Journal of Mathematics,
Osaka City University,
Vol. 13, No. 2

On Bott-Samelson K-cycles associated with symmetric spaces

By Shôrô ARAKI

(*Received Nov. 16, 1962*)

Introduction

In 1958, R. Bott and H. Samelson [8] defined the notion of K-cycles for every smooth complete Riemannian manifold M on which a compact connected Lie group K of isometries operates variationally completely, and showed that some K-cycles form a homology basis (mod 2 in general and integral in K-orientable cases) of some type of spaces of paths in M. They proved three kinds of variational completeness of K-actions related to symmetric spaces, and obtained many direct results.

In case K operates on itself or on its Lie algebra by adjoint actions, they determined moreover the integral cohomology of used K-cycles completely by making use of Cartan integers and applied it to several cohomological and homotopical problems of Lie groups [8], Chap. III.

The aim of the present work is to get an analogy (Theorems 2.10 and 6.4) of this for K-cycles associated with symmetric spaces, a partial result of which is used in determining the cohomology mod 2 of the compact exceptional group $\boldsymbol{E}_8$ [3].

§ **1** is preliminaries about symmetric pairs, their Cartan subalgebras, restricted root systems, etc., including the definition of symmetric pairs of splitting rank. In § **2** we discuss basic properties of K-cycles associated with symmetric pairs. It is proved that every K-cycle associated with a symmetric pair is an iterated sphere bundle over a sphere (Cor. 2.5). Theorem 2.10 asserts that the cohomology rings mod 2 of K-cycles, associated with pairs (G, K) with simply connected G, are determined completely by Cartan integers of restricted roots. In §§ **3** and **4** we compute the number of connected components of centralizers in K of maximal tori and singular tori of symmetric pairs (G, K) with simply connected G. § **5** is a preparation for subsequent two sections.

In § **6** we discuss symmetric spaces of splitting rank. These behave themselves very similarly to compact Lie groups as symmetric spaces from homological point of view; for example, there holds an analogy (Prop. 6.3) of a well known result of J. Leray [10], Prop. 11.1. Here we prove Theorem 6.4 which asserts that the integral cohomology rings of K-cycles, associated with symmetric pairs (G, K) of splitting rank with simply connected G, are determined completely analogously to [8], Chap. III, Prop. 4.2, by Cartan integers of restricted roots.

Though there are many other symmetric pairs for which their K-cycles are all orientable, this integral form can not be extended to them since each one of them has at least one restricted root of odd multiplicity by virtue of Prop. 1.2 and some K-cycles associated with it have exterior tensor factors of Prop. 2.9 in their integral cohomologies.

Finally §**7** is devoted to the proof of Theorem 2.10.

§1. Symmetric pairs.

1. 1. Let G be a compact connected Lie group, σ an involutive automorphism of G, K the e-component of the group $\hat{K}$ consisting of all fixed elements under σ. The pair (G, K) is called a *symmetric pair* [8], and the homogeneous space $G/\tilde{K}$ $(K \subseteqq \tilde{K} \subseteqq \hat{K})$ a compact symmetric space, $\hat{K}$ the fixed group of the pair. If G is simply connected, then $K=\hat{K}$ by [8, 9] and G/K is simply-connected. Conversely every compact simply-connected symmetric space can be expressed as a homogeneus space of a simply-connected group G.

Let (G, K) be a symmetric pair, and $\mathfrak{g}$, $\mathfrak{k}$ denote Lie algebras of G, K respectively. We choose once and for all a positive definite invariant metric on $\mathfrak{g}$. The scalar products defined canonically by this metric on $\mathfrak{g}$, subspaces of $\mathfrak{g}$, and their dual spaces, will be denoted by $<\ ,\ >$.

The involution σ of G induces an involutive automorphism of $\mathfrak{g}$ denoted by the same letter σ. The pair $(\mathfrak{g}, \mathfrak{k})$ with σ is called the *infinitesimal symmetric pair* of (G, K). $\mathfrak{k}$ is the eigenspace of σ with eigenvalue 1. Let $\mathfrak{m}$ be the eigenspace of σ with eigenvalue -1, then we have the well-known orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} \tag{1.1}$$

with respect to $<\ ,\ >$, satisfying

$$[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \ [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k} \tag{1.2}$$

which characterize the infinitesimal involution σ conversely.

Put $M=\exp \mathfrak{m}$. It is a closed submanifold of G, which can be regarded as a symmetric space identified with $G/\hat{K}$ in a well-known fashion. (Cf., [7] or others.)

1. 2. Let $\mathfrak{t}^-$ be a maximal abelian subalgebra of $\mathfrak{m}$, $\mathfrak{t}$ that of $\mathfrak{g}$ containing $\mathfrak{t}^-$. $T_-=\exp \mathfrak{t}^-$ is a maximal torus of M, and $T=\exp \mathfrak{t}$ a maximal torus of G containg T_-.

Since we are concerned only with compact ones, we mean by "roots" the angular parameters in the sense of E. Cartan. Let $\mathfrak{r}$ be the system of all non-zero roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$. We have the Cartan orthogonal decomposition

$$\mathfrak{g} = \mathfrak{t} + \sum \mathfrak{e}_\alpha \tag{1.3}$$

where the summation runs over all positive roots α of $\mathfrak{r}$ with respect to a linear order in $\mathfrak{t}^*$ (dual space of $\mathfrak{t}$). The space $\mathfrak{e}_\alpha$ is of dimension 2 and invariant under the adjoint actions of T (or of $\mathfrak{t}$). The adjoint action of $\exp H$, $H \in \mathfrak{t}$, on $\mathfrak{e}_\alpha$ is a rotation through the angle $2\pi\alpha(H)$.

By our choice of $\mathfrak{t}$, it is closed by the involution σ and $\sigma \mid \mathfrak{t}$ induces an involution in $\mathfrak{t}^*$ defined by

$$\sigma^*\alpha(H)=-\alpha(\sigma H) \qquad \text{for } \alpha\in\mathfrak{t}^* \text{ and } H\in\mathfrak{t}.$$

$\mathfrak{r}$ is closed by σ^* and becomes a σ-system of roots (normally extendable in the sense of [2]).

The set

$$\mathfrak{r}_0=\{\alpha\in\mathfrak{r}\,;\ \sigma^*\alpha=-\alpha\}$$

is a closed subsystem of roots of $\mathfrak{r}$. Let $\mathfrak{r}^-$ be the set of linear forms on $\mathfrak{t}^-$ obtained by restricting $\mathfrak{r}-\mathfrak{r}_0$ to $\mathfrak{t}^-$. It is a root system in the sense of [2], 2.1, and is called the *restricted root system* of $\mathfrak{m}$ (or of (G, K)) with respect to $\mathfrak{t}^-$. The elements of $\mathfrak{r}^-$ are called the *restricted roots*. About the properties of restricted roots we refer to [2]. One of the characteristic properties of restricted roots, different from those of roots of Lie groups, is that two times of a restricted root can be a restricted root (but four times of it is not so). Cf., [2], 2.1.2°).

For any $\lambda\in\mathfrak{r}^-$, $\mathfrak{r}_\lambda$ denotes the set of all $\alpha\in\mathfrak{r}$ such that $\alpha\mid\mathfrak{t}^-=\lambda$. The number of elements of $\mathfrak{r}_\lambda$ is called the *multiplicity of* λ, denoted by $m(\lambda)$; put

(1.4) $$\tilde{\mathfrak{e}}_\mu=\sum_{\alpha\in\mathfrak{r}_\lambda}\tilde{\mathfrak{e}}_\alpha,$$

then dim $\tilde{\mathfrak{e}}_\lambda=2m(\lambda)$. By [9], p. 353, or [1], p. 47, $\tilde{\mathfrak{e}}_\lambda$ has an ortho-normal basis

(1.5′) $$\{A_1,\ B_1,\ A_2,\ B_2,\ \cdots,\ A_{m(\lambda)},\ B_{m(\lambda)}\}$$

such that

(1.5) $$\sigma A_i=A_i,\ \sigma B_i=-B_i,$$
$$[H,\ A_i]=2\pi\lambda(H)B_i,\ [H,\ B_i]=-2\pi\lambda(H)A_i$$

for $H\in\mathfrak{t}^-$ and $1\leqq i\leqq m(\lambda)$. In particular,

(1.6) $$\dim(\mathfrak{k}\cap\tilde{\mathfrak{e}}_\lambda)\ =\ \dim(\mathfrak{m}\cap\tilde{\mathfrak{e}}_\lambda)\ =\ m(\lambda).$$

1. 3. For any pair (λ, n), $\lambda\in\mathfrak{r}$ or $\in\mathfrak{r}^-$ and n an integer, we define a singular plane p in $\mathfrak{t}$ or in $\mathfrak{t}^-$ by

$$p=\{H\in\mathfrak{t}\,(\text{or}\in\mathfrak{t}^-)\,;\ \lambda(H)=n\}.$$

We shall write $p=(\lambda, n)$. Thus

$$(\lambda,\ n)\ =\ (-\lambda,\ -n)$$

as a set, and in case p is a singular plane in $\mathfrak{t}^-$ such that $2\lambda\in\mathfrak{r}^-$,

$$p\ =\ (\lambda,\ n)\ =\ (2\lambda,\ 2n).$$

Define two subsystems $\mathfrak{r}^{-\prime}$ and $\mathfrak{r}^{-\prime\prime}$ of $\mathfrak{r}^-$ by

$$\mathfrak{r}^{-\prime}\ =\ \{\lambda\in\mathfrak{r}^-;\ \lambda/2\notin\mathfrak{r}^-\},$$

and

$$\mathfrak{r}^{-\prime\prime}=\{\lambda\in\mathfrak{r}^-;\ 2\lambda\notin\mathfrak{r}^-\}$$

respectively. Then every singular plane p in $\mathfrak{t}^-$ can be expressed as

(1.7) $$p\ =\ (\lambda,\ n),\ \lambda\in\mathfrak{r}^{-\prime\prime}.$$

Hereafter we express singular planes in $\mathfrak{t}^-$ always in this form.

If p is expressed as (1.7), then we say that λ is a representative root of p

which is determined up to sign. When it becomes necessary to orient a singular plane $p=(\lambda, n)$ in $\mathfrak{t}^-$ as will occur in discussions of integral cohomology of some K-cycles, then we distinguish (λ, n) from $(-\lambda, -n)$ as to denote oppositely oriented ones. As far as we are concerned to cohomology mod 2 of K-cycles, this convention is not necessary.

For any singular plane p in $\mathfrak{t}^-$ the number of distinct singular planes in $\mathfrak{t}$ containing p is called the *multiplicity of* p, denoted by $m(p)$. For a root $\lambda\in\mathfrak{r}^{-\prime\prime}$ such that $p=(\lambda, n)$,

$$(1.8)\qquad \begin{aligned} m(p) &= m(\lambda) && \text{if } \lambda/2\notin\mathfrak{r}^- \text{ or } n \text{ odd,}\\ &= m(\lambda)+m(\lambda/2) && \text{if } \lambda/2\in\mathfrak{r}^- \text{ and } n \text{ even.}\end{aligned}$$

In subsequent discussions we shall always mean by singular planes only "singular planes in $\mathfrak{t}^-$".

1. 4. For any subset L of G and any closed subgroup U of G, we denote by U_L the centralizer of L in U. If the Lie algebra of U is denoted by $\mathfrak{u}$, we denote the Lie algebra of U_L by $\mathfrak{u}_L$; thus

$$\mathfrak{u}_L=\{X\in\mathfrak{u};\ \text{ad}\ \ell\cdot X=X \ \text{ for all } \ell\in X\}.$$

exp p, p singular planes in $\mathfrak{t}^-$, are called singular tori in T_- though they are generally not subgroups. The centralizer of exp p in U and its Lie algebra are ad $\ell\cdot X$ denoted by U_p and $\mathfrak{u}_p$ respectively for the sake of shortness.

Let

$$(1.1')\qquad \mathfrak{t}=\mathfrak{t}^+ + \mathfrak{t}^-$$

be the orthogonal decomposition of $\mathfrak{t}$ with respect to $<,>$. And put $T_+=\exp\mathfrak{t}^+$. As is easily seen T_+ is a maximal torus of $K_{T_-}^0$. (For any subgroup U of G, we denote by U^0 the e-component of U.) The Lie algebra of G_{T_-}, $\mathfrak{g}_{T_-}$, is described by the decomposition (1.3) as follows:

$$(1.9)\qquad \mathfrak{g}_T=\mathfrak{t}+\sum_{\alpha\in\mathfrak{r}_0}\mathfrak{e}_\alpha.$$

i. e., $\mathfrak{r}_0$ is the root system of $\mathfrak{g}_T$ with respect to $\mathfrak{t}$. Since

$$\sigma\mathfrak{e}_\alpha=e_{\sigma^*\alpha} \text{ and } \mathfrak{e}_\alpha=\mathfrak{e}_{-\alpha}$$

for all $\alpha\in\mathfrak{r}$ (cf., [1], 1.2, or some others), we see that

$$\mathfrak{e}_\alpha\subset\mathfrak{k}\qquad\text{for all } \alpha\in\mathfrak{r}_0.$$

Therefore

$$(1.10)\qquad \mathfrak{k}_{T_-}=\mathfrak{t}^+ + \sum_{\alpha\in\mathfrak{r}_0}\mathfrak{e}_\alpha,$$

and $\mathfrak{r}_0$ becomes the root system of $\mathfrak{k}_{T_-}$ with respect to $\mathfrak{t}^+$.

For any singular plane $p=(\lambda, n)$, $\lambda\in\mathfrak{r}^{-\prime\prime}$, in $\mathfrak{t}^-$, we put

$$(1.11)\qquad \begin{aligned}\tilde{\mathfrak{e}}_p&=\tilde{\mathfrak{e}}_\lambda && \text{if } \lambda/2\notin\mathfrak{r}^- \text{ or } n \text{ odd,}\\ &=\tilde{\mathfrak{e}}_\lambda+\tilde{\mathfrak{e}}_{\lambda/2} && \text{otherwise.}\end{aligned}$$

And discuss the adjoint actions of exp p on each $\tilde{\mathfrak{e}}_\mu$, $\mu\in\mathfrak{r}^-$, by (1.5), then we see easily that

$$(1.12)\qquad \begin{aligned}\mathfrak{g}_p&=\mathfrak{g}_{T_-}+\tilde{\mathfrak{e}}_p,\\ \mathfrak{k}_p&=\mathfrak{t}_{T_-}+\mathfrak{k}\cap\tilde{\mathfrak{e}}_p.\end{aligned}$$

Then, by (1.6), (1.8) and (1.11) we see that

(1.13) $$\dim (K_p/K_{T_-}) = \dim \mathfrak{k}_p - \dim \mathfrak{k}_{T_-} = m(p).$$

1. 5. A linear order in $\mathfrak{t}^*$ satisfying that for any positive root α of $\mathfrak{r}-\mathfrak{r}_0$ $\sigma^*\alpha$ is also positive, is called a σ-order. A σ-fundamental system Δ of $\mathfrak{r}$ is a fundamental system with respect to some σ-order. About the properties of σ-fundamental systems, we refer to [12].

Let Δ be a σ-fundamental system of $\mathfrak{r}$, then $\Delta_0=\mathfrak{r}_0\cap\Delta$ is a fundamental system of $\mathfrak{r}_0$; on the other hand, Δ^-, defined as the subset of $\mathfrak{r}^-$ obtained by restricting $\Delta-\Delta_0$ to $\mathfrak{t}^-$, is a fundamental system of $\mathfrak{r}^-$, called the restricted fundamental system.

By W, W_0 and W^- we denote Weyl groups of $\mathfrak{r}$, $\mathfrak{r}_0$ and $\mathfrak{r}^-$ respectively, i.e., finite groups of orthogonal transformations on $\mathfrak{t}$, $\mathfrak{t}^+$ and $\mathfrak{t}^-$ respectively generated by reflections across singular planes $(\alpha, 0)$, $\alpha\in\mathfrak{r}$, $\in\mathfrak{r}_0$ or $\in\mathfrak{r}^-$. They operates also on dual spaces $\mathfrak{t}^*$, $\mathfrak{t}^{+*}$ and $\mathfrak{t}^{-*}$ by their transposed actions.

As is well known, every action of Weyl groups on $\mathfrak{t}^*$, $\mathfrak{t}^{+*}$ or $\mathfrak{t}^{-*}$, transforms roots to roots, fundamental systems of roots to themselves, and permutes the set of fundamental systems simply transitively.

Let W_σ be the subgroup of W consisting of all $s\in W$ commuting with σ. As is easily seen, every action of W_σ transforms σ-fundamental systems of $\mathfrak{r}$ to themselves. For every $w_0\in W_0$, extend the action of w_0 on $\mathfrak{t}^+$ to that on $\mathfrak{t}$ so that $w_0|\mathfrak{t}^-$=identity map. Thus W_0 becomes a subgroup, actually a normal subgroup, of W_σ. For any $w\in W_\sigma$, $w\mathfrak{t}^-=\mathfrak{t}^-$, i.e., $w|\mathfrak{t}^-$ is an orthogonal transformation of $\mathfrak{t}^-$. By [12], p. 107, lemmas 1 and 2, we can easily conclude that

(1.14) $$w|\mathfrak{t}\in W^- \quad \textit{for all } w\in W_\sigma.$$

and that

(1.15) *thus obtained natural homomorphism* $\rho: W_\sigma \to W^-$ *is surjective with* W_0 *as its kernel.*

Therefrom, furthermore, we see that

(1.16) W_σ *permutes the set of* σ*-fundamental systems of* $\mathfrak{r}$ *simply transitively.*

1. 6. As is well known, there is a canonical identity

(1.17) $$N(T)/T = W$$

in the sense that the adjoint actions of the left side on $\mathfrak{t}$ coincide with operations of W, where $N(T)$ is the normalizer of T in G.

Now we assume that G *is simply conncted.*

In the same sense as above, denoting by $N_K(T_-)$ and $N_0(T_+)$ the normalizers of T_- in K and of T_+ in $K_{T_-}^0$ respectively, we know the following identities

(1.17′) $$N_0(T_+)/T_+ = W_0. \quad N_K(T_-)/K_{T_-} = W^-.$$

Cf., [8] or others.

Put

$$\tilde{T}_+ = K \cap T$$

of which the e-component is T_+ since $\mathfrak{t}^+ = \mathfrak{k} \cap \mathfrak{t}$. Denoting by $s+1$ the number of connected components of T_+, we put

$$\tilde{T}_+ = T_+ + a_1 T_+ + \cdots + a_s T_+.$$

Since $T = T_+ \cdot T_-$ by (1.1′), we may choose the set of representatives $\{e, a_1, \cdots, a_s\}$ in T_-, which implies that

$$K \cap T_- / T_+ \cap T_- \cong \tilde{T}_+ / T_+.$$

as isomorphism induced by the inclution map $K \cap T_- \subset \tilde{T}_+$. On the other hand, by [1], Prop. 1.5, p. 48,

$$K_T / K_{T_-}^0 \cong K \cap T_- / T_+ \cap T_-.$$

Therefore we obtain the following decomposition into connected components:

$$K_{T_-} = K_{T_-}^0 + a_1 K_{T_-}^0 + \cdots + a_s K_{T_-}^0.$$

Then, denoting by $\tilde{N}_0(T_+)$ the normalizer of T_+ in $K_{T_-}^0$, we obtain easily the following identities

$$\tilde{N}_0(T_+) = N_0(T_+) + a_1 N_0(T_+) + \cdots + a_s N_0(T_+),$$

and

$$\tilde{N}_0(T_+)/\tilde{T}_+ = N_0(T_+)/T_+ = W_0 \tag{1.17″}$$

in the same sense as (1.17).

LEMMA 1.1. *The inclusion map $N_K(T) \subset N_K(T_-)$ induces an isomorphism*

$$N_K(T)/\tilde{N}_0(T_+) \cong N_K(T_-)/K_{T_-}.$$

Proof. As is easily seen

$$N_K(T) \cap K_{T_-} = \tilde{N}_0(T_+),$$

which proves the injectivity. To prove the surjectivity, take any element $a \in N_K(T_-)$, and put

$$a^{-1} T_+ a = T_+', \quad a^{-1} T a = T'.$$

Since $a^{-1} T_- a = T_-$, T' is a maximal torus of G containing T_-. Hence

$$T_+' \subset K_{T_-},$$

i.e., T_+' is a maximal torus of $K_{T_-}^0$. By the conjugacy of maximal tori of $K_{T_-}^0$, we have an element $b \in K_{T_-}^0$ such that

$$b^{-1} T_+' b = T_+.$$

Then

$$(ab)^{-1} T ab = T,$$

i.e., a is congruent to an elemen of $N_K(T)$ modulo K_{T_-}. Thereby was proved the lemma.

On the other hand we have natural inclusions

$$W_0 = \tilde{N}_0(T_+)/\tilde{T}_+ \subset N_K(T)/\tilde{T}_+ \subset N(T)/T = W$$

and the projection

(1.18) $\rho': N_K(T)/T_+ \rightarrow N_K(T)/N_0(T_+)$ ($=W^-$ by the above lemma)

of which the kernel is $\tilde{N}_0(T_+)/\tilde{T}_+ = W_0$. Comparing (1.18) with (1.15) we obtain an identity

(1.19) $$N_K(T)/\tilde{T}_+ = W_\sigma$$

in the same sense as (1.17).

1. 7. dim T_- is called the *(restricted) rank* of the symmetric pair (G, K), denoted by "rank (G, K)." By the conjugacy of maximal tori of the pair (G, K), the restricted rank is well defined. We say that the symmetric pair (G, K) has *splitting rank* if the relation

$$\text{rank } G = \text{rank } K + \text{rank } (G, K)$$

is satisfied. In this case T_+ becomes a maximal torus of K. And $T_+ = K \cap T = \tilde{T}_+$, whence K_{T_-} is connected if G is simply connected.

For any compact connected Lie group K considered as a symmetric space, its symmetric pair $(K \times K, K)$ has splitting rank as is easily seen. Thus the terms "symmetric spaces of splitting rank" form a category of symmetric spaces including compact Lie groups. They have many similar properties with compact Lie groups as symmetric spaces.

PROPOSITION 1.2. *The symmetric pair (G, K) has splitting rank if and only if its all restricted roots have even multiplicity.*

Proof. If $\lambda \in \mathfrak{r}^-$ has odd multiplicity, then $\lambda \in \mathfrak{r}$ by [2], Prop. 2.2. Then by [1], Prop. 1.1, p. 45, $\mathfrak{e}_\lambda$ has a basis $\{U_\lambda, V_\lambda\}$ such that

$$\sigma U_\lambda = U_\lambda, \ \sigma V_\lambda = -V_\lambda.$$

In particular $U_\lambda \in \mathfrak{k}$. Since $\lambda(H)=0$ for all $H \in \mathfrak{t}^+$, we see that

$$[\mathfrak{t}^+, U_\lambda]=0,$$

i.e., rank $K >$ dim $\mathfrak{t}^+$, which proves the "only if" part.

