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A type of separable algebras

Dedicated to Professor K. Shoda on his sixtieth birthday.

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Let Λ be an algebra over a commutative ring K with unite element, and $\Lambda^e = \Lambda\otimes_K\Lambda^\circ$ be the enveloping algebra of Λ. We define a Λ^e-homomorphism φ of Λ^e onto Λ by $\varphi(x\otimes y^\circ)=xy$, and we denote the right annihilator of the kernel of φ in Λ^e by A. It was shown by M. Auslander and O. Goldman [1] that Λ is separable (in the usual sense when K is a field) if and only if $\varphi(A)$ coincides with the center of Λ.

In this note, we are concerned with the case where K is a field. Let * be the anti-automorphism of Λ^e defined by $(x\otimes y^\circ)^* = y\otimes x^\circ$. In the first section we shall show that with any $a \neq 0$ in A (or A^*) there is associated a non-zero right (or left) ideal $\mathfrak{r}_a$ (or $\mathfrak{l}_a$) of finite rank over K in Λ (Proposition 1), and as a corollary we have that if Λ is right primitive and $A\neq 0$ then Λ is a simple algebra of finite rank over K (Corollary 2). In the second section we consider those algebras Λ for which $\varphi(A^*)=\Lambda$, and we shall show that $\varphi(A^*)=\Lambda$ if and only if Λ is a direct sum of simple algebras of finite rank over K whose degrees over the centers are all prime to the characteristic of K. Among several corollaries to the theorem, we shall show that if Λ is an algebra as above, then Λ is the direct sum of the center C and the K-submodule $[\Lambda,\Lambda]$ which is generated by the commutators $xy-yx$ in Λ (Corollary 4).

1. Let Λ be an algebra of finite or infinite rank over a field K with unit element. In the enveloping algebra $\Lambda^e=\Lambda\otimes_K\Lambda^\circ$ where Λ° is anti-isomorphic to Λ by correspondence $x\leftrightarrow x^\circ$ we set $J=\{x\otimes 1^\circ - 1\otimes x^\circ \mid x\in\Lambda\}$. Then $\operatorname{Ker}\varphi = \Lambda^e J$ and the right annihilator of J coincides with the right annihilator A of the kernel of φ, and A is the left annihilator of J in Λ^e. Let $\{x_1, x_2, \ldots, x_\tau, \ldots\}$ be a K-basis of Λ and let $x_i x_j=\sum_\sigma \gamma_{ij\sigma} x_\sigma$, $\gamma_{ij\sigma}\in K$.

LEMMA 1. *An element $a=\sum\alpha_{ij}\, x_i\otimes x_j^\circ$ $(\alpha_{ij}\in K)$ of Λ^e is contained in A if and only if $\sum_i \alpha_{i\sigma}\gamma_{\tau i\kappa} = \sum_j \alpha_{\kappa j}\gamma_{j\tau\sigma}$ for every κ, σ and τ.*

Proof. An element a of Λ^e is contained in A if and only if
$(x_\tau\otimes 1^\circ)\, a = (1\otimes x_\tau^\circ)\, a$ for all τ.
Since $(x_\tau\otimes 1^\circ)\, a = \sum_{i\kappa\sigma} \alpha_{i\sigma}\gamma_{\tau i\kappa}\, x_\kappa\otimes x_\sigma^\circ$ and $(1\otimes x_\tau^\circ)\, a = \sum_{j\kappa\sigma}\alpha_{\kappa j}\gamma_{j\tau\sigma}\, x_\kappa\otimes x_\sigma^\circ$,
$(x_\tau\otimes 1^\circ)\, a = (1\otimes x_\tau^\circ)\, a$ if and only if $\sum_i \alpha_{i\sigma}\gamma_{\tau i\kappa} = \sum_j \alpha_{\kappa j}\gamma_{j\tau\sigma}$ for all κ and σ.

