Journal of Mathematics,
Osaka City University
Vol. 14, No. 2.

HEREDITARY ORDERS IN GENERALIZED QUATERNIONS D_τ

Manabu HARADA

(Received September 11, 1963)

Let R be a Dedekind domain and K the qnotient field of R. Let Σ be a central simple K-algebra of finite dimension over K. By an order over R in Σ we mean a subring A in Σ such that A is a finitely generated R-module which spans Σ over K. We shall call A a hereditary order (briefly h-order) if A is a hereditary ring.

The author has recently studied the structure theory in h-orders in [4] and [5]. In this note we shall determine h-orders in D_τ; $D_\tau = K+Ki+Kj+Kij$, $i^2=-1, j^2=\tau$, and $ij=-ji$, K the field of rationals.

We shall give, in §§1 and 2, a criterion for hereditarity of intersection of two maximal orders by means of the norm in D_τ, and we obtain the complete type of h-orders in D_τ. By using this criterion, in §3, we shall consider isomorphisms between h-orders in D_τ, and show that they are isomorphic if and only if they are locally isomorphic in D_{+1}, however this fact is not true in D_{-1}, (see the difinition in §3).

In this note we shall denote the ring of rational integers by Z and the field of rational numbers by K. Let D_τ be a generalized quaternion. By [3], p. 185 we may assume that $\tau=p_1p_2\cdots p_t$, $p_i=4a_i+3$, $p_i\neq p_j$ if $i\neq j$.

Let $A^0=Z+Zi+Zj+Zij$ and $A_p^0=Z_p+Z_pi+Z_pj+Z_pij$. Then it is clear that the radical N of A_p contains pA_p, and N/pA_p is nilpotent.

Some parts of the following lemmas are already known (cf. [3], p. 160, Satz 1).

LEMMA 1.1. *If $p\neq 2$, and $p\not|\tau$, then A_p^0/pA_p^0 is a simple ring which is not a field. Hence A_p^0 is a maximal order.*

Proof. Let $x=a_1+a_2i+a_3j+a_4ij$; $a_i\in Z$ be in N. If $a_1\not\equiv 0 \pmod{pZ_p}$,

---

Almost results of this note were proved first by direct computations and presented to the Amer. Math. Soc. in August in 1962. However, recently the author has found structure theorems of h-order in [5], and the note is rewritten along this line.

then we may assume $a_1=1$. Since $xi+ix=2(-a_2+i)\in N$, $(-a_2+i)(-a_2-i)=a_2^2+1\in N\frown Z$. Hence, $(1+a_2i)=-(-a_2+i)i(\in N)$ is idempotent modulo pA_p, which is a contradiction. By using of the similar method, we obtain $a_i\equiv 0$ (mod pZ_p) for $i=2,3,4$ since $p\not|\tau$ and hence $N=pA_p^0$. Since A_p^0/pA_p^0 is not commutative and $[A_p^0/pA_p^0: Z/p]=4$, A_p^0/pA_p^0 is a simple ring which is not a field. Since $N=pA_p^0$ is A_p^0-projective, A_p^0 is hereditary by [4], Lemma 3.6. Therefore, A_p^0 is maximal by [4], Corollary 3.5.

LEMMA 1.2. *If $p|\tau$, then A_p^0/pA_p^0 is a field, and hence, A_p^0 is a unique maximal order in D_τ.*

Proof. Since $j^2=\tau\in pZ_p$, $N'=pZ_p+Z_pj+Z_pij\subseteqq N$. By assumption on τ, -1 is not a quadratic residue modulo p. Hence, A_p^0/N' is a field, and $N'=N$. It is clear that $N=jA_p^0$. Therefore, A_p^0 is a unique maximal order in D_τ by [1], Theorem 3.11.

Put $A^1=Z+Zi+Zj+Zt$; $t=\frac{1}{2}(1+i+j+ij)$ for $\tau\equiv -1$ (mod 4), and $A^{1'}=Z+Zi+Zh+Zl$; $h=\frac{1}{2}(i+j)$, $l=\frac{1}{2}(1+ij)$ for $\tau\equiv +1$ (mod 4) (cf. [3], p. 159).

LEMMA 1.3. *A_2^1 is a unique maximal order over Z_2 in D_τ and $A_2^{1'}/2A_2^{1'}$ is a simple ring which is not a field. Hence, $A_2^{1'}$ is also a maximal order in D_τ.*

Proof. 1) $\tau\equiv -1$ (mod 4). It is clear that $(1+i)A_2^1$ is contained in N. Let $e_1\equiv t$, $e_2\equiv 1+t$ (mod N), respectively. Then we have $e_1^2\equiv e_2$, $e_2^2\equiv e_1$, and $e_1e_2\equiv e_2e_1\equiv 1$. Therefore, $(1+i)A_2^1=N$ and A_2^1/N is a division ring. Hence, A_2^1 is a unique maximal order over Z_2.