Next assume that every root λ of $\mathfrak{r}^-$ has even multiplicity. Then, by [1], (1.9) and (1.11), p. 47, we see that the basis (1.5′) of $\mathfrak{k} \cap \tilde{\mathfrak{e}}_\lambda$:

$$\{A_1, A_2, \cdots, A_{m(\lambda)}\}$$

can be chosen so as to satisfy that the 2-planes generated by $\{A_{2i+1}, A_{2i}\}$, $1 \leqq i \leqq m(\lambda)/2$, are invariant and non-trivially rotated by the adjoint actions of T_+: whence $\mathfrak{k}$ is decomposed orthogonally as a direct sum of $\mathfrak{k}_{T_-}$ and the 2-planes as above. Finally, by the above and (1.10), $\mathfrak{k}$ can be decomposed orthogonally as a direct sum of $\mathfrak{t}^+$ and 2-planes which are invariant and non-trivially rotated by the adjoint actions of T_+, which shows immediately that $\mathfrak{t}^+$ is maximal abelian in $\mathfrak{k}$.

COROLLARY 1.3 *For any symmetric pair (G, K) of splitting rank its restricted root system $\mathfrak{r}^-$ is a proper root system* (in the sense of [2], 2.1).

Because, if $2\lambda \in \mathfrak{r}^-$ for a $\lambda \in \mathfrak{r}^-$, $m(2\lambda)$ must be odd by [2], Prop. 2.4.

Now by [2], the table at the end, we can list all irreducible symmetric pairs

(G, K) such that G is simply connected, except group cases, as follows:

(1.20) $(\mathbf{SU}(2n), \mathbf{Sp}(n))$, $(\mathbf{Spin}(2n), \mathbf{Spin}(2n-1))$, $(\mathbf{E}_6, \mathbf{E}_4)$.

PROPOSITION 1.4. *For every symmetric pair (G, K) of splitting rank with simply connected G, there is an isomorphism*

$$W_\sigma \cong W_K$$

obtained by restricting the operations of W_σ to $\mathfrak{t}^+$, where W_K denotes the Weyl group of K operating on $\mathfrak{t}^+$.

Proof. By Prop. 1.2 every restricted root of (G, K) has even multiplicity, which implies that, for every $\alpha \in \mathfrak{r}$, $\alpha \mid \mathfrak{t}^+$ is a non-zero linear form on $\mathfrak{t}^+$. (Cf. [2], Prop. 2.2.) Therefor $\mathfrak{t}^+$ contains a regular element of $\mathfrak{t}$. Thus, for any $n \in N_K(T_+)$, its adjoint operation sends a regular element of $\mathfrak{t}$ into $\mathfrak{t}$. Therefore

$$\text{ad } n \cdot \mathfrak{t} = \mathfrak{t},$$

which shows that

$$N_K(T_+) = N_K(T).$$

Then by (1.19) we finish our proof of the proposition.

1. 8. Let us consider the case that (G, K), G simply connected, has splitting rank and K_{T_-} is semi-simple, to which belongs every symmetric pair of (1.20).

Denote by A_0 the finite group of orthogonal transformations of $\mathfrak{t}^+$ obtained by restricting the group of all automorphisms of $\mathfrak{k}_{T_-}$ preserving $\mathfrak{t}^+$, to $\mathfrak{t}^+$. Since every action of W_σ transforms $\mathfrak{r}_0$ onto itself, by Prop. 1.4 every action of Weyl group W_K of K transforms $\mathfrak{r}_0$ onto itself; on the other hand $\mathfrak{r}_0$ is the root system of $\mathfrak{k}_{T_-}$ with respect to $\mathfrak{t}^+$. Hence

$$W_K \subset A_0. \tag{1.21}$$

Denote by D_0 the group of *particular rotations* on $\mathfrak{t}^+$ of $\mathfrak{k}_{T_-}$ preserving a fundamental system Δ_0 of $\mathfrak{r}_0$. As is well known since Dynkin, there is a splitting extension

$$0 \to W_0 \to A_0 \underset{\bar{\mu}}{\overset{\bar{\rho}}{\rightleftarrows}} D_0 \to 0 \tag{1.22}$$

where the splitting map $\bar{\mu}$ is a map making D_0 a subgroup of A_0 in the natural sense. Then, by (1.15), Prop. 1.4 and (1.22), we have an injective homomorphism: $W^- \to D^0$ so that the following diagram of homomorphisms is commutative.

$$\begin{array}{ccccccccc} 0 & \to & W_0 & \to & W_K & \to & W^- & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & W_0 & \to & A_0 & \to & D_0 & \to & 0. \end{array}$$

In particular, the upper extention is also splittable.

In each symmetric pair of (1.20) we see that

$$W^- \cong D_0, \; W_K \cong A_0$$

as will be seen by the form of its root systems $\mathfrak{r}_0$ and $\mathfrak{r}^-$.

1. 9. Let W^- operate on the homogeneous space K/K_{T_-} from right by choosing a representative in $N_K(T)$ for each element of W^- (by Lemma 1.1) Then W^- operates on K/K_{T_-} without fixed points.

These operations may be viewed as analogous ones to the Weyl group operations on G/T.

In fact, in case of the symmetric pair $(K\times K, K)$, $K_{T_-}=T^+$ and $W_0=\{1\}$ (trivial group). Then, by (1.15) and Prop. 1.4, $W_K\cong W^-$. Through this isomorphism the operations of W^- on $K/K_{T_-}=K/T_+$ coincide with the usual Weyl group operations.

In each case of (1.20) we may regard as if the group D_0 of particular rotations of $\mathfrak{k}_{T_-}$ operates on K/K_{T_-}, *via* the isomorphism $D_0\cong W^-$. Their homological effects will be discussed in a later section (§**6**).

§2. K-cycles.

2. 1. Let (G, K) be a symmetric pair. We fix every notation of §**1** once and for all.

Let $P=\{p_1, \cdots, p_n\}$ be a finite sequence of singular planes in $\mathfrak{t}^-$. As far as we are concerned to K-cycles we abbreviate K_{p_i} to K_i, and K_{T_-} to K_0. Put

$$W_P=K_1\times\cdots\times K_n,$$

and let the n-fold direct product $(K_0)^n$ of K_0 operate on W_P from the right by rule

(2.1) $$(x_1,\cdots, x_n)\cdot(k_1,\cdots, k_n)=(x_1k_1,\ k_1^{-1}x_2k_2,\cdots,\ k_{i-1}^{-1}x_ik_i,\cdots,\ k_{n-1}^{-1}x_nk_n)$$

for $(x_1, \cdots, x_n)\in W_P$, $(k_1, \cdots, k_n)\in(K_0)^n$. The quotient space of W_P by these operation of $(K_0)^n$ is by definition the K-cycle Γ_P associated with (G, K) corresponding to the sequence P [8]. It is also described as

$$\Gamma_P\ =\ K\times_{K_0}K_2\times_{K_0}\cdots\times_{K_0}(K_n/K_0),$$

the n-ple $\times_{K_0}$-product of $K_1, K_2, \cdots, K_{n-1}, K_n/K_0$.

Evidently, by the above operations $(K_0)^n$ operates on W_P without fixed points. Hence W_P is a principal $(K_0)^n$-bundle over Γ_P,

The discussions of cohomologies of Γ_P is the subject of the present work. Bott and Samelson [8] proved the variational completeness of the adjoint actions of K on $\mathfrak{m}$ as well as on G/K, and showed some K-cycles of the above type gave a basis for the homology mod 2, in general and the integral homology in some special cases, of spaces such as the loop space of G/K, or $K/K_{T'}$ where T' is a torus subgroup of T_-.

The projection: $W_P\rightarrow W_{P'}$, $P'=\{p_1, \cdots, p_{n-1}\}$, dropping off the last factor, induces a fibre bundle

(2.2) $$(\Gamma_P,\ \Gamma_{P'},\ \pi,\ K_n/K_0)$$

which has a canonical cross section [8]. Hence Γ_P can be endowed with the structure of an $(n-1)$-fold iterated fibre bundle, admitting cross-sections, over K_1/K_0 with successive fibres $K_2/K_0, \cdots, K_n/K_0$. In particular we see that

(2.3) $$\dim \Gamma_P = m(p_1) + \cdots + m(p_n).$$

2. 2. Let p be a singular plane in $\mathfrak{t}^-$. Clearly $G_p{}^0$ is closed by σ, and $\sigma \mid G_p{}^0$ gives a symmetric pair $(G_p{}^0, K_p{}^0)$, $(\mathfrak{g}_p, \mathfrak{k}_p)$ its infinitesimal symmetric pair. Let

$$\mathfrak{g}_p = \mathfrak{k}_p + \mathfrak{m}_p$$

be its decomposition (1.1); $\mathfrak{k}_p = \mathfrak{k} \cap \mathfrak{g}_p$ and $\mathfrak{m}_p = \mathfrak{m} \cap \mathfrak{g}_p$. By (1.12) we know that

$$\mathfrak{g}_p = \mathfrak{g}_{T_-} + \tilde{\mathfrak{e}}_p, \ \mathfrak{k}_p = \mathfrak{k}_{T_-} + \mathfrak{k} \cap \tilde{\mathfrak{e}}_p.$$

Further, by (1.9) and (1.10) we se that

(2.4) $$\mathfrak{m}_p = \mathfrak{t}^- + \mathfrak{m} \cap \tilde{\mathfrak{e}}_p.$$

Now it is clear that $\mathfrak{t}^-$ is a Cartan subalgebra (maximal abelian subalgebra) of $\mathfrak{m}$ and its restricted root system is of restricted rank 1. (The rank of a root system is defined as the number of roots of one of its fundamental systems, which may be different from the rank of its ambient group or symmetric pair.)

Let $\mathfrak{c}_p$ be the center of $\mathfrak{g}_p$, and put $\mathfrak{c}_p{}^- = \mathfrak{c}_p \cap \mathfrak{m}_p$. Clearly $\mathfrak{c}_p{}^- \subset \mathfrak{t}^-$. Let p be expressed as $p = (\lambda, n)$, $\lambda \in \mathfrak{r}^{-\prime\prime}$, and τ_λ the basic translation corresponding to λ, i.e., an element of $\mathfrak{t}^-$ which is perpendicular to the plane $(\lambda, 0)$ and satisfies $\lambda(\tau_\lambda) = 2$. Then we have an orthogonal decomposition

$$\mathfrak{t}^- = \mathbf{R}\{\tau_\lambda\} + \mathfrak{c}_p{}^-,$$

where $\mathbf{R}$ denotes the field of real numbers and $\mathbf{R}\{\ \}$ the linear subspace over $\mathbf{R}$ generated by elements in the parentheses. Let us denote by $\mathfrak{m}_p{}'$ the orthogonal complement of $\mathfrak{c}_p{}^-$ in $\mathfrak{m}_p$, i.e.,

(2.5) $$\mathfrak{m}_p{}' = \mathbf{R}\{\tau_\lambda\} + \mathfrak{m} \cap \tilde{\mathfrak{e}}_p.$$

Then

(2.6) $$\dim \mathfrak{m}_p{}' = \mathfrak{m}(p) + 1$$

by (1.6), (1.8) and (1.11).

2. 3. Put $p' = (\lambda, 0)$. Since $\exp p$ is contained in the group generated by $\exp p$, $G_p \subset G_{p'}$ and the latter is connected because $\exp p'$ is a torus subgroup of G. Use the notations of **2. 2** for p' in place of p. In particular

$$\mathfrak{c}_{p'}{}^- = \mathfrak{c}_p{}^-.$$

Now adjoint actions of $G_{p'} = \exp \mathfrak{g}_{p'}$ leave $\mathfrak{c}_p{}^-$ element-wise fixed, and hence those of G_p also do so as a subgroup of $G_{p'}$. On the other hand, through the adjoint actions G_p leaves $\mathfrak{g}_p$ invariant and K leaves $\mathfrak{m}$ invariant. Therefore $K_p = G_p \cap K$ leaves $\mathfrak{m}_p$ invariant and $\mathfrak{c}_p{}^-$ element-wise fixed, and hence $\mathfrak{m}_{p'}$ invariant.

By ad$'$ we denote the representation of K_p (and its subgroups $K \cap G_p{}^0$, $K_p{}^0$) on $\mathfrak{m}_{p'}$ obtained by restricting its adjoint representation to $\mathfrak{m}_{p'}$. Since adjoint representations are orthogonal ones, by (2.6) ad$'$ is a homomorphism

(2.7) $$\mathrm{ad}' \colon K_p \longrightarrow \mathbf{O}(\mathfrak{m}(p)+1).$$

$$(K \cap G_p{}^0 \longrightarrow \mathbf{O}(\mathfrak{m}(p)+1), \ K_p{}^0 \longrightarrow \mathbf{SO}(\mathfrak{m}(p)+1)).$$

Let $S^{m(p)}$ be the unit sphere of $\mathfrak{m}_{p'}$, i.e., the set of all $X \in \mathfrak{m}_{p'}$ such that $\langle X, X \rangle = 1$.

LEMMA 2.1 $\mathrm{ad}'K_p{}^0$ *operates transitively on* $S^{m(p)}$.

Proof. Let a and b be any two elements of $S^{m(p)}$, $\mathfrak{t}_a{}^-$ and $\mathfrak{t}_b{}^-$ Cartan subalgebras of the pair $(\mathfrak{g}_p, \mathfrak{k}_p)$ containing respectively a and b, i.e.,

$$\mathfrak{t}_a{}^- = \mathbf{R}\{a\} + \mathfrak{c}_p{}^-, \quad \mathfrak{t}_b{}^- = \mathbf{R}\{b\} + \mathfrak{c}_p{}^-.$$

By the conjugacy of Cartan subalgebras, there exists an element $k \in K_p{}^0$ such that

$$\mathrm{ad}\ k \cdot \mathfrak{t}_a{}^- = \mathfrak{t}_b{}^-.$$

Then

$$\mathrm{ad}'k \cdot \mathbf{R}\{a\} = \mathbf{R}\{b\}.$$

Now $<a, a> = <b, b> = 1$ and $\mathrm{ad}'k$ preserves length; consequently

$$\mathrm{ad}'k \cdot a = \pm b.$$

In case $\mathrm{ad}'k \cdot a = -b$, let k' be an element of $K_p{}^0$ representing the generator of the Weyl group of the pair $(\mathfrak{g}_p, \mathfrak{k}_p)$ with respect to $\mathfrak{t}_b{}^-$, then

$$\mathrm{ad}'k' \cdot (-b) = b$$

and

$$\mathrm{ad}'(k'k) \cdot a = b. \qquad \text{q.e.d.}$$

PROPOSITION 2.2. $K_p/K_{T_-} \approx K \cap G_p{}^0/K_{T_-} \approx K_p{}^0/K_p{}^0 \cap K_{T_-} \approx S^{m(p)}$, *where diffeomorphisms* $\approx$ *are induced by* ad' *and the natural inclusions*

$$K_p{}^0 \subset K \cap G_p{}^0 \subset K_p.$$

Proof. By the above lemma $K_p{}^0$, $K \cap G_p{}^0$ and K_p operates transitively on $S^{m(p)}$ through ad'. Since G_{T_-} is connected, $K_{T_-} \subset K \cap G_p{}^0$. Now every element fixing the point $\tau_\lambda/\sqrt{<\tau_\lambda, \tau_\lambda>}$ of $S^{m(p)}$ through ad', leaves $\mathfrak{t}^-$ element-wise fixed by its adjoint action, hence is contained in K_{T_-}, and *vice versa*; therefrom the proposition follows.

As a corollary of this proposition we see the

PROPOSITION 2.3 $\quad K_p = K \cap G_p{}^0$
for every singular plane p in $\mathfrak{t}^-$.

The author has no complete proof whether G_p in general is connected or not, so that the above proposition is interesting to him. (This problem will be partly discussed in **4. 1.**)

2. 4. Denote by $\widetilde{\mathrm{ad}}'$ the diffeomorphism $K_p/K_{T_-} \approx S^{m(p)}$ of Prop. 2.2. If we identify K_p/K_{T_-} with the unit sphere $S^{m(p)}$ of $\mathfrak{m}_{p'}$ by ad', then left translations of K_p on K_p/K_{T_-} change to ad' actions on $S^{m(p)}$.

Now we shall look at the bundle (2.2). Thie bundle is the associated bundle of the principal K_n-bundle $(\bar{\Gamma}_P, \Gamma_{P'}, \tilde{\pi})$ with the actions of K_n on K_n/K_0 by left translations, where

$$\bar{\Gamma}_P = K_1 \times_{K_0} K_2 \times_{K_0} \cdots \times_{K_0} K_n$$

the n-ple $\times_{K_0}$-product of $K_1, \cdots, K_n$ and the projection $\tilde{\pi}$ is the map to drop off the last factor. By the above mentioned remark, if we replace the fibre K_n/K_0 of the bundle by $S^{m(p_n)}$ *via* $\widetilde{\mathrm{ad}}'$, then K_n operates on $S^{m(p_n)}$ orthogonally through ad' of (2.7).

Thus we obtain

THEOREM 2.4. *For every finite sequence* $P=\{p_1, \cdots, p_n\}$ *of singular planes in* $\mathfrak{t}^-$, *the fibre* K_n/K_0 *of the bundle* $(\Gamma_P, \Gamma_{P'}, \pi, K_n/K_0)$ *is homeomorphic to an* $m(p_n)$*-sphere, where* $P'=\{p_1, \cdots, p_{n-1}\}$. *If we replace the fibre* K_n/K_0 *by* $S^{m(p_n)}$ *via* $\widetilde{\mathrm{ad}}'$, *then the obtained bundle* $(\Gamma_P, \Gamma_{P'}, S^{m(p_n)})$ *is a sphere bundle, of which the associated principal orthogonal bundle is the* ad′*-extension of the principal* K_n*-bundle* $(\bar{\Gamma}_P, \Gamma_{P'}, \bar{\pi})$.

(As to the extension of the structure group of a principal bundle by a homomorphism, we refer to [6], p. 477.)

COROLLARY 2.5. *Every K-cycle* Γ_P, $P=\{p_1, \cdots, p_n\}$, *associated with a symmetric pair, is endowed with a structure of a* $(n-1)$*-fold iterated sphere bundle, admitting canonical cross-sections, over* $S^{m(p_1)}$ *with successive fibres* $S^{m(p_2)}, \cdots, S^{m(p_n)}$.

2. 5. The canonical cross-section of the bundle (2.2) gives in a standard way a reduction of the structure group of the principal K_n-bundle $(\bar{\Gamma}_P, \Gamma_{P'}, \bar{\pi})$ to K_0 as well as that of the principal orthogonal bundle to $\mathbf{O}(m(p_n))$, where $\mathbf{O}(m(p_n))$ is the subgroup of $\mathbf{O}(m(p_n)+1)$ keeping $\tau_\lambda/\sqrt{<\tau_\lambda, \tau_\lambda>}$ invariant, where $p_n=(\lambda, m)$. The former reduced K_0-bundle is $(\tilde{\Gamma}_{P'}, \Gamma_{P'}, \tilde{\pi})$,

$$\tilde{\Gamma}_{P'}=K_1\times_{K_0}K_2\times_{K_0}\cdots\times_{K_0}K_{n-1}$$

the $(n-1)$-ple $\times_{K_0}$-product of $K_1, \cdots, K_{n-1}$ and the prodjection $\tilde{\pi}$ is induced by factorization of the last factor $K_{n-1}\longrightarrow K_{n-1}/K_0$. And the latter reduced bundle is the ad″-extension of the former one, where

$$\mathrm{ad}'':\ K_0\longrightarrow \mathbf{O}(m(p))$$

is the homomorphism obtained by restricting ad′ to K_0.

The map $\widetilde{\mathrm{ad}}'$ induces an isometry (up to a positive constant multiple) of tangent spaces at distinguished elements of K_n/K_0 and $S^{m(p_n)}$ for $p=p_n$, denoted by $\widetilde{\mathrm{ad}}'_*$. Identitify $S^{m(p_n)}$ with the homogeneous space $\mathbf{O}(m(p_n)+1)/\mathbf{O}(m(p_n))$ canonically and let us denote isotropy representations of homogeneous spaces K_n/K_0 and $\mathbf{O}(m(p_n)+1)/\mathbf{O}(m(p_n))$ respectively by ι_n and ι_n'. As is easily seen, $\widetilde{\mathrm{ad}}'_*$ gives an equivalence between two representations ι_n and $\iota_n'\circ\mathrm{ad}''$ of K_0, and ι_n' is equivalent to the identity map representation of $\mathbf{O}(m(p_n))$. Thus we have seen that the represention ad″ of K_0 is equivalent to ι_n.

Then, by the above discussions we obtain the following

PROPOSITION 2.6 *The reduced* $\mathbf{O}(m(p_n))$*-bundle over* $\Gamma_{P'}$, *defined by the canonical cross-section of the bundle* $(\Gamma_P, \Gamma_{P'}, S^{m(p_n)})$ *in the standard way, is the* ι_n*-extension of the principal* K_0*-bundle* $(\tilde{\Gamma}_{P'}, \Gamma_{P'}, \tilde{\pi})$, *where* ι_n *is the isotropy representation of the homogeneous space* K_n/K_0.

2. 6. We say that a K-cycle Γ_P associated with a symmetric pair (G, K) is *totally orientable* if, considering Γ_P as an iterated sphere bundle over a sphere (by Cor. 2.5), the sphere bundles at each stage are all orientable.

The following statement is well known and easily proved by observing the top dimensional term of the Gysin sequence of integral cohomology:

(2.8) *for every orientable sphere bundle over an orientable manifold, the bundle space is also an orientable manifold.*

As an immediate consequence of (2.8) and Cor. 2.5 we obtain

PROPOSITION 2.7. *If a K-cycle Γ_P of a symmetric pair (G, K) is totally orientable, then Γ_P is an orientable manifold and $H^*(\Gamma_P; Z)$ has no torsion.*

Since the sphere bundles at each stage of Γ_P have cross-sections, their Gysin sequences of integral cohomologies split, which shows the second assertion of Prop. 2.7 by a stage-wise argument.

2. 7. Let $P=\{p_1, \cdots, p_n\}$ be a finite sequence of singular planes in $\mathfrak{t}^-$. For any subsequence $P'=\{p_{i_1}, \cdots, p_{i_s}\}$ of P, we shall embed its K-cycle $\Gamma_{P'}$ as a submanifold of Γ_P. Let

$$i: W_{P'} \longrightarrow W_P$$

be an injection defined by

$$\begin{aligned} \pi_t \circ i(x_{i_1}, \cdots, x_{i_s}) &= x_t \quad \text{in case} \quad t \in \{i_1, \cdots, i_s\} \\ &= e \quad \text{otherwise} \end{aligned}$$

for $(x_{i_1}, \cdots, x_{i_s}) \in W_{P'}$, where π_t: $W_P \longrightarrow K_t$ is the natural projection onto the t-th factor and e the neutral element of K. Let

$$h: (K_0)^s \longrightarrow (K_0)^n$$

be a homomorphism defined by

$$\begin{aligned} \pi_t \circ h(k_{i_1}, \cdots, k_{i_s}) &= e \quad \text{if} \quad t<i_1 \\ &= k_{i_1} \quad \text{if} \quad i_1 \leqq t < i_2 \\ &\cdots\cdots\cdots\cdots \\ &= k_{i_r} \quad \text{if} \quad i_r \leqq t < i_{r+1} \quad \text{for} \quad r<s \\ &= k_{i_s} \quad \text{if} \quad i_s \leqq t \end{aligned}$$

for $(k_{i_1}, \cdots, k_{i_s}) \in (K_0)^s$. As is easily seen, the pair (i, h) is a homomorphism of principal bundles and induces and injection map

$$\bar{i}: \Gamma_{P'} \longrightarrow \Gamma_P$$

of base spaces. This inclusion is a natural one in a sense and, if $P'=\{p_1, \cdots, p_{n-1}\}$, coincide with the canonical cross-section of the bundle $\Gamma_P \longrightarrow \Gamma_{P'}$.

