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On Pontrjagin Classes I

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Introduction

In a series of papers which starts with the present one, we are concerned with the problem of invariance of the Pontrjagin classes of differentiable manifolds. We prove first, in the present paper, a main theorem whose application will be made in the sequel. The main theorem is stated in terms of Riemannian metrics of manifolds.

Throughout the present paper, a compact orientable n-dimensional differentiable manifold M is fixed. Given a differentiable structure $\mathcal{D}$ on M, we denote by $p(\mathcal{D})$ the Pontrjagin classes of $(M, \mathcal{D})$. For two differetiable structures $\mathcal{D}$, $\mathcal{D}'$ on M, we write $\mathcal{D}\sim\mathcal{D}'$ if there are a metric function d on M, d' on M and an onto-homeomorphism $h: M\to M$ such that

i) d, d' are induced from Riemannian metrics g, g' on $(M, \mathcal{D})$, $(M, \mathcal{D}')$ respectively.

ii) with $1\leqq s<3$ we have

$$d^2(x, y)/s\leqq d'^2(h(x), h(y))\leqq sd^2(x, y)$$

for any $(x, y)\in U$, a neighborhood of the diagonal of $M\times M$.

MAIN THEOREM *If* $\mathcal{D}\sim\mathcal{D}'$, *then* $p(\mathcal{D})=p(\mathcal{D}')$.

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§1. Notations and Method

Given topological spaces X and Y on which a group H operates to the left, we denote by $\Omega_H(X, Y)$ the totality of continuous maps of X to Y which are compatible with the H-operations. We topologize $\Omega_H(X, Y)$ by the compact open topology. If H is the identity group the notation $\Omega_H(X, Y)$ is simplified to $\Omega(X, Y)$.

Let $\mathcal{B}=\{Y, \pi, B\}$ be a bundle with structure group H. (By bundle we understand the E-F bundle in [3]). We then consider the subspace $\Omega_H'(X, Y)\subset\Omega_H(X, Y)$ consisting of all $\varphi\in\Omega_H(X, Y)$ such that $\pi\circ\varphi$ are constant maps, and define a map π': $\Omega_H'(X, Y)\to B$ by attaching of $\varphi\in\Omega_H'(X, Y)$ to point which is the image of $\pi\circ\varphi$. It follows that $\{\Omega_H'(X, Y), \pi', B\}$ is a bundle with structure group H. This bundle is denoted $\Omega_H(X, \mathcal{B})$.

Let $\mathfrak{T}=(T, p, B)$ be a principal bundle with structure group G, and Y a topological space on which G operates to the right. Then we consider the space $Y\times_G T$ obtained from $Y\times T$ by the identification

$$(y,\ t)=(y\cdot g,\ g^{-1}\cdot t),\ y\in Y,\ t\in T,\ g\in G,$$

and define a map p': $Y\times_G T\longrightarrow B$ by $p'(y,\ t)=p(t)$. Then it follows that $Y\times_G\mathfrak{T}=(Y\times_G T,\ p,\ B)$ is a fiber space.

Assume now that Y has a right G-operation and a left H-operation which are compatible. Then we can regard naturally $Y\times_G\mathfrak{T}$ a bundle with structure group H. It follows moreover that, for any topological space X having a left H-operation, the natural map defines a fibre-preserving homeomorphism

$$\Omega_H(X,\ Y)\times_G\mathfrak{T}\rightarrow\Omega_H(X,\ Y\times_G\mathfrak{T}). \tag{1. 1)}$$

Hereafter $\mathfrak{T}$ will exclusively denote the principal tangent bundle of the manifold $(M,\ \mathcal{D})$, and G the real general linear group $GL(n,\ R)$ of degree $n=dim\ M$.

We shall sketch in what follows our method to prove the main theorem.

Let C^q denote the complex q-space on which the norm is defined as usual, and consider the subspace $L(C^m,\ C^n)\subset\Omega(C^m,\ C^n)$ consisting of all non-degenerate complex linear map φ: $C^m\rightarrow C^n$, where $m\leqq n$. Since G operates on C^n as linear transformations, we can define a right G-operation on $L\ (C^m,\ C^n)$ by

$$(\varphi\cdot g)(y)=g^{-1}\cdot(\varphi(y)),\ g\in G,\ y\in C^m.$$

It follows then that the Pontrjagin classes $p(\mathcal{D})$ are given the characteristic classes of the bundle

$$L(C^m,\ C^n)\times_G\mathfrak{T} \tag{I}$$

Let S^1 denote the group of complex numbers with norm 1. Then the scalar multiplication on the left vector space C^q define a left S^1-operation on C^q-0, where 0 is the origin of C^q. Together with the right G-operation on C^n, this defines the bundle

$$\Omega_{S^1}(C^m\text{-}0,\ C^n\text{-}0)\times_G\mathfrak{T} \tag{II}$$

We prove in Theorem 1 that $p(\mathcal{D})$ are the characteristic classes of the bundle (II).

Let R^n denote the Euclidean n-space, and consider the space $\Omega(S^1,\ R^n)$. We define a continuous map

$$\rho:\ \Omega(S^1,\ R^n)\ \rightarrow\ C^n \tag{1. 2}$$

by

$$\rho\ (\varphi)=(\int_{-\pi}^{\pi}e^{-it}\varphi^1(t)dt,\ \cdots,\ \int_{-\pi}^{\pi}e^{-it}\ \varphi^n(t)\ dt), \tag{1. 3}$$

where for $\varphi\in\Omega(S^1,\ R^n)$ we put

$$\varphi\ (e^{it})=(\varphi^1(t),\ \cdots,\ \varphi^n\ (t)). \tag{1. 4}$$

Define a left S^1- and right G-operation on $\Omega(S^1,\ R^n)$ by

$$(z\cdot\varphi)(z')=\varphi(zz'),\ (\varphi\cdot g)(z')=g^{-1}\cdot\varphi(z'),$$

where $z,\ z'\in S^1$, $g\in G$ and G operates on R^n as linear transformations. Then it follows that ρ is compatible with the S^1- and G-operations.