2) $\tau\equiv 1$ (mod 4). Let $x=a_1+a_2i+a_3h+a_4l$ be in $N/2A_2^{1'}$; $a_i\in Z$. Then $xi+ix=2(a_1-a_2)+a_3+a_4i\in N$. If $a_3\not\equiv 0$ (mod N), $(1+(a_4/a_3)i)^2\equiv 1-(a_4/a_3)^2=0$ (mod N). Hence $l(1+i)-(1+i)h=1\in N$, which is a contradition. Similarly, we obtain $a_i\equiv 0$ (mod N) for all i. Therefore, $N=2A_2^{1'}$ and $A_2^{1'}/N$ is a simple ring which is not a field.

THEOREM 1.1. *Every nonmaximal h-order over Z_p in D_τ is a minimal and hence principal*[1] *h-order, and is written as an intersection of two maximal orders, and they are isomorphic. The number of nonmaximal h-orders in a maximal order is equal to $(1+p)$ or 0.*

Proof. If there exist non-maximal h-orders in D_τ, then $p\neq 2$ for $\tau\equiv -1$ (mod 4) and $p\not|\tau$ for any case. In this case we have by the above lemmas that $B/N=\Delta_2$ for any maximal order B and its radical N,

1) See the definition in [5], §§1 and 4.

where Δ is a division ring. Therefore, nonmaximal h-order is minimal by [5], Theorem 3.2, and is written as an intersection of two maximal orders by [4], Theorem 3.3. The rest of the theorem is true by [5], Theorems 4.2 and 6.1.

Let $\hat{Z}_p$ be the completion of Z_p with respect to pZ_p and $\hat{K}$ the quotient field of $\hat{Z}_p$. Let B be a maximal order over Z_p in D_τ, then $\hat{B}=B\otimes\hat{Z}_p$ is a maximal order in $\hat{D}_\tau$ by [1], Proposition 2.5, and which is isomorphic to a matrix ring over a unique maximal order in a division ring by [2], p. 100, Satz 12.

LEMMA 1.4. *If $p=2$, $\tau\equiv-1$ (mod 4) or $p|\tau$, then $\hat{D}_\tau=D_\tau\otimes\hat{K}$ is a division ring. In other cases, $\hat{D}_\tau$ is a matrix ring over $\hat{K}$ of degree two.*

Proof. If p and τ satisfy the first condition of the lemma then there exists a unique maximal order B over Z_p in D_τ such that B/N is a division ring by Lemmas 1.2 and 1.3. Hence $B/\hat{N}\approx B/N$ is also a division ring. Since B is a maximal order in $\hat{D}_\tau$, $\hat{D}_\tau$ is a division ring. In other cases D_τ contains at least two maximal orders by Leammas 1.1 and 1.3 and [4], Theorem 3.2. Hence, D_τ is not a division ring.

Since every maximal order B over Z_p is isomorphic by an inner-automorphism, every h-order A is written as $A=B\cap\alpha^{-1}B\alpha$, and A is isomorphic to an h-order contained in a maximal order which contains A_p^0. Hence, we may assume $B\supseteqq A_p^0$.

Let $\alpha'=a_1'+a_2'u+a_3'v+a_4'w$; $\{1, u, v, w\}$ is a base of D_τ over K. Then we can write $\alpha'=a\alpha$, where $a\in K$, $\alpha=a_1+a_2u+a_3v+a_4w$, and $(a_1, a_2, a_3, a_4)=1$. We shall call such an element "*normalized*". Since $A=B\cap\alpha'^{-1}B\alpha'=B\cap\alpha^{-1}B\alpha$, we may assume that α is normalized if we consider a form $A=B\cap\alpha^{-1}B\alpha$. If $\alpha'=a\alpha$; $a\in Z$, α normalized, we always assume $a>0$.

We define

$$N(\alpha) = a^2+b^2-\tau(c^2+d^2)$$

for $\alpha=a+bi+cj+dij$ in D_τ.

We note that if $p\neq2$, then $A_p^1=A_p^{1'}=A_p^0$.

Now, we can prove the following theorem:

THEOREM 1.2. *Let A be an h-order over Z_p in D_τ and $\alpha=a+bi+cj+dt$, $(\alpha'=a'+b'i+c'h+d'l)$ such that $N(\alpha)=pq$, $(p, q)=1$ $(N(\alpha')=pq'$, $(p, q')=1)$. Then A is isomorphic to one of the following forms:*

Case 1. $p=2$, $\tau\equiv-1$ (*mod* 4) $A=Z_2+Z_2i+Z_2j+Z_2t=A_2^1$ (*unique maximal*),

Case 2. $p=2$, $\tau\equiv 1$ (*mod* 4)   *a*) $A=Z_2+Z_2 i+Z_2 h+Z_2 l=A_2^{1'}$ (*maximal*)
b) $A=A_2^{1'}\cap\alpha'^{-1}A_2^{1'}\alpha'$,

Case 3. $p|\tau$ $A=Z_p+Z_p i+Z_p j+Z_p ij=A_p^0$ (*unique maximal*)

