Journal of Mathematics,
Osaka City University,
Vol. 13, No. 1.

# On a conjecture of Brauer for p-solvable groups

*Dedicated to Professor K. Shoda on his sixtieth birthday.*

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(*Received May 28, 1962*)

## 1. Introduction.

In the modular representation theory of groups for a prime $p$, the ordinary irreducible characters of a finite group are partitioned into $p$-blocks, and to every block $B$ there belong a defect $d$ and a defect group $\mathfrak{D}$ of order $p^d$. One of the conjectures which were asked by Brauer in [3] states as follows:

(*) *The number m of ordinary irreducible characters in a p-block of defect d is not larger than* $p^d$.

It was shown by Brauer and Feit [2], [4] that $m \leqq p^{2d-2}$ for $d>2$, and $m \leqq p^d$ if $d \leqq 2$ or $\mathfrak{D}$ is cyclic.

In this note, we shall consider the conjecture for p-solvable groups and prove the following:

THEOREM *The conjecture* (*) *for p-solvable groups is true if and only if the following statement is true:*

(**) *Suppose that a $p'$-group $\mathfrak{S}$, a $p'$-group being one of order prime to $p$, acts faithfully and irreduciblly on an elementary abelian $p$-group $\mathfrak{P}$ of order $p^a$, and let $\mathfrak{G}=\mathfrak{S}\mathfrak{P}$ be the semi-direct product of $\mathfrak{P}$ by $\mathfrak{S}$. Then the number of classes of conjugate elements of $\mathfrak{G}$ is not larger than $p^a$.*

Combining the reduction contained in the theorem and the result by Brauer for $d \leqq 2$, we have that the conjecture is true for a $p$-block *of* a p-solvable group with defect group which is metacyclic. To prove the theorem we use a reduction step by Fong and Osima and an inequality between the numbers of classes of conjugate elements.

## 2. An inequality between the class numbers.

Denote by $k(\mathfrak{G})$ the number of classes of conjugate elements of a group $\mathfrak{G}$.

LEMMA 1. *Let $\mathfrak{N}$ be a normal subgroup of a group $\mathfrak{G}$. Then $k(\mathfrak{G}) \leqq k(\mathfrak{N})\ k(\mathfrak{G}/\mathfrak{N})$.*

To prove the lemma, we need a lemma by Ado [1] and Ree [8]. For the completeness we shall give here a simple proof of the lemma which was outlined in [8].

Let $\sigma$ be an endomorphism of a group $\mathfrak{G}$. Two elements $A$ and $B$ are called $\sigma$-conjugate when there exists an element $X$ in $\mathfrak{G}$ such that $A=X^{-\sigma}BX$. The elements in $\mathfrak{G}$ are partitioned into the classes of $\sigma$-conjugate elements.

LEMMA 2 (Ado and Ree). *The number of classes of $\sigma$-conjugate elements in a finite group is equal to the number of $\sigma$-invariant classes of conjugate elements.*

*Proof.* Let $k_\sigma$ be the number of classes of $\sigma$-conjugate elements and $k^{(\sigma)}$ be the number of $\sigma$-invariant classes of conjugate elements. Consider the number $n$ of pairs $(A, X)$ of elements $A$ and $X$ in $\mathfrak{G}$ satisfying

(1) $A = X^{-\sigma}AX$, i.e.

(2) $AXA^{-1} = X^{\sigma}$.

For each $A$, the elements $X$ satisfying (1) form a subgroup $\mathfrak{N}_\sigma(A)$ and the index $[\mathfrak{G} : \mathfrak{N}_\sigma(A)]$ gives the number of elements $\sigma$-conjugate to $A$. Therefore the number of solutions $(A, X)$ of (1) with $A$ in a given class of $\sigma$-conjugate elements is equal to the order $g$ of $\mathfrak{G}$. Thus we have $n=gk_\sigma$.