$\Gamma_{P'}$, identified with a submanifold of Γ_P by $\bar{i}$, is called a *sub-K-cycle* of Γ_P corresponding to the subsequence $P'=\{p_{i_1}, \cdots, p_{i_s}\}$.

2. 8. If Γ_P is totally orientable, then evidently every *sub-K-cycle* $\Gamma_{P'}$ of it is also totally orientable.

Every *sub-K-cycle* $\Gamma_{P'}$, $P'=\{p_{i_1}, \cdots, p_{i_s}\}$, forms a cycle, mod 2 in general and integral in case of Γ_P being totally orientable after choosing a suitable orientation of $\Gamma_{P'}$, of Γ_P of degree $m(p_{i_1})+\cdots+m(p_{i_s})$. The homology class of Γ_P, represented by the cycle $\Gamma_{P'}$, is denoted by $[i_1, \cdots, i_s]_2$ in general case as a mod 2 class, or by $[i_1, \cdots, i_s]$ in totally orientable case as an integral class.

PROPOSITION 2.8. *For any* $P=\{p_1, \cdots, p_n\}$ i) *the set of all* $[i_1, \cdots, i_s]_2$, $1 \leqq i_1 < \cdots < i_s \leqq n$, *forms an additive base of* $H_*(\Gamma_P ; Z_2)$, *and* ii) *if* Γ_P *is totally orientable, then the set of all* $[i_1, \cdots, i_s]$, $1 \leqq i_1 < \cdots < i_s \leqq n$, *forms an additive base of* $H_*(\Gamma_P ; Z)$, *where we consider the generator of* $H_0(\Gamma_P ; Z_2)$ *or* $H_0(\Gamma_P ; Z)$ *represented by a point, denoted by* 1, *as represented by a sub-K-cycle corresponding to a void subsequence.*

Proof by induction on the length n of P. The case $u=1$ is evident since Γ_P itself is a sphere.

Put $P'=\{p_1, \cdots, p_{n-1}\}$. Since the sphere bundle $\Gamma_P \longrightarrow \Gamma_{P'}$ has the canonical cross-section κ, its Gysin sequence splits ints a direct sum decomposition

$$H_i(\Gamma_P)=\natural_* H_{i-m(p_n)}(\Gamma_{P'})+\kappa_* H_i(\Gamma_{P'}), \tag{2.9}$$

where the ceofficient group is Z_2 or Z according to the cases i) or ii), and $\natural_*$ is the dual of integration over the fibre of cohomology [6].

Now by induction hypothesis a basis of $H_*(\Gamma_{P'})$ is given by homology classes represented by sub-K-cycles of $\Gamma_{P'}$, denoted by $[i_1, \cdots, i_s]_2{}'$ or $[i_1, \cdots, i_s]'$. We see easily that

$$\kappa_*[i_1, \cdots, i_s]'=[i_1, \cdots, i_s] \quad \text{for} \quad i_s<n, \tag{2.10}$$

where the suffices 2 are dropped in case i). (The same convention is used in what follows since discussions in both cases i) and ii) are very parallel.)

In case $i=\dim(\Gamma_{P'})$ we have $\natural_* H_i(\Gamma_{P'})=H_{i+m(p_n)}(\Gamma_P)$ which implies that

$$\natural_*[1, \cdots, n-1]' = \pm[1, \cdots, n]. \tag{2.11$'$}$$

For any subsequence $P''=\{p_{i_1}, \cdots, p_{i_s}\}$ such that $i_s<n$, we put $P'''=\{p_{i_1}, \cdots, p_{i_s}, p_n\}$. In the following diagram

$$\begin{array}{ccc} \Gamma_{P'''} & \longrightarrow & \Gamma_P \\ \downarrow & & \downarrow \\ \Gamma_{P''} & \longrightarrow & \Gamma_{P'} \end{array}$$

vertical arrows are projections of bundles and horizontal arrows are natural inclusions as sub-K-cycles. As is easily seen the pair of horizontal arrows is a bundle map, then the naturality of $\natural_*$ and the formula (2.11′) implies that

$$\natural_*[i_1, \cdots, i_s]'=\pm[i_1, \cdots, i_s, n] \tag{2.11}$$

for every basis element $[i_1, \cdots, i_s]'$ of $H_*(\Gamma_{P'})$. (2.9), (2.10) and (2.11) complete the proof.

2. 9. We consider a basis of $H^*(\Gamma_P)$, $P=\{p_1, \cdots, p_n\}$ (the coefficient group is Z_2 or Z according as the considered case is general or totally orientable one), dual to the homology basis of Prop. 2.8. Let $x_{i_1 \cdots i_s}$ be the dual element to $[i_1, \cdots, i_s]_2$ or $[i_1, \cdots, i_s]$. First we note that x_n, restricted to the fibre, gives a generator of the top-dimensional fibre cohomology of the bundle $\Gamma_P \longrightarrow \Gamma_{P'}$, $P'=\{p_1, \cdots, p_{n-1}\}$, and that $\kappa^* x_n=0$, where $\kappa : \Gamma_{P'} \longrightarrow \Gamma_P$ is the canonical cross-section of the bundle. Secondly we note that the cohomology map π^* and the map consisting of π^* followed by cup-product with x_n are injective, and define a direct sum decomposition

(2.12) $$H^*(\Gamma_P)=\pi^*H^*(\Gamma_{P'})+x_n\cdot\pi^*H^*(\Gamma_{P'}).$$

(Cf., [8]. p. 998, or [11], p. 273.) Then, by a more or less parallel discussion to the proof of Prop. 2.8 using an induction on the length n of P, we see that

(2.13) $$x_{i_1\cdots i_s}=\pm x_{i_1}\cdots x_{i_s}$$

for all $1\leqq i_1<\cdots<i_s\leqq n$, which means that i) the cohomology ring $H^*(\Gamma_P)$ is generated by

$$\{x_1,\ \cdots,\ x_n\},$$

and that ii) an additive base of $H^*(\Gamma_P)$ is given by

(2.14) $$\{1,\ x_{i_1}\cdots x_{i_s},\ 1\leqq i_1<\cdots<i_s\leqq n\}.$$

Thus, if we obtain relations ρ_k to describe $x_k{}^2$ as linear combinations of basis elements (2.14) for $1\leqq k\leqq n$, then the cohomology ring $H^*(\Gamma_P)$ is determined completely.

In case Γ_P is totally orientable, summarizing the above and remarking that $x_k{}^2=0$ if deg x_k $(=m(p_k))$ odd, we obtain

PROPOSITION 2.9 *Assume that a K-cycle Γ_P, $P=\{p_1, \cdots, p_n\}$, is totally orientable, and that singular planes $p_{j_1}, \cdots, p_{j_r}$ of P have odd multiplicities and the rests $p_{t_1}, \cdots, p_{t_{n-r}}$ have even multiplicities; then*

$$H^*(\Gamma_P\,;Z)=\wedge_Z(x_{j_1},\ \cdots,\ x_{j_r})\otimes Z[x_{t_1},\ \cdots,\ x_{t_{n-r}}]/I_P$$

where $\wedge_Z$ denotes an exterior algebra over Z with generators denoted in parentheses, and I_P is the ideal generated by the elements ρ_k, $1\leqq k\leqq n-r$, which represent relations to describe $x_{t_k}^2$ as linear combinations of basis elements (2.14).

The same proposition holds also for the cohomology mod 2 of every K-cycle Γ_P without the exterior tensor factor. In this case the relations ρ_k can be determined completely if G is simply connected (cf., Theorem 2.10 below).

For each symmetric pair of (1.20), its all K-cycles are totally orientable and their relations ρ_k will be determined in §**6**.

2. 10. Take any K-cycle Γ_P, $P=\{p_1, \cdots, p_n\}$. For two singular planes $P_i=(\lambda_i,\ m_i)$ $p_j=(\lambda_j,\ m_j)$, $\lambda_i,\ \lambda_j\in\mathfrak{r}^{-\prime\prime}$, of P, using the Cartan integer

(2.15) $$a_{ij}=2<\lambda_i,\ \lambda_j>/<\lambda_j,\ \lambda_j>$$

we define two numbers mod 2 b_{ij} and c_{ij} as follows:

(2.16) $$b_{ij}\equiv 0 \pmod 2 \quad \text{if } m(p_i)\neq m(p_j)$$
$$\equiv a_{ij} \quad \text{otherwise},$$

(2.17) $$c_{ij}\equiv 0 \pmod 2 \quad \text{if } m(p_i)=1 \text{ or } m(p_j)\neq 1$$
$$\equiv a_{ij} \quad \text{otherwise}.$$

Now we shall state a theorem which will be proved in §**7**.

THEOREM 2.10. *Let (G, K) be a symmetric pair with G simply-connected, and $P=\{p_1, \cdots, p_n\}$ a finite sequence of singular planes in $\mathfrak{t}^-$. The cohomology ring* mod 2 *of the K-cycle* Γ_P *have* n *generators* $x_1, \cdots, x_n$ *with* deg $x_i = m(p_i)$, *and*

$$H^*(\Gamma_P ; Z_2) = Z_2[x_1, \cdots, x_n]/I_P$$

where I_P *is the ideal generated by the elements*

$$\rho_k = x_k(x_k + \sum_{i=1}^{k-1} b_{ki}x_i + (\sum_{i=1}^{k-1} c_{ki}x_i)^{m(p_k)})$$

for $1 \leqq k \leqq n$.

§ 3. Connected components of K_{T_-}.

3. 1. Let (G, K) be a symmetric pair, and use the notations of § **1**. To each element $\alpha \in \mathfrak{t}^*$, we associate an element $H_\alpha \in \mathfrak{t}$ defined by

$$<H_\alpha, H> = \alpha(H) \qquad \text{for all} \quad H \in \mathfrak{t},$$

and $\tau_\alpha \in \mathfrak{t}$ defined by

$$\tau_\alpha = 2H_\alpha / <H_\alpha, H_\alpha>.$$

When $\alpha \in \mathfrak{r}$ or $\in \mathfrak{r}^-$, τ_α is called a basic translation of $\mathfrak{t}$ or of $\mathfrak{t}^-$ corresponding to α. For each subset $\mathfrak{s} \subset \mathfrak{t}^*$ we put

$$\tilde{\mathfrak{s}} = \{\tau_\alpha ; \alpha \in \mathfrak{s}\}.$$

Let e be the neutral element of G. Discrete subgroups of $\mathfrak{t}$, $\exp^{-1}(e) \cap \mathfrak{t}$, $\exp^{-1}(e) \cap \mathfrak{t}^+$ and $\exp^{-1}(e) \cap \mathfrak{t}^-$, are called the unit lattices of T, T_+ and T_- respectively. The lattices generated by $\tilde{\mathfrak{r}}$, $\tilde{\mathfrak{r}}_0$ and $\tilde{\mathfrak{r}}^-$, are contained in the unit lattices of T, T_+ and T_- respectively. If G is simply-connected, then the lattice generated by $\tilde{\mathfrak{r}}$, or $\tilde{\mathfrak{r}}^-$, coincides with the unit lattice of T, or T_- [15, 7].

Let Δ be a fundamental system of $\mathfrak{r}$. Since $\tilde{\Delta}$ is a basis of the lattice generated by $\tilde{\mathfrak{r}}$,

(3.1) *the set* $\tilde{\Delta}$ *form a basis of the unit lattice of* T *if and only if* G *is simply connected.*

In **3. 3.** we obtain a basis of the lattice generated by $\tilde{\mathfrak{r}}^-$.

3. 2. Let us denote the ranks of $\mathfrak{r}$ and $\mathfrak{r}_0$ respectively by l and l_0. Let Δ be a σ-fundamental system of $\mathfrak{r}$, and put

$$\Delta = \{\alpha_1, \cdots, \alpha_l\}, \ \Delta_0 = \{\alpha_{l-l_0+1}, \cdots, \alpha_l\}.$$

Here we recall Lemma 1 of Satake [12], p. 80.

(3.2) *There exists an involutive permutation* $\bar{\sigma}$ *of the set of indices* $\{1, \cdots, l-l_0\}$ *such that*

$$\sigma^*\alpha_i = \alpha_{\bar{\sigma}(i)} + \sum_{j=l-l_0+1}^{l} c_j^{(i)}\alpha_j, \ c_j^{(i)} \geqq 0, \quad \textit{for} \quad 1 \leqq i \leqq l - l_0.$$

According to this, we can choose the numbering of elements of $\Delta - \Delta_0$ in such a way that

$$\begin{aligned} \sigma(i) &= i && \text{for} \quad 1 \leqq i \leqq p_1, \\ &= i + p_2 && \text{for} \quad p_1 + 1 \leqq i \leqq p_1 + p_2, \\ &= i + p_2 && \text{for} \quad p_1 + p_2 + 1 \leqq i \leqq p_1 + 2p_2 \end{aligned}$$

as in [12], p. 80. Then $l - l_0 = p_1 + 2p_2$. Putting

$$p_1 + p_2 = p,$$

and $\lambda_i=\alpha_i|\mathfrak{t}^-$ for $1\leqq i\leqq p$, we see that

$$\Delta^-=\{\lambda_1, \cdots, \lambda_p\}.$$

Let p' be the number of roots of Δ^- of multiplicity 1, then $p'\leqq p_1$, and we can choose the numbering of roots of Δ and of Δ^- further to satisfy that

$$\begin{aligned} m(\lambda_i)&=1 \quad \text{for } 1\leqq i\leqq p', \\ &>1 \quad \text{for } p'+1\leqq i\leqq p. \end{aligned}$$

3. 3. The following assertions are routine proofs.

(3.3) *The set of basic translations* $\tilde{\mathfrak{r}}^-$ *is a root system* (in the sense of [2]).

(3.4) *If* $\mathfrak{r}^-$ *is a proper root system* (in the sense of [2]), *then* $\tilde{\mathfrak{r}}^-$ *is also a proper root system, and* $\tilde{\Delta}^-$ *is a fundamental system of* $\tilde{\mathfrak{r}}^-$.

For any root system $\mathfrak{s}$, put

$$\mathfrak{s}'=\{\lambda\in\mathfrak{s}\,;\,\lambda/2\notin\mathfrak{s}\},$$
$$\mathfrak{s}''=\{\lambda\in\mathfrak{s}\,;\,2\lambda\notin\mathfrak{s}\},$$

then

(3.5) $\mathfrak{s}'$ *and* $\mathfrak{s}''$ *are proper root systems, and fundamental systems of* $\mathfrak{s}$ *coincide with those of* $\mathfrak{s}'$. *Furthermore, if a set* $F=\{\gamma_1, \cdots, \gamma_q\}$ *is a fundamental system of* $\mathfrak{s}$, *them the set* $F''=\{\varepsilon_1\gamma_1, \cdots, \varepsilon_q\gamma_q\}$ *defined by*

$$\begin{aligned} \varepsilon_i&=1 \quad \text{if } 2\gamma_i\notin\mathfrak{s} \\ &=2 \quad \text{if } 2\gamma_i\in\mathfrak{s} \end{aligned}$$

is a fundamental system of $\mathfrak{s}''$. *Every fundamental system of* $\mathfrak{s}''$ *can be obtained in this way.*

$\mathfrak{s}'$ is called a canonical proper subsystem of $\mathfrak{s}$ in [2].

(3.6) $(\tilde{\mathfrak{r}}^-)'=\widetilde{\mathfrak{r}^{-\prime\prime}}$ and $(\tilde{\mathfrak{r}}^-)''=\widetilde{\mathfrak{r}^{-\prime}}$.

Finally, from (3.3)-(3.6), we conclude

PROPOSITION 3.1. *The set* $\widetilde{\Delta^{-\prime\prime}}=\{\tau_{\varepsilon_1\lambda_1}, \cdots, \tau_{\varepsilon_p\lambda_p}\}$, *where*

$$\begin{aligned} \varepsilon_i&=1 \quad \text{if } 2\lambda_i\notin\mathfrak{r}^- \\ &=2 \quad \text{if } 2\lambda_i\in\mathfrak{r}^-, \end{aligned}$$

is a fundamental system of $\tilde{\mathfrak{r}}^-$ *and, if* G *is simply connected, forms a basis of the units lattice of* T_-.

In the sequel we abbreviate $\tau_{\varepsilon_i\lambda_i}$ to $\bar{\tau}_i$ for $1\leqq i\leqq p$, and τ_{α_i} to τ_i for $1\leqq i\leqq l$. Thus

$$\Delta^{-\prime\prime}=\{\bar{\tau}_1, \cdots, \bar{\tau}_p\} \quad\text{and}\quad \tilde{\Delta}=\{\tau_1, \cdots, \tau_l\}.$$

3. 4. We shall express $\bar{\tau}_i$ by basic translations of $\tilde{\Delta}$.

i) *The case* $\alpha_i=\lambda_i$, which is equivalent to saying that $m(\lambda_i)=1$. We see immediately that

$$\bar{\tau}_i = \tau_i = -\sigma\tau_i.$$

ii) *The case* $<\alpha_i,\sigma^*\alpha_i>=0$, which is equivalent to saying that $2\lambda_i\notin\tilde{\mathfrak{r}}^-$ and $m(\lambda_i)\neq 1$ by [2]. In this case

$$\lambda_i=(\alpha_i+\sigma^*\alpha_i)/2,$$

and

$$\bar{\tau}_i=\tau_{\lambda_i}=H_{\alpha_i+\sigma^*\alpha_i}/<(\alpha_i+\sigma^*\alpha_i)/2,\ (\alpha_i+\sigma^*\alpha_i)/2>$$
$$=2H_{\alpha_i}/<\alpha_i,\ \alpha_i>+2H_{\sigma^*\alpha_i}/<\sigma^*\alpha_i,\ \sigma\alpha_i>$$
$$=\tau_{\alpha_i}+\tau_{\sigma^*\alpha_i}.$$

Thus

$$\bar{\tau}_i = \tau_i-\sigma\tau_i$$

because $\tau_{\sigma^*\alpha}=-\sigma\tau_\alpha$ for any $\alpha\in\mathfrak{t}^*$.

iii) *The case* $<\alpha_i,\ \sigma^*\alpha_i><0$, which is equivalent to saying that $2\lambda_i\in\mathfrak{r}^-$ by [2]. Then

$$2\lambda_i=\alpha_i+\sigma^*\alpha_i.$$

And

$$\bar{\tau}_i=\tau_{2\lambda_i}=2H_{\alpha_i+\sigma^*\alpha_i}/<\alpha_i+\sigma^*\alpha_i,\ \alpha_i+\sigma^*\alpha_i>$$
$$=2H_{\alpha_i+\sigma^*\alpha_i}/<\alpha_i,\ \alpha_i>$$
$$=\tau_{\alpha_i}+\tau_{\sigma^*\alpha_i}.$$

Thus

$$\bar{\tau}_i=\tau_i-\sigma\tau_i.$$

3. 5. In this and the next subsection we assume that G *is simply connected*, To obtain a basis of the unit lattice of T_+, first we change the basis $\tilde{\Delta}$ of the unit lattice of T.

By (3.2) we see that

$$\sigma\tau_{p_1+j}=-\tau_{p_1+p_2+j}+a_{p_1+j}\quad\text{for}\quad 1\leqq j\leqq p_2,$$

where a_{p_1+j} is an integral linear combination of elements of $\tilde{\Delta}_0$. Therefore, putting

(3.7) $$\tilde{\Delta}_1=\{\tau_1,\ \cdots,\ \tau_p,\ \sigma\tau_{p_1+1},\ \cdots,\ \sigma\tau_p,\ \tau_{l-l_0+1},\ \cdots,\ \tau_l\},$$

the coefficient matrix of the change of bases: $\tilde{\Delta}\longrightarrow\tilde{\Delta}_1$ is a triangular integral matrix whose diagonal elements are ±1, hence is unimodular. And we conclude that

(3.8) *the set* $\tilde{\Delta}_1$ *is a basis of the unit lattice of* T.

Next we put

(3.9) $$\tilde{\Delta}_2=\{\tau_1,\ \cdots,\ \tau_p,\ \sigma\tau_{p_1+1}+\tau_{p_1+1},\ \cdots,\ \sigma\tau_p+\tau_p,\ \tau_{l-l_0+1},\ \cdots,\ \tau_l\}.$$

The coefficient matrix of the change of bases: $\tilde{\Delta}_1\longrightarrow\tilde{\Delta}_2$ is also unimodular as is easily seen, and we conclude that

(3.10) *the set* $\tilde{\Delta}_2$ *is a basis of the unit lattice of* T.

Now

(3.11) *the set* $\tilde{\Delta}_2\cap\mathfrak{t}^+=\{\sigma\tau_{p_1+1}+\tau_{p_1+1},\ \cdots,\ \sigma\tau_p+\tau_p,\ \tau_{l-l_0+1},\ \cdots,\ \tau_l\}$ *is a linear basis of* $\mathfrak{t}^+$,

since the number of elements of $\tilde{\Delta}_2\cap\mathfrak{t}^+$ is equal to $l-p=\dim\mathfrak{t}^+$.

By (3.10)-(3.11) we obtain

PROPOSITION 3.2 *The set* $\{\tau_{p_1+1}+\sigma\tau_{p_1+1}, \cdots, \tau_p+\sigma\tau_p, \tau_{l-l_0+1}, \cdots, \tau_l\}$ *forms a basis of the unit lattice of* T_+.

3. 6. Now we shall discuss $K_T/K_T{}^0$. By [1], Prop. 1.5,

$$K_{T_-}/K_{T_-}{}^0 \cong K\cap T_-/T_+\cap T_- . \tag{3.12}$$

$K\cap T_-=\{t\in T_-\,;\, t^2=1\}$ is the image of the half unit lattice of T_- by the exponential map. Hence, by Prop. 3.1 we see that

(3.13) $K\cap T_-\cong(Z_2)^p$ *with generators* $\exp(\bar{\tau}_i/2)$. $1\leqq i\leqq p$.

Next we prove the following

PROPOSITION 3.3. $T_+\cap T_-\cong(Z_2)^{p-p'}$ *with generators* $\exp(\bar{\tau}_i/2)$, $p'+1\leqq i\leqq p$.