Therefore, putting

$$N=\rho^{-1}(0)$$

we can consider the bundle

$$\text{(III)} \qquad \Omega_{S^1}(C^m-0,\ \Omega(S^1,\ R^n)-N)\times_G\mathfrak{T}$$
$$=\Omega_{S^1}(C^m-0,\ (\Omega(S^1,\ R^n)-N)\times_G\mathfrak{T})$$

(see (1. 1)). We prove in Theorem 2 that ρ induces a fiber homotopy equivalence (III) to the bundle (II).

In connection with a Riemannian metric g introduced on $(M,\ \mathcal{D})$ we define for any positive number r and k a subbundle $\mathfrak{T}(r,\ k)$ of $(\Omega(S^1,\ R^n)-N)\times_G\mathfrak{T}$ (see §§3 and 4), and prove in Theorem 3 that there is a fibre homotopy equivalence between the bundles (III) and

$$\text{(IV)} \qquad \Omega_{S^1}(C^m-0,\ \mathfrak{T}(r,\ k))$$

if $1/2\leqq k<3/2$.

The above arguments show that $p(\mathcal{D})$ are given as the characteristic classes of the bundle (IV). On the other hand we show that if r is sufficiently small there is a homeomorphism λ of the total space of $\mathfrak{T}(r,\ k)$ onto a subspace of $\Omega(S^1,\ M)\times M$. The proof of the main theorem consists in comparing the bundle (IV) for $\mathcal{D}$ and $\mathcal{D}'$ by making use of the homeomorphism λ and $h: M\to M$. (see the final part of §4).

§2 Reduction to S-operation

Let

$$\iota:\ L(C^m,\ C^n)\times_G\mathfrak{T}\to\Omega_{S^1}(C^m-0,\ C^n-0)\times_G\mathfrak{T}$$

denote the map induced by the inclusion

$$\iota:\ L(C^m,\ C^n)\to\Omega_{S^1}(C^m-0,\ C^n-0).$$

We have

THEOREM 1. *Let $n>m$ and put $q_0=2(n-m)+1$, then*

$$\pi_i(L(C^m,\ C^n))=0 \qquad \textit{if } i<q_0$$
$$=Z \qquad \textit{if } i=q_0;$$

the homomorphism

$$\iota_*:\pi_i(L(C^m,\ C^n))\to\pi_i(\Omega_{S^1}(C^m-0,\ C^n-0))$$

is an isomorphism for $i\leqq 2(q_0-1)$: the isomorphism

$$\iota_*: H^{q_0+1}(M,\ \pi_{q_0}(L(C^m,\ C^n)))\to$$
$$H^{q_0+1}(M,\ \pi_{q_0}(\Omega_{S^1}(C^m-0,\ C^n-0)))$$

maps the characteristic class of $L(C^m,\ C^n)\times_G\mathfrak{T}$ to that of

$$\Omega_{S^1}(C^m,-0,\ C^n-0)\times_G\mathfrak{T}$$

Proof Since $L(C^m,\ C^n)$ is the complex Stiefel manifold, the first part of the theorem is the well-known fact. The last part of the theorem follows quickly from the naturality of the characteristic classes and the second part of the theorem. The proof of the second part is as follows.*

Regarding the sphere S^{2k-1} of odd dimension as the set of all elements of C^k with norm 1, we denote by $V_{m,n}$ the set of all linear maps which send S^{2m-1} to S^{2n-1}, and by $W_{m,n}$ the set of all maps $\varphi: S^{2m-1} \to S^{2n-1}$ compatible with the S^1-operation. Then it follows that the natural maps $V_{m,n} \to L(C^m, C^n)$ and $W_{m,n} \to \Omega_{S^1}(C^m-0, C^n-0)$ are homotopy equivalences.

Therefore the problem is to prove that the homomorphism

$$\kappa_m{}^*: \pi_q(V_{m,n}) \to \pi_q(W_{m,n})$$

induced by the inclusion $\kappa_m: V_{m,n} \to W_{m,n}$ is an isomorphism for $q \leqq 4(n-m)$.

Regard C^{m-1} as a subspace of C^m by identifying $(c_1,\cdots,c_{m-1})$ with $(c_1,\cdots,c_{m-1}, 0)$, and define continuous maps $p: V_{m,n} \to V_{m-1,n}$ and $q: W_{m,n} \to W_{m-1,n}$ to be the duals to the inclusion $C^{m-1} \subset C^m$. Then it follows that $(V_{m,n}, p, V_{m-1,n})$ and $(W_{m,n}, q, W_{m-1,n})$ are fibre spaces. In fact, the former is well known, and the covering homotopy property for the latter follows directly from the homotopy extension if we notice the following: the classified space of S^{2m-1} by the action of S^1 is homeomorphic with the space E_+^{2m-2} of all elements $y=(c_1,\cdots,c_m) \in C^m$ with norm 1 and real $c_m \geqq 0$, and the boundary of E_+^{2m-2} is S^{2m-3}.

Let $v \in V_{m-1,n}$ be a point, and put $w=\kappa(v) \in W_{m-1,n}$. Then it follows that $p^{-1}(v)$ is homeomorphic with $S^{2n-2m-1}$ and that $q^{-1}(w)$ is homotopy equivalent to the iterated loop space $\Omega^{2m-2}(S^{2n-1})$. Furthermore it follows that the homomorphism $\kappa_{m*}^0: \pi_i(p^{-1}(v)) \to \pi_i(q^{-1}(w))$ induced by the restriction $\kappa_m^0 = \kappa_m | (p^{-1}(v))$ is the iterated suspension homomorphism

$$E^{2m-2}: \pi_i(S^{2n-2m+1}) \to \pi_i(\Omega^{2m-2}(S^{2n-1})) = \pi_{i+2m-2}(S^{2n-1})$$

under the identifications $p^{-1}(v) = S^{2n-2m-1}$ and $q^{-1}(w) = \Omega^{2m-2}(S^{2n-1})$. Therefore κ_{m*}^0 is isomorphic onto if $i < 2(2n-2m+1)-1 = 4(n-m)+1$. Consider now the commutative diagram

$$\begin{tikzcd}
\arrow[r] & \pi_{i+1}(V_{m-1,n}) \arrow[r] \arrow[d, "\kappa_{m-1*}"] & \pi_i(p^{-1}(v)) \arrow[r] \arrow[d, "\kappa_{m*}^0"] & \pi_i(V_{m,n}) \arrow[r] \arrow[d, "\kappa_{m-1*}"] & \pi_i(V_{m-1,n}) \arrow[r] \arrow[d, "\kappa_{m-1*}"] & \pi_{i-1}(p^{-1}(v)) \arrow[r] \arrow[d, "\kappa_{m*}^0"] & {} \\
\arrow[r] & \pi_{i+1}(W_{m-1,n}) \arrow[r] & \pi_i(q^{-1}(w)) \arrow[r] & \pi_i(W_{m,n}) \arrow[r] & \pi_i(W_{m-1,n}) \arrow[r] & \pi_{i-1}(q^{-1}(w)) \arrow[r] & {}
\end{tikzcd}$$

in which the rows are exact. Then, in virtue of the five lemma, the induction on m proves that κ_{m*} is an isomorphism if $i \leqq 4(n-m)$. This completes the proof of Theorem 1.

