Journal of Mathematics,
Osaka City University,
Vol. 13, No. 1.

On root systems and an infinitesimal classification of irreducible symmetric spaces

by Shôrô ARAKI

(*Received May 8, 1962*)

Introduction. The classification of real simple Lie algebras was given first by E.Cartan [2] in 1914. Though his first classification lacked in general theorems, Cartan himself [5] established in 1929 a general theorem suitable to simplify the classification. Then Gantmacher [6] in 1939 gave a simplified classification depending on Cartan's general theorem by making use of his theory on canonical representation of automorphisms of complex semi-simple Lie groups.

In his earlier papers [5, 3] Cartan established *a priori* a one-one correspondence between non-compact real simple Lie algebras and irreducible infinitesimal symmetric spaces (compact or non-compact), where "infinitesimal" means locally isomorphic classes. Hence the infinitesimal classification of irreducible symmetric spaces is the same thing as the classification of non-compact real simple Lie algebras.

Let $\mathfrak{g}$ be a real semi-simple Lie algebra and $\mathfrak{k}$ a maximal compact subalgebra of $\mathfrak{g}$. Then we have a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k}+\mathfrak{p},$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form. In the classical theories of classification of real simple Lie algebras due to E.Cartan and Gantmacher, one used a Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{g}$ whose torus part $\mathfrak{h}_1 \cap \mathfrak{k}$ is maximal abelian in $\mathfrak{k}$, whereas certain geometric objects (such as roots, geodesics etc.) of symmetric spaces are related to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ whose vector part $\mathfrak{h} \cap \mathfrak{p}$ is maximal abelian in $\mathfrak{p}$ (cf., Cartan [4], Bott-Samelson [1] and Satake [7]). The two types of Cartan subalgebras mentioned above are not the same and even non-conjugate to each other in general. So it seems preferable to the author to have a classification theory by making use of the latter Cartan subalgebra, so as to connect it more closely with the theory of roots of symmetric spaces, and this will be developed in the present work.

If we denote by $\mathfrak{g}_C$ and $\mathfrak{h}_C$ the complexifications of $\mathfrak{g}$ and $\mathfrak{h}$ respectively, the conjugation of $\mathfrak{g}_C$ with respect to $\mathfrak{g}$ defines an involutive automorphism σ of the system $\mathfrak{r}$ of non-zero roots of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$. In **§1** are discussed some basic properties (Props. 1.1 and 1.3) of $\mathfrak{r}$ endowed with the invlution σ which are more or less known.

In **§2** we define root systems and σ-systems of roots in the abstract, and are sketched briefly some basic properties of them. Here is defined the notion that a σ-system of roots $\mathfrak{r}$ is normal, on the basis of Prop. 1.3, which covers the basic properties of the root system of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$ such that the Cartan subalgebra $\mathfrak{h}$ has a maximal vector part. If $\mathfrak{r}$ is normal, then the set $\mathfrak{r}^-$ (No. 2.4) forms a root system called a restricted root system (Prop. 2.1). In Nos. 2.6 and 2.7 are discussed some relations between multiplicities of $\lambda\in\mathfrak{r}^-$ and the inner products of roots of $\mathfrak{r}$ associated with λ for a normal σ-system of roots $\mathfrak{r}$. These relations are basic tools in the classification of normal σ-systems of roots with restricted rank 1 given in **§3.** In Nos. 2.9 and 2.10 are used the notions of a σ-order and a σ-fundamental system of $\mathfrak{r}$ due to Satake [7]. It is shown that two real simple Lie algebras are isomorphic to each other if and only if their σ-fundamental systems are σ-isomorphic, *via* Cor. 2.15 and Prop. 1.2. Consequently our task is reduced to classify all normally extendable (No. 2.3) σ-irreducible σ-fundamental systems up to σ-isomorphisms.

Now it becomes important to determine whether any given σ-fundamental system is normally extendable or not. In **§3** this problem is rduced to the case of normal σ-systems of roots with restricted rank 1 (Theo. 3.6). In **§4** first σ-fundamental systems of normal σ-systems of roots with restricted rank 1 are all classified, and then it is determined whether they are normally extendable or not, based on two Lemmas (Lemmas 4.6 and 4.7). Finally in **§5** the complete classification is achieved.