Case 4. $p\neq 2$, $p\not|\tau$ *a*) $A=A_p'$ (*maximal*)
b) $A=A_p^0\cap\alpha^{-1}A_p^0\alpha=(A_p^0\cap\alpha'^{-1}A_p^0\alpha'^{-1})$,

and those types are h-orders. In cases 2) *and* 4) *we assume that* β, β' *are normalized with respect to basis* 1, i, j, ij *and* 1, i, h, l, *respectively. Then* $F=A_p^0\cap\beta^{-1}A_p^0\beta$ $(=A_2^{1'}\cap\beta'^{-1}A_2^{1'}\beta')$ *is a nonmaximal h-order if and only if* $N(\beta)=pq'$, $(p, q')=1$, $(N(\beta')=2q''$, $(2, q'')=1)$, *where* $t=\frac{1}{2}(1+i+j+ij)$, $h=\frac{1}{2}(i+j)$ *and* $l=\frac{1}{2}(1+ij)$.

Proof. Let A be an h-order. In cases 1) and 3) there exists a unique maximal order in D_τ by Lemmas 1.2 and 1.3. Hence A is the maximal order, we have already shown that types 2, a) and 4, a) are maximal. Thus, we may assume that A is not maximal. Then $A\approx B\cap\alpha_1^{-1}B\alpha_1$ by [5], Proposition 6.3, where $\alpha_1^2=pu$; B is one of 1, a) and 4, a), and u is unit in B. Hence $N(\alpha_1)^2=N(\alpha_1^2)=p^2N(u)$. Since $N(\gamma)\in Z_p$ for all $\gamma\in B$, $N(u)$ is a unit in Z_p. Therefore, $N(\alpha_1)=pq$. Next, we assume $F=A_p^0\cap\beta^{-1}A_p^0\beta$ is an h-order. Then $A_p^0\cap\beta^{-1}A_p^0\beta=A_p^0\cap\alpha_1^{-1}A_p^0\alpha_1$ from the above argument. By [4], Theorem 3.3 we obtain $\beta^{-1}A_p^0\beta=\alpha_1^{-1}A_p^0\alpha_1$. Hence, $\alpha_1\beta^{-1}A_p^0$ (or $\beta\alpha_1^{-1}A_p^0$) is a two-sided ideal in A_p^0. Therefore, $\alpha_1\beta^{-1}A_p^0=p^tA_p^0$, $t\geqslant 0$ by Lemma 1.1, which implies $\alpha_1\beta^{-1}=p^tu'$; u' a unit in A_p. Since α_1 is normalized, $t=0$. Therefore, $N(\beta)=N(\alpha_1)N(u')=pq'$, $(p, q')=1$. We have the case 2 by the similar argument. In order to complete the proof of the theorem, it is sufficient to prove the "if part". We know by Lemma 1.4 that $\hat{D}_\tau=(\hat{K})_2$. Since the norm of regular representation of β is equal to $N(\beta)^2$, $N(\beta)$ is equal to the determinant of β as an element in $(\hat{K})_2$. It is clear that $\hat{F}=\hat{B}\cap\beta^{-1}\hat{B}\beta$, and $B=(T)_2$, where B is equal to A_p^0 or $A_2^{1'}$, and $T\approx\hat{Z}_p$. We can find unit elements u_1, u_2 in B such that $u_1\beta u_2=\begin{pmatrix} b_1 & 0 \\ 0 & b_2\end{pmatrix}=\beta'$, since $\beta\in B=(T)_2$. Hence $u_2^{-1}\hat{F}u_2=\begin{pmatrix} Z_p & pZ_p \\ Z_p & Z_p\end{pmatrix}$, which is an h-order with radical $\begin{pmatrix} pZ_p & pZ_p \\ Z_p & pZ_p\end{pmatrix}$. Therefore, F is an h-order by [5], Lemma 1.1.

We are only interested in the cases 2, b) and 4, b) in Theorem 1.2. In those cases we shall consider a condition under which the intersection of two maximal orders is an h-order.

Let $B=A_2^{1'}$ or A_p^1. Then every maximal order is written as $\alpha^{-1}B\alpha$. $A=\alpha_1^{-1}B\alpha_1\cap\alpha_2^{-1}B\alpha_2$ is an h-order *i.e.* $(\alpha_1\alpha_2)^{-1}B\alpha_1\alpha_2\cap B$ is an h-order if and only if $\alpha_1\alpha_2=(\alpha_1(\bar{\alpha}_2-d_2))/N(\alpha_2)=(a/N(\alpha_2))\beta$; $a\in Z$, β normalized in B, and $N(\beta)=pq$, $(p, q)=1$, where the (a_i, b_i, c_i, d_i)'s are coefficients of α_i. $N(\alpha_1\alpha_2^{-1})=N(\alpha_1)N(\alpha_2)^{-1}=N(\alpha_2)^{-2}a^2N(\beta)$, and hence $a^2N(\beta)=p^{e_1+e_2}q_1q_2$, where $N(\alpha_i)=p^{e_i}q_i$. If A is an h-order, then $N(\beta)=pq$. Hence, $e_1+e_2\equiv 1 \pmod 2$. We assume $e_1>e_2$. Since $\alpha_1=(a/N(\alpha_2))\gamma$, $\gamma\in B$, if α_1 is normalized, $e_1=1+e_2$. Conversely, if $e_1=e_2+1$ and $\alpha_1(\bar{\alpha}_2-d_2)\equiv 0 \pmod{p^{e_2}B}$ then A is a nonmaximal h-order. Thus we have,