COROLLARY. *The Pontrjagin classes $p(\mathcal{G})$ are the characteristic classes of the bundle $\Omega_{S^1}(C^m-0, C^n-0) \times_G \mathfrak{X}$.*

Define a map

$$\sigma: C^n \to \Omega(S^1, R^n)$$

by

* This is a complex analogue to the proof of the fundamental lemma (1.1) in Haefliger-Hirsch [1].

(2. 1) $(\sigma(c_1, \cdots, c_n))(z) = (1/\pi\ \mathcal{R}e(c_1 z), \cdots, 1/\pi\ \mathcal{R}e\ (c_n z)),$

where $\mathcal{R}e\ (c_i z)$ denotes the real part of the complex number $c_i z$. As is easily seen, σ is continuous and is compatible with the S^1-and G-operations. By a direct calculation from (1. 3) and (2. 1) we see

(2. 2) $\rho\circ\sigma=$the identity.

Define a homotopy $H_t: \Omega\ (S^1, R^n) \to \Omega\,(S^1, R^n)$ by

$$(H_t\,(\varphi))\ (z) = (1-t)((\sigma\circ\rho)\ (\varphi))\ (z) + t\ \varphi(z).$$

Then H_t is compatible with the S^1-and G-operations, and it holds that

$$H_0 = \sigma\circ\rho \qquad H_1 = \text{the identity}, \qquad \rho\circ H_t = \rho$$

Thus we obtain

PROPOSITION 1. The maps

$$\rho: \Omega\,(S^1,(R^n) - N \to C^n - 0,$$
$$\sigma: C^n - 0 \quad \to \quad \Omega(S^1, R^n) - N$$

are homotopy equivalences which are compatible with the S^1-and G-operations

This Proposition shows

THEOREM 2. *The map ρ induces a fibre homotopy equivalence*

$$\rho: \Omega_{S^1}\,(C^m - 0,\ \Omega(S^1, R^n) - N)\times_G\mathfrak{X} \to \Omega_{S^1}(C^m - 0,\ C^n - 0)\times_G\mathfrak{X}$$

Together this theorem with Corollary to Theorem 1, we have

COROLLARY. *The Pontrjagin classes $p(\mathcal{G})$) are the characteristic classes of the bundle $\Omega_{S^1}(C^m - 0,\ (\Omega(S^1, R^n) - N)\times_G\mathfrak{X})$.*

§ 3. The space $L\ (r, k;\ X, x_0)$

Let X be a metric space, and d its metric function. Then we associate each element $\varphi\in\Omega\ (S^1, X)$ with sequences $\{S_n(\varphi)\}$ and $\{s_n(\varphi)\}$ of real numbers defined as follows:

(3. 1) $$S_n(\varphi) = \pi/2^{n-1} \sum_{k=-2^{n-1}}^{2^{n-1}} d^2(\varphi(\nu_n{}^k),\ \varphi(-\nu_n{}^k))),$$

$$s_n(\varphi) = 2^{n-1}/\pi \sum_{k=-2^{n-1}}^{2^{n-1}} d^2(\varphi(\nu_n{}^k),\ \varphi(\nu_n{}^{k-1})),$$

where

$$\nu_n = exp(\pi i/2^{n-1}).$$

Since a function $r(t) = d^2(\varphi(-e^{it}))$, is continuous, it follows that for any $\varphi\in\Omega\ (S^1, X)$ the sequence $\{S_n(\varphi)\}$ converges and we have

(3. 2) $$S(\varphi) = \lim_{n\to\infty} S_n(\varphi) = \int_{-\pi}^{\pi} d^2(\varphi(e^{it}),\ \varphi(-e^{it}))\ dt.$$

We say that $\varphi\in\Omega\,(S^1, X)$ is a *Lipschitz map* if the following condition is satisfied: There exists a constant $K = K(\varphi)$ such that

(3. 3) $$d\,(\varphi(e^{it}),\ \varphi(e^{it'}))/\ |\ t - t'\ |\ \leqq K,$$

for any distinct real numbers t and t'.

LEMMA 1. *If $\varphi\in\Omega\,(S^1, X)$ is a Lipschitz map, then $\{s_n(\varphi)\}$ is a convegent* *sequence.*

Proof. The triangle inequality implies

$$d^2(x, z) \leqq 2(d^2(x, y)+d^2(y, z)).$$

By making use this and $\nu_{n+1}^2=\nu_n$, it follows that

$$s_n(\varphi) \leqq s_{n+1}(\varphi).$$

On the other hand, (3. 3) implies $d(\varphi(\nu_n{}^k), \varphi(\nu_n{}^{k-1}))\leqq\pi K/2^{n-1}$, and hence we have

$$s_n(\varphi) \leqq 2\pi K^2$$

Thus the sequence $\{s_n(\varphi)\}$ converges, and the lemma is proved.

For each Lipschitz map $\varphi\in\Omega(S^1, X)$ we put

$$s(\varphi)=\lim_{n\to\infty} s_n(\varphi). \tag{3. 4}$$

Given positive numbers r, k and a point $x_0\in X$, we denote by $L(r, k; X, x)$ the totality of all Lipschitz maps $\varphi\in\Omega(S^1, X)$ satisfying the following two conditions:

(L 1) $\qquad d^2(\varphi(z), x_0) \leqq r \qquad$ for all $z\in S^1$,

(L 2) $\qquad 0 < s(\varphi)/k \leqq kS(\varphi)$.