PROPOSITION 1. *If A (or A^*) contains a non-zero element a (or a^*), then there exists a non-zero right ideal $\mathfrak{r}_a$ (or non-zero left ideal $\mathfrak{l}_a$) of finite rank over K in Λ.*

Proof. Suppose that $a=\sum_{ij}\alpha_{ij}\,x_i\otimes x_j^\circ$ is a non-zero element of A, then only a finite number of α_{ij}'s are non-zero elements of K, and at least one of α_{ij}'s is not zero. If we put $y_\kappa=\sum_j \alpha_{\kappa j}\,x_j$ for each κ, there exists only a finite number of non-zero y_κ. Let $\mathfrak{r}_a$ be a K-submodule $\sum_\kappa Ky_\kappa$ of Λ generated by $\{y_\kappa\}$, then we have $[\mathfrak{r}_a : K]<\infty$ and $y_\kappa x_\tau=\sum_j\alpha_{\kappa j}\,x_jx_\tau=\sum_{j\sigma}\alpha_{\kappa j}\,\gamma_{j\tau\sigma}\,x_\sigma=\sum_{\sigma i}\alpha_{i\sigma}\,\gamma_{\tau i\tau}\,x_\sigma=\sum_i\gamma_{\tau i\tau}\,y_i$. Therefore $\mathfrak{r}_a$ is a non-zero right ideal of Λ of finite rank over K. Similarly for a non-zero element $a^*=\sum_{ij}\alpha_{ij}\,x_j\otimes x_i^\circ$ of A^* if we put $z_\kappa=\sum_j\alpha_{j\kappa}\,x_j$ then there is only a finite number of non-zero z_κ's, and we have $x_\tau z_\kappa=\sum_j\gamma_{j\tau\kappa}\,z_j$ for every x_τ, z_κ. The K-submodule $\mathfrak{l}_a=\sum Kz_\kappa$ is a non-zero left ideal of Λ and $[\mathfrak{l}_a : K]<\infty$.

COROLLARY 1. *If the right annihilator of the kernel of φ in Λ^e is a non-zero right ideal of Λ^e, then Λ has a non-zero right ideal and a non-zero left ideal of finite rank.*

COROLLARY 2. *Let Λ be a right (or left) primitive algebra.*[1] *If $A\neq 0$, then Λ is a simple algebra of finite rank over K.*

Proof. We assume that Λ is a right primitive algebra. From Corollary 1, there exists a non-zero right ideal $\mathfrak{r}$ of Λ such that $[\mathfrak{r} : K]<\infty$. Let M be a faithful irreducible Λ-right module. Since $M\mathfrak{r}\neq 0$, there exists a non-zero element x in M such that $M=x\mathfrak{r}$. Hence $[M : K]\leqq[\mathfrak{r} : K]<\infty$. Therefore Λ is a simple algebra of finite rank over K.

COROLLARY 3. *Let Λ be an algebra over a field K. If $\varphi(A^*)=\Lambda$ then $[\Lambda : K]<\infty$.*

Proof. If $\varphi(A^*)=\Lambda$, then there exists a non-zero element a^* in A^* such that $\varphi(a^*)=1$. Let $a^*=\sum\alpha_{ji}\,x_i\otimes x_j^\circ$. The left ideal $\mathfrak{l}$ which is generated by the finite set of non-zero elements $z_\kappa=\sum_j\alpha_{j\kappa}x_j$ has a finite rank over K and $1=\varphi(a^*)=\sum_{ij}\alpha_{ji}\,x_ix_j=\sum_i x_iz_i$ is contained in $\mathfrak{l}$. Hence we get $\mathfrak{l}=\Lambda$ and $[\Lambda : K]<\infty$.

For an element $a=\sum_{ij}\alpha_{ij}\,x_i\otimes x_j^\circ$ in Λ^e let $P_a=(\alpha_{ij})$ be a matrix with (i,j)-component α_{ij}. Let S be the left regular representation of Λ and R be the right regular representation of Λ, then for the infinite low vector $(x_1, x_2, \ldots, x_\tau, \ldots)$ consisting of basis elements of Λ and the infinite column vector $\begin{pmatrix}x_1\\x_2\\\vdots\\x_\tau\\\vdots\end{pmatrix}$

$$x\cdot(x_1, x_2, \ldots, x_\tau, \ldots)=(x_1, x_2, \ldots, x_\tau, \ldots)S(x),\qquad \begin{pmatrix}x_1\\x_2\\\vdots\\x_\tau\\\vdots\end{pmatrix}\cdot x=R(x)\begin{pmatrix}x_1\\x_2\\\vdots\\x_\tau\\\vdots\end{pmatrix}.$$