Proof. Take an index i such that $p'+1\leqq i\leqq p$. By cases ii), iii) of **3.4**

$$\bar{\tau}_i/2=(\tau_i-\sigma\tau_i)/2=(\tau_i+\sigma\tau_i)/2-\sigma\tau_i$$
$$\equiv(\tau_i+\sigma\tau_i)/2 \quad \text{modulo the unit lattice of } T,$$

whence

$$\exp(\bar{\tau}_i/2)=\exp((\tau_i+\sigma\tau_i)/2)\in T_+ \quad \text{for} \quad p'+1\leqq i\leqq p.$$

On the other hand, if we assume that

$$\Pi_{s=1}^k \exp(\bar{\tau}_{i_s}/2)\in T_+,$$

then $(\bar{\tau}_{i_1}+\cdots+\bar{\tau}_{i_k})/2$ is congruent to an element of $\mathfrak{t}^+$ modulo the unit lattice of T, which implies by (3.10) that there exists an element $\tau\in\mathfrak{t}^+$ such that

$$\tau=(\bar{\tau}_{i_1}+\cdots+\bar{\tau}_{i_k})/2+\Sigma_{i=1}^p n_i\tau_i,$$

n_i are integers. (Remark that last $l-p$ elements of $\tilde{\Delta}_2$ belongs to $\mathfrak{t}^+$.) Now

$$\tau=\sigma\tau=-(\bar{\tau}_{i_1}+\cdots+\bar{\tau}_{i_k})/2+\Sigma_{i=1}^p n_i\sigma\tau_i.$$

Therefore

$$\bar{\tau}_{i_1}+\cdots+\bar{\tau}_{i_k}+\Sigma_{i=1}^p n_i(\tau_i-\sigma\tau_i)=0.$$

Here we put

$$\varepsilon_i=0 \qquad \text{for} \quad i\notin\{i_1, \cdots, i_k\}$$
$$=1 \qquad \text{otherwise,}$$

then, using the identities of **3.4,** we see that

$$\Sigma_{i=1}^{p'}(2n_i+\varepsilon_i)\bar{\tau}_i+\Sigma_{i=p'+1}^p(n_i+\varepsilon_i)\bar{\tau}_i=0.$$

Finally, the linear independence of $\bar{\tau}_1, \cdots, \bar{\tau}_p$ shows that

$$\varepsilon_i=0 \qquad \text{for} \quad 1\leqq i\leqq p'.$$

q.e.d.

By (3.12), (3.13) and Prop. 3.3 we obtain

THEOREM 3.4. *Let* (G, K) *be a symmetric pair such that* G *is simply connected,* p' *the number of roots of multiplicity* 1 *in a restricted fundamental system of the pair* (G, K), *and* $\bar{\tau}_i$, $1\leqq i\leqq p'$, *the corresponding basic translations; then*

$$K_{T_-}/K_{T_-}^0\cong(Z_2)^{p'}$$

whose p' *generators are represented by* $\exp(\bar{\tau}_i/2)$, $1\leqq i\leqq p'$.

COROLLARY 3.5. *Under the same assumptions as in the above theorem, the number of connected components of* K_{T_-} *is equal to* $2^{p'}$.

§ 4. Centralizers in K of singular tori in T_-.

4. 1. Let (G, K) be a symmetric pair. It is well known that, for any torus subgroup T' of G, $G_{T'}$ is connected. We shall first discuss whether G_p is connected or not for each singular plane p in $\mathfrak{t}^-$.

Put $p=(\lambda, n)$, $\lambda \in \mathfrak{r}^{-\prime\prime}$.

i) In case $n=0$, exp p is a torus subgroup of G; hence G_p *is connected* by the above remark.

ii) In case $n=2m$ (even), the fact that $\lambda(m\tau_\lambda)=n$ and $\exp \tau_\lambda=e$, implies that $\exp p=\exp(\lambda, 0)$, whence G_p *is connected.*

There remains the case $n=2m+1$ (odd) to be discussed. In this case

$$\exp(\lambda, 2m+1) = \exp(\lambda, 1)$$

by the same reason an in case ii), and the group generated by this set contains $\exp(\lambda, 0)$. Therefore

$$G_p \subset G_{(\lambda, 0)}.$$

This case is further divided into two cases.

iii) If $\lambda/2 \notin \mathfrak{r}^-$, then by (1.11)-(1.12) their Lie algebras are

$$\mathfrak{g}_p = \mathfrak{g}_{(\lambda, 0)} = \mathfrak{g}_{T_-} + \tilde{\mathfrak{e}}_\lambda.$$

In particular

$$\dim G_p = \dim G_{(\lambda, 0)}.$$

Hence G_p is open and closed in $G_{(\lambda, 0)}$, and the latter is connected. Therefore

$$G_p = G_{(\lambda, 0)},$$

and also in this case G_p *is connected.*

iv) If $\lambda/2 \in \mathfrak{r}^-$, then we put $\lambda/2=\lambda'$. By (1.11)-(1.12). Lie algebras of G_p and of $G_{(\lambda, 0)}$ are respectively expessed as

$$\mathfrak{g}_p = \mathfrak{g}_T + \tilde{\mathfrak{e}}_\lambda,$$

$$\mathfrak{g}_{(\lambda, 0)} = \mathfrak{g}_{T_-} + \tilde{\mathfrak{e}}_{\lambda'} + \tilde{\mathfrak{e}}_\lambda.$$

Put $a=\exp(\tau_\lambda/2)$. $a^2=e$. Discuss the adjoint action of a on $\mathfrak{g}_{(\lambda, 0)}$ by (1.5) and (1.9), then we see that

$$\text{ad } a \,|\, \mathfrak{g}_p = \text{identity map}$$

and

$$\text{ad } a \,|\, \tilde{\mathfrak{e}}_{\lambda'} = -\text{ identity map},$$

which imply that

(4.1) $(\mathfrak{g}_{(\lambda, 0)}, \mathfrak{g}_p, \text{ad } a)$ *is an infinitesimal symmetric pair.*

Correspondingly we obtain

(4.2) $(G_{(\lambda, 0)}, G_p{}^0, \text{ad } a)$ *is a symmetric pair with the fixed group* G_p.

The last assertion of (4.2) can be proved as follows: let an element b of $G_{(\lambda,0)}$ be commutative with a. For any element $x \in \exp p$, $xa \in \exp(\lambda, 0)$. Thereby

$$xa = bxab^{-1} = bxb^{-1}a.$$

Therefore, $x=bxb^{-1}$ and $b \in G_p$; and *vice versa.*

Let $\mathfrak{g}_{(\lambda,0)}{}^{ss}$ denote the semi-simple part of $\mathfrak{g}_{(\lambda,0)}$ and $\mathfrak{t}_1$ its center. $G_{(\lambda,0)}{}^{ss} = \exp(\mathfrak{g}_{(\lambda,0)}{}^{ss})$ and $T_1 = \exp \mathfrak{t}_1$ are respectively the semi-simple part and the connected center of $G_{(\lambda,0)}$. As is well known

(4.3) $$G_{(\lambda,0)} = G_{(\lambda,0)}{}^{ss} \cdot T_1,$$

and clearly

(4.4) $$T_1 \subset G_p.$$

(4.2) implies that

(4.5) $(G_{(\lambda,0)}{}^{ss}, (G_p \cap G_{(\lambda,0)}{}^{ss})^0, \mathrm{ad}\ a)$ *is a symmetric pair with the fixed group* $G_p \cap G_{(\lambda,0)}{}^{ss}$.

Let $\mathfrak{t}_2$ be the Cartan subalgebra of $\mathfrak{g}_{(\lambda,0)}{}^{ss}$ contained in $\mathfrak{t}$. $T_2 = \exp \mathfrak{t}_2$ is a maximal torus of $G_{(\lambda,0)}{}^{ss}$. Since $\lambda' \in \mathfrak{r}^{-\prime}$ we can choose a σ-fundamental system Δ of $\mathfrak{r}$ such that $\lambda' \in \Delta^-$. We put

$$\Delta_{\lambda'} = \{\alpha \in \Delta\ ;\ \alpha | \mathfrak{t}^- = \lambda'\}.$$

A slight modification of the proof of Prop. 3.4 of [2] shows that $\Delta_0 \cup \Delta_{\lambda'}$ is a fundamental system of roots of $G_{(\lambda,0)}{}^{ss}$.

Here we assume that G *is simply connected;* then the fact that a fundamental system of roots of $G_{(\lambda,0)}{}^{ss}$ is a part of a fundamental system of roots of G shows that a basis of the unit lattice of T_2 is given by basic translations corresponding to roots of a fundamental system of $G_{(\lambda,0)}{}^{ss}$, which in turn proves that $G_{(\lambda,0)}{}^{ss}$ is simply connected. Then by [8, 9] the fixed group $G_p \cap G_{(\lambda,0)}{}^{ss}$ of (4.5) is connected. Now, since

$$G_p = (G_p \cap G_{(\lambda,0)}{}^{ss}) \cdot T_1$$

as is easily seen from (4.3)-(4.4), G_p is connected as a product of two connected groups.

By the above discussions we obtain the following

PROPOSITION 4.1. *If G is simply connected, then G_p is connected for any singular plane p in* $\mathfrak{t}^-$.

4. 2. We shall associate with each $\lambda \in \mathfrak{r}^-$ an irreducible symmetric pair $(G(\lambda), K(\lambda))$ of rank 1, which will play an important rôle in our subsequent sections.

Put

$$\tilde{\mathfrak{r}}_\lambda = \text{the union of } \mathfrak{r}_{m\lambda} \text{ such that } m\lambda \in \mathfrak{r}^-,\ m \text{ an integer.}$$

$\mathfrak{r}_0 \cup \tilde{\mathfrak{r}}_\lambda$ is the root system of $G_{(\lambda,1)}$ by (1.12), and clearly closed by σ. Using terminologies of [2] $\tilde{\mathfrak{r}}_\lambda$ is σ-connected (Lemma 3.2 of [2]). By $\bar{\mathfrak{r}}_\lambda$ we denote the σ-component of $\mathfrak{r}_0 \cup \tilde{\mathfrak{r}}_\lambda$ containing $\tilde{\mathfrak{r}}_\lambda$. Corresponding to the decomposition of $\mathfrak{r}_0 \cup \tilde{\mathfrak{r}}_\lambda$ into σ-components, we have the decomposition of $\mathfrak{g}_{(\lambda,1)}{}^{ss}$, the semi-simple part of $\mathfrak{g}_{(\lambda,1)}$, into the direct sum of σ-irreducible factors.

Let $\mathfrak{g}(\lambda)$ denote the σ-irreducible factor having $\bar{\mathfrak{r}}_\lambda$ as its root system. The pair $(\mathfrak{g}(\lambda), \mathfrak{k}(\lambda))$, $\mathfrak{k}(\lambda) = \mathfrak{k} \cap \mathfrak{g}(\lambda)$, is an infinitesimal symmetric pair of rank 1. Its associated symmetric pair $(G(\lambda), K(\lambda))$, where $G(\lambda) = \exp \mathfrak{g}(\lambda)$ and $K(\lambda) = \exp \mathfrak{k}(\lambda)$, is the above mentioned one, of which the involution is $\sigma | G(\lambda)$ and, if G is simply connected, the fixed group is $G(\lambda)\cap K$. $\mathfrak{t}(\lambda)=\mathfrak{t}\cap\mathfrak{g}(\lambda)$ and $\mathfrak{t}(\lambda)^-=\mathfrak{t}(\lambda)\cap\mathfrak{t}^-$ are Cartan subalgebras of $\mathfrak{g}(\lambda)$ and the pair $(\mathfrak{g}(\lambda), \mathfrak{k}(\lambda))$ respectively. $\mathfrak{t}(\lambda)^-$ is one-dimensional and generated by τ_λ.

PROPOSITION 4.2. *If G is simply-connected, then $G(\lambda)$ is simply connected for each $\lambda\in\mathfrak{r}^-$, and $K(\lambda)=G(\lambda)\cap K$, the fixed group.*

Proof. Once was proved the simply-connected-ness of $G(\lambda)$, then the last assertion follows from [8, 9].

i) *In case* $\lambda\in\mathfrak{r}^{-\prime}$; by [2], Prop. 3.4, $\Delta^\lambda=\Delta\cap\mathfrak{r}_\lambda$ is a σ-fundamental system of roots of $G(\lambda)$ with respect to $\mathfrak{t}(\lambda)$ for any σ-fundamental system Δ of roots of G with respect to $\mathfrak{t}$. Then the same reasoning as in **4.1.** iv) shows that basic translations of $\mathfrak{t}(\lambda)$ corresponding to roots of Δ^λ form a basis of the unit lattice of $\mathfrak{t}(\lambda)$, i.e., $G(\lambda)$ is simply connected.

ii) *In case* $\lambda\notin\mathfrak{r}^{-\prime}$. There exists a root $\lambda\in\mathfrak{r}^{-\prime}$ such that $\lambda=2\lambda'$. First we remark that the symmetric pair $(G(\lambda'), K(\lambda'))$ has λ' and λ as its restricted roots, and their multiplicities for the pair $(G(\lambda'), K(\lambda'))$ are the same as those for the pair (G, K). Secondly we remark that we can define $(G(\lambda), K(\lambda))$ by starting from $(G(\lambda'), K(\lambda'))$ instead of (G, K), i.e., $G(\lambda)=G(\lambda')(\lambda)$, and that $G(\lambda')$ is simply connected as the result of case i).

By discussions of multiplicities of restricted roots [2], §§**2** and **4**, $m(\lambda)=1$, 3 or 7.

a) The case $m(\lambda)=1$. Then $\lambda\in\mathfrak{r}$ and $\mathfrak{g}(\lambda)=\mathbf{R}\{\tau_\lambda\}+\mathfrak{e}_\lambda$. It is well known that, if G is simply connected, exp $(\mathbf{R}\{\tau_\alpha\}+\mathfrak{e}_\alpha)$ is a 3-sphere for any $\alpha\in\mathfrak{r}$. Thus $G(\lambda)$ is a 3-sphere, in particular simply connected.

b) The case $m(\lambda)=3$. By the classification of infinitesimal symmetric pairs of rank 1 ((cf., [2], §4), $\mathfrak{g}(\lambda')=C_l$, $l\geqq 3$, and $m((\lambda')=4(l-2)$. Since $G(\lambda')$ is simply connected, $G(\lambda')\cong\mathbf{Sp}(l)$, $l\geqq 3$. Consider the symmetric pair (4.2) for the group $G(\lambda')$, then we obtain the symmetric pair

$$(G(\lambda'),\ G(\lambda')_{(\lambda,\,1)},\ \mathrm{ad}\ a),\quad a=\exp(\tau_\lambda/2).$$

Here

$$\dim\ (G(\lambda')/G(\lambda')_{(\lambda,\,1)})=\dim\ \tilde{\mathfrak{e}}_{\lambda'}=8(l-2).$$

By the classification of compact symmetric spaces, if $G\cong\mathbf{Sp}\,(l)$ and dim G/K $=8(l-2)$, then we must conclude that $K\cong\mathbf{Sp}\,(2)\times\mathbf{Sp}\,(l-2)$, i.e.,

$$G(\lambda')_{(\lambda,\,1)}\cong\mathbf{Sp}\,(2)\times\mathbf{Sp}\,(l-2).$$

Therefrom we see that $G(\lambda)$ is simply connected since it is the semi-simple part of $G(\lambda')_{(\lambda,\,1)}$.

c) The case $m(\lambda)=7$; then $\mathfrak{g}(\lambda')=F_4$ and $m(\lambda')=8$. Consider the symmetric pair similar to the above; then

$$\dim\ (G(\lambda')/G(\lambda')_{(\lambda,\,1)})=\dim\ \tilde{\mathfrak{e}}_{\lambda'}=16.$$

Therefrom by the classification of compact symmetric spaces we conclude that

$$G(\lambda)=G(\lambda')_{(\lambda,\,1)}\cong\mathbf{Spin}\ (9).$$

In particular, $G(\lambda)$ is simply connected.

4. 3. Let $p=(\lambda, n)$, $\lambda\in\mathfrak{r}^{-\prime\prime}$, be a singular plane in $\mathfrak{t}^{-}$. λ can be expressed as $\lambda=\varepsilon\lambda'$, $\lambda'\in\mathfrak{r}^{-\prime}$, $\varepsilon=1$ or 2. The symmetric pair $(G(\lambda'), K(\lambda'))$ has λ', and possibly $2\lambda'$, as its restricted roots. Now by the definition of the pair $(G(\lambda'), K(\lambda'))$, $m(\lambda')$ and $m(2\lambda')$ for the pair $(G(\lambda'), K(\lambda'))$ are the same as those for the pair (G, K). Consequently $m(p)$ for $(G(\lambda'), K(\lambda'))$ is the same as that for (G, K).

By the inclusion $G(\lambda')\subset G$ is induced the inclusions

$$K(\lambda')_{p}\subset K_{p} \quad \text{and} \quad K(\lambda')_{T(\lambda')_{-}}\subset K_{T_{-}},$$

where $T(\lambda')_{-}=\exp\mathfrak{t}(\lambda')^{-}$. Clearly

$$K(\lambda')_{T(\lambda')_{-}} = K_{T_{-}}\cap K(\lambda')_{p}.$$

Hence the map

$$K(\lambda')_{p}/K(\lambda')_{T(\lambda')_{-}} \longrightarrow K_{p}/K_{T_{-}}$$

induced by the natural inclusion is injective, and both homogeneous spaces of this map are same dimensional by the above remark. Furthermore $K_{p}/K_{T_{-}}$ is connected since it is homeomorphic to $S^{m(p)}$ by Prop. 2.2. Therefrom we can conclude that the above map is bijective, i.e., we obtained

PROFOSITION 4.3. *$K(\lambda')_{p}/K(\lambda')_{T(\lambda')_{-}}\approx K_{p}/K_{T_{-}}$, diffeomorphic by the map induced by the natural inclusion $G(\lambda')\subset G$.*

If $\lambda=\lambda'$ or if $\lambda=2\lambda'$ and n even, then $K(\lambda')=K(\lambda')_{p}$. Thus

$$K(\lambda')/K(\lambda')_{T(\lambda')_{-}} \approx K_{p}/K_{T_{-}} \tag{4.6}$$

by the natural map, if $\lambda=\lambda'$ or if $\lambda=\lambda'$ and n even.

If $\lambda=2\lambda'$ and n odd, then $K(2\lambda')=K(2\lambda')_{p}$; and the similar discussions as above show that

$$K(2\lambda')/K(2\lambda')_{T(\lambda')_{-}} \approx K_{p}/K_{T_{-}} \tag{4.7}$$

by the natural map, if $\lambda=2\lambda'$ and n odd.

4. 4. Now we shall assume that G *is simply connected,* and determine the number of connected components of K_{p} for every singular plane p in $\mathfrak{t}^{-}$.

LEMMA 4.4. *Let $p=(\lambda, n)$ satisfy $\lambda\in\mathfrak{r}^{-\prime}$ and $m(\lambda)=1$, then K_{p}^{0} contains at least two connected components of $K_{T_{-}}$.*

Proof. If we take a σ-fundamental system Δ of $\mathfrak{r}$ such that $\Delta^{-}\ni\lambda$, then we see that

$$\exp(\tau_{\lambda}/2) \notin K_{T_{-}}^{0}$$

by Theorem 3.4. Hence, to prove the lemma it is enough to show that

$$\exp(\tau_{\lambda}/2)\in K_{p}^{0}.$$

Because of $m(\lambda)=1$, $\lambda\in\mathfrak{r}$ and $\mathfrak{g}(\lambda)=\mathbf{R}\{\tau_{\lambda}\}+\mathfrak{e}_{\lambda}$. Let $(U_{\lambda}, V_{\lambda})$ be an orthogonal frame of $\mathfrak{e}_{\lambda}$ such that $\sigma U_{\lambda}=U_{\lambda}$ and $\sigma V_{\lambda}=-V_{\lambda}$ by (1.5). Then $U_{\lambda}\in\mathfrak{k}_{p}$, and

$$\exp(\mathbf{R}\{U_{\lambda}\}) \subset K_{p}^{0}.$$

Now $G(\lambda)$ is a 3-sphere, and $\exp(\mathbf{R}\{U_{\lambda}\})$ is the one of the great circles of $G(\lambda)$, which passes through the anti-pode of e in $G(\lambda)$. On the other hand, the anti-pode of e is idntical with $\exp(\tau_{\lambda}/2)$. Hence

$$\exp(\tau_\lambda/2) \in K^0_p.$$

PROPOSITION 4.5. *$K(\lambda')_p$ is connected for every singular plane p in $\mathfrak{t}^-$, where $p=(\lambda, n)$, $\lambda \in \mathfrak{r}^{-''}$, $\lambda=\varepsilon\lambda'$, $\lambda' \in \mathfrak{r}^{-'}$, $\varepsilon=1$ or 2.*

Proof. Since G is simply connected, $G(\lambda')$ is simply connected by Prop. 4.2. Apply Theorem 3.4 to the pair $(G(\lambda'), K(\lambda'))$. Since the restricted fundamental system of $(G(\lambda'), K(\lambda'))$ is of rank 1 and consists only of λ, $K(\lambda')_{T(\lambda')_-}$ is connected if $m(\lambda')\neq 1$, and has exactly two connected components if $m(\lambda')=1$. Therefore, applying Lemma 4.4 to the pair $(G(\lambda'), K(\lambda'))$ in case $m(\lambda')=1$, we see that

$$K(\lambda')^0_p \supset K(\lambda')_{T(\lambda')_-} \tag{4.8}$$

in all cases.

By Prop. 2.2 applied to the pair $(G(\lambda'), K(\lambda'))$ we see that $K(\lambda')_p/K(\lambda')_{T(\lambda')_-}$ is connected, which implies that every connected component of $K(\lambda')_p$ contains at least one conneccted component of $K(\lambda')_{T(\lambda')_-}$. Then (4.8) implies the connected-ness of $K(\lambda')_p$.

PROPOSITION 4.6. *Let $p=(\lambda, n)$, $\lambda \in \mathfrak{r}^{-''}$, be a singular plane in $\mathfrak{t}^-$.* i) *In case $m(\lambda)=1$ and $\lambda \in \mathfrak{r}^{-'}$, K^0_p contains just two connected components of K_{T_-}:*

$$K^0_p \cap K_{T_-} = K_{T_-}{}^0 + \exp(\tau_\lambda/2)\cdot K_{T_-}{}^0 .$$

ii) *Otherwise*

$$K^0_p \cap K_{T_-} = K^0_{T_-}.$$

Proof. Let λ be expressed as $\lambda=\varepsilon\lambda'$, $\lambda' \in \mathfrak{r}^{-'}$, $\varepsilon=1$ or 2. In case i) $\lambda=\lambda'$.

Consider the following commutative diagram

$$\begin{array}{ccc} K(\lambda')^0_p/K(\lambda')_{T(\lambda')_-}{}^0 & \xrightarrow{\ \alpha\ } & K^0_p/K_{T_-}{}^0 \\ \big\downarrow \gamma & & \big\downarrow \delta \\ K(\lambda')_p/K(\lambda')_{T(\lambda')_-} & \xrightarrow{\ \beta\ } & K_p/K_{T_-} \end{array}$$

induced by natural inclusions.

β is a diffeomorphism by Prop. 4.3.

Clearly γ and δ are covering maps by definition of [14], p. 67. Then α must be locally homeomorphic by the commutativity of the above diagram, and the image of α is open and closed in the connected space $K^0_p/K_{T_-}{}^0$. Hence the image of α coincide with $K^0_p/K_{T_-}{}^0$, i.e., α is also a covering map.

Now by Prop. 4.5 $K(\lambda')_p=K(\lambda')^0_p$, and then by Theorem 3.4 applied to the pair $(G(\lambda'), K(\lambda'))$ we see that

$$\begin{aligned} \deg(\gamma) &= 2 \qquad \text{in case i)}, \\ &= 1 \qquad \text{in case ii)}, \end{aligned}$$

where deg ( ) denotes the degree (number of fibre elements) of the covering map in parentheses.

Next, by Lemma 4.4 we see that

$$\begin{aligned} \deg(\delta) &\geqq 2 \qquad \text{in case i)} \\ &\geqq 1 \qquad \text{in case ii)}. \end{aligned}$$

Therefore deg $(\gamma)\leqq$deg (δ) in both cases.