PROPOSITION 1.2. *Let p be as in 2) and 4) in Theorem 1.2. $B=A_2^{1\prime}$ if $p=2$ and $B=A_p^1$ if $p\neq 2$, and α_i be normalized elements in B with coefficients (a_i, b_i, c_i, d_i) such that $N(\alpha_i)=p^{e_i}q_i$, $(p, q_i)=1$. Then $\alpha_1^{-1}B\alpha_1 \cap\alpha_2^{-1}B\alpha_2$ is a nonmaximal h-order if and only if $e_1\sim e_2$[2)]$=1$ and either $\alpha_1(\bar{\alpha}_2-d_2)\equiv 0$ (mod $p^{e_2}B$) if $e_1>e_2$ or $\alpha_2(\bar{\alpha}_1-d_1)$ (mod $p^{e_1}B$) if $e_1<e_2$.*

2. *H*-orders over *Z* in D_T.

In this section we shall consider h-orders over Z in D_T. By the obervations in [4], Section 7 there exists a finite set P of prime numbers p_i for any h-order A over Z such that A_p is a nonmaximal h-order for $p\in P$ and A_q is a maximal order for $q\notin P$. If an h-order A satisfies the above conditions for a given P we shall call that A belongs to P-block.

We note that P consists of prime integers which appear in 2) and 4) in Theorem 1.2.

Let A be an h-order which belongs to P-block. Then A_p is an intersection of two maximal orders in D_T for $p\in P$ by Theorem 1.1. Hence we have by [4], Theorem 7.2

THEOREM 2.1. *Let $P=\{p_1,\cdots,p_n\}$ and A an h-order belonging to P-block. Then there exist precisely 2^n maximal orders B_i containing A and $2n$ minimal h-orders F_i containing A in D_T. For any h-order H containing A H is written uniquely as $H=\bigvee_j B_{ij}$ when $H\neq A$, and H has 2^{m-1} representations as $H=B_{i_1}\cap B_{i_2}$ when H belongs to $P'=\{p_{i_1},p_{i_2},\cdots,p_{i_m}\}$ and $m\neq 0$. Furthermore, there exist 3^n h-orders containing A in D_T.*

From [5], Theorems 1.2 and 6.3 we have

THEOREM 2.2. *Let B be a maximal order in D_T and P be a set of prime integers which are neither divisors of T nor even for $T\equiv 1$ (mod 4). For any given nonmaximal h-orders $A(p)$ in B_p, $p\in P$, there exists a unique*

2) $e_1\sim e_2$ means the difference of larger one and smaller one.

h-orders A in B such that $A_p = A(p)$ for $p \in P$ and A_q is maximal for $q \notin P$, and hence A belongs to P-block. Therefore, there exist precisely $\Pi_{i=1}^{n}(p_i+1)$ h-orders in B which belong to P.

If we assume that B is principal, then any two maximal orders are isomorphic under some inner-automorphism. Hence, if we want to find a condition under which an intersection of two maximal order is an h-order, we may restrict that one of those is equal to A^1 or $A^{1'}$. Let $B = A^1$ or $A^{1'}$. It is clear that $A = B \cap \alpha^{-1}B\alpha$ is an h-order if and only if $A_p = B_p \cap \alpha^{-1}B_p\alpha$ is an h-order for all p by [1], p. 8, Corollary. Thus, we have from Theorem 1.2.

PROPOSITION 2.1. *Let Q be the set of prime factors of τ and α a normalized element in A^1 or $A^{1'}$ such that $N(\alpha) = p_1^{e_1}p_2^{e_2}\cdots p_n^{e_n}$. For $\tau \equiv -1$ (mod 4) $A^1 \cap \alpha^{-1}A^1\alpha$ is a nonmaximal h-order if and only if $e_i = 1$ for $p_i \notin Q$ and there exists at least one odd prime $p \notin Q$. For $\tau \equiv 1$ (mod 4) $A^{1'} \cap \alpha^{-1}A^{1'}\alpha$ is a nonmaximal h-order if and only if $e_i = 1$ for $p_i \notin Q$ and there exists at least one prime $p_j \notin Q$.*

3. Isomorphisms of h-orders.

In this section we shall study isomorphisms of h-orders in $D_{\pm 1}$.

Let A, A' be h-orders in D_τ. If $A_p \approx A'_p$ for all p, we call "*A and A' are locally isomorphic*". Then we have from Theorem 1.1 for any τ

THEOREM 3.1. *Let A, A' be h-orders in D_τ. A and A' are locally isomorphic if and only if A and A' belong to the same block.*

Now we shall consider isomorphisms of h-orders in $\{p\}$-block in D_{-1}.