From Lemma 1, $a=\sum_{ij}\alpha_{ij}x_i\otimes x_j^\circ$ is in A if and only if $S(x_\tau)\,P_a=P_a\,R\,(x_\tau)$ for

1) For the definition, see [3] p. 4.

every x_τ. Therefore, we have

PROPOSITION 2. *An element* $a = \sum_{ij} a_{ij}\, x_i \otimes x_j^\circ$ *in* Λ^e *is contained in* A *if and only if* $S(x)\, P_a = P_a R(x)$ *for any* x *in* Λ.

2. LEMMA 2. $\varphi(A^*) = \Lambda$ *if and only if* $\Lambda^e = A^* \oplus \Lambda^e J$.

Proof. If $\Lambda^e = A^* \oplus \Lambda^e J$, then clearly $\varphi(A^*) = \Lambda$. We suppose $\varphi(A^*) = \Lambda$. Then from Corollary 3 $[\Lambda : K] = n < \infty$ and there exists $a^* = \sum_{ij} a_{ji} x_i \otimes x_j^\circ$ in A^* such that $\varphi(a^*) = 1$. As is shown in the proof of Corollary 3 Λ is generated by $z_\kappa = \sum_j a_{i\kappa} x_j$ $(\kappa = 1, 2, \ldots n)$ as K-module: $\Lambda = \sum_{\kappa=1}^{n} K z_\kappa$. Hence $z_1, z_2, \ldots z_n$ is a basis of Λ over K. Therefore $P_a = (a_{ij})$ is regular. Since a is contained in A, $S(x)\, P_a = P_a\, R(x)$ for any x in Λ. Hence Λ is a Frobenius algebra, and Λ^e is so (see [4], Th. 14). Since A is the right annihilator of $\Lambda^e J$ in Λ^e, we have $[A : K] + [\Lambda^e J : K] = [\Lambda^e : K]$. Since $*$ is anti-automorphism (also K-linear), we have $[A^* : K] = [A : K]$. From $\varphi(A^*) = \Lambda$ we have $\Lambda^e = A^* + \Lambda^e J$ and considering it over K we have $\Lambda^e = A^* \oplus \Lambda^e J$.

PROPOSITION 3. *Let* Λ *be an algebra over a field K. If* $\varphi(A^*) = \Lambda$, *then* Λ *is a separable algebra.*

Proof. Suppose $\varphi(A^*) = \Lambda$. By Lemma 2, $\Lambda^e = A^* \oplus \Lambda^e J$, and Λ is Λ^e-isomorphic to A^*. Therefore, Λ is Λ^e-projective, and hence, Λ is separable (see [2], Th. 7.10.).

Now we suppose that Λ is an algebra of finite rank over K, and $\bar{K}$ is the algebraic closure of K. If for Λ the base field K is extended to $\bar{K}$, we have $(\Lambda^{\bar{K}})^e = (\Lambda^e)^{\bar{K}}$, and $\{x \otimes 1^\circ - 1 \otimes x^\circ \mid x \in \Lambda^{\bar{K}}\} = J \otimes_K \bar{K} = J^{\bar{K}}$. By a theorem of simultaneous linear equations we can see that $(A^{\bar{K}})^*$ is the left annihilator of $J^{\bar{K}}$ in $(\Lambda^{\bar{K}})^e$. Therefore, we have

LEMMA 3. $\varphi(A^{\bar{K}*}) = \Lambda^{\bar{K}}$ *if and only if* $\varphi(A^*) = \Lambda$.

LEMMA 4. *Let* Λ *be the total matric algebra of degree n over K. If n is divisible by the characteristic of K then* $\varphi(A^*) = 0$, *and if n is not so then* $\varphi(A^*) = \Lambda$.