§1. Preliminaries.

1.1. Let **C** be the field of complex numbers and $\mathfrak{g}_C$ be a complex semi-simple Lie algebra. A real Lie subalgebra $\mathfrak{g}$ of $\mathfrak{g}_C$, whose complexification $\mathbf{C}\otimes\mathfrak{g}$ can be identified with $\mathfrak{g}_C$ by a map: $\alpha\otimes X\rightarrow\alpha X$ for $\alpha\in\mathbf{C}$ and $X\in\mathfrak{g}$, is usually called a real form of $\mathfrak{g}_C$.

Let $\mathfrak{g}$ be a real form of $\mathfrak{g}_C$. The conjugation σ of $\mathfrak{g}_C$ with respect to $\mathfrak{g}$, defined by $\sigma(\alpha X)=\bar{\alpha}X$ for $\alpha\in\mathbf{C}$ and $X\in\mathfrak{g}$, is an anti-involution of $\mathfrak{g}_C$, namely σ is an involutive automorphism of $\mathfrak{g}_C$ as a real Lie algebra and is anti-linear as a map of a vector space over **C.** Conversely, let σ be an anti-involution of $\mathfrak{g}_C$. Then the set $\mathfrak{g}$ consisting of all fixed elements of $\mathfrak{g}_C$ by σ is a real form of $\mathfrak{g}_C$, and σ is identical with the conjugation of $\mathfrak{g}_C$ with respect to $\mathfrak{g}$. Thus real forms and anti-involutions of $\mathfrak{g}_C$ are in a one-one correspondence in the above natural way. A real form corresponding to an anti-involution σ is denoted by $\mathfrak{g}_\sigma$. So, when we are considering a real form $\mathfrak{g}_\sigma$, σ always means the corresponding anti-involution of $\mathfrak{g}_C$.

1.2. Let $\mathfrak{h}_C$ be a Cartan subalgebra of $\mathfrak{g}_C$ and $\mathfrak{r}$ the system of non-zero roots of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$. We have the well known decomposition:

$$\mathfrak{g}_C=\mathfrak{h}_C+\sum_{\alpha\in\mathfrak{r}}\mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the eigenspace of $\alpha\in\mathfrak{r}$. A "Weyl base" $\{E_\alpha,\alpha\in\mathfrak{r}\}$ of $\mathfrak{g}_C$ is defined as to satisfy the following:

$$E_\alpha\in\mathfrak{g}_\alpha \text{ for } \alpha\in\mathfrak{r},$$
$$[E_\alpha, E_{-\alpha}] = -H_\alpha$$

where H_α $(\in\mathfrak{h}_C)$ is defined by $\langle H_\alpha, H\rangle=\alpha(H)$ for all $H\in\mathfrak{h}_C$,

$$[E_\alpha,\ E_\beta] = \begin{matrix} N_{\alpha,\beta}E_{\alpha+\beta} & \text{if } \alpha+\beta\in\mathfrak{r} \\ 0 & \text{if } \alpha+\beta\neq 0 \text{ and } \notin \mathfrak{r} \end{matrix}$$

such that $N_{\alpha,\beta} = N_{-\alpha,-\beta}$ is a real number for each pair $(\alpha,\ \beta)$.

Let $\mathfrak{g}_\tau$ be a compact real form of $\mathfrak{g}_C$, i.e., the Killing form is negative definite on it. We can choose a Cartan subalgebra $\mathfrak{h}_C$ and a Weyl base such that $\tau\mathfrak{h}_C = \mathfrak{h}_C$ and that $\tau E_\alpha = E_{-\alpha}$ for all $\alpha\in\mathfrak{r}$ (cf., [8], Exp. 11, théorème 2.2). Let us once for all fix $\mathfrak{g}_\tau$, $\mathfrak{h}_C$ and a Weyl base $\{E_\alpha,\ \alpha\in\mathfrak{r}\}$ in these relations.