Since two maximal orders in D_{-1} are isomorphic, we may consider h-orders in the maximal order $A^1 = Z + Zi + Zj + Zt$. We note that every isomorphism is given by an inner-automorphism in D_{-1}, and every h-order in $\{p\}$-block which is contained in A^1 is written as $A^1 \cap \alpha^{-1}A^1\alpha$, $N(\alpha) = p$, and $\alpha \in A^0$. Furthermore, we note the following facts: Let γ be a normalized element in A^0 such that $N(\gamma) = 2^n q$, $(2, q) = 1$, then $n \leqslant 2$, [6] p. 113, Satz 172.

We shall decide all h-orders in $\{p\}$-block in D_{-1} which are contained in A^1 and which are conjugate to a fixed h-order F in A^1, which means that we shall decide an isomorphic class in A^1. It is clear that those isomorphisms are induced by normalized element γ, and hence $N(\gamma) = 2^n q$, $n \leqslant 2$.

We shall denote A^1 by A and $A^1 \cap \alpha^{-1}A^1\alpha$ by F.

If $\gamma^{-1}(A\cap\alpha^{-1}A\alpha)\gamma=A\cap\beta^{-1}A\beta$, then $\gamma^{-1}A_q\gamma=A_q$ for $q\neq p$, and $\gamma^{-1}A_p\gamma\cap(\alpha\gamma)^{-1}A_p(\alpha\gamma)=A_p\cap\beta^{-1}A_p\beta$ where $\beta\in A^0$ and $N(\beta)=p$. Hence by [4], Theorem 3.3 we have two cases:

$$(1)\qquad A=(\alpha\gamma)^{-1}A\alpha\gamma,\ \gamma^{-1}A\gamma=\beta^{-1}A\beta,$$

$$(2)\qquad A=\gamma^{-1}A\gamma,\ \beta^{-1}A\beta=(\alpha\gamma)^{-1}A\alpha\gamma.$$

Let γ be a normalized element in A^0 which satisfies (1). Since $N(\beta)=p$, $N(\gamma)=2^n p$ by Proposition 1.2. Let $\alpha\gamma=2^t e\eta$ where $e\in Z$ such that $(e,2)=1$ and η is a normalized element in A^0. Since $A=(\alpha\gamma)^{-1}A(\alpha\gamma)$, $N(\eta)=2^{t'}$, and hence we know $e=p$ and $2t+t'=n$. Furthermore, since $\gamma=2^t\bar{\alpha}\eta$[3] is normalized, $t=0$. Therefore, $\gamma=\bar{\alpha}\eta$ and $N(\eta)=2^n$. Conversely if $\gamma=\bar{\alpha}\eta$, then $\gamma^{-1}F\gamma\subset A$. Therefore $\gamma^{-1}F\gamma$ is an h-order in A.

Next let γ' be a normalized element in A^0 as in (2). Since $A=\gamma'^{-1}A\gamma'$, $N(\gamma')=2^n$. Conversely if $N(\gamma')=2^n$, then $A=\gamma'^{-1}A\gamma'$. Therefore, $\gamma'^{-1}F\gamma'$ is also an h-order in A.

By $\Phi(\alpha)$ we denote the set of normalized elements in A^0 which induce conjugate h-orders to F in A. Then $\Phi(\alpha)$ consists of all normalized elements in A^0 which are either of

$$(3)\qquad N(\gamma)=2^n\quad\text{for}\quad n\leqslant 2$$

or expressed as

(4) $\gamma=\bar{\alpha}\eta$ where η is a normalized element in A such that $N(\eta)=2^{n'}$ for $n'\leqslant 2$.

Let $\Psi(\alpha)=\{\gamma|\in\Phi(\alpha),\ \gamma^{-1}F\gamma=F\}$. If $\gamma^{-1}F\gamma=\gamma'^{-1}F\gamma'$ for $\gamma,\gamma'\in\Phi(\alpha)$, then $\gamma\gamma'^{-1}=e/N(\gamma')\xi$ where $\xi\in\Phi(\alpha)$ and $e\in Z$ such that $e\xi=\gamma\gamma'^{-1}$. Conversely if $\gamma=\xi\gamma'$ for $\xi\in\Phi(\alpha)$, then $\gamma^{-1}F\gamma=\gamma'^{-1}F\gamma'$. Therefore, the number of conjugate h-orders to F in A is equal to $\sharp(\Phi(\alpha))/\sharp(\Psi(\alpha))$, where $\sharp(\ )$ means the number of elements in ( ).

Now, we shall decide $\sharp(\Phi(\alpha))$. In order to that we shall divide elements of $\Phi(\alpha)$ into two parts according to (3), and (4).

Let η be a normalized element in A^0 such that $N(\eta)=2^n$ for $n\leqslant 2$. There exist 48 elements η by [6], p. 113, Satz 172.

