-sided Λ^j-ideal.*

PROPOSITION 3.3. *$D^{ij}=(\Lambda^j\Lambda^i)^{-1}$ if and only if there exists ideal A^{ij} such that $A^{ij}\Lambda^i$ $(\Lambda^j A^{ij})$ is a normal ideal in Λ^i (Λ^j).*

We can define a path similarly to Definition 6.

DEFINITION 7′. *Let Λ^{α}, Λ^{β} be of the same type. The set of h-orders $\Lambda^{\alpha}=\Lambda^0, \Lambda^1, \cdots, \Lambda^n=\Lambda^{\beta}$ such that $\Lambda^i=\Lambda^l(L^{ii+1})$, $\Lambda^{i+1}=\Lambda^r(L^{ii+1})$ is called a path of Λ^{α} to Λ^{β} and $\prod_i L^{ii+1}$ is the ideal of this path where L^{ii+1} is a maximal left Λ^i-ideal.*

It is clear that $\Lambda_p^i \cap \Lambda_p^{i+1}$ is an h-order for some p and $\Lambda_q^i=\Lambda_q^{i+1}$ for $q \neq p$.

LEMMA 3.2. *Let A^{ij} be an integral ideal. A^{ij} is expressed as a characteristic product of maximal left ideals if and only if so is A_p^{ij} for all p.*

Proof. "Only if" part is clear from Lemma 3.1. Let $\{x \mid \in R, \Lambda_i x \subseteq \Lambda_j\}=p_1^{e_1}p_2^{e_2}\cdots p_t^{e_t}$. If $t=1$, "if part" is clear. We shall prove "if part" by induction on t. Let B^{ik} be an integral ideal such that $B_p^{ik}=A_p^{ij}$ for $p=p_1, p_2, \cdots, p_{t-1}$, and $B_q^{ik}=\Lambda_q^i$ for $q \neq p_1, \cdots, p_{t-1}$ and $C^{k'j}$ an integral ideal such that $C_{p_t}^{k'j}=A_{p_t}^{ij}$, $C_q^{k'j}=\Lambda_q^j$ for $q \neq p_t$. It is clear that $\Lambda^k=\Lambda^{k'}$. By assumption B^{ik}, C^{kj} are expressed as a product of maximal ideals and hence so is $A^{ij}=B^{ik}C^{kj}$.

THEOREM 3.1. *Let Λ^i, Λ^j be of the same type which satisfy the condition that if $\Lambda_p^i \neq \Lambda_p^i$ then they are not minimal. Then there exists paths of Λ^i to Λ^j, and the ideals of shortest paths coincide with each other.*

Proof. Let $P^{ij}(p)$ be the ideal of shortest path of Λ_p^i to Λ_p^j and $P^{ij}(p)=\Lambda_p^i$ if $\Lambda_p^i=\Lambda_p^j$. Then there exists a unique integral ideal P^{ij} in $\Lambda^i \cap \Lambda^j$ such tnat $P_p^{ij}=P^{ij}(p)$ for all p. P^{ij} is the ideal of a path of Λ^i to Λ^j by Lemma 3.2. Let A^{ij} be the ideal of path of Λ^i to Λ^j. Then $A_p^{ij}=P_p^{ij}B^{jj}(p)$ for all p. Let B^{jj} be a two-sided ideal in Λ^j such that $B_p^{jj}=B^{jj}(p)$, for all p, then $A^{ij}=P^{ij}B^{jj}$. Hence P^{ij} is uniquely determined.

COROLLARY 3.1. *If A^{ij} is a product of maximal left ideals, then $A^{ij}=P^{ij}B^{jj}$.*

THEOREM 3.2. *Let Λ^i, Λ^j be as in Theorem 3.1. Then every integral ideal A^{ij} except finite number is a (characteristic) product of normal and maximal left ideals.*

From Theorem 2.3 and Lemma 3.2 we have

THEOREM 3.3. *Let Λ^i, Λ^j be of the same type which contain the same h-order. Then every integral Λ^i, Λ^j-ideal is a product of normal and maximal left ideals.*

PROPOSITION 3.4. *If $D^{ij}=(\Lambda^j\Lambda^i)^{-1}$, then every Λ^i, Λ^j-ideal is a product of normal and maximal left ideals.*

It is clear from Poposition 2.7 and Lemma 3.2.