On the other hand

$$\deg(\gamma) = \deg(\delta\circ\alpha) = \deg(\delta)\cdot\deg(\alpha)$$

since β is bijective. Hence deg $(\gamma)\geqq$ deg (δ).

Thus

$$\deg(\gamma) = \deg(\delta),$$

and

$$\deg(\delta) = 2 \quad \text{in case i)}$$
$$= 1 \quad \text{in case ii)}.$$

Since K_p/K_{T_-} is connected by Prop. 2.2,

$$K_p/K_{T_-} \approx K_p^0/K_p^0\cap K_{T_-}.$$

Therefrom the conclusion of the proposition follows.

Now, by Theorem 3.4 and Prop. 4.6 the number of connected components of K_p can be counted immediately, and we obtain

THEOREM 4.7. *Let (G, K) be a symmetric pair such that G is simply connected, $p=(\lambda, n)$, $\lambda\in\mathfrak{r}^{-\prime\prime}$, a singular plane in $\mathfrak{t}^-$, $\lambda=\varepsilon\lambda'$ such that $\lambda'\in\mathfrak{r}^{-\prime}$ and $\varepsilon=1$ or 2. Choose a σ-fundamental system Δ of $\mathfrak{r}$ such that $\Delta^-\ni\lambda'$. Let p' be the number of restricted roots of multiplicity* 1 *of Δ^-, and $\bar{\tau}_i$, $1\leqq i\leqq p'$, the corresponding basic translations.*

i) *In case $m(\lambda')=1$, take $\bar{\tau}_1$ as the basic translation corresponding to λ', then*

$$K_p/K_p^0 \cong (Z_2)^{p'-1},$$

whose $p'-1$ generators are represented by $\exp(\bar{\tau}_i/2)$, $2\leqq i\leqq p'$.

ii) *In case $m(\lambda')\neq1$,*

$$K_p/K_p^0 \cong (Z_2)^{p'},$$

whose p' generators are represented by $\exp(\bar{\tau}_i/2)$, $1\leqq i\leqq p'$.

§5. Some reduction of K-cycles.

5. 1. We assume that G *is simply connected* for every symmetric pair (G, K) throughout this section.

Let (G, K) be a symmetric pair, and

$$G=G^1\times G^2 \tag{5.1}$$

be a decomposition of G into a direct product of two σ-invariant subgroups G^1 and G^2. Then we have a decomposition

$$K=K^1\times K^2 \tag{5.2}$$

of K into a direct product such that $K^i=K\cap G^i$, $i=1$, 2. The pairs (G^i, K^i), $i=1$ and 2, are symmetric pairs such that G^i are simply connected, with involutions $\sigma_i=\sigma|G^i$, and

$$G/K\cong G^1/K^1\times G^2/K^2 \tag{5.3}$$

as a symmetric space.

The infinitesimal symmetric pair $(\mathfrak{g}, \mathfrak{k})$ of (G, K) is also decomposed into a direct sum

(5.4) $$\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2,\ \mathfrak{k}=\mathfrak{k}_1+\mathfrak{k}_2,\ \mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2,$$

where $(\mathfrak{g}_i, \mathfrak{k}_i)$, $i=1$ and 2, are infinitesimal pairs of (G^i, K^i) and

$$\mathfrak{g}_1=\mathfrak{k}_1+\mathfrak{m}_1,\ \mathfrak{g}_2=\mathfrak{k}_2+\mathfrak{m}_2$$

are their decompositions (1.1). We have also a direct product decomposition

(5.5) $$M = M^1\times M^2,$$

where $M=\exp\mathfrak{m}$ and $M^i=\exp\mathfrak{m}_i$ for $i=1, 2$.

Cartan subalgebras $\mathfrak{t}^-$ of the pair $(\mathfrak{g}, \mathfrak{k})$ and $\mathfrak{t}$ of $\mathfrak{g}$ containing $\mathfrak{t}^-$ are also decomposed into direct sums

(5.6) $$\mathfrak{t}^-=\mathfrak{t}_1^-+\mathfrak{t}_2^-,\ \mathfrak{t}=\mathfrak{t}_1+\mathfrak{t}_2,$$

where $\mathfrak{t}_i^-=\mathfrak{t}^-\cap\mathfrak{m}_i$ are Cartan subalgebras of $(\mathfrak{g}_i, \mathfrak{k}_i)$ and $\mathfrak{t}_i=\mathfrak{t}\cap\mathfrak{g}_i$ are those of $\mathfrak{g}_i$ containing $\mathfrak{t}_i^-$ for $i=1, 2$. Correspondingly the maximal torus $T_-=\exp\mathfrak{t}^-$ of (G, K) is decomposed into a direct product

(5.7) $$T_-=T_-^{(1)}\times T_-^{(2)}$$

of maximal tori $T_-^{(i)}=\exp\mathfrak{t}_i^-$, $i=1$ and 2, of the pairs (G^i, K^i).

Root systems $\mathfrak{r}$ (of $\mathfrak{g}$ with respect to $\mathfrak{t}$) and $\mathfrak{r}^-$ (of $(\mathfrak{g}, \mathfrak{k})$ with respect to $\mathfrak{t}^-$) are decomposed into disjoint unions of mutually orthogonal subsystems

(5.8) $$\mathfrak{r}=\mathfrak{r}_1\cup\mathfrak{r}_2,\ \mathfrak{r}^-=\mathfrak{r}_1^-\cup\mathfrak{r}_2^-,$$

such that $\mathfrak{r}_i|\mathfrak{t}_j$ and $\mathfrak{r}_i^-|\mathfrak{t}_j^-$ are zero forms for $i\neq j$. $\mathfrak{r}_i$ and $\mathfrak{r}_i^-$, identified with $\mathfrak{r}_i|\mathfrak{t}_i$ and $\mathfrak{r}_i^-|\mathfrak{t}_i^-$ respectively, are root systems of $\mathfrak{g}_i$ and $(\mathfrak{g}_i, \mathfrak{k}_i)$ with respect to $\mathfrak{t}_i$ and $\mathfrak{t}_i^-$ for $i=1, 2$.

5. 2. Denote by pr_i, $i=1, 2$, the projection onto the i-th factor in (5.1) (or in (5.2), (5.3), (5.4) etc.).

LEMMA 5.1. *Let L be any subset of $M=\exp\mathfrak{m}$, then*

$$K_L = (K^1)_{L^1}\times(K^2)_{L^2},$$

where $L^i=\mathrm{pr}_iL$ for $i=1, 2$.

Proof. Put $g_i=\mathrm{pr}_i\, g$ for any $g\in G$, $i=1$ and 2; then $g=(g_1, g_2)$. For $k=(k_1, k_2)\in K$ and $l=(l_1, l_2)\in L$, k is commutative with l if and only if k_i are commutative with l_i for $i=1$ and 2, whence the lemma follows.

In particular, if $L=T_-$, then $L^i=T_-^{(i)}$ for $i=1$ and 2. Therefore by the above Lemma we obtain

PROPOSITION 5.2. $K_{T_-}=(K^1)_{T_-^{(1)}}\times(K^2)_{T_-^{(2)}}$.

Next, let $p=(\lambda, n)$ be a singular plane in $\mathfrak{t}^-$. By the decomposition (5.8) $\lambda\in\mathfrak{r}_1^-$ or $\mathfrak{r}_2^-$. If $\lambda\in\mathfrak{r}_1^-$, then p may also be regarded as a singular plane in $\mathfrak{t}_1^-$, denoted here by p' to distinguish it from the original one, and

$$\exp p=\exp p'\times T_-^{(2)}.$$

Similarly, if $\lambda\in\mathfrak{r}_2^-$, then we can regard p as a singular plane in $\mathfrak{t}_2^-$, and denoting it by p'',

$$\exp p=T_-^{(1)}\times\exp p''.$$

Thus by Lemma 5.1 we obtain

RROPOSITION 5.3. *Let $p=(\lambda, n)$ be a singular plane in* $\mathfrak{t}^-$. i) *If* $\lambda\in\mathfrak{r}_1^-$, *then*

$$K_p=(K^1)_p\times(K^2)_{T_-^{(2)}};$$

ii) *if* $\lambda\in\mathfrak{r}_2^-$, *then*

$$K_p=(K^1)_{T_-^{(1)}}\times(K^2)_p,$$

where p is regarded as a singular plane in $\mathfrak{t}^-$ as well as that in $\mathfrak{t}_1^-$ or $\mathfrak{t}_2^-$.

COROLLARY 5.4. *Let $p=(\lambda, n)$ be a singular plane in* $\mathfrak{t}^-$. i) *If* $\lambda\in\mathfrak{r}_1^-$, *then*

$$K_p/K_{T_-}\approx(K^1)_p/(K^1)_{T_-^{(1)}}$$

natural diffeomorphism induced by the inclusion $K_1\subset K$. Similarly, ii) *if* $\lambda\in\mathfrak{r}_2^-$, *then*

$$K_p/K_{T_-}\approx(K^2)_p/(K^2)_{T_-^{(2)}}.$$

5. 3. Let $p=\{p_1, \cdots, p_n\}$, $p_i=(\lambda_i, m_i)$ be a finite sequence of singular planes in $\mathfrak{t}^-$. Under the decomposition (5.1)-(5.8) we assume that

$$\lambda_i\in\mathfrak{r}_1^- \text{ if } i\in\{j_1, \cdots, j_r\} \quad (\subset\{1, \cdots, n\}),$$
$$\in\mathfrak{r}_2^- \text{ otherwise.}$$

Let $\{j_1, \cdots, j_r\}$ and its complement $\{k_1, \cdots, k_{n-r}\}$ be arranged in their ascending orders; and put $P'=\{p_{j_1}, \cdots, p_{j_r}\}$ and $P''=\{p_{k_1}, \cdots, p_{k_{n-r}}\}$, which are considered as finite sequences of singular planes in $\mathfrak{t}_1^-$ and $\mathfrak{t}_2^-$ respectively.

Let us consider K-cycles Γ_P, $\Gamma_{P'}$ and $\Gamma_{P''}$ of the pairs (G, K), (G^1, K^1) and (G^2, K^2) respectively. If Γ_P is totally orientable, then $\Gamma_{P'}$ and $\Gamma_{P''}$ are also totally orientable since they can be regarded as sub-K-cycles of Γ_P. Their homology bases described in Prop. 2.8 are denoted respectively by $[i_1, \cdots, i_s]_2$, $1\leqq i_1<\cdots<i_s\leqq n$, $[i_1, \cdots, i_s]_2'$, $1\leqq i_1<\cdots<i_s\leqq r$, and $[i_1, \cdots, i_s]_2''$, $1\leqq i_1<\cdots<i_s\leqq n-r$, or dropping suffices 2 in case that Γ_P is totally orientable and $H_*(\Gamma_P; Z)$ is discussed; and their dual cohomology bases are denoted by $\{x_{i_1\cdots i_s}\}$, $\{x'_{i_1\cdots i_s}\}$ and $\{x''_{i_1\cdots i_s}\}$ respectively.

PROPOSITION 5.5. *There exists a homeomorphism*

$$\Gamma_P\approx\Gamma_{P'}\times\Gamma_{P''} \quad \text{(direct product)},$$

which is natural in the sense that, denoting by π_1 and π_2 the projections onto the first and the second factors,

$$\pi_1^*(x_s')=x_{j_s} \qquad for \quad 1\leqq s\leqq r$$
$$\pi_2^*(x_t'')=x_{k_t} \qquad for \quad 1\leqq t\leqq n-r,$$

where π_i^* denotes the cohomology map (mod 2 or integral according to the cases) *induced by π_i for $i=1, 2$.*

Proof. Put

$$W'=\mathrm{pr}_1K_1\times\cdots\times\mathrm{pr}_1K_n,$$
$$W''=\mathrm{pr}_2K_1\times\cdots\times\mathrm{pr}_2K_n,$$

where K_i denotes K_{p_i} for $1\leqq i\leqq n$.

Abbreviating K_{T_-}, $(K^1)_{T_-^{(1)}}$ and $(K^2)_{T_-^{(2)}}$ respectively to K_0, K_0^1 and K_0^2, n-fold direct products $(K_0)^n$, $(K_0^1)^n$ and $(K_0^2)^n$ operate on W_P, W' and W''

respectively by the rules (2.1). The quotient spaces of W' and W'' by these operations are denoted by Γ' and Γ'' respectively; $W_P/(K_0)^n=\Gamma_P$ by definition.

The pairs of maps

$$((\mathrm{pr}_1)^n, (\mathrm{pr}_1)^n) : (W_P, (K_0)^n) \longrightarrow (W', (K_0^1)^n),$$
$$((\mathrm{pr}_2)^n, (\mathrm{pr}_2)^n) : (W_P, (K_0)^n) \longrightarrow (W'', (K_0^2)^n)$$

are respectively homomorphisms of principal bundles, and induce the maps of base spaces

$$\bar{\pi}_1 : \Gamma_P \longrightarrow \Gamma', \quad \bar{\pi}_2 : \Gamma_P \longrightarrow \Gamma''.$$

First we claim

(5.9) $$\Gamma_P \approx \Gamma' \times \Gamma'' \quad \text{(direct product)}$$

with $\bar{\pi}_1$ *and* $\bar{\pi}_2$ *as its projections onto the first and the second factors.*

Define the maps

$$q : W' \times W'' \longrightarrow W_P,$$
$$\tilde{q} : (K_0^1)^n \times (K_0^2)^n \longrightarrow (K_0)^n$$

by

$$q((y_1', \cdots, y_n'), (y_1'', \cdots, y_n''))=(y_1'y_1'', \cdots, y_n'y_n'')$$

for $y_i' \in \mathrm{pr}_1 K_{p_i}$ and $y_i'' \in \mathrm{pr}_2 K_{p_i}$, $1\leqq i\leqq n$, and by

$$\tilde{q}((t_1', \cdots, t_n'), (t_1'', \cdots, t_n''))=(t_1't_1'', \cdots, t_n' t_n'')$$

for $t_i' \in K_0^1$ and $t_i'' \in K_0^2$, $1\leqq i\leqq n$. It is a routine proof to see that the pair $(q, \tilde{q})$ is a homomorphism of principal bundles considering $W' \times W''$ as a product bundle, and that, denoting by

$$\bar{q} : \Gamma' \times \Gamma'' \longrightarrow \Gamma_P$$

the map of base spaces induced by q,

$$\bar{q} \circ (\bar{\pi}_1 \times \bar{\pi}_2) = \text{identity map},$$
$$(\bar{\pi}_1 \times \bar{\pi}_2) \circ \bar{q} = \text{identity map}.$$

Thus (5.9) is proved.

Next define maps

$$u_1 : W' \longrightarrow W_{P'}, \quad \tilde{u}_1 : (K_0^1)^n \longrightarrow (K_0^1)^r$$

by

$$u_1(y_1', \cdots, y_n')=(y_1' \cdots y_{j_1}', y_{j_1+1}' \cdots y_{j_2}', \cdots, y_{j_{r-1}+1}' \cdots y_{j_r}')$$
$$u_1(t_1', \cdots, t_n')=(t_{j_1}', \cdots, t_{j_r}')$$

for $y_i' \in \mathrm{pr}_1 K_{p_i}$, $t_i' \in K_0^1$, $1\leqq i\leqq n$. Also define maps

$$u_2 : W'' \longrightarrow W_{P''}, \ \tilde{u}_2 : (K_0^2)^n \longrightarrow (K_0^2)^{n-r}$$

similarly as aboves. Then we see easily that $(u_i, \tilde{u}_i)$, $i=1$ and 2, are homomorphisms of principal bundles, by which are induced the maps of base spaces

$$\bar{u}_1 : \Gamma' \longrightarrow \Gamma_{P'}, \quad \bar{u}_2 : \Gamma'' \longrightarrow \Gamma_{P''}.$$

Using Prop. 5.3 we see easily that

(5.10) $\bar{u}_1$ *and* $\bar{u}_2$ *are homeomorphisms.*

Put

$$\pi_1=\bar{u}_1 \circ \bar{\pi}_1 \quad \text{and} \quad \pi_2=\bar{u}_2 \circ \bar{\pi}_2.$$

Then, by (5.9)-(5.10) we see that

$$(5.11) \qquad \Gamma_P \approx \Gamma_{P'} \times \Gamma_{P''}$$

with π_1 *and* π_2 *as the projections onto the first and the second factors,* which is the first half of Prop. 5.5.

As the effect on the top dimensional homology of the homeomorphism $\pi_1 \times \pi_2$ we see that

$$(5.12') \qquad (\pi_1 \times \pi_2)_*[1, \cdots, n] = \pm[1, \cdots, r]' \otimes [1, \cdots, n-r]'',$$

where suffices 2 are dropped in case that Γ_P is general and $H_*(\Gamma_P ; Z_2)$ is discussed. Apply (5.12′) to every sub-K-cycle of Γ_P, and use appropriate commutative diagrams similar to that in the proof of (2.11), then we see that

$$(5.12) \qquad (\pi_1 \times \pi_2)_*[i_1, \cdots, i_s] = \pm[a_1, \cdots, a_t]' \otimes [b_1, \cdots, b_{s-t}]''$$

under the same convention as (5.12′) for $1 \leqq i_1 < \cdots < i_s \leqq n$, where

$$\{i_1, \cdots, i_s\} \cap \{j_1, \cdots, j_r\} = \{j_{a_1}, \cdots, j_{at}\},$$
$$\{i_1, \cdots, i_s\} \cap \{k_1, \cdots, k_{n-r}\} = \{k_{b_1}, \cdots, k_{b_{s-t}}\},$$

arranged in ascending orders. In particular

$$(5.12'') \qquad \begin{aligned} (\pi_1 \times \pi_2)_*[j_s] &= [s]' \otimes 1 \qquad \text{for } 1 \leqq s \leqq r, \\ (\pi_1 \times \pi_2)_*[k_t] &= 1 \otimes [t]'' \qquad \text{for } 1 \leqq t \leqq n-r \end{aligned}$$

under the same convention as above, where, in totally orientable case, signs become unnecessary by choosing the same orientations to K_s^1/K_0^1 and K_{j_s}/K_0, or to K_t^2/K_0^2 and K_{k_t}/K_0 *via* natural homeomorphisms of Cor. 5.4.

Therefrom the last half of Prop. 5.5 follows.

If we remarrk that $<\lambda, \mu> = 0$ for $\lambda \in \mathfrak{r}_1^-$ and $\mu \in \mathfrak{r}_2^-$, then we see easily the following

COROLLARY 5.6. *In the decomposition* (5.1), *assume that Theorem 2.10 holds for the pairs* (G^1, K^1) *and* (G^2, K^2), *then it holds also for the pair* (G, K).

5. 4. In every symmetric pair (G, K), G can be decomposed into the direct product of σ-irreducible factors

$$(5.13) \qquad G = G^1 \times \cdots \times G^s.$$

Correspondingly we have a decomposition

$$(5.14) \qquad K = K^1 \times \cdots \times K^s$$

of K into a direct product such that $K^i = K \cap G^i$ for $1 \leqq i \leqq s$. The pairs (G^i, K^i), $1 \leqq i \leqq s$, are irreducible symmetric pairs such that G^i are simply connected, called the irreducible factors of (G, K). Now the decomposition (5.13) can be achieved as a result of a finite number of successions of decompositions of type (5.1). Therefore, by Prop. 5.5 and Cor. 5.6 we obtain the following propositions.

PROPOSITION 5.7. *In any symmetric pair* (G, K) *with G simply connected, every K-cycle can be decomposed into a direct product of K-cycles of irreducible factors by choosing one from each factor.*

PROPOSITION 5.8. *If Theorem 2.10 is true for every irreducible symmetric pair,* *then it is true for all symmetric pairs.*

5. 5. We say that a symmetric pair (G, K) is *of totally orientable type* if all K-cycles associated with this pair are totally orientable. We shall discuss a condition under which a symmetric pair (G, K) (with simply connected G) is of totally orientable type.

The following assertion is evident by definitions.

(5.15) *In a symmetric pair we assume that, for every finite sequence* $P=\{p_1, \cdots, p_n\}$ *of singular planes in* $\mathfrak{t}^-$, *the sphere bundle* $(\Gamma_P, \Gamma_{P'}, S^{m(p_n)})$, $P'=\{p_1, \cdots, p_{n-1}\}$, *is orientable; then the pair is of totally orientable type.*

LEMMA 5.9. *In a symmetric pair* (G, K), *assume that any one of its restricted fundamental systems of roots contains no roots of multiplicity* 1, *then the pair is of totally orientable type.*

Proof. For any $P=\{p_1, \cdots, p_n\}$, finite sequence of singular planes in $\mathfrak{t}^-$, the principal orthogonal bundle associated with the sphere bundle $(\Gamma_P, \Gamma_{P'}, S^{m(p_n)})$, $P'=\{p_1, \cdots, p_{n-1}\}$, is the ad$'$-extension of the K_{p_n}-bundle $\bar{\Gamma}_P \longrightarrow \Gamma_{P'}$ by Theorem 2.4. On the other hand K_{p_n} is a connected group by Theorem 4.7. ii) since G is simply connected and $p'=0$ by the assumption of the lemma. Hence

$$\mathrm{ad}'(K_p\ ) \subset \mathbf{SO}(m(p_n)+1),$$

i.e., the structure group of the orthogonal bundle can be reduced to $\mathbf{SO}(m(p_n)+1)$ and the bundle $(\Gamma_P, \Gamma_{P'}, S^{m(p_n)})$ is orientable. Therefrom the lemma follows by (5.15).

Lemma 5.10. *Let* (G, K) *be a symmetric pair, and assume that any restricted fundamental system* Δ^- *of the pair contains only one root of multiplicity* 1 *and all other roots of* Δ^- *have even multiplicity. Then the pair is of totally orientable type.*

Proof. As in the proof of the above lemma, it is sufficient to show that

$$\mathrm{ad}'(K_p) \subset \mathbf{SO}(m(p)+1)$$

for all singular planes p in $\mathfrak{t}^-$.