Let η be as above. We denote the coefficients of α, η, and $\bar{\alpha}\eta$ by (a, b, c, d), (x_1, x_2, x_2, x_4) and (X_1, X_2, X_3, X_4), respectively. Then we have

---

3) $\bar{\alpha}$ means the conjugate element of α.

$$(5)\qquad \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}.$$

Since the determinant of the matrix of (5) is equal to p^2 and η is normalized, $2 \nmid X_i$ for all i. Therefore, $\bar{\alpha}\eta$ is normalized, since $N(\bar{\alpha}\eta)=2^n p$, $n\leqslant 2$. Thus we have

LEMMA 3.1. $\#(\Phi(\alpha))=98$ *for any normalized element in* A^0.

Now, we shall consider $\#(\Psi(\alpha))$. We denote the sets of elements of $\Phi(\alpha)$ which satisfy (3) and (4) by $\Phi_1(\alpha)$ and $\Phi_2(\alpha)$, respectively. It is clear that $\Phi(\alpha)=\Phi_1(\alpha)\cup\Phi_2(\alpha)$ and $\Phi_1(\alpha)\cap\Phi_2(\alpha)=\phi$.

Let $\gamma\in\Psi_1(\alpha)=\Phi_1(\alpha)\cap\Psi(\alpha)$ and $\gamma=\bar{\alpha}\eta$ with $N(\eta)=2^n$, $n\leqslant 2$. Let $\bar{\alpha}\eta\alpha^{-1}=(e/p)\zeta$, where $e\in Z$, is normalized. Since $\gamma^{-1}F\gamma=F$, $(\bar{\alpha}\eta)^{-1}A\bar{\alpha}\eta=A$. Therefore, $N(\zeta)=2^m$ by Theorem 1.2, and hence $e=2^{\frac{1}{2}(n-m)}p$. Thus $\eta=2^{\frac{1}{2}(n-m)}\bar{\alpha}^{-1}\zeta\alpha$. Since η is normalized and $N(\alpha)=p$, $\eta=\bar{\alpha}^{-1}\zeta\alpha$, and $n=m$. Conversely, if $\eta=\bar{\alpha}^{-1}\zeta\alpha$, then $(\bar{\alpha}\eta)^{-1}F\bar{\alpha}\eta=F$. Thus we obtain that $\#(\Phi_1(\alpha))$ is equal to the number of normalized elements ζ in A^0 such that $\bar{\alpha}^{-1}\zeta\alpha$ is normalized and $N(\zeta)=2^n$ for $n\leqslant 2$. Let $\bar{\alpha}^{-1}\zeta\alpha=(1/p)\alpha\zeta\alpha=(e/p)\xi$, where e, ξ are as above. Let (x_1, x_2, x_3, x_4), (X_1, X_2, X_3, X_4) be the coefficients of ζ and $\alpha\zeta\alpha$. Then

$$(6)\qquad \begin{pmatrix} a^2-b^2-c^2-d^2 & -2ab & -2ac & -2ad \\ 2ab & a^2-b^2+c^2+d^2 & -2bc & -2bd \\ 2ac & -2bc & a^2+b^2-c^2+d^2 & -2cd \\ 2ad & -2bd & -2cd & a^2+b^2+c^2-d^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}.$$

Since the determinant of matrix in (6) is equal to p^4 and ζ is normalized, $2\nmid X_i$ for all i. Hence $\alpha^{-1}\zeta\alpha$ is normalized if and only if $\alpha\zeta\alpha\equiv 0 \pmod{pA^0}$, which is equivalent to $ax_1-bx_2-cx_3-dx_4\equiv 0 \pmod{p}$ since $N(\alpha)=p$. On the other hand, since ζ is normalized and $N(\zeta)=2^n$ for $n\leqslant 2$, $|x_i|\leqslant 1$ for all i. Furthermore, $|ax_1-bx_2-cx_3-dx_4|\leqslant |a|+|b|+|c|+|d| < a^2+b^2+c^2+d^2=p$ if $p\neq 3$. Therefore,

$$(7)\qquad ax_1-bx_2-cx_3-dx_4 = 0\,.$$

If $|x_i|=1$ for all i, then $0=(ax_1-bx_2-cx_3-dx_4)^2=p+2e$, where $e\in Z$, which is a contradiction. Hence $N(\zeta)=1$ or 2 if $p\neq 3$. Therefore, either three or two of the x_i's are equal to zero. Thus we obtain

LEMMA 3.2. *$\sharp(\Psi_1(\alpha))$ is equal to the number of occasins for α that a coefficient of α is zero or the absolute values of two coefficients of α are the same, except $p=3$ and two of coefficients are zero. Hence $\sharp(\Psi(\alpha))=0, 2, 4$ or 6 except the following special cases. If $p=3$, $\sharp(\Psi(\alpha))=12$ and if two of coefficients are zero, then $\sharp(\Psi_1(\alpha))=8$.*