PROPOSITION 3.5. *If Λ_p^i is a non-minimal principal h-order for all p, then every one-sided integral ideal in Λ is a product of normal and maxemal one-sided ideals.*

It is clear from Corollary 2.2.

Finally we shall consider the uniqueness of representation as a product.

DEFINITION 8. *Let A^{ij}, $B^{i'j'}$ be integral ideals. If there exist integral ideals L^{ti}, $L^{ti'}$ such that $L^{ti}/L^{ti}A^{ij}$ is isomorphic to $L^{ti'}/L^{ti'}B^{i'j'}$ as left Λ^t-modle, then we call that A^{ij} is quasi-equivalent to $B^{i'j'}$. We shall denote this relation by $A^{ij} \underset{l}{\approx} B^{i'j'}$.*

From the definition we have

LEMMA 3.3. *If $A^{ij} \underset{l}{\approx} B^{i'j'}$, then $A_p^{ij}=\Lambda_p^i$ is equivalent to $B_p^{i'j'}=\Lambda_p^{i'}$ for any p.*

PROPOSITION 3. 5. *Let A^{ij}, $B^{i'j'}$ be maximal left ideals. Then $L^{ti}/L^{ti}A^{ij}$ is isomorphic to $L^{ti'}/L^{ti'}B^{i'j'}$ for any integral ideals L^{ti}, $L^{ti'}$ if and only if Λ_p^i, $\Lambda_p^{i'}$ are maximal and $A_p^{ij} \neq \Lambda_p^i$, $B_p^{i'j'} \neq \Lambda_p^{i'}$ for some p.*

Proof. We assume Λ_p^i, $\Lambda_p^{i'}$ are maximal, and L^{ti}, $L^{ti'}$ are ideals. Since A_p^{ij} is maximal in Λ_p^i, the annihilator of Λ_p^i/A_p^{ij} is equal to $N(\Lambda_p^i)$. It is clear that the annihlator of Λ_q^i/A_q^{ij} is equal to Λ_q^i for $q \neq p$. Hence the annihilator of $L^{ti}/L^{ti}A^{ij}$ is equal to $Q_{\Lambda^t}(p)$, where $Q_{\Lambda^t}(p)_p = N(\Lambda_p^t)$ and $Q_{\Lambda^t}(p)_q = \Lambda_q^t$ for $p \neq q$. Similarly we have the annihilator of $L^{ti'}/L^{ti'}B^{i'j}$ is equal to $Q_{\Lambda^t}(p)$. Hence $L^{ti}/L^{ti}A^{ij}$ and $L^{ti'}/L^{ti'}B^{i'j'}$ are $\Lambda^t/Q_{\Lambda^t}(p)$-module. Since those modules are simple and $\Lambda^t/Q_{\Lambda^t}(p)$ is simple, they are isomorphic. Conversely, we assume $L^{ti}/L^{ti}A^{ij}$ and $L^{ti'}/L^{ti'}B^{i'j'}$ are isomorphic. Since A_p^{ij} is maximal in Λ_p^i, the annihilator M of Λ_p^i/A_p^{ij} is a maximal two-sided in Λ_p^i. Furthermore by the assumption we have $L_p^{ti}M_p(L_p^{ti})^{-1} = (L_p^{ti'})M'_p(L_p^{ti'})^{-1}$. If we take $L'^{ti'} = L^{ti'}Q_{\Lambda^{i'}}(p)$, then we obtain $L_p^{ti}M_p(L_p^{ti})^{-1} = L_p^{ti'}M'_p(L_p^{ti'})^{-1} = L_p^{ti'}N(\Lambda_p^{i'})M'_pN(\Lambda_p^{i'})^{-1}(L_p^{ti'})^{-1}$. Hence $M_p = N(\Lambda_p^{i'})M'_pN(\Lambda_p^{i'})^{-1}$, which implies $M'_p = N(\Lambda_p^{i'})$ by [6], Theorem 2.1. Therefore, Λ_p^i is maximal by [5], Theorem 3.3. Hence $\Lambda_p^{i'}$ is also maximal.