On the other hand, for an element $X$, there is an element $A$ satisfying (2) if and only if the class of elements conjugate to $X$ is $\sigma$-invariant. For such an $X$ the order of the centralizer $\mathfrak{N}(X)$ of $X$ gives the number of elements $A$ satisfying (2). Therefore the number of solutions $(A, X)$ of (2) with $X$ in a given $\sigma$-invariant class of conjugate elements is $g$ and hence $n=gk^{(\sigma)}$. Thus we have $k_\sigma=k^{(\sigma)}$.

*Proof of Lemma* 1. Let $\{G_i\mathfrak{N}\}$ be a representative system of classes of conjugate elements in $\mathfrak{G}/\mathfrak{N}$, and denote by $\sigma(G)$ the automorphism of $\mathfrak{N}$ induced by the transformation by $G$. Since, for two elements $M$ and $N$ in $\mathfrak{N}$, $M^{-1}(G_iN)M = G_iM^{-\sigma(G_i)}NM$, the number of classes of conjugate elements which intersect a coset $G_i\mathfrak{N}$ is not larger than the number of classes of $\sigma(G_i)$ - conjugate elements in $\mathfrak{N}$ and hence by Lemma 2 it is not larger than $k(\mathfrak{N})$. Each class of conjugate elements in $\mathfrak{G}$ intersects an only coset $G_i\mathfrak{N}$, therefore we have the inequality.

### 3. Reduction step.

In the following, we assume that $\mathfrak{G}$ is always a $p$-solvable group of order $g=p^a g'$, where $(p, g') = 1$. Let $\mathfrak{H}$ be the maximal normal $p'$-subgroup of $\mathfrak{G}$. To a $p$-block $B$ there corresponds a family $\{\theta\}$ of irreducible characters of $\mathfrak{H}$ associated in $\mathfrak{G}$, and the restriction $\chi_\mu \mid \mathfrak{H}$ of any irreducible character $\chi_\mu$ in $B$ to $\mathfrak{H}$ decomposes as

$$\chi_\mu \mid \mathfrak{H}=r_\mu \sum \theta',$$

where the $\theta'$ are in $\{\theta\}$. When we want to prove the conjecture (*) for $p$-solvable groups by induction on the order, by Lemma 5 and 7 in [7] or Theorem (2B) in [5], we may assume that $\mathfrak{G}$ coincides with the inertial group of $\theta$, i.e. $\theta$ is the only irreducible character of $\mathfrak{H}$ associated in $\mathfrak{G}$ to $\theta$. By Theorem (2D) in [5] we may further assume that

(A) *$\mathfrak{H}$ is contained in the center of $\mathfrak{G}$.*

Now assume (A). As is shown in [5] §3, the defect of any p-block $B$ is $a$. Let $B = \{\chi_1, \ldots, \chi_m\}$. There corresponds a linear character $\theta$ of $\mathfrak{H}$ and $\chi_i \mid \mathfrak{H} = x_i\,\theta$ $(i = 1, 2, \ldots, m)$. If $G_1\mathfrak{H}, \ldots, G_{\bar{k}}\mathfrak{H}$ are representatives of classes of conjugate elements in $\mathfrak{G}/\mathfrak{H}$, any element $G$ in $\mathfrak{G}$ is conjugate to some $G_\mu H$ with $H$ in $\mathfrak{H}$ and then $\chi_i(G) = \chi_i(G_\mu)\,\theta(H)$. Consider the matrix $X = (\chi_i(G))$ with $i = 1, \ldots, m$ row indices, $G\,(\in\mathfrak{G})$ column indices. Since $m$ rows of $X$ are linearly independent, the rank of $X$ is $m$. On the other hand, each column is linearly dependent on $G_1$, . . . , $G_{\bar{k}}$-columns, therefore we have $m \leqq \bar{k} = k(\mathfrak{G}/\mathfrak{H})$. Thus in order to prove the conjecture (*) for p-solvable groups it is sufficient to prove the following statement:

*Under the assumption*

(B) $\mathfrak{G}$ *has only trivial normal* $p'$*-subgroup*,

*it holds that*

(***) *the number of classes of conjugate elements in* $\mathfrak{G}$ *is not larger than* $p^a$.