We shall next consider the case X is a real n-space R^n in which a metric function d is given by

$$d^2(x, y)=\textstyle\sum_{ij} g_{ij}(x^i-y^i)(x^j-y^j) \tag{3. 5}$$

in terms of a symmetric positive definite quadratic form g.

Given a function $f(t)$ defined on $[-\pi, \pi]$, we denote as usual by $a_k(f)$, $b_k(f)$ the Fourier coefficients of $f(t)$:

$$a_k(f)=1/\pi\int_{-\pi}^{\pi} f(t)\cos kt\,dt,\quad b_k(f)=1/\pi\int_{-\pi}^{\pi} f(t)\sin kt\,dt.$$

For $\varphi\in\Omega(S^1, R^n)$, we put

$$A_k(\varphi)=\textstyle\sum g_{ij}(a_k(\varphi^i)\,a_k(\varphi^j)+b_k(\varphi^i)\,b_k(\varphi^j)) \tag{3. 6}$$

(See (1. 4) for the definition of φ^i).

PROPOSITION 2. *For any $\varphi\in\Omega(S^1, R^n)$ we have*

$$S(\varphi)=\textstyle\sum_{k=1}^{\infty} 4A_{2k-1}(\varphi);$$

If φ is a Lipschitz map we have

$$s(\varphi)=\textstyle\sum_{k=1}^{\infty} k^2 A_k(\varphi).$$

(Proof is given in § 5.)

The following Lemma is fundamental.

LEMMA 2 *For any Lipschitz map $\varphi\in\Omega(S^1, R^n)$ satisfying*

$$0<s(\varphi) \leqq cS(\varphi)$$

with a constant c $(0<c<9/4)$, we have $\rho(\varphi)\neq 0$.

Proof. Suppose that $\rho(\varphi)=0$. Then we have

$$\int_{-\pi}^{\pi} e^{-it}\varphi^j(t)dt=a_1(\varphi^j)-ib_1(\varphi^j)=0$$

for any j, and hence $A_1(\varphi)=0$. Therefore, in virtue of Proposition 2, it follows from $s(\varphi) \leqq cS(\varphi)$ that

$$0 \geqq \sum_{k=2}^{\infty} 4k^2 A_{2k}(\varphi)+\sum_{k=1}^{\infty}((2k+1)^2-4c)A_{2k+1}(\varphi).$$

Since (g_{ij}) is positive definite, $A_k(\varphi) \geqq 0$. Consequently we obtain

$$A_2(\varphi)=A_3(\varphi)=\cdots=0 \text{ or } s(\varphi)=0$$

which contradicts with the assumption $0 < s(\varphi)$.

REMARK 1 Let $\varphi: S^1 \to S^n$ be a Lipschitz map, then

$$s(\varphi) = 0$$

if and only if φ is a constant map.

(See § 5 for the proof).

PROPOSITION 3 *For* $0<k<3/2$ *and any* $r>0$, *we have*

$$L(r, k; R^n, 0) \subset \Omega(S^1, R^n)=N;$$

$L(r, k: R^n, 0)$ *is invariant under the* S^1-*operation.*

Proof. The former is a direct consequence of Lemma 2. If φ is a Lipschitz map then $\varphi\cdot\alpha$ $(\alpha \in S^1)$ is also a Lipschitz map and we have

$$\operatorname*{Max}_{z\in S^1} d(\varphi(z), 0) = \operatorname*{Max}_{z\in S^1} d(\varphi\cdot\alpha(z), 0).$$

Furthermore direct calculations show

$$A_k(\varphi\cdot\alpha)=A_k(\varphi)$$

and hence by Proposition 2 we have

$$S(\varphi\cdot\alpha)=S(\varphi),\ s(\varphi\cdot\alpha)=s(\varphi). \tag{3.7}$$

Therefore the latter is obtained, and the proof completes.

For any element $y \in C^n$, put

$$\| y \| = \operatorname*{Max}_{z\in S^1} d(\sigma(y)(z), 0). \tag{3.8}$$

Then by direct calculation we can prove that $\| y \|$ is a norm in C^n. i) $\| y \|=0$ if and only if $y=0$, ii) $\| cy \| = |c|\ \| y \|$ for any $c \in C^1$, iii) $\| y_1+y_2 \| \leqq \| y_1 \| + \| y_2 \|$.

For any $r>0$, put

$$\mathrm{S}(r; R^n, 0)=\{\sigma(y) \mid y \in C^n,\ \| y \|=\sqrt{r}\} \tag{3.9}$$

LEMMA 3. *There is a deformation retraction of* $\Omega(S^1, R^n)-N$ *to* $S(r; R^n, 0)$ *which is compatible with the* S^1-*operation.*

Proof. Define a homotopy $q_t: C^n-0 \to C^n-0$ by

$$q_t(y)=(1-t+\sqrt{r}\,t/\| y \|)\, y$$

Then q_t is compatible with the S^1-operation and we have

$$q_0=1,\ q_1(C^n-0)=\mathfrak{S}(r),\ q_t \mid \mathfrak{S}(r)=1,$$

where $\mathfrak{S}(r)=\{y \in C^n / \| y \|=\sqrt{r}\}$. Therefore if we put

$$Q_t=\sigma\circ q_t\circ\rho:\ \Omega(S^1, R^n)-N \to \Omega(S^1, R^n)-N$$

then Q_t is a homotopy which is compatible with the S^1-operation, and we have

$$Q_0=\sigma\circ\rho,\quad Q_1(\Omega(S^1, R^n)-N)=S(r; R^n, 0),\quad Q_t \mid S(r; R^n, 0)=1.$$

Consequently, together this with Proposition 1, we obtain the desired result.