LEMMA 3. 4. *Let A^{ij}, $B^{i'j'}$ be maximal left ideals and Λ^i, $\Lambda^{i'}$ be of the same type. Then $A^{ij} \underset{l}{\approx} B^{i'j'}$ if and only if $A_p^{ij} \neq \Lambda_p^i$ implies $B_p^{i'j'} \neq \Lambda_p^{i'}$ and converse.*

Proof. "only if" part is clear from Lemma 3, 3. We assume, $A_p^{ij} \neq \Lambda_p^i$, Let M be a maximal two-sided Λ_p^i-module contained in A_p^{ij} and M' contained in $B_p^{i'j'}$. There exists an integral $\Lambda^i, \Lambda^{i'}$-ideal $C^{ii'}$. Since $C^{ii'}M'(C^{ii'})^{-1}$ is a maximal two-sided Λ_p^i-module in Λ_p^i, $C_p^{ii'}M(C_p^{ii'})^{-1} = (N_p^{ii})^tM(N_p^{ii})^{-t}$ for some t. Let $A = Q_{\Lambda^i}(p)^tC^{ii'}/Q_{\Lambda^i}(p)^tC^{ii'}B^{i'j'}$ and $B = \Lambda^i/A^{ij}$. The annihilators of A and B are equal to S, where $S_p = M$, $S_q = \Lambda_q^i$ for $q \neq p$. Since A and B are simple Λ^i-modules, they are isomorphic by the same argument in the proof of Proposition 3. 5.

COROLLARY 3. 2. *The left qusi-equivalent relation for maximal left ideals is an equivalent relation. $A^{ij} \underset{l}{\approx} B^{i'j'}$ if and only if $A^{ij} \underset{r}{\approx} B^{i'j'}$, where $\underset{r}{\approx}$ is defined similarly to $\underset{l}{\approx}$.*

LEMMA 3. 5. *Let A^{ij}, B^{jk} be maximal. Then there exists maximal left Λ^i-ideal B' and maximal right Λ^k-ideal A' such that $AB = B'A'$, and $A \underset{l}{\approx} A'$, $B \underset{l}{\approx} B'$.*

Proof. If $A^{ij} \underset{l}{\approx} B^{jk}$, then we can take $B' = A^{ij}$, $A' = B^{jk}$. Hence we assume $A^{ij} \underset{l}{} B^{jk}$. Then there exist distinct prime ideals p, q in R by

Lemma 3.4 such that $A_p^{ij} \neq \Lambda_p^i$, $B_q^{jk} \neq \Lambda_q^j$. Let C be a maximal left Λ^j-ideal such that $C_q = B_q^{jk}$, $C_p = \Lambda_p^j$ for $p \neq q$. Then $(AB)_t = A_t \cap C_t$ for all t. Hence $AB = A \cap C \subseteq C$. By Proposition 2.9 we obtain $A^{ij}B^{jk} = CD$ for some integral ideal D. $A_p^{ij}B_p^{jk} = C_p D_p = D_p \neq \Lambda_p^i$. Hence, $A^{ij} \underset{l}{\approx} D$ and $B^{jk} \underset{l}{\approx} C$.

THEOREM 3.4. *Let A^{ij} be an integral ideal which is expressed as a (characteristic) product of normal and maximal left ideals. Then the number of maximal left ideals which appear in this expression is the same and those ideals are uniquely detemined and commutative up to left quasi-equivalent.*

Proof. Let $A^{ij} = L_1 L_2 \cdots L_t$; the L's are maximal. Then $\Lambda^i = L_0 \supset L_1 \supset L_1 L_2 \supset \cdots \supset L_1 L_2 \cdots L_t = A^{ij}$ is a composition series of Λ^i / A^{ij}. Hence t is uniquely determined and $\{L_1 L_2 \cdots L_i / L_1 L_2 \cdots L_{i+1}\}$ are unique as a left Λ^i-module. Hence the theorem is true by Lemmas 3.3, 3.4 and 3.5.

From Theorem 2.5 we have

THEOREM 3.5. *Let A^{ij} be an integral ideal. Then $A^{ij} = B^{ik}C^{kj}$, where B^{ik} is expressed as a product of normal and maximal one-sided ideals, and C_p^{kj} is principal for all p and Λ^k, Λ^j are locally isomorphic.*

4. Ideal class.

We shall consider the ideal classes in h-orders following [4], p. 88. We define equivalent relations of left ideals A^{ij}, B^{ik} of Λ^i and of h-orders Λ^j, Λ^k, and Λ^i.