A real form $\mathfrak{g}_\sigma$ of $\mathfrak{g}_C$ is said to be *related* to $(\mathfrak{g}_\tau,\ \mathfrak{h}_C)$ if $\sigma\mathfrak{h}_C = \mathfrak{h}_C$ and $\sigma\tau = \tau\sigma$. Let $\mathfrak{g}_\sigma$ be related to $(\mathfrak{g}_\tau,\ \mathfrak{h}_C)$. Putting $\mathfrak{k}_\sigma = \mathfrak{g}_\sigma \cap \mathfrak{g}_\tau$ (a maximal compact subalgebra of $\mathfrak{g}_\sigma$), we obtain the Cartan decompositions.

(1.1) $$\mathfrak{g}_\sigma=\mathfrak{k}_\sigma+\mathfrak{p}_\sigma, \qquad \mathfrak{g}_\tau=\mathfrak{k}_\sigma+\sqrt{-1}\ \mathfrak{p}_\sigma,$$

where $\mathfrak{p}_\sigma$ is the orthogonal complement of $\mathfrak{k}_\sigma$ in $\mathfrak{g}_\sigma$ with respect to the Killing form. $\mathfrak{g}_\sigma$ and $\mathfrak{g}_\tau$ are invariant by τ and σ respectively, and $\sigma\,|\,\mathfrak{g}_\tau$ (or $\tau\,|\,\mathfrak{g}_\sigma$) is an involutive automorphism of $\mathfrak{g}_\tau$ (or of $\mathfrak{g}_\sigma$), where $|$ denotes the restriction of a map to a subset. $(\mathfrak{g}_\tau,\ \mathfrak{k}_\sigma,\ \sigma\,|\,\mathfrak{g}_\tau)$ (or $(\mathfrak{g}_\sigma,\ \mathfrak{k}_\sigma,\ \tau\,|\,\mathfrak{g}_\sigma)$) is an infinitesimal compact (or non-compact) symmetric pair corresponding to $\mathfrak{g}_\sigma$.

1.3. Let a real form $\mathfrak{g}_\sigma$ of $\mathfrak{g}_C$ be related to $(\mathfrak{g}_\tau,\ \mathfrak{h}_C)$ as in the above No. σ induces an anti-involution σ^* of $\mathfrak{h}_C^*$ (dual space of $\mathfrak{h}_C$) defined by

$$(\sigma^*\varphi)\ (H) = \overline{\varphi(\sigma H)} \qquad \text{for all } H\in\mathfrak{h}_C.$$

For any $\alpha\in\mathfrak{r}$, $\sigma^*\alpha\in\mathfrak{r}$ and $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma^*\alpha}$. Putting

$$\sigma \mathrm{E}_\alpha = \varrho_\alpha \mathrm{E}_{\sigma^*\alpha} \qquad \text{for each } \alpha\in\mathfrak{r},$$

we see that

(1.2) $$\bar{\varrho}_\alpha\varrho_{\sigma^*\alpha} = 1,\ \varrho_\alpha\varrho_{-\alpha} = 1 \qquad \text{for any } \alpha\in\mathfrak{r},$$

(1.3) $$\varrho_{\alpha+\beta}N_{\alpha,\beta} = \varrho_\alpha\varrho_\beta N_{\sigma^*\alpha,\sigma^*\beta} \quad \text{for}\quad \alpha,\ \beta,\ \alpha+\beta\in\mathfrak{r},$$

since σ is an anti-involution, and that

(1.4) $$\bar{\varrho}_\alpha = \varrho_{-\alpha} \qquad \text{for any } \alpha\in\mathfrak{r}$$

since $\sigma\tau = \tau\sigma$. By (1.2) and (1.4) we see that

(1.5) $$|\ \varrho_\alpha\ | = 1 \qquad \text{for any}\quad \alpha\in\mathfrak{r}.$$

Put

(1.6) $$\mathfrak{r}_o = \{\alpha\in\mathfrak{r};\ \ \sigma^*\alpha = -\alpha\}.$$

$\mathfrak{r}_o$ is a closed subsystem of roots of $\mathfrak{r}$. (1.2) and (1.5) imply that

(1.7) $$\varrho_\alpha = \varrho_{-\alpha} = \pm 1 \qquad \text{for}\quad \alpha\in\mathfrak{r}_o.$$

Let $\mathfrak{k}_C$ and $\mathfrak{p}_C$ be the complexifications of $\mathfrak{k}_\sigma$ and $\mathfrak{p}_\sigma$ in $\mathfrak{g}_C$. We have the following orthogonal decompositions