Proof. It is easily shown that $\varphi(A^*)$ is a two sided ideal of Λ. We denote the matrix units in Λ by e_{ij}, $i,j = 1,2, \ldots n$, and put

$$a^* = \sum_{ij=1}^{n} e_{ji} \otimes e^\circ_{ij}\ . \qquad \text{Then} \qquad \varphi(a^*) = \sum_{ij=1}^{n} e_{ij}\, e_{ji} = \sum_{i=1}^{n} n e_{ii} = n1\ .$$

Since $a^*(e_{\kappa l} \otimes 1^\circ - 1 \otimes e_{\kappa l}) = \sum_{ij=1}^{n} e_{ij} e_{\kappa l} \otimes e_{ji}^\circ - e_{ij} \otimes (e_{\kappa l} e_{ji})^\circ = \sum_{i=1}^{n} (e_{il} \otimes e_{\kappa i}^\circ - e_{il} \otimes e_{\kappa i}^\circ) = 0$ for every $e_{\kappa l}$, a^* is a non-zero element in A^*. If n is not divisible by the characteristic of K, then $\varphi(a^*)$ is non-zero element of K, and hence $\varphi(A^*) = \Lambda$. If n is divisible by the characteristic of K, then $\varphi(a^*) = 0$, hence a is contained in Ker $\varphi = \Lambda^e J$. Since Λ is a simple algebra, either $\varphi(A^*) = 0$ or $\varphi(A^*) = \Lambda$. If $\varphi(A^*) = \Lambda$, then from Lemma 2, $A^* \cap \Lambda^e J = 0$. This is impossible, since $a^* \in A^* \cap \Lambda^e J$. Hence $\varphi(A^*) = 0$.

LEMMA 5. *If* $\Lambda = \sum_{i=1}^{r} \oplus \Lambda_i$ *is a direct sum of two sided ideals* Λ_i, *and if* A_i *is the right annihilator of* $J = \{x \otimes 1^\circ - 1 \otimes x^\circ \mid x \in \Lambda_i\}$ *in* Λ_i^e, *then* $\varphi(A^*) = \sum_{i=1}^{r} \varphi(A_i^*)$.

Proof. It is sufficient to prove for $r=2$. We suppose $\Lambda = \Lambda_1 \oplus \Lambda_2$, then $\Lambda^e = \Lambda_1 \otimes_K \Lambda_1^\circ \oplus \Lambda_1 \otimes_K \Lambda_2^\circ \oplus \Lambda_2 \otimes_K \Lambda_1^\circ \oplus \Lambda_2 \otimes_K \Lambda_2^\circ$ is two sided ideal decomposition of Λ^e. Since A^* is a left ideal

$A^* = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \mathfrak{l}_3 \oplus \mathfrak{l}_4$ where $\mathfrak{l}_1 = A^* \cap \Lambda_1 \otimes \Lambda_1^\circ$,

$\mathfrak{l}_2 = A^* \cap \Lambda_1 \otimes \Lambda_2^\circ$, e.c.t. . But $A^* \cap \Lambda_1 \otimes \Lambda_1^\circ = A^* \cap \Lambda_1^e = A_1^*$, therefore $\mathfrak{l}_1 = A_1^*$, similarly $\mathfrak{l}_4 = A_2^*$. It follows that

$$\varphi(A^*) = \sum_{i=1}^{4} \varphi(\mathfrak{l}_i) = \varphi(A_1^*) + \varphi(A_2^*),$$

since $\varphi(\mathfrak{l}_2) \subset \varphi(\Lambda_1 \otimes \Lambda_2^\circ) = \Lambda_1\Lambda_2 = 0$ and similarly $\varphi(\mathfrak{l}_3) = 0$.

Now let Λ be a separable algebra over K and let $\Lambda = \Lambda_1 \oplus \dots \oplus \Lambda_r$ be its decomposition into simple ideals. If the degree of normal simple algebra Λ_i over its center (square root of rank of Λ_i over its center) is not multiple of the characteristic of K for every Λ_i, for convenience we call Λ a separable algebra with non divisible degrees by the characteristic of K (simply denoted by S.N.D.C.). If Λ_i is not so for every Λ_i, we call Λ a separable algebra with divisible degrees by the characteristic of K (S.D.C.).

THEOREM. *An algebra* Λ *over a field* K *is S.N.D.C. if and only if* $\varphi(A^*) = \Lambda$. *If* Λ *is a separable algebra and* $\varphi(A^*) = \mathfrak{A}$ *then* $\Lambda = \mathfrak{A} \oplus \mathfrak{A}'$ *where* $\mathfrak{A}$ *is S.N.D.C. and* $\mathfrak{A}'$ *is S.D.C. over K.*

Proof. We can assume that Λ is separable over K. From Lemma 3, we can suppose that K is algebraically closed. Then the theorem follows from Lemma 4.