LEMMA 4 *If* $1/2 \leqq k < 3/2$ *there exists a deformation retraction of* $L(r, k; R^n$ $0)$ *to* $S(r; R^n, 0)$ *which is compatible with the* S^1-*operation.*

Proof Define a homotopy $H_t : \Omega(S^1, R^n)-N \to \Omega(S^1, R^n)$ by

$$H_t(\varphi)(z)=(1-t)\varphi(z)+t(Q_1(\varphi)))z), \ z\in S^1.$$

Then $H_t(\varphi)$ is compatible with the S^1-operation, and we have

$$H_0=1, \ H_1(\Omega(S^1, R^n)-N)=S(r; R^n, 0), \ H_t \mid S(r; R^n, 0)=1.$$

Therefore, in virtue of Proposition 3, the problem is to prove

$$(3.10) \qquad H_t(L(r, k; R^n, 0))\subset L(r, k; R^n, 0).$$

This is proved as follows.

If we notice that sin θ, cos θ satisfy the Lipschitz condition, it is easily proved that $\sigma(y)$ is a Lipschitz map for any $y\in C^n$. From this it follows that if φ is a Lipschitz map then so is $H_t(\varphi)$. Let φ satisfy $d^2(\varphi(z), 0)\leqq r$ for any $z\in S^1$, then it holds that

$$\begin{aligned} d(H_t(\varphi)(z), 0) &\leqq (1-t)d(\varphi(z), 0)+t\ d(Q_1(\varphi(z)), 0) \\ &\leqq (1-t)\sqrt{r}+t\sqrt{r}/\|\rho(\varphi)\|\, d((\sigma\circ\rho(\varphi)(z), 0) \\ &\leqq (1-t)\sqrt{r}+t\sqrt{r}=\sqrt{r}. \end{aligned}$$

Hence we have

$$d^2(H_t(\varphi)(z), 0)\leqq r$$

for any $z\in S^1$ and $t\in[0, 1]$.

Observe next that

$$A_1(\sigma(y))>0, \ A_j(\sigma(y))=0 \ (j\geqq 2)$$

for any $0\neq y\in C^n$. Then, it follows that

$$\begin{aligned} &A_1(H_t(\varphi))\geqq(1-t)^2A_1(\varphi), \\ &A_j(H_t(\varphi))=(1-t)^2A_j(\varphi) \ (j\geqq 2). \end{aligned}$$

Therefore, in virtue of Proposition 2, direct calculations show that if $0<r(\varphi)/k \leqq kS(\varphi)$ with $1/2\leqq k<3/2$, then $0<s(H_t(\varphi))/k\leqq ks(H_t(\varphi))$. Thus we have (3.10), and the proof is completed.

Together with Lemmas 3 and 4, we obtain

PROPOSITION 4 *For* $1/2\leqq k<3/2$ *and* $0<r$, *there is a homotopy equivalence of* $L(r, k; R^n, 0)$ *to* $\Omega(S^1, R^n)-N$ *which is compatible with the* S^1*-operation.*

PROPOSITION 5 *Let* $1/2\leqq k<3/2$, $1<u<3/2k$, *then the homomorphism*

$$i_*: \pi(\Omega_{S^1}(C^m-0, L(r, k; R^n, 0)))\to\pi_q(\Omega(C^m-0, L(ur, uk; R^n, 0)))$$

induced by the inclusion $i: L(r, k; R^n, 0)\to L(ur, uk; R^n, 0)$ *is an isomorphism.*

Proof. It is easily seen that a homotopy

$$f_t: S(r; R^n, 0) \to L(ur, uk; R^n, 0)$$

can be defined by

$$f_t(\varphi)=((1-t)+ut)\varphi, \ \varphi\in S(r; R^n, 0).$$

It follows that f_t is compatible with the S^1-operation, and that

$$f_0(\varphi)=\varphi, \ f_1(\varphi)\in S(ur; R^n, 0).$$

Therefore the diagram

$$\begin{array}{ccc}\Omega_{S^1}(C^m-0,\ S(r;\ R^n,0)) & \xrightarrow{f_1} & \Omega_{S^1}(C^m-0,\ S(ur;\ R^n,\ 0)) \\ \downarrow i & & \downarrow i \\ \Omega_{S^1}(C^m-0, L(r,k;R^n,\ 0)) & \xrightarrow{i} & \Omega_{S^1}(C^m-0, L(ur;\ uk;\ R^n,\ 0))\end{array}$$

is commutative, where i are the inclusions. Obviously f_1 is an onto-homeomorphism. Therefore the proposition is a direct consequence of Lemma 4.

§ 4. Proof of the main theorem.

Take a Riemannian metric g on M. Then g determines a metric function d_p on each fibre $T_p=\pi^{-1}(p)$ of the tangent bundle $R^n\times_G\mathfrak{T}$. Therefore T_p is a real linear n-space having a metric function defined in terms of a symmetric positive definite matrix $g_{ij}(p)$, so that we can apply the arguments in § 3 with T_p instead of R^n.

As is well known, g defines also a metric function on M. We denote this by d_M. Consider a system of normal coordinates $(x^1, \cdots, x^n)$ in a neighborhood U_p of $p\in M$. Then the correspondence of points $q=(x^1, \cdots, x^n)\in U_p$ to points $\sum_{i=1}^n x^iL_i(p)$, $L_i(p)=(\partial/\partial x^i)_p$, defines a homeomorphism of U_p into T_p. This homeomorphism is denoted by λ_p

LEMMA 5 *For any sufficiently small* $\varepsilon>0$, *there exists* $\delta(\varepsilon)\geqq 0$ *such that*

i) $\delta(\varepsilon)\leqq k\varepsilon$

ii) *for any* $p,\ q,\ q'\varepsilon M$ *with* $d_M(q,\ p)\leqq\varepsilon,\ d_M(q',\ p)\leqq\varepsilon$ *we have*

$$d_p^2(\bar{q},\ \bar{q}')/(1+\delta(\varepsilon))\leqq d_M^2(q,\ q')\leqq(1+\delta(\varepsilon))d_p^2(\bar{q},\ \bar{q}')$$

where we put $\bar{q}=\lambda_p(q)$ *and* $\bar{q}'=\lambda_p(q')$.