Put $p=(\lambda, n)$, $\lambda=\varepsilon\lambda'$, $\lambda' \in \mathfrak{r}^{-\prime}$, $\varepsilon=1$ or 2.

i) In case $m(\lambda')=1$, K_p is connected by Theorem 4.7. i) since $p'=1$ by the assumption of the lemma. Hence

$$\mathrm{ad}'(K_p) \subset \mathbf{SO}(m(p)+1).$$

ii) In case $m(\lambda')\neq 1$,

$$K_p = K_p^0+\exp(\tau_\mu/2)\cdot K_p^0$$

by Theorem 4.7. ii) after choosing a σ-fundamental system such that $\Delta^- \ni \lambda'$ and denoting by μ the root of multiplicity 1 of Δ^-. Now

$$\mathfrak{m}_p' = \mathbf{R}\{\tau_\lambda\}+\tilde{\mathfrak{e}}_p\cap\mathfrak{m}$$

by (2.5) and

$$\tilde{\mathfrak{e}}_p\cap\mathfrak{m} = \tilde{\mathfrak{e}}_{2\lambda'}\cap\mathfrak{m} \quad \text{in case} \quad \varepsilon=2 \quad \text{and } n \text{ odd}$$
$$= \tilde{\mathfrak{e}}_{\lambda'}\cap\mathfrak{m}+\tilde{\mathfrak{e}}_{2\lambda'}\cap\mathfrak{m} \quad \text{otherwise.}$$

Using the basis (1.5$'$) for $\tilde{\mathfrak{e}}_{\lambda'}$ and $\tilde{\mathfrak{e}}_{2\lambda'}$, compute $\mathrm{ad}'(\exp(\tau_\mu/2))$ on $\tilde{\mathfrak{e}}_{\lambda'}\cap\mathfrak{m}$, and

$\tilde{\mathfrak{e}}_{2\lambda'}\cap\mathfrak{m}$ (if $2\lambda'\in\mathfrak{r}^-$). Then, remarking that $\lambda'(\tau_\mu)$ is an integer and $2\lambda'(\tau_\mu)$ is even, we see that

$$\mathrm{ad}'(\exp(\tau_\mu/2))|\tilde{\mathfrak{e}}_{\lambda'}\cap\mathfrak{m} = \pm\text{identity map},$$
$$\mathrm{ad}'(\exp(\tau_\mu/2))|\tilde{\mathfrak{e}}_{2\lambda'}\cap\mathfrak{m} = \text{identity map}.$$

Furthermore

$$\mathrm{ad}'(\exp(\tau_\mu/2))|\mathbf{R}\{\tau_\lambda\} = \text{identity map}$$

as is immediately seen.

By the assumption of the lemma $m(\lambda')$ is even. Therefore, by the above discussions we see that

$$\mathrm{ad}'(\exp(\tau_\mu/2))\in\mathbf{SO}(m(p)+1).$$

On the other hand

$$\mathrm{ad}'(K_p^0)\subset\mathbf{SO}(m(p)+1)$$

Hence

$$\mathrm{ad}'(K_p)\subset\mathbf{SO}(m(p)+1)$$

also in case ii). Thereby is proved the lemma.

By Prop. 5.7 and Lemmas 5.9, 5.10 we obtain the following

THEOREM 5.11. *Let (G, K) be a symmetric pair such that G is simply connected. And assume that every restricted fundamental system Δ^- of all irreducible factors of (G, K) satisfies that either it contains no root of multiplicity* 1, *or contains exactly one root of multiplicity* 1 *and every other root of Δ^- has even multiplicity. Then the pair (G, K) is of totally orientable type.*

By the classification of irreducible infinitesimal symmetric pairs (cf., [2], the table at the end), we obtain the following

COROLLARY 5.12. *Let G/K be a compact symmetric space such that G is simply connected and that every irreducible factor of G/K is isomorphic to one of the following spaces: compact Lie groups, complex grassmann manifolds* (type AIII, AIV), *quaternion grassmann manifolds* (type CII), *spheres* (type BII, DII), $\mathbf{SO}(2n+2)/\mathbf{SO}(2)\times\mathbf{SO}(2n)$ (type DI of restricted rank 2), $\mathbf{SU}(2n)/\mathbf{Sp}(n)$ (type AII), $\mathbf{SO}(2n)/\mathbf{U}(n)$ (type DIII), $\mathbf{E}_6/\mathbf{Spin}(10)\cdot\mathbf{T}^1$ (type EIII), $\mathbf{E}_6/\mathbf{F}_4$ (type EIV), $\mathbf{E}_7/\mathbf{E}_6\cdot\mathbf{T}^1$ (type EVII) *and octanion projective plane* (type FII). *Then every K-cycle associated with (G, K) is totally orientable.*

We can see *via* classification and case-by-case discussions that the condition of Theorem 5.1 is also sufficient for a symmetric pair with simply connected G to be of totally orientable type.

Finally, applying Theorem 5.11 to the theory of [8], Theorem I and its consequences, we see that, for every symmetric pair (G, K) of totally orientable type, the integral cohomologies of the loop space $\Omega(G/K)$ and any space $K/K_{T'}$, T' a torus subgroup of $M=\exp\mathfrak{m}$, have no torsion.

5. 6. For any singular plane $p=(\lambda, n)$, $\lambda\in\mathfrak{r}^{-\prime\prime}$, we put again $\lambda=\varepsilon\lambda'$, $\lambda'\in\mathfrak{r}^{-\prime}$, $\varepsilon=1$ or 2. Further we put

$$(5.16)\quad \begin{aligned}(G(p),\ K(p)) &= (G(\lambda'),\ K(\lambda')) \text{ if } \varepsilon=1 \text{ or if } \varepsilon=2 \text{ and } n \text{ even}\\ &= (G(2\lambda'),\ K(2\lambda')) \text{ if } \varepsilon=2 \text{ and } n \text{ odd,}\end{aligned}$$

using the notations of **4.3.** Then by (4.6)-(4.7)

$$(5.17)\qquad K(p)/K(p)_{T_-} \approx K_p/K_{T_-},$$

by the natural map; and since G is simply connected Prop. 4.2 implies that

$$(5.18)\qquad K(p) = K\cap G(p).$$

LEMMA 5.13. *For every* $k\in K_{T_-}$, $k\cdot K(p)\cdot k^{-1}=K(p)$.

Proof. $\mathfrak{g}(p)$ and $\mathfrak{k}(p)$ denote Lie algebras of $G(p)$ and $K(p)$ respectively.

Using a standard argument with Weyl base of $\mathfrak{g}^C$, the complexification of $\mathfrak{g}$, we see that

$$[\mathfrak{e}_\alpha,\ \mathfrak{e}_\beta] \subset \mathfrak{e}_{\alpha+\beta}+\mathfrak{e}_{\alpha-\beta}$$

for α, $\beta\in\mathfrak{r}$, where $\mathfrak{e}_{\alpha+\beta}=0$ (or $\mathfrak{e}_{\alpha-\beta}=0$) if $\alpha+\beta$ (or $\alpha-\beta$)$\notin\mathfrak{r}$. Further, for $\alpha\in\mathfrak{r}_0$ and $\beta\in\bar{\mathfrak{r}}_{\lambda'}$ (or $\bar{\mathfrak{r}}_{2\lambda'}$)

$$\alpha\pm\beta\in\bar{\mathfrak{r}}_{\lambda'} \text{ (or } \bar{\mathfrak{r}}_{2\lambda'})$$

if they belong to $\mathfrak{r}$, beause "$\beta\in\bar{\mathfrak{r}}_{\lambda'}$ (or $\bar{\mathfrak{r}}_{2\lambda'}$)" means that β is connected with $\bar{\mathfrak{r}}_{\lambda'}$ (or $\bar{\mathfrak{r}}_{2\lambda'}$) in $\mathfrak{r}_0$, and then $\alpha\pm\beta$ is connected with $\bar{\mathfrak{r}}_{\lambda'}$ (or $\bar{\mathfrak{r}}_{2\lambda'}$) in $\mathfrak{r}_0$.

Hence by (1.9) adjoint operations of $\mathfrak{g}_{T_-}$ in $\mathfrak{g}$ make the space $\mathfrak{n}=\sum\mathfrak{e}_\beta$ invariant, where the summation runs over all roots of $\bar{\mathfrak{r}}_{\lambda'}$ (or $\bar{\mathfrak{r}}_{2\lambda'}$), and consequently adjoint actions of exp $(\mathfrak{g}_{T_-})=G_{T_-}$ make $\mathfrak{n}$ invariant. Since the latter adjoint actions are homomorphisms, they make invariant the Lie algebra generated by $\mathfrak{n}$, which is equal to $\mathfrak{g}(p)$.

Finally the adjoint operations of $G_{T_-}\cap K$ make $\mathfrak{g}(p)\cap\mathfrak{k}=\mathfrak{k}(p)$ invariant, and do also $K(p)=\exp\mathfrak{k}(p)$ invariant. Thus the lemma is proved.

5. 7. Let $P=\{p_1, \cdots, p_n\}$ be a sequence of singular planes in $\mathfrak{t}^-$. Using the notations of **2. 1**, $K_i=K_{p_i}$ and $K_0=K_{T_-}$ for $1\leqq i\leqq n$. Here we put

$$K(i)=K(p_i),\quad K(i)_0=K(p_i)_{T_-}$$

for $1\leqq i\leqq n$. For any subgroup L of K_0

$$(5.19)\qquad LK(i)=K(i)L \quad \textit{and} \quad LK(i)_0=K(i)_0L$$

by Lemma 5.13, which are respectively subgroups generated by $\{L, K(i)\}$ and $\{L, K(i)_0\}$, for $1\leqq i\leqq n$.

Next we put

$$(5.20)\qquad K_{(i)}=K(1)_0K(2)_0\cdots K(i-1)_0K(i),\quad K_0^{(i)}=K(1)_0K(2)_0\cdots K(i)_0$$

for $1\leqq i\leqq n$, which are respectively subgroups generated by $\{K(1)_0, \cdots, K(i-1)_0, K(i)\}$ and $\{K(1)_0, \cdots, K(i)_0\}$ by the above remarks.

$$(5.21)\qquad K_{(i)}/K_0^{(i)}\approx K_i/K_0 \qquad \textit{for}\quad 1\leqq i\leqq n,$$

by the maps induced by the natural inclusions.

Proof. Consider the map

$$\alpha_i : K(i)/K(i)_0 \longrightarrow K_{(i)}/K_0^{(i)}$$

induced by the natural inclusion $K(i)\subset K_{(i)}$. Because of (5.17) it is sufficient to see that α_i is bijective. Since

$$K_{(i)}=K(i)K(1)_0K(2)_0\cdots K(i-1)_0$$

by (5.19), α_i is surjective. On the other hand

$$K(i)_0\subseteqq K(i)\cap K_J^{(i)}\subseteqq K(i)\cap K_0=K(i)_0.$$

Thus $K(i)\cap K_J^{(i)}=K(i)_0$, which shows that α_i is injective. q.e.d.

Put

$$\Gamma'_P=K_{(1)}\times_{K_0^{(1)}}K_{(2)}\times_{K_0^{(2)}}\cdots\times_{K_0^{(n-1)}}(K_{(n)}/K_J^{(n)}).$$

By dropping off the last factor we obtain a fibre bundle $(\Gamma'_P, \Gamma'_{P'}, K_{(n)}/K_J^{(n)})$, where $P'=\{p_1, \cdots, p_{n-1}\}$. The inclusion $K_{(1)}\times K_{(2)}\times\cdots\times K_{(n)}\subset W_P$ induces a map $\beta_n:\Gamma'_P\longrightarrow\Gamma_P$. And the pair (β_n, β_{n-1}) is a bundle map with the inclusion $K_0^{(n-1)}\subset K_0$ as homomorphism of structure groups, where we regard their associated principal bundles as reduced ones as in **2.5.** In this bundle map the fibres are mapped homemorphic onto by (5.21). Therefore, if β_{n-1} is a homeomorphism, then β_n is also so. By an induction on the length n of P and making use of (5.21) we can see that β_n is homeomorphic.

Thus the pair (β_n, β_{n-1}) is an isomorphism of fibre bundles, and we see the following

PROPOSITION 5.14. *The structure group of the bundle* $(\Gamma_P, \Gamma_{P'}, K_n/K_0)$ *is reducible to* $K_J^{(n-1)}$.

§6. Symmetric pairs of splitting rank and K-cycles.

6. 1. In this section we shall discuss K-cycles associated with symmetric pairs (G, K) of splitting rank with simply connected G.

As an immediate corollary of Prop. 1.2 and Theorem 5.11 we obtain

PROPOSITION 6.1. *For every symmetric pair (G, K) of splitting rank with simply connected G, all singular planes in $\mathfrak{t}^-$ have even multiplicities and the K-cycles associated with it are all totally orientable and even dimensional.*

This proposition, combined with the theory of [8], implies

COROLLARY 6.2. *For every symmetric pair (G, K) of splitting rank with simply connected G, $H^*(K/K_{T_-}; Z)$ and $H^*(\Omega(G/K); Z)$ have no torsion, and their subgroups of odd degrees vanish.*

6. 2. Let (G, K) be a symmetric pair of splitting rank with simply connected G. We shall consider the operations of W^- on K/K_{T_-} derived from right translations as in **1.9,** and the representation of W^- on $H^*(K/K_{T_-}; \mathbf{R})$ induced by these operations.

By Cor. 6.2 every odd dimensional cohomology of K/K_{T_-} vanishes, and

$$\dim H^*(K/K_{T_-}; \mathbf{R}) = \dim H^*(K/K_{T_-}; Z_2).$$

On the other hand

$$\dim H^*(K/K_{T_-}; Z_2) = \textit{order of } W^-$$

by [8], Chap. IV, Cor. 2.13, p. 1022. Since W^- operates on K/K_{T_-} without fixed points, we get a proof of the following proposition in entirely the same manner as that of Leray [10], Prop. 11.1, p. 113, by making use of the above facts and the Lefshetz fixed point theorem:

PROPOSITION 6.3. *For every symmetric pair* (G, K) *of splitting rank with simply connected* G, *the representation of* W^- *on* $H^*(K/K_{T_-}; \mathbf{R})$ *is equivalent to the regular representation of* W^-.

In case the symmetric space is a Lie group, this proposition reduces to Prop. 11.1 of [10] as will be seen by a remark in **1.9.** Therefore this proposition is an extension of Prop. 11.1 of [10].

6. 3. For every irreducible symmetric pair of the considered type, the multiplicities of its restricted roots are all the same as will be seen from [2], the table, which we denote by $2m$; that is, i) $m=1$ in group cases, ii) $m=2$ for $(\mathbf{SU}(2n), \mathbf{Sp}(n))$, iii) $m=n-1$ for $(\mathbf{Spin}(2n), \mathbf{Spin}(2n-1))$, and iv) $m=4$ for $(\mathbf{E}_6, \mathbf{F}_4)$.

Here we shall distinguish singular planes $p=(\lambda, n)$ and $-p=(-\lambda, -n)$ as oppositely oriented ones. (How to orient them is immaterial.)

THEOREM 6.4. *For every symmetric pair* (G, K) *of splitting rank with simply connected* G, *we can orient* K_p/K_{T_-} *for each singular plane* p *in* $\mathfrak{t}^-$ *in a suitable way so that* K_p/K_{T_-} *and* K_{-p}/K_{T_-} *are oppositely oriented and that, for each* K*-cycle* Γ_P, $P=\{p_1, \cdots, p_n\}$ *and* $p_i=(\lambda_i, n_i)$,

$$H^*(\Gamma_P; Z) = Z[x_1, \cdots, x_n]/I_P,$$

where I_P *is the ideal generated by the elements*

$$\rho_k=x_k(x_k+\textstyle\sum_{i=1}^{k-1}a_{ki}x_i), \quad 1\leqq k\leqq n,$$

$a_{ki}=2<\lambda_k, \lambda_i>/<\lambda_i, \lambda_i>$, *and* $x_1, \cdots, x_n$ *are generators described in Prop.* 2.9, *and, if* (G, K) *is irreducible, form a basis of* $H^{2m}(\Gamma_P; Z)$.

In case the symmetric space is a Lie group, this theorem reduces to Prop. 4.2 of [8], Chap. III, p. 996.

By virtue of Props. 5.5 and 5.7 and the fact that $a_{ki}=0$ if λ_k and λ_i belong to mutually different irreducible factors, to prove Theorem 6.4 it is sufficient to prove the following

PROPOSITION 6.5. *Theorem* 6.4 *holds for every irreducible symmetric pair.*

This will be proved in **6.6** after some preparations.

6. 4. In the present discussed cases, for each singular plane $p=(\lambda, n)$ in $\mathfrak{t}^-$, we have $K_p=K_{(\lambda, 0)}$, independent of n, as is easily seen by Cor. 1.3 and **4.1**, i)-iii), so that we shall write it simply as K_λ.

Let $w\in W^-$ and n be a representative of w in $N_K(T_-)$. Denote by φ_n the conjugation of K with respect to n^{-1}. By an easy calculation we see that

$$\varphi_n(K_\lambda)=K_{w^*\lambda} \quad\text{and}\quad \varphi_n(K_{T_-})=K_{T_-}.$$

Then, passing to quotients we obtain homeomorphisms

$$\varphi_n' : K_\lambda/K_{T_-} \approxeq K_{w^*\lambda}/K_{T_-} \quad\text{and}\quad \varphi_n'' : K/K_{T_-} \approxeq K/K_{T_-}.$$

φ_n'' is homotopic to the action of w on K/K_{T_-} induced by right translation. If we change the representative n of w by another one n', then φ_n' and $\varphi_{n'}'$ is homotopic to each other; hence the induced homology map is determined only by w, denoted by φ_w. Therefore we obtain the commutativity of the following diagram

$$
(6.1)\qquad \begin{array}{ccc}
H_*(K/K_{T_-};Z) & \xrightarrow{w_*} & H_*(K/K_{T_-};Z) \\
\uparrow i_\lambda & & \uparrow i_{w^*\lambda} \\
H_*(K_\lambda/K_{T_-};Z) & \xrightarrow{\varphi_w} & H_*(K_{w^*\lambda}/K_{T_-};Z)
\end{array}
$$

for $w\in W^-$ and $\lambda\in\mathfrak{r}^-$, where w_* denotes the homology map induced by the action w on K/K_{T_-} and i_λ, $i_{w^*\lambda}$ those induced by natural inclusions.

We say that the set $\{K_\lambda/K_{T_-};\lambda\in\mathfrak{r}^-\}$ is *coherently oriented* when every K_λ/K_{T_-} is oriented in such a way that i) K_λ/K_{T_-} and $K_{-\lambda}/K_{T_-}$ are oppositely oriented and ii) φ_w is orientation preserving for all $w\in W^-$ and $\lambda\in\mathfrak{r}^-$.

PROPOSITION 6.6. *For every irreducible symmetric pair (G, K) of splitting rank with simply connected G, we can give a coherent orientation for the set $\{K_\lambda/K_{T_-};\lambda\in\mathfrak{r}^-\}$.*

Proof. i) *Group cases.* $\mathfrak{r}^-$ and W^- can be identified with the root system and Weyl group of K with respect to T_+. For each $\lambda\in\mathfrak{r}^-$ (considered as a root of K) we orient K/T_+ by the rule of [8], Chap. III, § 4, i.e., the image by the homology transgression of its fundamental class is τ_λ, considered as an element of $H_1(T_+;Z)$. The pair of maps

$$(\varphi_w, w^{-1}):(K_\lambda, T_+)\longrightarrow(K_{w^*\lambda}, T_+)$$

is a homorphism of principal bundles $(K_\lambda, K_\lambda/T_+, T_+)$ to $(K_{w^*\lambda}, K_{w^*\lambda}/T_+, T_+)$, and $w^{-1}\tau_\lambda=\tau_{w^*\lambda}$ for every $\{w, \lambda\}$, which show the proposition in this case.

ii) $(\mathbf{SU}(2n), \mathbf{Sp}(n))$. Express every element of $\mathbf{Sp}(n)$ by $n\times n$ unitary matrix of quaternions. The inclusion $\mathbf{Sp}(n)\subset\mathbf{SU}(2n)$ is interpreted as a map sending (s, t)-elements a_{st} of $A\in\mathbf{Sp}(n)$ to (s, t)-boxes of the forms

$$\begin{pmatrix} x_{st} & -\bar{y}_{st} \\ y_{st} & \bar{x}_{st}\end{pmatrix}$$

by partizing elements of $\mathbf{SU}(2n)$ into 2×2 boxes, where $a_{st}=x_{st}+j\cdot y_{st}$, x_{st} and y_{st} are complex numbers and j a usual quaternion unit.

Let T be the maximal torus of $\mathbf{SU}(2n)$ consisting of all diagonal matrices, and $T_+=T\cap\mathbf{Sp}(n)$. Every element H of the Cartan subalgebra $\mathfrak{t}$, tangential to T, is expressed as

$$H=(t_1, \cdots, t_{2n}),\ t_i\in\mathbf{R},\ t_1+\cdots+t_{2n}=0,$$

and

$$\exp H=\begin{pmatrix} e^{2\pi\sqrt{-1}t_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi\sqrt{-1}t_{2n}}\end{pmatrix}.$$

Then

(6.2) *$H\in\mathfrak{t}^+$ if and only if $t_{2i-1}=-t_{2i}$ for all $1\leqq i\leqq n$,*
as will be seen by the inclusion $T_+\subset T$. Consequently
(6.3) *$H\in\mathfrak{t}^-$ if and only if $t_{2i-1}=t_{2i}$ for all $1\leqq i\leqq n$,*
since $\mathfrak{t}^-$ is the orthogonal complement of $\mathfrak{t}^+$ and the invariant metric on $\mathfrak{t}$ is given by the quadratic form $t_1^2+\cdots+t_{2n}^2$.

Let $\omega_1, \cdots, \omega_{2n}$ be the weights of identity map representation of $\mathbf{SU}(2n)$ with respect to $\mathfrak{t}$, i.e.,

$$\omega_i(H)=t_i \quad \text{for all} \quad H\in\mathfrak{t} \quad \text{and} \quad 1\leqq i\leqq 2n.$$

Then $\mathfrak{r}=\{\omega_i-\omega_j\,;\, i\neq j\}$ and $\mathfrak{r}_0=\{\pm(\omega_{2i-1}-\omega_{2i}),\ 1\leqq i\leqq n\}$ as will be seen by (6.3).

Any linear order in $\mathfrak{t}^*$ satisfying

$$\omega_1>\omega_2>\cdots>\omega_{2n}$$

is a σ-order as will be seen by (6.2)-(6.3), and the σ-fundamental system Δ of $\mathfrak{r}$ with respect to this order is

$$\Delta=\{\omega_1-\omega_2,\ \omega_2-\omega_3,\ \cdots,\ \omega_{2n-1}-\omega_{2n}\},$$

and

$$\Delta_0=\{\omega_{2i-1}-\omega_{2i},\ 1\leqq i\leqq n\},$$
$$\Delta^-=\{\lambda_1,\ \cdots,\ \lambda_{n-1}\}$$

by putting $\lambda_i=\omega_{2i}-\omega_{2i+1}|\mathfrak{t}^-$. $\mathfrak{r}^-$ is of type A_{n-1}, and every positive root $\lambda\in\mathfrak{r}^-$ can be written as

$$\lambda=\lambda_i+\cdots+\lambda_j,\ 1\leqq i\leqq j\leqq n-1.$$

Then

$$\mathfrak{r}_\lambda=\{\omega_{2i}-\omega_{2j+1},\ \omega_{2i-1}-\omega_{2j+1},\ \omega_{2i}-\omega_{2j+2},\ \omega_{2i-1}-\omega_{2j+2}\}.$$

By an easy computation we see that K_{T_-} is the subgroup of $\mathbf{Sp}(n)$ consisting of all diagonal matrices. Let $Sp^i(1)$ donote 3-dimensional subgroup of $\mathbf{Sp}(n)$ consisting of diagonal matrices whose elements are all 1 except the i-th. Then

$$K_{T_-}\ =\ Sp^1(1)\times\cdots\times Sp^n(1).$$

For each (i, j), $1\leqq i<j\leqq n$, we denote by $Sp^{(i,j)}(2)$ the subgroup of $\mathbf{Sp}(n)$ consisting of such matrices that their matrix elements are the same as the unit matrix except the i-th and j-th rows and columns, which is isomorphic to $\mathbf{Sp}(2)$. Now by a short calculation we see that

$$K_\lambda=Sp^1(1)\times\cdots\hat{i}\cdots\widehat{j+1}\cdots\times Sp^n(1)\times Sp^{(i,j+1)}(2)$$

if $\pm\lambda=\lambda_i+\cdots+\lambda_j$, $1\leqq i\leqq j\leqq n-1$, where $\hat{i}$ means to omit the i-th factor. Furthermore

$$K(\lambda)=Sp^{(i,j+1)}(2) \quad \text{and} \quad K(\lambda)_{T_-}=Sp^i(1)\times Sp^{j+1}(1)$$

for $\pm\lambda=\lambda_i+\cdots+\lambda_j$, $1\leqq i\leqq j\leqq n-1$.