Finally, we shall consider $\sharp(\Phi_2(\alpha))$, where $\Psi_2(\alpha)=\Phi_2(\alpha)\cap\Psi(\alpha)$. Let $\gamma\in\Phi_2(\alpha)$ and $\alpha\gamma\alpha^{-1}=(e/p)\xi$ where $\alpha\gamma\bar{\alpha}=e\xi$. Then $N(\xi)=2^n$ since $\alpha^{-1}A\alpha=(\alpha\gamma)^{-1}A\alpha\gamma$. Hence $p\mid e$. Conversely, if $\gamma\in\Phi_2(\alpha)$ and $p\mid e$, then $\gamma\in\Psi_2(\alpha)$. Hence for $\gamma\in\Phi_2(\alpha)$, $\gamma\in\Psi_2(\alpha)$ if and only if

$$\alpha\gamma\bar{\alpha}\equiv 0 \pmod{pA^0}. \tag{8}$$

Let $\gamma=x_1+x_2i+x_3j+x_4ij$, then (8) is equal to

$$\begin{pmatrix} a^2+b^2 & ad+bc & bd-ac \\ bc-ad & a^2+c^2 & ab+cd \\ ac+bd & cd+ab & a^2+d^2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix}\equiv 0 \pmod{p} \tag{9}$$

Since $N(\alpha)=p$, we may assume $a^2+b^2\not\equiv 0 \pmod{p}$. Then $x_2\equiv Ax_3+Bx_4 \pmod{p}$ is solutions of (9) where x_3, $x_4=0, \pm 1$ and $A\equiv-(ad+bc)/(a^2+b^2)$, $B\equiv-(bd-ac)/(a^2+b^2) \pmod{p}$. We may assume that A, B are integers whose absolute values are equal to or less than $(p-1)/2$. We first note that the following cases do not appear:

a) $A=B=0$,

b) $A=0$, $|B|=1$ or $|A|=1$, $B=0$.

Noting the above remark and $N(\gamma)=2^n$ for $n\leqslant 2$, we can check easily that

LEMMA 3.3. *$\sharp(\Psi_2(\alpha))$ is equal to the following:*

I) *2 if $|A|$, $|B|\neq 0, 1$ and $|A|\sim|B|\geqslant 2$,*

II) *4 if $|A|=|B|$, $|A|$, $|B|\neq 0, 1$ and $|A|+|B|\neq p-1$,*
  or if $|A|=1$, or $|B|=1$ and $|A|\sim|B|\geqslant 2$,

III) *6 if $|A|\sim|B|=1$,*

IV) *8 if $p\neq 3$, $|A|=|B|=1$*
  or if $|A|=|B|=1$ and $|A|+|B|=p-1$
  or if $|A||B|=0$ and $|A|\sim|B|\geqslant 2$,

V) *12 if $p=3$,*

where $A\equiv-(ad+bc)/(a^2+b^2)$, $B\equiv-(bd-ac)/(a^2+b^2)$ if $a^2+b^2\not\equiv 0$ (mod p), and $A\equiv-(ab+cd)/(a^2+c^2)$, $B\equiv-(bc-ad)/(a^2+c^2)$ if $a^2+c^2\not\equiv 0$ (mod p), *and* $A\equiv-(ac+bd)/(a^2+d^2)$ $B=-(cd-ab)/(a^2+d^2)$ *if* $a^2+d^2\not\equiv 0$ (*mod* p), *and* $\alpha=a+bi+cj+dij$ *with* $N(\alpha)=p$.

Since $\#(\Phi(\alpha))=98$ and $\#(\Psi(\alpha))=\#(\Psi_1(\alpha))+\#(\Psi_2(\alpha))$ we have from Lemmas 3.1, 3.2, and 3.3

THEOREM 3.2. *Let F be an h-order in D_{-1} which belongs to $\{p\}$-block, and A a maximal order containing F. Then the number of conjugate h-orders to F in A is equal to* 6, 8, 12, 16, 24, *or* 48 *if* $p\neq 3$. *If* $p=3$ *it is equal to* 4.

THEOREM 3.3. *Any pair of two h-orders in D_{-1} belonging to $\{p\}$-block is isomorphic if and anly if* $p=3$, 5, 7, 11 *or* 23.

Proof. Since any pair of two maximal order is isomorphic, we may consider h-orders in a maximal order A in D_{-1}. The number of h-orders of $\{p\}$-block in A is equal to $p+1$ by Theorem 2.1. If $p=3$, then the theorem is true by Theorem 3.2. We put $\alpha=1+i+2j+6ij$ for $p=47$, then the number of conjugate h-orders to $A\cap\alpha^{-1}A\alpha$ in A is less than 48 by Lemmas 3.1, 3.2 and 3.3. Similarly, let $\alpha=1+2i$, $1+i+j+2ij$, $1+i+3ij$ and $3+3i+2j+ij$ for $p=5$, 7, 11 and 23, respectively. Then the number of conjugate h-orders to $A\cap\alpha^{-1}A\alpha$ in A is equal to $p+1$.