Proof Let $x=(x^i)$, $x'=(x'^i)$ be normal coordinates of q, q' in U_p. Then it follows that there are functions $a_{ij}(x,\ x',\ p)$ such that

$$d_M^2(q,\ q')=\textstyle\sum_{ij}a_{ij}(x,\ x',\ p)(x^i-x'^i)(x^j-x'^j)$$

$$d_p^2(q,\ q')=\textstyle\sum_{ij}a_{ij}(0,\ 0,\ p)(x^i-x'^i)(x^j-x'^j)$$

if $q,\ q'\in U_p$. We can take $a_{ij}(x, y, p)$ in such a way that they are continuous on p and are differentiable on x and y. Consider now a quadratic form

$$A_{(x,\ y,\ p)}(\xi)=\textstyle\sum_{ij}a_{ij}(x,\ y,\ p)\ \xi^i\xi^j.$$

Then we have

$$A_{(x,\ y,\ p)}(x-x')=d_M^2(q,q'),\ A_{(0,\ 0,\ p)}(x-x')=d_p^2(\bar{q},\ \bar{q}').$$

It follows that there is a function $c(p,\ \xi)$ which is upper semi-continuous on p and ξ and which satisfy

$$|A_{(x,\ x',\ p)}(\xi)-A_{(0,\ 0,\ p)}(\xi)|\leqq c(p,\ \xi)(d_M(q,\ p)+d_M(q',\ p)).$$

Put

$$c(p)=\operatorname*{Max}_{\xi\in\mathfrak{S}}\ c(p,\ \xi),$$

where $\mathfrak{S}=\{\xi/A_{(o,o,p)}(\xi)=1\}$. Since,

$$(x-x')/\sqrt{A_{(o,o,p)}(x-x')}\in\mathfrak{S}.$$

if $d_M(q,p)\leqq\varepsilon$ and $d_M(q',p)\leqq\varepsilon$ then we have

$$|A_{(x,x',p)}(x-x')-A_{(0,0,p)}(x-x')|\leqq 2\varepsilon c(p)A_{(0,0,p)}(x-x'),$$

namely,

$$(1-2\varepsilon c(p))\,d_p^2(q,q')\leqq d_M^2(q,q')\leqq(1+2\varepsilon c(p))d_p^2(q,q').$$

Therefore, putting

$$\delta(\varepsilon)=2\varepsilon c/(1-2\varepsilon c)\quad\text{with } c=\operatorname*{Max}_{p\in M}c(p),$$

we have the desired result.

PROPOSITION 6 *For sufficiently small $\varepsilon>o$, λ_p induces maps*

$$\lambda_p: L(\varepsilon,k;M,p)\to L((1+\delta(\varepsilon))\varepsilon,(1+\delta(\varepsilon))k;T_p,0)$$
$$\lambda_p^{-1}: L(\varepsilon,k;T_p,0)\to L((1+\delta(\varepsilon))\varepsilon,(1+\delta(\varepsilon))k;M,p)$$

Proof Obvious from Lemma 5 and the definition of $L(r,k;X,x_0)$.

For any $r>o$ and $k>o$, we define a subspace $T(r,k)$ by

$$T(r,k)=\bigcup_{p\in M}L(r,k:T_p,0)\subset\Omega(S^1,R^n)\times_G\mathfrak{T}$$

and a continuous map $\pi_0: T(r,k)\to M$ by

$$\pi_0(r,k;T_p,0))=p.$$

PROPOSITION 7 *$\mathfrak{T}(r,k)=\{T(r,k),\pi_0,M\}$ is a bundle with structure group S^1.*

Proof Let (x^i) be a system of normal coordinates in U_p, and let $x=(x^1)$ be the coordinate of $x\in U_p$. Then there is a matrix $(f_j^i(x))$ such that

$$g_{\mu,\nu}(x)=\sum g_{ij}(0)f_\mu^i(x)f_\nu^j(x)$$

and $f_j^i(x)$ are continuous on x. We have a homeomorphism

$$\xi: T_p\times U_p\to\pi^{-1}(U_p)$$

defined by

$$\xi(\textstyle\sum_i x^iL_i(p),q)=\sum_{ij}x^jf_j^i\cdot L_i(q).$$

It follows that $d_p(\bar{x},\bar{x}')=d_p(\xi(\bar{x},q),\xi(\bar{x}',q))$ for any $x,x'\in T_p$. Therefore if we define

$$\eta:\Omega(S^1,T_p)\times U_p\to\pi_0^{-1}(U_p)\subset\Omega(S^1,R^n)\times_G\mathfrak{T}$$

by

$$(\eta(\varphi,q))(z)=\xi(\varphi(z),q),\ z\in S^1,\ \varphi\in\Omega(S^1,T_p),$$

then it follows that η is a homeomorphism such that $\pi_0\circ\eta(\varphi,q)=q$, and that $\eta(L(r,k;T_p,0)\times U_p)=\bigcup_{q\in U_p}L(r,k;T_q,0)=\pi_0^{-1}(U_p)$. Thus we have the lemma.

THEOREM 3 *For $1/2\leqq k<3/2$ and $r>o$, there is a fibre homotopy equivalence of the bundle $\mathfrak{T}(r,k)$ to the bundle $(\Omega(S^1,R^n)-N)\times_G\mathfrak{T}$ which is compatible with the S^1-operation.*

Proof. We mean by N_p the set N defined for $R^n=T_p$. Then the fibre on p of the bundle $(\Omega(S^1,R^n)-N)\times_G\mathfrak{T}$ is $\Omega(S^1,T_p)-N_p$. On the other hand, the fibre space $\mathfrak{T}(r, k)$ is $L(r, k; T_p, 0)$. Therefore the desired result follows from Proposition 4. Here it is to be noticed that the fibre homotopy equivalence ξ_p: $L(r, k; T_p, 0) \to \Omega(S^1, T_p)-N_p$ can be taken in such a way that ξ_p is continuous on p. (See the proof of Lemmas 3 and 4).

Together this theorem with Corollary to Theorem 2, we have

COROLLARY *The Pontrjagin classes $p(\mathcal{D})$ are characteristic classes of the bundle $\Omega_{S^1}(C^m-0, \mathfrak{T}(r, k))$, where $1/2 \leqq k < 3/2$ and $r>0$.*

We now proceed to proving the main theorem.

In the following, the notations with $'$ denote the corresponding notions defined for $(M, \mathcal{D}')$. We assume that $\varepsilon>0$ is sufficiently small.