DEFINITION 9. *$A^{ij} \sim B^{ik}$ if there exists an elment α in Σ such that $A^{ij} = B^{ik}\alpha$. $\Lambda^i \sim \Lambda^k$ if there exists α in Σ such that $\Lambda^k = \alpha\Lambda^i\alpha^{-1}$. $A^{ij} \overset{w}{\sim} B^{ik}$ if there exists two-sided regular ideal C^{ii} of Λ^i and an element α such that $A^{ij} = C^{ii}B^{ik}\alpha$.*

It is clear that those relations are equivalent ones. We can define the classes of right ideals. The classes of left ideals correspond to those for right ideals. By the same argument as in [4], p. 89, we have the following proposition, however we give the proof for the completeness.

PROPOSITION 4.1. *There exists a one-to-one correspondence of classes of h-orders which are of the same type as a fixed h-order Λ^i to the classes of left Λ^i-ideals with respect to relation $\overset{\omega}{\sim}$.*

Proof. We assume $\Lambda^k = \alpha^{-1}\Lambda^i\alpha$. Since Λ^j, Λ^k are of the same type, there exist ideals A^{ij}, B^{ik} by Theorem 1.6. Then $(B^{ik})^{-1}B^{ik}=\Lambda^k=\alpha^{-1}\Lambda^j\alpha=\alpha^{-1}(A^{ij})^{-1}A^{ij}\alpha$. Hence $B^{ik}=B^{ik}\alpha^{-1}(A^{ij})^{-1}A^{ij}\alpha$. Let $C=B^{ik}\alpha^{-1}(A^{ij})^{-1}$, then $CA^{ij}\alpha(B^{ik})^{-1}=B^{ik}\alpha^{-1}\Lambda^j\alpha(B^{ik})^{-1}=\Lambda^i$. Hence C is a two-sided regular ideal of Λ^i by Corollary 1.2. Conversely, if $B^{ik}=C^{ii}A^{ij}\alpha$, then $\Lambda^k=(B^{ik})^{-1}B^{ik}=\alpha^{-1}(A^{ij})^{-1}(C^{ii})^{-1}C^{ii}A^{ij}\alpha=\alpha^{-1}\Lambda^j\alpha$.

THEOREM 4.1. *Let R be the ring of integers, and K the field of rationals. Then one-sided ideal classes of an h-order over R is finite, and hence the isomorphic classes of h-orders which are of the same type is finite.*

Proof. Let Λ^j be an h-order which belongs to $\Phi=\{p_i\}_{i=1}^n$-block, and $\Phi'=\{p_i\}_{i=1}^m$ the set of prime factors of the different of a maximal order Ω containing Λ. Let P be a maximal one[4] among normal two-sided ideal in Λ. If P does not divide any p_i in $\Phi\cup\Phi'$, then $P_r=(\Lambda r)_r$ and $P_q=(\Lambda r)_q=\Lambda_q$ for $q\neq r$ by [4], p. 84, Satz 3. Hence $P=\Lambda r$. If P divides either p in Φ or p' in Φ', then $P^{\rho(p)}=p\Lambda$ or $p'\Lambda'$ for some $\rho(p)$ by [6], Theorem 2.2 and [4], p. 89. By the assumption and [6], Theorems 1.2 and 6.3 there exists a finite number of h-orders in a maximal order whith belongs to Φ. Since the class number of maximal order in Σ is finite, hence the class number of h-orders in Σ which are of the same type is also finite. Therefore, we can find finite representations A^{ij} ($j=1,\cdots,t$) of ideal class with respect to the relation $\overset{\omega}{\sim}$ by Proposition 4.1. Let B^{ik} be any ideal, then there exist a two-sided ideal C^{ii} and a regular element α in Σ snch that $B^{ik}=C^{ii}A^{ij}\alpha$. We may assume that C^{ii} is a normal ideal in Λ^i. Then $C^{ii}=P_1'^{e_1}P_2'^{e_2}\cdots P_r'^{e_r}Q_1^{f_1}\cdots Q_s^{f_s}$ by Theorem 1.4. Let $\rho=\max\{\rho(p)\}$. On the other hand we know the above argument that $P_i'^{\rho}$ is a principal ideal generated by an element in R. Therefore, $\{\Pi P_i^{e_i}Q_j^{f_j}A^{ik};\ 0\leqslant e_i, f_i\leqslant\rho-1\}$ can represnt any class of ideals with respect to the relation $\sim$, where P_i is a maximal ideal dividing a prime in $\{\Phi'-\Phi'\cap\Phi\}$ and Q_j is maximal normal ideal in Λ dividing a prime in $\{\Phi-\Phi'\cap\Phi\}$. Hence the class number is finite.