Using the same argument as above we have

THEOREM 3.3′ *If $\{p_1, p_2,\cdots,p_n\}$ are large enough, then there exist always h-orders in $\{p_1,\cdots,p_n\}$-block in D_{-1} which are not ismorphic.*

REMARK 1. Let $\alpha=1+3i+5j+6ij$, then the number of conjugate h-orders to $A\cap\alpha^{-1}A\alpha$ in A is equal to 48.

REMARK 2. $F=A\cap\alpha^{-1}A\alpha$ has exactly 16 h-orders in A which are conjugate to F if only if $abcd\neq 0$ and two of $|a|$, $|b|$, $|c|$ and $|d|$ are not the same, and III) is satisfied, when $\alpha=a+bi+ij+dij$ with $N(\alpha)=p$.

Finally, we shall consider the same problem in $D_{+1}=(K)_2$. Let $A=(Z)_2$, then $F=\begin{pmatrix} Z & lZ \\ Z & Z\end{pmatrix}$ is an h-order in A which belongs to $\{p_1, p_2,\cdots,p_n\}$-block, where $l=\prod_{i=1}^{n} p_i$, and $F_p=\begin{pmatrix} Z_p & pZ_p \\ Z_p & Z_p\end{pmatrix}$; $p=p_i$. Let $\alpha=\begin{pmatrix} 1 & i \\ 0 & 1\end{pmatrix}$, $i=0,\cdots,p-1$, and $\alpha_p=\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$. Then $\{\alpha_i^{-1}F_p\alpha_i\}_{i=0}^{p}$ consists of all non-maximal h-orders in A_p. Furthermore, we note that for unit elements β_1, β_2 in A_p $\beta_1^{-1}F_p\beta_1=\beta_2^{-1}F_p\beta_2$ if and only if $\beta_1\beta_2^{-1}$ is a unit in F_p by [5], Lemma 6.1. Hence if $\beta_1\beta_2^{-1}=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, then $\beta_1^{-1}F_p\beta_1=\beta_2^{-1}F_p\beta_2$ if and only if $b\equiv 0$ (mod pZ_p).

Let F' be an h-order in A belonging to $(p_1,\cdots,p_n)$-block such that $F'_{p_i}=\begin{pmatrix}1 & j_i\\ 0 & i\end{pmatrix}^{-1}F_{p_i}\begin{pmatrix}1 & j_i\\ 0 & 1\end{pmatrix}$ for $1\leq i\leq s$, and $0\leqq j_i\leqq p_i-1$, and $F'_{p_j}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1}F_{p_j}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ for $s+1\leq j\leq n$. Let $\alpha=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ in B such that $a=\prod_{j=s+1}^{n} p_j$ and $ad-bc=1$. Then we obtain from the above remark that $\alpha^{-1}F_{p_j}\alpha=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1}F_{p_j}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ since $a\equiv 0 \pmod{p_j}$ for $s+1\leq j\leq n$, and $\alpha^{-1}F_{p_i}\alpha=\begin{pmatrix}1 & j_i\\ 0 & 1\end{pmatrix}^{-1}A_{p_i}\begin{pmatrix}1 & j_i\\ 0 & 1\end{pmatrix}$ if and only if $b\equiv aj_i \pmod{p_i}$ for $1\leq i\leq s$. We can find a solution b_0 of $x\equiv aj_i \pmod{p_i}$ $(i=1,\cdots,s)$ such that $(a, b_0)=1$ since if b_1 is a solution of $x\equiv j_i \pmod{p_i}$ $(i=1,\cdots,s)$, then $b_0=ab_1+\prod_{i=1}^{s} p_i$. Hence there exist integers b_0, d_0 such that $ad_0-c_0b_0=1$. Therefore, if we take $\alpha_0=\begin{pmatrix}a & b_0\\ c_0 & d_0\end{pmatrix}$, then $\alpha^{-1}F\alpha=F'$. Thus we obtain

THEOREM 3.4.[4] *Any pair of two h-orders in a maximal order A in* D_1 *which belong to the same block is isomorphic under some innerautomorphism in A and hence, any pair of two h-orders of the same type is isomorphic.*

Bibliography

[1] M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc., **97** (1960), 1-24.
[2] M. Deuring, Algebren, Springer, (1935).
[3] L. Dickson, Algebren und ihre Zahlentheorie, Zürich, (1927).
[4] M. Harada, *Hereditary orders*, Trans. Amer. Math. Soc., **107** (1963) 273-290.
[5] ———, *Structure of hereditary orders over local rings*, J. Math. Osaka City Univ. **14** (1963) 1-22.
[6] E. Landau, Vorlesungen und Zahlentheorie, Band 1, Leipzig, (1927).

4) Added in proof. This theorem is generalized easily by Corollary to Theorem 4.7 of A. Brumer; The structure of hereditary orders, Doctoral dissertation Princeton Univ., 1963.

(Let R be a principal Dedekind domain and $\Sigma=K_n$. Then for two h-oders Λ, Γ in Σ Λ and Γ are isomorphic if and only if Λ_p and Γ_p are isomorphic, where p runs through all prime ideals in R).