(1.8) $$\mathfrak{g}_C = \mathfrak{k}_C+\mathfrak{p}_C,\quad \mathfrak{h}_C = \mathfrak{h}_C^+ + \mathfrak{h}_C^-,$$

where $\mathfrak{h}_C^+ = \mathfrak{h}_C\cap\mathfrak{k}_C$ and $\mathfrak{h}_C^- = \mathfrak{h}_C\cap\mathfrak{p}_C$. $\sigma\tau X = X$ for $X\in\mathfrak{k}_C$, whereas $\sigma\tau X = -X$ for $X\in\mathfrak{p}_C$. Let $\alpha\in\mathfrak{r}_o$; if $\varrho_\alpha = 1$, then $E_\alpha,\ E_{-\alpha}\in\mathfrak{k}_C$ since $\sigma\tau(E_{\pm\alpha}) = E_{\pm\alpha}$; similarly, if $\varrho_\alpha = -1$,then $E_\alpha,\ E_{-\alpha}\in\mathfrak{p}_C$. In case $\alpha\in\mathfrak{r}-\mathfrak{r}_o$, we see easily that $E_\alpha+\sigma E_{-\alpha}\in\mathfrak{k}_C$ and $E_\alpha-\sigma E_{-\alpha}\in\mathfrak{p}_C$, and that

$$\mathfrak{g}_\alpha+\mathfrak{g}_{-\sigma^*\alpha} = \mathbf{C}\{E_\alpha+\sigma E_{-\alpha}\}+\mathbf{C}\{E_\alpha-\sigma E_{-\alpha}\}$$

where $\mathbf{C}\{X\}$ denotes a 1-dimensional vector space over $\mathbf{C}$ generated by X. From these we have the following decompositions

$$(1.9) \qquad \mathfrak{k}_C = \mathfrak{h}_C^+ + \sum_{\alpha\in\mathfrak{r}_o^+} \mathfrak{g}_\alpha + \sum_{\alpha\in\mathfrak{r}-\mathfrak{r}_o} \mathbf{C}\{E_\alpha+\sigma E_{-\alpha}\},$$

$$(1.10) \qquad \mathfrak{p}_C = \mathfrak{h}_C^- + \sum_{\alpha\in\mathfrak{r}_o^-} \mathfrak{g}_\alpha + \sum_{\alpha\in\mathfrak{r}-\mathfrak{r}_o} \mathbf{C}\{E_\alpha-\sigma E_{-\alpha}\},$$

where $\mathfrak{r}_o^+ = \{\alpha\in\mathfrak{r}_o;\ \varrho_\alpha = 1\}$, $\mathfrak{r}_o^- = \{\alpha\in\mathfrak{r}_o;\ \varrho_\alpha = -1\}$ and the last summations of both formulas run over all unordered pairs $(\alpha, -\sigma^*\alpha)$ such that $\alpha\in\mathfrak{r}-\mathfrak{r}_o$. (1.9) and (1.10) imply immediately the

PROPOSITION 1.1 *$\mathfrak{h}_C^-$ is maximal abelian in $\mathfrak{p}_C$ if and only if $\varrho_\alpha = 1$ for all $\alpha\in\mathfrak{r}_o$.*

The "only if" part of this proposition is the same as the Lemma 5 of [8], Exp. 11.

DEFINITION. Let a real form $\mathfrak{g}_\sigma$ be related to $(\mathfrak{g}_\tau, \mathfrak{h}_C)$. When $\mathfrak{h}_C^-$ is maximal abelian in $\mathfrak{p}_C$, $\mathfrak{g}_\sigma$ is called to be *normally related* to $(\mathfrak{g}_\tau, \mathfrak{h}_C)$.