COROLLARY 4. *If* Λ *is S.N.D.C. and C is the center of* Λ, *then*

$$\Lambda = C \oplus [\Lambda, \Lambda]$$

where $[\Lambda, \Lambda]$ *is the K-submodule of* Λ *generated by* $\{xy - yx \mid x, y \in \Lambda\}$

Proof. From Theorem and Lemma 2 we have $\varphi(A^*) = \Lambda$ and $\Lambda^e = A^* \oplus \Lambda^e J$. Hence $\Lambda^e = \Lambda^{e*} = A^{**} + (\Lambda^e J)^* = A + J\Lambda^e$. Since Λ is a separable algebra, $\varphi(A) = C$, and $\varphi(J\Lambda^e) = J$. $\varphi(\Lambda) = [\Lambda, \Lambda]$. Therefore, $\Lambda = C + [\Lambda, \Lambda]$.

Now extending the ground field K to its algebraic closure, and taking a simple component, we may assume that Λ is a total matric algebra over K of degree n which is prime to the characteristic of K. If $C \cap [\Lambda, \Lambda] \ni a \neq 0$ then

$$a = \begin{pmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & \alpha \end{pmatrix} \qquad (\alpha \neq 0).$$

Therefore, $\mathrm{Tr}\, a = n\alpha \neq 0$. On the other hand, since $a \in [\Lambda, \Lambda]$ $\mathrm{Tr}\, a = 0$. This is a contradiction. Thus we have $C \cap [\Lambda, \Lambda] = 0$, and $\Lambda = C \oplus [\Lambda, \Lambda]$.

COROLLARY 5. *Let* Λ *be a simple algebra over* K *with center* C.
Then $\varphi(A^*) = \Lambda$ *or* $\varphi(A^*) = 0$, *and*
a) if $\varphi(A^*) = \Lambda$, *then* $\Lambda = [\Lambda, \Lambda] \oplus C$,

b) *if* $\varphi(A^*)=0$, *then* $C\subset[\Lambda,\Lambda]$ *or* $A^2=A^*A=0$.

Proof. *a*) follows from Theorem and Corollary 4.

b) Since $\varphi(A)$ is the ideal of the center C (see [1], p. 369), $\varphi(A)=C$ or $\varphi(A)=0$. If $\varphi(A)=C$ and $\varphi(A^*)=0$ then $A^*\subset\Lambda^eJ$, $A=A^{**}\subset(\Lambda^eJ)^*=J\Lambda^e$ hence $\varphi(A)\subset\varphi(J\Lambda^e)$, $C\subset[\Lambda,\Lambda]$. If $\varphi(A)=\varphi(A^*)=0$ then $A\subset\Lambda^eJ$ and $A^*\subset\Lambda^eJ$. From the latter we have $A\subset J\Lambda^e$ and hence $A^2=A^*A=0$.

COROLLARY 6. *Let n be a positive integer which is not divisible by the characteristic of K. If a matrix X of degreen n has trace zero then X can be expressed as a sum of commutator of matrices of degree n* :

$$X=\sum_i(X_iY_i-Y_iX_i)$$

Proof. From Corollary 4, X can be expressed uniquely as a sum of a scalar αE_n and an element Y in $[K_n, K_n]$. Since $\mathrm{Tr}(X)=\mathrm{Tr}\ Y=0$, we have $\mathrm{Tr}(\alpha E_n)=n\alpha=0$. From the assumption on n, we have $\alpha=0$ and hence $X=Y\in[K_n, K_n]$.

References

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[3] N. JACOBSON, Structure of rings. (Amer. Math. Soc. Colloq. Publ. no. 37, 1956).

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Added in proof.

Let K be any commutative ring (not necessarily field) with unit element. Then concerning an algebra Λ over a commutative ring K, it is proved that some results of the above are true, that is, Lemma 2., Prop. 3. and Cor. 4. If Λ is K-projective as K-module, then Prop. 1 is also true.