REMARK 5. It is clear that if two h-orders are not of the same type, then they are not isomorphic, and hence the number of isomorphic classes of h-orders in general is infinite.

REMARK 6. Let D_τ be the generalized quaternions. By [3], Theorem 3.2′ the number of isomorphic classes in D_{-1} which are of the same

4) We call those ideals maximal normal ideals.

type, in general, is larger than that of maximal orders, however the latter coincides with the former in D_{+1}.

5. Norm and different.

We shall use the same definitions as in [1], [4].

Let M be a finitely generated R-module with generator $(u_2, u_2, \cdots, u_n)$. Let

$$R_i = a_{i1}u_1 + \cdots + a_{in}u_n = 0 \ (i = 1, 2, \cdots, n)$$

be relations in M. We shall denote $R(M)$ the R-module generated by $|a_{ij}|$ where R_i runs through all relations.

Let N be a finitely generated R-torsion module and $N=N_0 \supset N_1 \supset \cdots \supset N_t=(0)$ be a composition series of N. Let $A(N_i/N_{i+1})=\{x| \in R,\ xN_i \subset N_{i+1}\}$, then $A(N_i/N_{i+1})$ is a prime ideal in R. Then $\Pi A(N_i/N_{i+1})$ in uniquely determined. By the usual argument (cf. [1] p. 261, [4] P. 79, §4) we can prove $R(N)=\Pi A(N_i/N_{i+1})$.

DEFINITION 10. *Let Λ^i be an h-order and A^{ii} a normal two-sided ideal. We denote $R(\Lambda^i/A^{ii})$ by NA^{ii}.*

LEMMA 5.1. *Let Q be a maximal normal two-sided ideal then $NQ=q^f$ where $q=Q\cap R$.*

Proof. If Q is maximal then $q=Q\cap R$ is prime and Λ_q is a maximal order. If Q is not maximal, $Q\cap R=q^t$. Since Q_q is the radical of Λ_q, $t=1$ In any case, since $\Lambda/Q=(R/q)u_1\oplus\cdots\oplus(R/Q)u_f$, $NQ=q^f$.

LEMMA 5.2. *If $(A^{ii}, B^{ii})=\Lambda^i$, then $NA^{ii}B^{ii}=NA^{ii}NB^{ii}$.*

Proof. Since $A^{ii}B^{ii}=B^{ii}A^{ii}$, $A^{ii}B^{ii}=A^{ii}\cap B^{ii}$. Therefore, $\Lambda/A^{ii}B^{ii}=\Lambda/A^{ii}\oplus\Lambda/B^{ii}$.

PROPOSITION 5.1. *Let A, B be a normal two-sided ideal in Λ. Then $NAB=NANB$.*

Proof. By Theorem 1.2. $A=\Pi Q_\Lambda(p_i)^{e_i}$, $B=\Pi Q_\Lambda(p_i)^{f_i}$. It is clear from the construction of $Q_\Lambda(p)$ that $(Q_\Lambda(p), Q_\Lambda(q))=\Lambda$ if $p\neq q$. Therefore, the proposition is clear from Lemmas 5.1 and 5.2.

We can naturally extend the definition of norm to the fractional two-sided ideals and it satisties the relation in the proposition 5.1.

Let Λ be an h-order belonging to Φ-block and Ω a maximal order containing Λ. Let Φ' be the set of prime ideals in R which divide the disdriminant of Ω. If $p\notin\Phi\cup\Phi'$, then $\Lambda_p=\Omega_p$ and p is unramified. Therefore, Λ_p is the different of Ω_p. If $p\in\Phi\cup\Phi'$, there exists a unique maximal regular twosided ideal $\tilde{D}(p)$ of Λ such that $S(\tilde{D}(p))\subset R_p$, because $S(\Lambda_p)\subseteq R_p$ and regular two-sided ideals over R_p are linearly ordered by Theorem 1.2, where $S(\ )$ means the trace of Σ over K. Then there exists a unique normal two-sided ideal D in Λ such that $D_p^{-1}=\tilde{D}(p)$ for $p\in\Phi\cup\Phi'$ and $D_q^{-1}=\Lambda_q=\Omega_q$ for $q\notin\Phi\cup\Phi'$. It is clear that D is a unique maximal normal two-sided ideal in Λ such that $S(D^{-1})\subseteq R$.