In virtue of the condition ii) of Introduction, h induces a map

$$h_p: L(\varepsilon, k; M, p) \to L(s\varepsilon, sk; M, h(p)).$$

Therefore, by Proposition 6, the composition

$$\zeta_p = \lambda_{h(p)} h \lambda_p^{-1}: L(\varepsilon, k; T_p, 0) \to L(\varepsilon s\Delta, ks\Delta; T'_{h(p)}, 0)$$

can be defined, where

$$\Delta = (1+\delta_0')(1+\delta_0)$$

with $\delta_0 = \delta(\varepsilon)$, $\delta_0' = ((1+\delta_0)\varepsilon s)$. It follows that the maps ζ_p for all $p \in M$ give rise to a map $\zeta: T(s, k) \to T'(\varepsilon s\Delta, ks\Delta)$ such that $\pi_0' \circ \zeta = h \circ \pi_0$. By Proposition 7, ζ is a bundle map of $\mathfrak{T}(\varepsilon, k)$ to $\mathfrak{T}'(\varepsilon s\Delta, ks\Delta)$, so that ζ induces a bundle map

$$\zeta: \Omega_{S^1}(C^m-0, \mathfrak{T}(\varepsilon, k)) \to \Omega_{S^1}(C^m-0, \mathfrak{T}'(\varepsilon s\Omega, ks\Delta))$$

such that $\pi_0' \circ \zeta = h \circ \pi_0$. Therefore, in virtue of Corollary to Theorem 3 and the fact

$$\lim_{\varepsilon\to 0} \Delta = 1,$$

the main theorem is a direct consequence of the following: If $1 \leqq s^2 < 3$ then the homomorphism

$$\zeta_*: \pi_{2(n-m)+1}(\Omega_{S^1}(C^m-0, L(\varepsilon, 1/2; T_p, 0))) \to \pi_{2(n-m)+1}$$
$$(\Omega_{S^1}(C^m-0, L(\varepsilon s\Delta, 1/2s\Delta; T'_{h'(p)}, 0)))$$

induced by ζ is an isomorphism. To prove this we consider the composition

$$\zeta'_{h(p)} = \lambda_p \circ h^{-1} \circ \lambda'^{-1}_{h(p)}: L_1 = L(\varepsilon s\Delta, s\Delta/2; T'_{h(p)} 0) \to L_2 = L(\varepsilon s^2\Delta\Delta', s^2\Delta\Delta'/2: T_p, 0),$$

where we put

$$\Delta' = (1+\delta_1)(1+\delta_1')$$

with $\delta_1' = \delta'(\varepsilon s\Delta)$, $\delta_1 = \delta((1+\delta_1')\varepsilon s^2\Delta)$. Then the composition

$$\zeta'_{h(p)} \circ \zeta_p: L_0 = L(\varepsilon, 1/2; T_p, 0) \to L_2$$

is the inclusion. Since

$$\lim_{\varepsilon\to 0} \Delta\Delta' = 1$$

it follows from Proposition 5 that, if $1 \leqq s^2 < 3$, then

$$\zeta'_{h(p)} \circ \zeta_{p*}: \pi_{q_0}(\Omega_{S^1}(C^m-0, L_0)) \to \pi_{q_0}(\Omega(C^m-0, L_2))$$

is an isomorphism. Thus for $1 \leqq s < \sqrt{3}$, $\zeta'_{h(p)*}$ is an epimorphism. On the other hand, by Theorem 1, Propositions 2 and 4, we have

$$\pi_{q_0}(\Omega_{S^1}(C^m-0,\ L_i))\approx Z$$

($i=0$, 1, 2). Therefore $\zeta_{h(p)*}$, and hence ζ_{p*}, is an isomorphism if $1\leqq s<3$. This completes the proof of the main theorem.

§5 Proof of Proposition 2

We first prepare from theory of real functions some theorems whose proofs are referred to, for example, the book of Natanson [2].

Let $f(t)$ be a (real valued) function defined on a closed interval $[a, b]$, and let

$$a=t_0^n<t_1^n\cdots<t_n^n=b,\quad t_k^n=a+k(b-a)/n,$$

be a partition of $[a, b]$. Then we define a function $D_nf(t)$ by

$$D_nf(t)=\begin{cases} n(f(t_{k+1}^n)-f(t_k^n))/(b-a) & \text{for } t_k^n<t<t_{k+1}^n, \\ 0 & \text{for } t=t_k^n \end{cases} \tag{5.1}$$

THEOREM A ([2, p. 257]). *Let $f(t)$ be a function defined on $[a, b]$, and assume that there exists a constant $K=K(f)$ which depends only on f and*

$$\sum_{k=0}^{n-1}(f(t_{k+1})-f(t_k))^2/(t_{k+1}-t_k)\leqq K$$

for any partition $a=t_0<t_1<\cdots<t_n=b$ of $[a, b]$. The sequence of the functions $\{D_nf(t)\}$ converges almost everywhere, and for the limit fuction

$$Df(t)=\lim_{n\to\infty} D_nf(t)$$

it holds that

$$Df(t)\in L^2[a, b],\ f(t)=\text{const.}+\int_a^t Df(s)ds.$$

where L^2 $[a, b]$ stands for the totality of measurable function $f(t)$ defined on $[a, b]$ such that

$$\int_a^b f(t)^2dt<\infty.$$

REMARK 2 If $f(t)$ satisfies the Lipschitz condition, namely if there is a constant $c=c(f)$ such that

$$|f(t)-f(t')|/|t-t'|\leqq c$$

for any $t, t'\in[a, b]$ $(t\neq t')$, then the assumption in Theorem A is satisfied. In fact,

$$\sum_{k=0}^{n-1}(f(t_{k+1})-f(t_k))^2/(t_{k+1}-t_k)\leqq\sum_{k=0}^{n-1}c^2(t_{k+1}-t_k)=(b-a)c^2$$