1.4. Let $\mathfrak{g}_{\sigma'}$ be a real form of $\mathfrak{g}_C$ and $\mathfrak{k}'$ be a maximal compact subalgebra of $\mathfrak{g}_{\sigma'}$. Then, as is well known, there exists a uniquely determined compact form $\mathfrak{g}_{\tau'}$ of $\mathfrak{g}_C$ such that $\mathfrak{g}_{\sigma'}\cap\mathfrak{g}_{\tau'}=\mathfrak{k}'$ and that $\sigma'\tau'=\tau'\sigma'$. Let $\mathfrak{p}'$ be the orthogonal complement of $\mathfrak{k}'$ in $\mathfrak{g}_{\sigma'}$ with respect to the Killing form, and $\mathfrak{h}'^-$ be a maximal abelian subalgebra of $\mathfrak{p}'$. Choose a Cartan subalgebra $\mathfrak{h}_{C'}$ of $\mathfrak{g}_C$ so that it contains $\mathfrak{h}'^-$. By conjugacies of compact forms and of Cartan subalgebras of $\mathfrak{g}_C$ we have an inner automorphism φ of $\mathfrak{g}_C$ such that $\varphi\mathfrak{g}_{\tau'}=\mathfrak{g}_\tau$ and that $\varphi\mathfrak{h}'_C=\mathfrak{h}_C$. Then, putting $\sigma=\varphi\sigma'\varphi^{-1}$, we see that $\varphi\mathfrak{g}_{\sigma'}=\mathfrak{g}_\sigma$, $\sigma\tau=\tau\sigma$, $\sigma\mathfrak{h}_C = \mathfrak{h}_C$, and that $\mathfrak{h}_C^- = \varphi\mathfrak{h}_C'^-$ is maximal abelian in $\mathfrak{p}_C = \varphi\mathfrak{p}_C'$, i.e., $\mathfrak{g}_\sigma$ is normally related to $(\mathfrak{g}_\tau, \mathfrak{h}_C)$. Hence we have the

PROPOSITION 1.2. *Any real form of $\mathfrak{g}_C$ is conjugate to a real form which is normally related to the fixed $(\mathfrak{g}_\tau, \mathfrak{h}_C)$.*

This proposition justifies us to discuss only the real forms which are normally related to the fixed $(\mathfrak{g}_\tau, \mathfrak{h}_C)$.

1.5. The following proposition, essentially due to Satake [7], is important for our later discussions.

PROPOSITION 1.3. *Let a real form $\mathfrak{g}_\sigma$ be normally related to $(\mathfrak{g}_\tau, \mathfrak{h}_C)$. Then*

$$\sigma^*\alpha-\alpha\notin\mathfrak{r} \qquad \textit{for all } \alpha\in\mathfrak{r}.$$

Proof. In case $\alpha\in\mathfrak{r}_o$, then $\sigma^*\alpha = -\alpha$ and

$$\sigma^*\alpha-\alpha = -2\alpha \notin \mathfrak{r}.$$

In case $\alpha\in\mathfrak{r}-\mathfrak{r}_o$: suppose that $\sigma^*\alpha-\alpha\in\mathfrak{r}$, and hence $\in\mathfrak{r}_o$, then

$$(\sharp) \qquad \sigma E_{\alpha-\sigma^*\alpha} = E_{\sigma^*\alpha-\alpha}$$

by Prop. 1.1. Now

$$[E_\alpha, \sigma E_{-\alpha}] = [E_\alpha, \varrho_{-\alpha}E_{-\sigma^*\alpha}] = \varrho_{-\alpha}N_{\alpha,-\sigma^*\alpha}E_{\alpha-\sigma^*\alpha}.$$

Applying σ on both sides of this identity, we see that

$$[\sigma E_\alpha, E_{-\alpha}] = \overline{\varrho_{-\alpha}}N_{\alpha,-\sigma^*\alpha}E_{\sigma^*\alpha-\alpha}$$

by ($\sharp$). On the other hand we have that

$$[\sigma E_\alpha, E_{-\alpha}] = \varrho_\alpha N_{\sigma^*\alpha,-\alpha}E_{\sigma^*\alpha-\alpha}.$$

Hence we see that

$$\overline{\varrho_{-\alpha}}N_{\alpha,-\sigma^*\alpha} = \varrho_\alpha N_{\sigma^*\alpha,-\alpha} = -\varrho_\alpha N_{\alpha,-\sigma^*\alpha}.$$

Therefore, $\overline{\varrho_{-\alpha}} = -\varrho_\alpha$, which, combined with (1.4), implies that $\varrho_\alpha = 0$. This contradicts to the fact "σ is bijective". Consequently

$$\sigma^*\alpha - \alpha \notin \mathfrak{r}.$$

§ 2. σ-system of roots.