Choose an orientation of $\mathbf{Sp}(1)$ once and for all fixed. Since $\mathbf{Sp}(1)$ has no outer automorphism the group of all automorphisms of $\mathbf{Sp}(1)$ is connected, which means that every automorphism of $\mathbf{Sp}(1)$ is orientation preserving. Now we can orient $Sp^i(1)$, $1\leqq i\leqq n$, such that every isomorphism $\mathbf{Sp}(1)\cong Sp^i(1)$ is orientation preserving. Then every isomorphism $Sp^i(1)\cong Sp^j(1)$ is also orientation preserving by the same reason as above. Denote by s^i, $1\leqq i\leqq n$, the homology fundamental class of thus oriented $Sp^i(1)$.

For each $\lambda\in\mathfrak{r}^-$, $\pm\lambda=\lambda_i+\cdots\lambda_j$, denote by J_λ a matrix of $Sp^{(i,j+1)}(2)$ which has 0 as the (i, i)-th and $(j+1, j+1)$-th element and 1 as the $(i, j+1)$-th and $(j+1, i)$-th. Clearly

$$J_\lambda\in N_K(T).$$

By an easy computation we see that $(\mathrm{ad}\,J_\lambda)^*$ permutes ω_{2i-1} with ω_{2j+1} and ω_{2i} with ω_{2j+2}, and fixes all other weights, which implies that J_λ is a representative of R_λ, an element of W^- defined by the reflection across the plane $(\lambda, 0)$, in $N_K(T_-)$.

The conjugation by J_λ, denoted by $\tilde{J}_\lambda$, is involutive, mapping $Sp^i(1)$ isomorphic onto $Sp^{j+1}(1)$ and leaving $Sp^k(1)$ invariant for $k\notin\{i, j+1\}$. Hence

(6.4) $$\tilde{J}_{\lambda*}s^i=s^{j+1}, \quad \textit{and} \quad \tilde{J}_{\lambda*}s^k=s^k \quad \textit{for} \quad k\notin\{i, j+1\}$$

by our choice of orientations of $Sp^k(1)$ as above. On the other hand

$$\tilde{J}_{\lambda*}: H_*(Sp^{(i,j+1)}(2)\,; Z)\longrightarrow H_*(Sp^{(i,j+1)}(2)\,; Z)$$

is an identity map since $\tilde{J}_\lambda$ is an inner automorphism of the connected group $Sp^{(i,j+1)}(2)$. Thus

(6.5) $$k_{\lambda*}(s^i-s^{j+1})=0$$

where $k_\lambda: Sp^i(1)\times Sp^{j+1}(1)\subset Sp^{(i,j+1)}(2)$ is the inclusion.

Here we put

$Sp^{(i,j+1)}(2)/Sp^i(1)\times Sp^{j+1}(1)=S_4^\lambda$, 4-spheres, for $\lambda\in\mathfrak{r}^-$ such that $\pm\lambda=\lambda_i+\cdots+\lambda_j$. (6.5) implies that

(6.6′) $$\partial_*H_4(S_4^\lambda\,; Z)=\textit{the subgroup generated by } s^i-s^{j+1},$$

where ∂_* is the homology transgression of the bundle $Sp^{(i,j+1)}(2)\longrightarrow S_4^\lambda$. We shall orient S_4^λ such that its homology fundamental class, denoted by s_4^λ, satisfies

(6.6) $$\begin{aligned}\partial_*s_4^\lambda&=s^i-s^{j+1} \qquad \textit{if}\ \ \lambda>0\\ &=s^{j+1}-s^i \qquad \textit{if}\ \ \lambda<0.\end{aligned}$$

Thus $S_4^{\pm\lambda}$ are oppositely oriented. By the canonical homeomorphism

$$K_\lambda/K_{T_-}\approx K(\lambda)/K(\lambda)_{T_-}\approx S_4^\lambda,$$

we orient K_λ/K_{T_-} so that the above map becomes orientation preserving.

Now the pair of maps

$$(\tilde{J}_\lambda, \tilde{J}_\lambda): (K_\mu, K_{T_-})\longrightarrow (K_{R_\lambda\mu}, K_{T_-})$$

is a homomorphism of bundles $(K_\lambda, S_4^\mu, K_{T_-})$ to $(K_{R_\lambda\mu}, S_4^{R_\lambda\mu}, K_{T_-})$ for each λ, $\mu\in\mathfrak{r}^-$. By an easy discussion of the induced homomorphism of integral homology spectral sequences making use of (6.4) and (6.6), we see that the induced map of base spaces is orientation preserving. Thus the proposition was proved in case ii).

iii) (**Spin**$(2n)$, **Spin**$(2n-1)$). This is a symmetric pair of rank 1; $K=K_\lambda$ $=$**Spin**$(2n-1)$ for each $\lambda\in\mathfrak{r}^-$, $K_{T_-}=$**Spin**$(2n-2)$, and K/K_{T_-} is a $2(n$-$1)$-sphere.

In this case $W^-\cong Z_2$, and by Prop. 6.3 the operations of W^- on $H^{2n-2}(K/K_{T_-}; Z)\cong Z$ is non-trivial. Hence the operation of the generator of W^- on K/K_{T_-}

must be orientation reversing. Therefore, by the commutativity of (6.1)

$$\varphi_J : H_*(K_\lambda/K_{T_-}; Z) \approx H_*(K_{-\lambda}/K_{T_-}; Z)$$

is orientation reversing, where J is the generator of W^-, the reflection across the plane $(\lambda, 0)$, $\lambda \in \mathfrak{r}^-$. Thus, if we orient K_λ/K_{T_-} and $K_{-\lambda}/K_{T_-}$ oppositely, then the set $\{K_\lambda/K_{T_-}, K_{-\lambda}/K_{T_-}\}$ is coherently oriented.

iv) $(\mathbf{E}_6, \mathbf{F}_4)$. This is a symmetric pair of rank 2, whose restricted root system is of type A_2. For each $\lambda \in \mathfrak{r}^-$, $K_\lambda = K(\lambda) \cong \mathbf{Spin}(9)$, and $K_{T_-} \cong \mathbf{Spin}(8)$.

In this case $\mathfrak{r}^-$ contains 6 roots, and the operations of W^- permute the roots of $\mathfrak{r}^-$ transitively since the roots of $\mathfrak{r}^-$ have all the same length. Now the order of W^- is 6, which implies that W^- permutes the roots of $\mathfrak{r}^-$ simply transitively, i.e., for each pair $\{\lambda, \mu\} \subset \mathfrak{r}^-$, there is only one element $w \in W^-$ such that $w^*\lambda = \mu$. Hence, choosing a root $\lambda \in \mathfrak{r}^-$ we take and fix an orientation of K_λ/K_{T_-}; and then for each $\mu \in \mathfrak{r}^-$ take a unique $w \in W^-$ such that $w^*\lambda = \mu$ and define an orientation of K_μ/K_{T_-} so that

$$\varphi_w : K_\lambda/K_{T_-} \longrightarrow K_\mu/K_{T_-}$$

becomes orientation preserving. Thus we could define an orientation for each K_μ/K_{T_-} such that φ_w is orientation preserving for every $w \in W^-$ and $\mu \in \mathfrak{r}^-$.

Next, for every $\lambda \in \mathfrak{r}^-$, $G(\lambda) \cong \mathbf{Spin}(10)$, and the symmetric pair $(G(\lambda), K(\lambda))$ becomes isomorphic to the one of case iii) for $n=5$. Its restricted root system becomes the subsystem of $\mathfrak{r}^-$ consisting of $\pm\lambda$, and its restricted Weyl group becomes the subgroup of order 2 generated by the reflection across the plane $(\lambda, 0)$. By the discussion of case iii) we know that $\varphi_J : K(\lambda)/K_{T_-} \longrightarrow K(\lambda)/K_{T_-}$ is orientation reversing, where J is the element of W^-, defined as the reflection across the plane $(\lambda, 0)$. Therefore K_λ/K_{T_-} and $K_{-\lambda}/K_{T_-}$ is oppositely oriented for every $\lambda \in \mathfrak{r}^-$. And the set $\{K_\lambda/K_{T_-}, \lambda \in \mathfrak{r}^-\}$ is coherently oriented. q.e.d.

6. 5. Let (G, K) be an irreducible symmetric pair of splitting rank with simply connected G. We shall orient every K_λ/K_{T_-} coherently by Prop. 6.6.

For every $\lambda \in \mathfrak{r}^-$, consider the natural inclusion

$$i_\lambda : K_\lambda/K_{T_-} \subset K/K_{T_-}.$$

The image of the fundamental class of K_λ/K_{T_-} by $i_{\lambda *}$ defines an element of $H_{2m}(K/K_{T_-}; Z)$, denoted by $[[\lambda]]$. By the definition of coherent orientations

(6.7) $$-[[\lambda]] = [[-\lambda]],$$

and by the commutativity of (6.1)

(6.8) $$[[w^*\lambda]] = w_*[[\lambda]],$$

for each $w \in W_-$.

Choose a fundamental system $\Delta^- = \{\lambda_1, \cdots, \lambda_p\}$ of $\mathfrak{r}^-$. If we realize the additive basis of $H(K/K_{T_-}; Z)$, [8], Theorem VI and Cor. 2.13, p. 1022 (interpreted as K-orientable case by our Prop. 6.1), by cycles in K/K_{T_-} directly, then we see that

(6.9) $\{[[\lambda_1]], \cdots, [[\lambda_p]]\}$ *forms an additive base of* $H_{2m}(K/K_{T_-}; Z)$.

We denote basic translations in $\mathfrak{t}^-$ corresponding to λ_i by τ_i.

PROPOSITION 6.7. *For each* $\lambda\in\mathfrak{r}^{-}$, *express the basic translation* τ_{λ} *corresponding to* λ *as an integral linear combination*

$$\tau_{\lambda}=a_{1}\tau_{1}+\cdots+a_{p}\tau_{p}.$$

Then

$$[[\lambda]]=a_{1}[[\lambda_{1}]]+\cdots+a_{p}[[\lambda_{p}]].$$

Proof. It is enough to prove the proposition for a suitably chosen Δ^{-} since the Weyl group W^{-} permutes restricted fundamental systems transitively and we can apply (6.8), and for a suitably chosen coherent orientation since the change of coherent orientation changes $[[\lambda]]$ to its minus for all $\lambda\in\mathfrak{r}^{-}$ at the same time. So we use as Δ^{-} and the coherent orientation those used in the proof of Prop. 6.6.

i) *Group cases.* In the fibre bundle $(K, K/T_{+}, T_{+})$ the homology transgression

$$\partial_{*}: H_{2}(K/T_{+}\,; Z)\longrightarrow H_{1}(T_{+}\,; Z)$$

is bijective. And

$$\begin{aligned}\partial_{*}[[\lambda]]&=\tau_{\lambda}\quad\text{by our choice of orientations}\\&=\sum a_{i}\tau_{i}=\sum a_{i}\partial_{*}[[\lambda_{i}]],\end{aligned}$$

i.e.,
$$[[\lambda]]=a_{1}[[\lambda_{1}]]+\cdots+a_{p}[[\lambda_{p}]].$$

ii) $(\mathbf{SU}(2n), \mathbf{Sp}(n))$. In the bundle $(K, K/K_{T_{-}}, K_{T_{-}})$ the homology transgression

$$\partial_{*}: H_{4}(K/K_{T_{-}}\,; Z)\longrightarrow H_{3}(K_{T_{-}}\,; Z)$$

is injective. And, if $\lambda>0$ and $\lambda=\lambda_{i}+\cdots+\lambda_{j}$,

$$\begin{aligned}\partial_{*}[[\lambda]]&=\partial_{*}s_{4}^{\lambda}=s^{i}-s^{j+1}\\&=(s^{i}-s^{i+1})+(s^{i+1}-s^{i+2})+\cdots+(s^{j}-s^{j+1})\\&=\partial_{*}[[\lambda_{i}]]+\partial_{*}[[\lambda_{i+1}]]+\cdots+\partial_{*}[[\lambda_{j}]],\end{aligned}$$

i.e.,
$$[[\lambda]]=[[\lambda_{i}]]+\cdots[[\lambda_{j}]].$$

On the other hand

$$\tau_{\lambda}=\tau_{i}+\cdots+\tau_{j}$$

since all roots of $\mathfrak{r}^{-}$ have the same length. That is, Prop. 6.7 was proved in case ii) for $\lambda>0$. The case $\lambda<0$ can be also discussed in the same way.

iii) $(\mathbf{Spin}(2n), \mathbf{Spin}(2n-1))$. In this case $\mathfrak{r}^{-}=\{\lambda, -\lambda\}$, and (6.7) completes the proof.

iv) $(\mathbf{E}_{6}, \mathbf{F}_{4})$. Put $\Delta^{-}=\{\lambda_{1}, \lambda_{2}\}$. Then

$$\mathfrak{r}^{-}=\{\pm\lambda_{1}, \pm\lambda_{2}, \pm(\lambda_{1}+\lambda_{2})\}.$$

Put

$$\alpha=[[\lambda_{1}]]+[[\lambda_{2}]]-[[\lambda_{1}+\lambda_{2}]],$$

and apply every operation of W^{-} to α. Then, by making use of (6.7) and (6.8), we see that the set $\{\alpha, -\alpha\}$ is closed by the operations of W^{-}. Hence, if $\alpha\neq 0$, α generates a one dimensional W^{-}-invariant subspace of the 2 dimensional space $H_{8}(\mathbf{F}_{4}/\mathbf{Spin}(8)\,;\mathbf{R})$. Now $H_{*}(\mathbf{F}_{4}/\mathbf{Spin}(8)\,;\mathbf{R})$ is the space of the regular representation of W^{-} by Prop. 6.3 and the representation of W^{-} on $H_{8}(\mathbf{F}_{4}/\mathbf{Spin}(8)\,;\mathbf{R})$ is one of the irreducible components of the regular representation of W^{-}. (Cf., also [5], p. 333.) Therefore $\alpha=0$. That is,

$$[[\lambda_1]]+[[\lambda_2]]=[[\lambda_1+\lambda_2]],$$

which completes the proof in case iv). q.e.d.

The above proposition implies that, if there holds a linear equation among τ_λ, then there holds a linear equation among $[[\lambda]]$ with the same coefficients. In particular,

COROLLARY 6.8. *For any λ, $\mu\in\mathfrak{r}^-$, there holds the equality*

$$[[R_\mu^*\lambda]]=[[\lambda]]-\mu(\tau_\lambda)[[\mu]],$$

where R_μ denotes the reflection across the plane $(\mu, 0)$.

6. 6. *Proof of Proposition* 6.5. By Prop. 2.9 it is sufficient only to prove the relations ρ_k, $1\leqq k\leqq n$. Choose a coherent orientation for the set $\{K_\lambda/K_{T_-}; \lambda\in\mathfrak{r}^-\}$. Then, for every K-cycle Γ_P, $P=\{p_1, \cdots, p_n\}$ and $p_i=(\lambda_i, n_i)$, the basis $[1], \cdots, [n]$ of $H_{2m}(\Gamma_P; Z)$ is well defined, and also its dual basis $x_1, \cdots, x_n$. Furthermore, for every sub-K-cycle $\Gamma_{P'}$, $P'=\{p_{i_1}, \cdots, p_{i_r}\}$, of Γ_P,

$$\bar{i}_*[k]'=[i_k] \qquad \text{for } 1\leqq k\leqq r,$$

and

$$\begin{aligned} \bar{i}^*x_j&=x'_t && \text{if } j=i_t \\ &=0 && \text{if } j\notin\{i_1, \cdots, i_r\}, \end{aligned}$$

where $\bar{i}: \Gamma_{P'}\longrightarrow\Gamma_P$ is the natural inclusion of **2.7** and the $2m$-dimensional basis elements of homology and cohomology of $\Gamma_{P'}$ are expressed with $'$ added.

Thus, if we prove Prop. 6.5 for every K-cycle $\Gamma_{P''}$ with P'' of length 2, then we can see that Prop. 6.5 is true for every K-cycle Γ_P by evaluating x_k^2, $1\leqq k\leqq n$, on each sub-K-cycle of dimension $4m$.

Now we shall consider a K-cycle Γ_P with $P=\{p_1, p_2\}$, $p_1=(\mu, n_1)$ and $p_2=(\nu, n_2)$.

Since $x_1\in\pi^*H_{2m}(K_\mu/K_{T_-}; Z)$ where $\pi:\Gamma_P\longrightarrow K_\mu/K_{T_-}$ is the projection, it follows that

$$x_1^2=0.$$

To prove the relation ρ_2 we proceed the more or less parallel way to the corresponding proof of [8], Chap. III, § 5. First we remark that

$$(G(\nu), K(\nu))\cong(\mathbf{Spin}(2m+2), \mathbf{Spin}(2m+1))$$

as symmetric pairs, and its restricted Weyl group can be identified with a subgroup of W^-, generated by R_ν, the reflection across the plane $(\nu, 0)$. Then we can choose a representative j of R_ν in $N_K(T_-)\cap K(\nu)$. Let $\bar{J}: W_P\longrightarrow W_P$ be the map sending (y_1, y_2) to $(y_1, y_2 j)$. This is a homomorphism of the bundle $W_P\longrightarrow\Gamma_P$ into itself relative to the homomorphism $\tilde{J}: (K_{T_-})^2\longrightarrow(K_{T_-})^2$, defined by $\tilde{J}(k_1, k_2)=(k_1, j^{-1}k_2 j)$. The induced map of Γ_P into itself is denoted by J. Since $j\in K_\nu$ by our choice and K_ν is connected, J map the sub-K-cycle K_ν/K_{T_-} into itself and $J|K_\nu/K_{T_-}$ is homotopic to φ'_j, the map defined at the biginning of **6.4.** Thus $J|K_\nu/K_{T_-}$ is orientation reversing, i.e., (6.10) $$J_*[2] = -[2].$$

Next we discuss $J_*[1]$. The map $\bar{\rho}$: $W_P \to K$, defined by $\bar{\rho}(y_1, y_2) = y_1 y_2$, induces a map $\rho : \Gamma_P \longrightarrow K/K_{T_-}$, and

(6.11) $$\rho_*[1] = [[\mu]],\ \rho_*[2] = [[\nu]].$$

Since $\rho \circ J = \varphi''_j \circ \rho$ evidently, where $\varphi''_j : K/K_{T_-} \longrightarrow K/K_{T_-}$ is the map defined by the right translation by j, we obtain

$$\rho_* J_*[1] = R_{\nu *}[[\mu]] = [[\mu]] - \nu(\tau_\mu)[[\nu]]$$

by (6.11), Cor. 6.8 and (6.8). Thus

$$\rho_* J_*[1] = \rho_*([1] - \nu(\tau_\mu)[2]).$$

If $\mu \neq \pm\nu$, then μ and ν are linear independent, and ρ_* is injective in degree $2m$. Therefore we obtain

(6.12) $$J_*[1] = [1] - \nu(\tau_\mu)[2].$$

In case $\mu = \pm\nu$, the map $\bar{\xi}$: $W_P \longrightarrow W_P$, defined by $\bar{\xi}(y_1, y_2) = (y_1, y_1 y_2)$, induces a homeomorphism

$$\xi : \Gamma_P \approx \Gamma'_P$$

where $\Gamma'_P = (K_\nu/K_{T_-}) \times (K_\mu/K_{T_-})$, the direct product. Denote the elements of $H_{2m}(\Gamma'_P\,; Z)$, represented by the first and the second factors as oriented ones, by $[1]'$ and $[2]'$ respectively. Then it is easy to see that

$$\xi_*[1] = [1]' \pm [2]' \quad \text{and} \quad \xi_*[2] = [2]',$$

where the sign $\pm$ coincides with the sign of $\mu = \pm\nu$. Let j operate on Γ'_P as a right translation of the second factor. The obtained map we denote by ψ. Then

$$\psi_*[1]' = [1]' \quad \text{and} \quad \psi_*[2]' = -[2]'.$$

And evidently $J = \xi^{-1} \circ \psi \circ \xi$. Therefore

$$\begin{aligned} J_*[1] &= \xi_*^{-1}([1]' \mp [2]') \\ &= [1] \mp 2[2] = [1] - \nu(\tau_\mu)[2], \end{aligned}$$

i.e., (6.12) holds also for the case $\mu = \pm\nu$.

Now by the same way as in [8], p. 999, we see that

$$x_2 \cdot J^* x_2 = 0.$$

And, (6.10) and (6.12) implies that

$$J^* x_2 = -x_2 - \nu(\tau_\mu) x_1\,;$$

consequently $$x_2(x_2 + \nu(\tau_\mu) x_1) = 0.$$ q.e.d.

§7. Proof of Theorem 2.10.

7. 1. This section is directed to the proof of Theorem 2.10. Hence we assume that G is simply connected throughout the section. The only task is to prove the relations ρ_k for $1 \leqq k \leqq n$.

Let $P = \{p_1, \cdots, p_n\}$ be a sequence of singular planes in $\mathfrak{t}^-$, and put $P'' = \{p_1, \cdots, p_k\}$, $k \leqq n$. $(\Gamma_P, \Gamma_{P''}, \pi)$ is a fibre bundle with a cross section, where $\pi : \Gamma_P \longrightarrow \Gamma_{P''}$, the projection of the bundle, is a map obtained by dropping off the last $(n-k)$ factors. Using cohomology (mod 2) bases (2.14) for Γ_P and $\Gamma_{P''}$, and considering their relations with respect to π^*, we see that the relation ρ_k to describe x_k^2 as the linear combination of basis elements (2.14) of $H^*(\Gamma_P; Z_2)$ is obtained as the π^*-image of the corresponding one for $\Gamma_{P''}$. In particular, x_k^2 is described as a linear combination of $x_{i_1}\cdots x_{i_s}$ such that $1\leqq i_1<\cdots<i_s\leqq k$.

On the other hand, for any $x_{i_1}\cdots, x_{i_s}$ such that $1\leqq i_1<\cdots i_s<k$, its restriction to the sub-K-cycle $\Gamma_{P'''}$, $P'''=\{p_{i_1},\cdots,p_{i_s}\}$, is non-zero, and the restriction of x_k^2 to $\Gamma_{P'''}$ is zero since $x_k|\Gamma_{P'''}$ is zero. Hence $x_{i_1}\cdots x_{i_s}$, $1\leqq i_1<\cdots<i_s<k$, do not appear in ρ_k. Thus we obtain

Lemma 7.1 *x_k^2 is expressed as a linear combination of*

$$x_{i_1}\cdots x_{i_s}x_k$$

such that $1\leqq i_1<\cdots<i_s<k$ *and* $\deg(x_{i_1}\cdots x_{i_s})=\deg x_k$. *In particular*

$$x_1^2 = 0.$$

Next we state the following

LEMMA 7.2 *For any K-cycle Γ_P, $P=\{p_1,\cdots,p_r\}$, associated with an irreducible symmetric pair (G, K) such that*

$$m(p_1)+\cdots+m(p_{r-1})=m(p_r),$$

the relation ρ_r holds in the same form as that of Theorem 2.10.