THEOREM 5.1. (*Different's theorem*). *Normal maximal two-sided ideal Q in Λ divides D if and only if Q is either ramified*[5] *or a second kind prime ideal.*

Proof. Let Q be a normal maximal ideal, and $p=Q\cap R$. Then $p=Q^eT$ and $(Q, T)=\Lambda$ by Theorem 2.2. If $\Lambda_p=R_pu_1\oplus\cdots\oplus R_pu_n$, then $\Lambda_p/p\Lambda_p=R_p/pR_pu_1\oplus\cdots\oplus R_p/pR_pu_n$. Hence for any element a in Λ_p the regular representation over K of $\bar{a}$ induces naturally the regular representation over R/p of $\bar{a}$, If, $e>1$ and $a\in QT$, then $a^e\in p\Lambda_p$, and hence $\bar{a}^e=\bar{o}$. Therefore, $S(\bar{a})=(\bar{o})$, which implies $S(QT)\equiv 0 \pmod{p}$. $S(Q^{1-e})S(p^{-1}QT)=p^{-1}S(QT)\subseteq R$. Hence $Q^{1-e}\subset D^{-1}$ and $Q^{e-1}\supseteq D$. If $e=1$, then $p\notin\Phi$ by [6], Theorem 2.2. Hence $\Lambda_p=\Omega_p$ and Q is a second kind prime ideal if $Q|D$, and the converse is true by §1 and [4], p. 84, Satz 3. Finally we assume $Q|D$. If $p\notin\Phi$, then we have proved the thorem. If $p\in\Phi$, then Q is ramified by [6], Theorem 2.2.

DEFINITION 11. *Let δ be the ideal in R generated by $|S(a_ia_j)|$ where a_i's run through all n elements in an order Λ in Σ and $[\Sigma:K]=n$. We call δ the discriminant of Λ.*

LEMMA 5.3. *Let $(u_1, u_2, \cdots, u_n)$ be a minimal basis of Λ_p over R_p, then $\delta=|S(u_iu_j)|$.*

By Proposition 5.1 and the proof of [4], p. 81, Satz 4, we have

LEMMA 5.4. *Let A be a regular two-sided ideal of an h-order Λ_p over R_p. If we have, for minimal basis (w_i), (a_i) of Λ_p and A over R_p respectively,*

$$(a_i)=(w_i)M,\quad M\in K_n$$

then $NA=|M|R_p$.

THEOREM 5.2. *Let Λ be an h-order over R. A prime ideal p in R divides δ if and only if maximal normal two-sided ideal Q in Λ which divides p is either ramified or a second kind prime ideal.*

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5) In h-order Λ we have $p=Q_1^{e_1}\cdots Q_t^{e_t}$ by Theorem 1.4, where the Q_i's are maximal normal two-sided ideals in Λ. If $e_i\geqslant 2$, then we call Q_i is ramified.

Proof. Let $(u_1, u_2, \cdots, u_n)$ be a minimal basis of Λ_p over R_p. Then it is well known that

$$(\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_n) = (u_1, u_2, \cdots u_n)(S(u_iu_j))^{-1}$$

is a minimal basis of $\bar{D}=\{x \mid \in \Sigma, S(x\Lambda_p) \subseteq R_p\}$. Since $\tilde{D} \subset \bar{D}$, we obtain for a minimal basis (w_i) of $\tilde{D}$ over R_p that $(w_1, w_2, \cdots, w_n)=(\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_n)(a_{ij})$, $a_{ij} \in R_p$. Hence, $(w_1, \cdots, w_n)=(u_1, u_2, \cdots, u_n)(S(u_iu_j))^{-1}(a_{ij})$. Therefore, from Lemma 5.4 we have $N(\tilde{D}_p^{-1})=\delta^{-1}|a_{ij}|$, and hence $N(D_p)=|a_{ij}|^{-1}\delta$ by the remark after Proposition 5.1. Let p be a factor of δ and P a prime ideal in Λ such that $P \cap R=p$. If P is unramified, then Λ_p is a maximal order by [6], Theorem 2.2. Hence P is second kind by [4], p. 88, satz 2.

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