Now assume (B). Let $\mathfrak{P}_0$ be the maximal normal $p$-subgroup and $\mathfrak{M}$ the subgroup such that $\mathfrak{M}/\mathfrak{P}_0$ is the maximal normal $p'$-subgroup of $\mathfrak{G}/\mathfrak{P}_0$. Since $\mathfrak{G}/\mathfrak{M}$ has only trivial normal $p'$-subgroup and $k(\mathfrak{G}) \leqq k(\mathfrak{M})\ k(\mathfrak{G}/\mathfrak{M})$, when we want to prove (***) under the assumption (B) by induction on the order of $\mathfrak{G}$ we may assume that $\mathfrak{G} = \mathfrak{M}$, i.e.

(C) $\mathfrak{G}$ *has a normal Sylow p-subgroup* $\mathfrak{P}$.

Now assume (B) and (C), and let $\Phi(\mathfrak{P})$ be the Frattini subgroup of $\mathfrak{P}$. By Lemma 1. 2. 5. in [6], we can easily see that $\mathfrak{G}/\mathfrak{P}$ acts faithfully on $\mathfrak{P}/\Phi(\mathfrak{P})$. Since $k(\mathfrak{G}) \leqq k(\Phi(\mathfrak{P}))\ k(\mathfrak{G}/\Phi(\mathfrak{P}))$ by Lemma 1, in order to prove (***) under the assumption (B) we may assume that $\Phi(\mathfrak{P}) = 1$, i.e. we may prove (***) under the following assumptions:

(D) $\mathfrak{G}$ *has a normal Sylow p-subgroup* $\mathfrak{P}$ *which is elementary abelian*,

(B′) $\mathfrak{G}/\mathfrak{P}$ *acts faithfully on* $\mathfrak{P}$.

Now assume (B′), (D) and that $\mathfrak{P}$ has a proper subgroup $\mathfrak{P}_1$ ($\neq 1$) which is normal in $\mathfrak{G}$. Let $\mathfrak{S}$ be a $p$-complement in $\mathfrak{G}$, then there is an $\mathfrak{S}$-invariant subgroup $\mathfrak{P}_2$ such that $\mathfrak{P} = \mathfrak{P}_1 \times \mathfrak{P}_2$. Let $\mathfrak{C}(\mathfrak{P}_1)$ be the centralizer of $\mathfrak{P}_1$ in $\mathfrak{G}$. Then $\mathfrak{C}(\mathfrak{P}_1) = \mathfrak{S}_1\mathfrak{P}$ where $\mathfrak{S}_1 = \mathfrak{S} \cap \mathfrak{C}(\mathfrak{P}_1)$ and $\mathfrak{S}_1$ acts faithfully on $\mathfrak{P}_2$. Let $\mathfrak{M} = \mathfrak{S}_1\mathfrak{P}_2$. Then $\mathfrak{M}$ is a normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{M}$ and $\mathfrak{M}$ satisfy the assumptions (B′) and (D). Again by Lemma 1, we can see that to prove (***) by induction on the order of $\mathfrak{G}$ we may assume further

(E) $\mathfrak{G}$ *acts irreduciblly on* $\mathfrak{P}$.

Thus "if" part of the theorem is proved.

To show the "only if" part of the theorem, let $\mathfrak{G}$ be a group as in Theorem. Then $\mathfrak{G}$ has only trivial normal $p'$-subgroup and by Lemma (3A) in [5], $\mathfrak{G}$ has only one block $B$ of defect $a$. Therefore the statement (**) in Theorem is the conjecture itself for such a group $\mathfrak{G}$. Thus if the conjecture (*) is true for $p$-solvable groups then (**) is also true.

Now suppose that B is a $p$-block *of* a p-solvable group $\mathfrak{G}$ with defect group cyclic or metacyclic. Since in the reduction by Fong and Osima the defect group is left invariant and each group occurring in the reduction from (A) to (E) has Sylow $p$-subgroups which are also cyclic or metacyclic, we can apply the result of Brauer for $d \leqq 2$ to have

COROLLARY. *If a defect group of a p-block of p-solvable group is cyclic or metacyclic, then the conjecture* (*) *is true.*

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**Added in proof:** The author was informed by N. Ito that P. Fong obtained the same reduction theorem as in this note independently.