Once were proved Lemma 7.2, then Theorem 2.10 would hold for every K-cycle Γ_P associated with irreducible symmetric pairs as is easily seen by evaluating the values of ρ_i, $1\leqq i\leqq n=$the length of P, on each sub-K-cycle of Γ_P of dimension $2m(p_i)$ using Lemmas 7.1 and 7.2. And then Theorem 2.10 is proved in its full generality by Prop. 5.8.

Proof of Lemma 7.2. We shall divide our discussions into five cases: A) $r=2$ and $m(p_2)=1$; B) $r\geqq 3$ and $m(p_1)=\cdots=m(p_{r-1})=1$; C) $r\geqq 3$ and $m(p_i)>1$ for at least one i, $1\leqq i<r$; D) $r=2$ and $m(p_2)$ odd>1; E) $r=2$ and $m(p_2)$ even. We put $p_i=(\lambda_i, n_i)$, $\lambda_i\in\mathfrak{r}^{-''}$, $\lambda_i=\varepsilon_i\lambda_i'$, $\varepsilon_i=1$ or 2, and $\lambda_i'\in\mathfrak{r}^{-'}$ for $1\leqq i\leqq r$, and $P'=\{p_1,\cdots,p_{r-1}\}$.

7. 2. *Case A*). In this case, using notations of **5.7**, $G(i)$ is a 3-sphere and $K(i)$ is a circle $\{\exp tU_i, t\in\mathbf{R}\}$ for $i=1, 2$, where $\{U_i, V_i\}$ is an ortho-normal basis of $\mathfrak{e}_{\lambda_i}$ such that $\sigma U_i=U_i$ and $\sigma V_i=-V_i$ (by (1.5)). Furthermore $K_0^{(1)}\cong Z_2$ generated by $\exp(\tau_{\lambda_1}/2)$ (cf., also Theorem 4.7). By Props. 2.6 and 5.14 the structure group of the circle bundle $(\Gamma_P, \Gamma_{P'}, S^1)$ is reducible to $\iota_2(K_0^{(1)})$, where ι_2 is the isotropy representation of K_2/K_0.

Now, since $\mathrm{ad}(\exp(\tau_{\lambda_1}/2))|\mathfrak{e}_{\lambda_2}$ is a rotation through the angle $\pi\lambda_2(\tau_{\lambda_1})$ in $\mathfrak{e}_{\lambda_2}$, we see that

$$\mathrm{ad}(\exp(\tau_{\lambda_1}/2))\cdot U_2=U_2 \text{ or } -U_2$$

according as the Cartan integer $a_{21}=\lambda_2(\tau_{\lambda_1})$ is even or odd. That is, $\iota_2(K_0^{(1)})$ is trivial or non trivial, and hence the bundle $(\Gamma_P, \Gamma_{P'}, S^1)$ is orientable or not according as a_{21} is even or odd.

As is well known the first whitney class w_1 of the bundle $(\Gamma_P, \Gamma_{P'}, S^1)$ is zero or non-zero according as the bundle is orientable or not, (e.g., cf., [14], p. 197). And by Massey [11], p. 274, Theorem III (which is true also for non-orientable sphere bundles and their cohomology mod 2 as is easily seen from his proof, through it is stated for orientable sphere bundles),

$$x_2^2 = \pi^*(w_1)\cdot x_2$$

where $\pi : \Gamma_P \longrightarrow \Gamma_{P'}$ is the projection of the bundle. Therefore

$$x_2^2 = b_{21}x_2x_1, \tag{7.1}$$

which proves Lemma 7.2 in case A).

7. 3. *Case B*). Similarly to the above case $K(i)_0 \cong Z_2$ with generators $\exp(\tau_{\lambda_i}/2)$ for $1 \leqq i \leqq r-1$. And $K_0^{(r-1)} = K(1)_0 \cdots K(r-1)_0$ is a finite group generated by $\exp(\tau_{\lambda_i}/2)$, $1 \leqq i \leqq r-1$. By Props. 2.6 and 5.14 the structure group of the sphere bundle $(\Gamma_P, \Gamma_{P'}, S^{m(p_r)})$ is reducible to $\iota_r(K_0^{(r-1)})$, where ι_r is the isotropy representation of K_r/K_0.

Using the bases (1.5) of $\tilde{e}_{\lambda'_r}$ and $\tilde{e}_{2\lambda'_r}$ we see that

$$\mathrm{ad}(\exp(\tau_{\lambda_i}/2))\,|\,\tilde{e}_{\lambda'_r} = (-1)^{a'_{ri}} \text{ identity map}$$

for $1 \leqq i \leqq r-1$, where $a'_{ri} = \lambda'_r(\tau_{\lambda_i})$ as in the above case, and that

$$\mathrm{ad}(\exp(\tau_{\lambda_i}/2))\,|\,\tilde{e}_{2\lambda'_r} = \text{identity map}$$

since $2\lambda'_r(\tau_{\lambda_i})$ is even always, which implies firstly that,

i) if $m(2\lambda'_r) \neq 0$, then the structure group of the sphere bundle $(\Gamma_P, \Gamma_{P'}, S^{m(p_r)})$ can be further reduced to $\mathbf{O}(m(p_r)-1)$ and $m(p_r)$-th Whitney class $W_{m(p_r)}$ (mod 2) vanishes. Then, by [11], Theorem III,

$$x_r^2 = \pi^*(w_{m(p_r)})\cdot x_r = 0. \tag{7.2}$$

Now this case i) is possible only for the irreducible symmetric pairs with the following types of infinitesimal structures: AIII, AIV, DIII, EIII, as will be seen from [2], the table. Furthermore, in each possible symmetric pair, all roots of odd multiplicities, up to signs, are mutually orthogonal. In particular $a_{ri} = \lambda_r(\tau_{\lambda_j}) = 0$ or ± 2, i.e.,

$$c_{ri} = 0 \tag{7.3}$$

for $1 \leqq i \leqq r-1$. Thus, by (7.2)-(7.3), Lemma 7.2 was proved in this case B)i).

ii) If $m(2\lambda'_r) = 0$, then $\lambda_r = \lambda'_r$, and the vector bundle, associated with reduced $S^{m(p_r)-1}$-bundle over $\Gamma_{P'}$ (by the canonical cross-section $\nu : \Gamma_{P'} \longrightarrow \Gamma_P$), splits as a Whitney sum of $m(p_r)$ copies of a real line bundle with the following actions of $K_0^{(r-1)}$ on $\mathbf{R}$:

$$\exp(\tau_{\lambda_i}/2)\cdot t = (-1)^{a_{ri}}t, \ t \in \mathbf{R}.$$

for $1 \leqq i \leqq r-1$, where $a_{ri} = \lambda_r(\tau_{\lambda_i})$. Denote this line bundle by γ, and its first Whitney class by w_1. Then $m(p_r)$-th Whitney class $w_{m(p_r)}$ of the sphere bundle $(\Gamma_P, \Gamma_{P'}, S^{m(p_r)})$ is

$$w_{m(p_r)} = (w_1)^{m(p_r)} \tag{7.4}$$

by the Whitney duality theorem. On the other hand, consider the restriction of γ on each K_i/K_0 regarded as a sub-K-cycle of $\Gamma_{P'}$, of which the structure group is reduced to $K(i)_0$ with the operation

$$\exp(\tau_{\lambda_i}/2)\cdot t = (-1)^{a_{ri}}t,\ t\in\mathbf{R}.$$

Therefrom we conclude as in case A) that

$$w_1 = \sum_{i=1}^{r-1} c_{ri}x_i', \tag{7.5}$$

where x_i' are the basis elements (2.14) of $H^*(\Gamma_{P'}; Z_2)$. Thus, by [11], Theorem III, and the fact that $\pi^*x_i'=x_i$, (7.4)-(7.5) implies

$$x_r^2 = x_r(\sum_{i=1}^{r-1} c_{ri}x_i)^{m(p_r)}, \tag{7.6}$$

which proves Lemma 7.2 in case B)ii).

7. 4. *Case C*). This case is possible only for such an irreducible symmetric pair that $m(p)$ may take three different values. Therefore by [2], the table, its type must be either one of the following fours: AIII, CII, DIII and EIII. Then, by Cor. 5.12 we see that every K-cycle Γ_P of case C) is totally orientable and consequently that $H^*(\Gamma_P; Z)$ has no torsion; on the other hand we see also that $m(p_r)$ is odd, because it is the largest multiplicity and hence $\varepsilon_r=2$. From these two facts we conclude immediately that

$$x_r^2 = 0. \tag{7.7}$$

Now the result of case A) implies that for every 1-class $w\in H^1(\Gamma_P; Z)$ its t-th power w^t can be expressed as a linear combination of $x_{i_1}\cdots x_{i_t}$, $1\leqq i_1<\cdots<i_t<r$, such that deg $x_{i_s}=1$ for $1\leqq s\leqq t$, which means, in particular, that $w^{m(p_r)}=0$ since the number of singular planes of multiplicity 1 in P is smaller than $m(p_r)$ by our assumption, whence we have

$$\rho_r = x_r^2. \tag{7.8}$$

By (7.7)-(7.8) was proved Lemma 7.2 in case C).

7. 5. *Case D*). First we remark that, in the present case, $K_0^{(1)}=K(1)_0$ is connected by Theorem 3.5 applied to the pair $(G(1), K(1))$. Since the structure group of the bundle $(\Gamma_P, \Gamma_{P'}, S^{m(p_2)})$ is reducible to the connected group $\iota(K_0^{(1)})$ by Props. 2.6 and 5.14, we see that Γ_P is orientable and $H^*(\Gamma_P; Z)$ has no torsion. On the other hand, deg x_2 is odd by the assumptions. Therefore we see that

$$x_2^2 = 0 \tag{7.9}$$

as (7.7).

The case D) is possible only for the following types of irreducible symmetric pairs: AIII, AIV, BI, BII, CII, DIII, EIII, and FII. In either case $\mathfrak{r}^{-\prime\prime}$ is of type C_l or B_l (doubly laced) except the case of restricted rank 1; and λ_i, $i=1$ and 2, are long roots of $\mathfrak{r}^{-\prime\prime}$ if it is of type C_l, and are short ones otherwise. Therefore

$$a_{21}=0 \quad\text{or}\quad \pm 2 \tag{7.10}$$

From (7.9)-(7.10) follows Lemma 7.2 in case D).

7. 6. *Case E*). By the assumption of case E), $\lambda_i\in\mathfrak{r}^{-\prime}$ for $i=1$ and 2.

i) If $\lambda_1=\pm\lambda_2$, then

$$(G(1), K(1))=(G(2), K(2)) \tag{7.11}$$

is a symmetric pair of splitting rank. And we can regard Γ_P as a K-cycle of the pair (7.11) by natural homeomorphisms and identifications. Then, we can apply Theorem 6.4 to Γ_P. Since $a_{21}=\pm 2$, by reducing mod 2 the integral relation ρ_2 we obtain the proof for case E)i).

ii) In case $\lambda_1 \neq \pm\lambda_2$, we shall reduce our discussion to the case of rank$(G, K) = 2$. Denoting dim $\mathfrak{t}^-$ by p, choose a basis $\{H_1, \cdots, H_p\}$ of $\mathfrak{t}^-$ so as to satisfy

$$\{H_1, \cdots, H_{p-2}\} \subset (\lambda_1, 0) \cap (\lambda_2, 0).$$

The lexicographic order with respect to the basis $\{H_1, \cdots, H_p\}$ defines a fundamental system Δ^- of $\mathfrak{r}^-$. By our definition and assumption Δ^- contains two simple roots, denoted by μ_1 and μ_2, such that λ_i is a linear combination of μ_1 and μ_2 for $i=1, 2$. Let Δ be a σ-fundamental system of $\mathfrak{r}$ such that its restricted fundamental system becomes Δ^-. By notations of **4.2** $\bar{\mathfrak{r}}_{\mu_i}$ is the root system of $G(\mu_i)$, $i=1, 2$. And $\Delta^{\mu_i}=\Delta \cap \bar{\mathfrak{r}}_{\mu_i}$ is the σ-fundamental system of $\bar{\mathfrak{r}}_{\mu_i}$ by [2], Prop. 3.4.

Let $\mathfrak{s}$ denote the subsystem of $\mathfrak{r}$ generated by $\Delta^{\mu_1} \cup \Delta^{\mu_2}$, i.e., the set of all roots of $\mathfrak{r}$ which can be expressible as linear combinations of roots of $\Delta^{\mu_1} \cup \Delta^{\mu_2}$. Clearly $\mathfrak{s}$ is a σ-system of roots with induced involution, and has $\Delta^{\mu_1} \cup \Delta^{\mu_2}$ as its σ-fundamental system and hence the set $\{\mu_1, \mu_2\}$ as its restricted fundamental system.

Let G' denote the semi-simple part of the centralizer in G of the intersections of all planes $(\alpha, 0)$ such that $\alpha \in \mathfrak{s}$. Since G' has $\Delta^{\mu_1} \cup \Delta^{\mu_2}$ as its fundamental system of roots which is a part of Δ, we see that G' is simply connected. G' is clearly σ-invariant and the pair (G', K') with the induced involution, where $K'=K \cap G'$, is a symmetric pair of restricted rank 2 with $\{\mu_1, \mu_2\}$ as its restricted fundamental system.

Considering $\mathfrak{s}^-$, the restricted root system of (G', K'), as a subsystem of $\mathfrak{r}^-$, we see easily that

$$(G(\nu), K(\nu))=(G'(\nu), K'(\nu))$$

for each $\nu \in \mathfrak{s}^-$, which proves diffeomorphisms

$$K'_{(\nu, n)}/K'_{T'_-} \approx K_{(\nu, n)}/K_{T_-}$$

induced by natural inclusions for all $\nu \in \mathfrak{s}^-$ and n integer *via* (4.6)-(4.7), where T'_- is the maximal torus of the pair (G', K') contained in T_-, which in turn defines natural isomorphism

$$\Gamma_P \cong K'_{p_1} \times_{K'_{T'_-}} (K'_{p_2}/K'_{T'_-}).$$

Thus we can regard Γ_P as a K-cycle associated with the pair (G', K').

Therefore it becomes sufficient to prove Lemma 7.2 in case E)ii) under the assumption that rank$(G, K) = 2$ (where (G, K) is not always irreducible), so we assume it hereafter.

a) If the pair (G, K) is of splitting rank, then we can apply Theorem 6.4 to Γ_P, and by reducing mod 2 the integral relation ρ_2, we obtain the desired proof.

Here we remark that, if (G, K) is reducible, then it is necessarily of splitting rank since each irreducible factor is isomorphic to $(\mathbf{Spin}(2m+2), \mathbf{Spin}(2m+1))$

as symmetric pairs by putting $m(p_i)=2m$.

b) Symmetric pairs of restricted rank 2, not of splitting rank and having Γ_P of case E)ii), are as follows (cf., [2], the table): AIII ($l\geqq 3$, $p=2$), CII ($l\geqq 4$, $p=2$), DI ($l\geqq 4$, $p=2$), DIII ($l=4$, 5) and EIII, where l = rank G and p = rank (G, K). For each symmetric pair listed here, its root system $\mathfrak{r}^{-\prime\prime}$ is of type B_2; furthermore long roots of $\mathfrak{r}^{-\prime\prime}$ have odd multiplicites and short roots have the same even multiplicities, say $2m$. Thus $\lambda_i, i=1, 2$, must be short roots of $\mathfrak{r}^{-\prime\prime}$ since $m(p_i)=m(\lambda_i)$ even. Then, since short roots up to signs are mutually orthogonal, the assumption $\lambda_1\neq\pm\lambda_2$ implies

$$a_{21} = 0. \tag{7.12}$$

As one of the properties of root systems of type B_2, there exists a long root λ' of $\mathfrak{r}^{-\prime\prime}$ such that

$$\lambda_2 = \lambda_1+\lambda'.$$

Then $<\lambda_1, \lambda'><0$. Now we put

$$\mathfrak{r}_{\lambda_i}=\{\alpha_1^i, \cdots, \alpha_m^i, \sigma^*\alpha_1^i, \cdots, \sigma^*\alpha_m^i\}$$

for $i=1, 2$. Since $m(\lambda')$ is odd, $\lambda'\in\mathfrak{r}$ and, for each j, $1\leqq j\leqq m$, $<\alpha_j^1, \lambda'>=<\lambda_1, \lambda'><0$, i.e., $\alpha_j^1+\lambda'\in\mathfrak{r}_{\lambda_2}$. Thus we can choose α_j^2, $1\leqq j\leqq m$, so that $\alpha_j^2=\alpha_j^1+\lambda'$. Then $\sigma^*\alpha_j^2=\sigma^*\alpha_j^1+\lambda'$. Therefore

$$\alpha_j^1-\sigma^*\alpha_j^1 = \alpha_j^2-\sigma^*\alpha_j^2 \qquad \text{for } 1\leqq j\leqq m. \tag{7.13}$$

Consider $\mathfrak{g}(\lambda_i)$. Its root system $\bar{\mathfrak{r}}_{\lambda_i}$ is decomposed as

$$\bar{\mathfrak{r}}_{\lambda_i}=(\mathfrak{r}_0\cap\bar{\mathfrak{r}}_{\lambda_i})\cup\mathfrak{r}_{\lambda_i}\cup\mathfrak{r}_{-\lambda_i}.$$

Hence its Cartan subalgebra $\mathfrak{t}(\lambda_i)$, contained in $\mathfrak{t}$ of $\mathfrak{g}$, is generated by

$$\{\tau_\gamma\ ;\ \gamma\in\mathfrak{r}_0\cap\bar{\mathfrak{r}}_{\lambda_i}\}\cup\{\tau_{\alpha_1^i}, \cdots, \tau_{\alpha_m^i}, \sigma\tau_{\alpha_1^i}, \cdots, \sigma\tau_{\alpha_m^i}\}.$$

And $\mathfrak{t}(\lambda_i)^+ = \mathfrak{t}(\lambda_i)\cap\mathfrak{k}$ is generated by

$$\{\tau_\gamma, \gamma\in\mathfrak{r}_0\cap\bar{\mathfrak{r}}_{\lambda_i}\}\cup\{\tau_{\alpha_j^i-\sigma^*\alpha_j^i}, 1\leqq j\leqq m\}. \tag{7.14}$$

Next we show

$$\mathfrak{r}_0\cap\bar{\mathfrak{r}}_{\lambda_1}=\mathfrak{r}_0\cap\bar{\mathfrak{r}}_{\lambda_2}. \tag{7.15}$$

Let $\gamma\in\mathfrak{r}_0\cap\bar{\mathfrak{r}}_{\lambda_1}$. By [2], Lemma 4.1, there exists an element $\alpha\in\mathfrak{r}_{\lambda_1}$ such that $<\gamma, \alpha>\neq 0$. Then $<\gamma, \alpha+\lambda'>\neq 0$ since $<\gamma, \lambda'>=0$, and $\alpha+\lambda'\in\mathfrak{r}_{\lambda_2}$, i.e., $\gamma\in\mathfrak{r}_0\cap\bar{\mathfrak{r}}_{\lambda_2}$, and *vice versa*.

Now

$$\mathfrak{k}(\lambda_i)_{T_-} = \mathfrak{t}(\lambda_i)^+ + \sum\mathfrak{e}_\gamma,$$

where the summation runs over all $\gamma\in\mathfrak{r}_0\cap\bar{\mathfrak{r}}_{\lambda_i}$. Then (7.13), (7.14) and (7.15) imply that $\mathfrak{k}(\lambda_1)_{T_-}=\mathfrak{k}(\lambda_2)_{T_-}$, i.e.,

$$K(\lambda_1)_{T_-} = K(\lambda_2)_{T_-}. \tag{7.16}$$

Let n be an element of $N_K(T_-)$ representing $R_{\lambda'}$ of W^-. (7.16) implies that the conjugation φ_n' with respect to n^{-1}, gives an equivalence between two representations $\iota_1|K(\lambda_1)_{T_-}$, and $\iota_2|K(\lambda_1)_{T_-}$ where ι_i is the isotropy representation of homogeneous space K_{p_i}/K_{T_-} for each $i=1, 2$. Therefore by Prop. 5.14 we see that the principal bundle associated with the bundle $(\Gamma_P, \Gamma_{P'}, S^{m(p_2)})$ is equivalent to ι_1-extension of the bundle $(K_{p_1}, K_{p_1}/K_{T_-}, K_{T_-})$, which is in turn equivalent to the principal tangent bundle of K_{p_1}/K_{T_-} $(\approx S^{m(p_1)})$ by [6], p. 481. Thus the $m(p_2)$-th Whitney class $w^{m(p_2)}$ of the bundle $(\Gamma_P, \Gamma_{P'}, S^{m(p_2)})$ vanishes as a mod 2 class. Hence, by [11], Theorem III,

$$x_2^2 = \pi^*(w^{m(p_2)})x_2 = 0. \tag{7.17}$$

(7.12) and (7.17) prove Lemma 7.2 for the case E)ii)b). q.e.d.

References

[1] S. Araki, *Cohomology modulo 2 of the compact exceptional groups E_6 and E_7*, Jour. of Math., Osaka City Univ., **12** (1961), 43–65.

[2] ——, *On root systems and an infinitesimal classification of irreducible symmetric spaces*, Jour. of Math., Osaka City Univ., **13** (1962), 1–34.

[3] S. Araki and Y. Shikata, *Cohomology mod 2 of the compact exceptional group* E_8, Proc. of Japan Acad., **37** (1961), 619–622.

[4] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math., **57** (1953), 115–207.

[5] ——, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math., **76** (1954), 273–342.

[6] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*. I, Amer. J. Math., **80** (1958), 458–538.

[7] R. Bott, *The stable homotopy of the classical groups*, Ann. of Math., **70** (1959), 313–337.

[8] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math., **80** (1958), 964–1029.

[9] E. Cartan, *Sur certaines formes riemanniennes remarquables* des *géométries à groupe fondamental simple*, Ann. Ecole Norm. Sup., **44** (1927), 345–467.

[10] J. Leray, *Sur l'homologie des groupes de Lie, des espaces homogènes et des espaces fibrés principaux*, Colloque de Topologie (espaces fibrés), Bruxelles, 1950, 101–115.

[11] W.S. Massey, *On the cohomology ring of a sphere bundle*, Jour. of Math. and Mech., **7** (1958), 265–289.

[12] I. Satake, *On representations and compactifications of symmetric Riemannian spaces*, Ann. of Math., **71** (1960), 77–110.

[13] J.-P. Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math., **54** (1951), 425–505.

[14] N.E. Steenrod, The topology of fibre bundles, Princeton Univ. Press, 1951.

[15] E. Stiefel, *Ueber eine Beziehung zwischen geschlossenen Lie'schen Gruppen und diskontinuierlichen Bewegungsgruppen euklidischer Räume und ihre Anwendung auf die Aufzählung der einfachen Lie'schen Gruppen*, Comment. Math. Helv., **14** (1941), 350–380.

[16] R. Thom, *Espaces fibrés en sphères et carrés de Steenrod*, Ann. Ecole Norm. Sup., **69** (1952), 109–182.