THEOREM B ([2, p. 266]). *Let $f(t)$ be an integrable function defined on $[a, b]$, and put*

$$F(t)=\int_a^t f(s)\,ds.$$

Then, for a differentiable function $h(t)$ defined on $[a, b]$, the following formula holds:

$$\int_a^b h(t)f(t)\,dt=[h(t)F(t)]_a^b-\int_a^b h'(t)F(t)dt.$$

THEOREM C (Lebesgue's theorem, [2, p. 127]) *Let $f_n(t)$ be a sequence of functions which converges almost everywhere to f, and such that $|f_n(t)|\leqq K<\infty$ for* *all n and t. Then we have*

$$\lim_{n\to\infty}\int f_n(t)\,dt=\int f(t)\,dt.$$

THEOREM D (Parseval formula, [2, p. 179]). *For any $f(t),\ g(t)\in L^2[-\pi,\pi]$ it holds that*

$$\int_{-\pi}^{\pi}f(t)g(t)dt=4a_0(f)a_0(g)+\sum_{k=1}^{\infty}a_k(f)a_k(g)+b_k(f)b_k(g).$$

We shall prove

LEMMA 6 *Let $f(t)$ be function defined on $[-\pi,\pi]$ which satisfies the Lipschitz condition, and such that $f(-\pi)=f(\pi)$. Then, for the function $Df(t)$, we have*

$$a_k(Df)=kb_k(f),\ b_k(Df)=-ka_k(f).$$

Proof. By Theorem A and Remark 2 we have

$$f(t)=c+\int_{-\pi}^{t}D(f(s))ds.$$

Therefore the assumption $f(-\pi)=f(\pi)$ implies

$$\pi a_0(f)=\int_{-\pi}^{\pi}Df(s)ds=o.$$

Hence, in virtue of Theorem B, we have

$$\begin{aligned}a_k(D_f)&=1/\pi\int_{-\pi}^{\pi}Df(s)\cos ks\,ds\\&=1/\pi\left\{[(\cos kt)\int_{-\pi}^{t}Df(s)ds]_{-\pi}^{\pi}+k\int_{-\pi}^{t}(\sin kt)\,(\int_{-\pi}^{t}Df(s)ds)\,dt\right.\\&=k/\pi\int_{-\pi}^{\pi}(\sin kt)(\int_{-\pi}^{t}Df(s)ds)\,dt\\&=k/\pi\int_{-\pi}^{\pi}(\sin kt)(f(t)-c)dt\\&=k/\pi\int_{-\pi}^{\pi}f(t)\sin kt\,dt=kb_k(f).\end{aligned}$$

Similarly we have $b_k(Df)=-ka_k(f)$, and the proof completes.

LEMMA 7 *If $\varphi: S^1\to R^n$ is a Lipschitz map, then each function $\varphi^i(t)$ satisfies the Lipschitz condition.*

Proof. It follows that there is a constant c_i such that

$$|x^i-y^i|\leqq c_id(x,y)$$

for any $x,y\in R^n$. Therefore we have

$$|\varphi^i(t)-\varphi^i(t')|/|t-t'|\leqq c_i\,d(\varphi(e^{it}),\ \varphi(e^{it'}))/|t-t'|\leqq c_iK(\varphi)$$

We now proceed to

Proof of Proposition 2. By (3.2) we have

$$\begin{aligned}S(\varphi)=&\int_{-\pi}^{\pi}d^2(\varphi(-e^{it}),\ \varphi(e^{it}))\,dt.\\=&\int_{-\pi}^{\pi}\sum_{ij}g_{ij}(\varphi^i(t+\pi)-\varphi^i(t))(\varphi^j(t+\pi)-\varphi^j(t))dt\end{aligned}$$

It follows that

$$a_k(\varphi^i(t+\pi))=(-1)^ka_k(\varphi^i(t)),$$
$$b_k(\varphi^i(t+\pi))=(-1)^kb_k(\varphi^i(t)).$$

Therefeore, in virtue of Theorem D, we obtain

$$S(\varphi)=\sum_{ij}g_{ij}\sum_{k=1}^{\infty}, 4(a_{2k-1}(\varphi^{i})a_{2k-1}(\varphi^{j})+b_{2k-1}(\varphi^{i})b_{2k-1}(\varphi^{j}))$$
$$=\sum_{k=1}^{\infty} A_{2k-1}(\varphi)$$

Next, assume φ is a Lipschitz map, then it follows from (3.1) and (5.1) that

$$s_{n}(\varphi)=2^{n-1}/\pi \sum_{k=-2^{n-2}}^{2^{n-1}} d^{2}(\varphi(\nu_{n}{}^{k}), \varphi(\nu_{n}^{k-1}))$$
$$=\sum_{ij}g_{ij}\int_{-\pi}^{\pi}D_{n}\varphi^{i}D_{n}\varphi^{j}dt.$$

Since $\varphi^{i}(t)$ satisfies the Lipschitz condition by Lemma 7, it follows from Theorem A and Remark 2 that the sequence $D_{n}\varphi^{i}(t)$ converges almost everywhere to $D\varphi^{i}(t)\in L^{2}[-\pi, \pi]$. Furthermore, in the notation of the proof of Lemma 7, we have

$$|D_{n}\varphi^{i}(t)D_{n}\varphi^{j}(t)| \leqq c_{i}c_{j}K^{2}(\varphi)<\infty.$$

Therefore, in virtue of Theorems C, D and Lemma 6, we obtain

$$s(\varphi)=\lim s_{n}(\varphi)=\sum_{ij} g_{ij}\int_{-\pi}^{\pi}D\varphi^{i}(t)D\varphi^{i}(t)dt$$
$$=\sum_{ij}g_{ij}\sum_{k=1}^{\infty}(k^{2}b_{k}(\varphi^{i})b_{k}(\varphi^{j})+k^{2}a_{k}(\varphi^{i})a_{k}(\varphi^{j}))$$
$$=\sum_{k=1}^{\infty}k^{2}A_{k}(\varphi)$$

This completes the proof of Proposition 2.

We shall here prove Remark 1.

Proof of Remark 1. If $s(\varphi)=0$ then $A_{k}(\varphi)=0$ for all $k\geqq 1$, so that $a_{k}(\varphi^{j})=b_{k}(\varphi^{j})=0$ for all $k\geqq 1$ and j. Therefore, by Lemma 6 and Theorem D, we have

$$\int_{-\pi}^{\pi}(D\varphi^{j}(t))^{2}dt=0.$$

This implies $D\varphi^{j}(t)=0$ almost everywhere. Therefore, by Theorem A, we obtain

$$\varphi^{j}(t)=c+\int_{-\pi}^{t}D\varphi^{j}(s)ds=c.$$

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