2.1. Let V be a finite dimensional real vector space with a positive definite inner product , and $\mathfrak{r}$ a finite set of non-zero vectors in V. $\mathfrak{r}$ is called a *root system* if it satisfies the conditions: for any $\alpha,\beta\in\mathfrak{r}$, a) the number $a_{\alpha,\beta}=2\langle\alpha,\beta\rangle/\langle\beta,\beta\rangle$ is an integer (named *Cartan integer*), and b) $\alpha-a_{\alpha,\beta}\beta\in\mathfrak{r}$.

The following properties of a root system $\mathfrak{r}$ are well known from the usual theory of classification of complex simple Lie algebras.

1°) For any $\alpha,\beta\in\mathfrak{r}$, $0\leqq a_{\alpha,\beta}a_{\beta,\alpha}\leqq 4$. If α and β are not parallel to each other, then $0\leqq a_{\alpha,\beta}a_{\beta,\alpha}\leqq 3$.

2°) Let $\alpha\in\mathfrak{r}$ and $m\alpha\in\mathfrak{r}$ $(m\in\mathbf{R})$, then $m=\pm 1/2$, ± 1 or ± 2.

(To be seen from the fact that $a_{\alpha,m\alpha}=2/m$ and $a_{m\alpha,\alpha}=2m$ are both integers.)

3°) Let $\alpha,\beta\in\mathfrak{r}$ be such that $\langle\alpha,\beta\rangle\neq 0$, and that α and β are not parallel to each other. a) If $a_{\alpha,\beta}a_{\beta,\alpha}=1$, then $\langle\alpha,\alpha\rangle=\langle\beta,\beta\rangle$; b) if $a_{\alpha,\beta}a_{\beta,\alpha}=2$, then $\langle\alpha,\alpha\rangle/\langle\beta,\beta\rangle=2$ or $1/2$; c) if $a_{\alpha,\beta}a_{\beta,\alpha}=3$, then $\langle\alpha,\alpha\rangle/\langle\beta,\beta\rangle=3$ or $1/3$.

4°) For $\alpha,\beta\in\mathfrak{r}$, consider the series

$$\ldots,\ \alpha-2\beta,\ \alpha-\beta,\ \alpha,\ \alpha+\beta,\ \alpha+2\beta,\ldots,$$

then every non-zero vector between α and $\alpha-a_{\alpha,\beta}\beta$ is contained in $\mathfrak{r}$.

(This property can be proved for an abstractly defined root system by making use of the property 3°)).

5°) If $\alpha\in\mathfrak{r}$, then $-\alpha\in\mathfrak{r}$.

A root system $\mathfrak{r}$ is called a *proper* root system if it satisfies the condition that "if α, $m\alpha\in\mathfrak{r}$ $(m\in\mathbf{R})$, then $m=\pm 1$". The systems of non-zero roots of complex semi-simple Lie algebras are proper root systems. For a root system $\mathfrak{r}$, the following set

$$\mathfrak{r}'=\{\alpha\in\mathfrak{r};\ \alpha/2\notin\mathfrak{r}\}$$

is a proper root system which is called a *canonical* proper subsystem of $\mathfrak{r}$.

2.2. Let $\mathfrak{r}$ be a root system in V. A linear base of V^* (dual space of V) defines a lexicographic order in V and hence in $\mathfrak{r}$. The set Δ of all simple roots (in the usual sense due to Dynkin) with respect to a lexicographic order is called a fundamental system of $\mathfrak{r}$ as usual. For any two $\alpha,\beta\in\Delta$, $\langle\alpha,\beta\rangle\leqq 0$ because $\alpha-\beta\notin\mathfrak{r}$. This property of Δ shows firstly that the elements of Δ are linearly independent and any element of $\mathfrak{r}$ can be expressed uniquely as a linear combination of simple roots with integers of the same signs as coefficients, and secondly that Δ is isomorphic to a fundamental system of roots of a complex semi-simple Lie algebra $\mathfrak{g}_C$ up to a homography, by the usual arguments in the classification theory of complex simple Lie algebras. The image of the root system of $\mathfrak{g}_C$ by this isomorphism is the canonical proper subsystem $\mathfrak{r}'$ of $\mathfrak{r}$. In particular fundamental systems of $\mathfrak{r}'$ and of $\mathfrak{r}$ with respect to the same linear order coincides to each other. And any proper root system is isomorphic to a root system of a suitable complex semi-simple Lie algebra up to a homography.

For a fundamental system of $\mathfrak{r}$ the set $\{X\in V^*;\ \alpha(X)>0$ for all $\alpha\in\Delta\}$ is an open Weyl chamber. Weyl group of $\mathfrak{r}$ is the group operating on V^* (or dually on V)

generated by reflections across the plane $\alpha = 0$ for all $\alpha \in \mathfrak{r}$, which is a finite group and permutes simply transitively the Weyl chambers and henceforce the fundamental systems of $\mathfrak{r}$. In particular the fundamental systems of a fixed $\mathfrak{r}$ are isomorphic to each other.

Two proper root systems $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are isomoprphic to each other (up to a homography) if and only if their fundamental systems are so.

2.3. Let $\mathfrak{r}$ be a proper root system in V such that $\mathfrak{r}$ generates V. When we are given a linear isometry $\sigma: V \rightarrow V$ such that σ is involutive and $\sigma\mathfrak{r} = \mathfrak{r}$, then we say that the pair $(\mathfrak{r}, \sigma)$ (or simply $\mathfrak{r}$) is a *σ-system of roots.*

Every σ-system of roots $\mathfrak{r}$ can be identified with the root system of a complex semi-simple Lie algebra $\mathfrak{g}_C$ up to a homography. By this identification V is identified with a real subspace of $\mathfrak{h}_C{}^*$ which is generated by roots and metrized by the Killing form, where $\mathfrak{h}_C{}^*$ is the dual space of a Cartan subalgebra $\mathfrak{h}_C$, and V^* is identified with a real subspace $\mathfrak{h}_o$ defined by

$$\mathfrak{h}_o = \{H \in \mathfrak{h}_C;\ \alpha(H) \in \mathbf{R} \text{ for all } \alpha \in \mathfrak{r}\}.$$

$\mathfrak{h}_C$ becomes the complexification of $\mathfrak{h}_o$. Hereafter we use this identification without any comments. σ can be extended to an anti-involution of $\mathfrak{h}_C{}^*$ and induces an anti-involution $\tilde{\sigma}$ of $\mathfrak{h}_C$ such that $\tilde{\sigma}\mathfrak{h}_o = \mathfrak{h}_o$ and $\tilde{\sigma}^* = \sigma$. Since $\tilde{\sigma} \mid \mathfrak{h}_o$ is involutive, it has eigenvalues ± 1. Let $\mathfrak{h}_o^+$ be the eigenspace of the value -1 and $\mathfrak{h}_o^-$ be that of $+1$, then we have an orthogonal decomposition

$$\mathfrak{h}_o = \mathfrak{h}_o^+ + \mathfrak{h}_o^-$$

with respect to the Killing form.

Let a Weyl base $\{E_\alpha, \alpha \in \mathfrak{r}\}$ and a compact form $\mathfrak{g}_\tau$ of $\mathfrak{g}_C$ be given so as to satisfy the relations in No. 1.2. When for a σ-system of roots $\mathfrak{r}$ the induced involution $\tilde{\sigma}$ of $\mathfrak{h}_o$ is extendable to an anti-involution of $\mathfrak{g}_C$ such that it is normally related to $(\mathfrak{g}_\tau, \mathfrak{h}_C)$, then we say that σ (or $\mathfrak{r}$) is *normally extendable* for the sake of simplicity. We want to obtain some sufficient conditions for σ to be normally extendable. A necessary condition for this (Prop. 1.3) is that

(ν) *for any* $\alpha \in \mathfrak{r}$, $\sigma\alpha - \alpha \notin \mathfrak{r}$.

Any σ-system of roots satisfying the condition (ν) is called a *normal* σ-system of roots.

2.4. Let $\mathfrak{r}$ be σ-system of roots. We shall denote by $\mathfrak{r}^-$ the set of linear forms on $\mathfrak{h}_o^-$ obtained by restricting the elements of $\mathfrak{r} - \mathfrak{r}_o$ to $\mathfrak{h}_o^-$, where $\mathfrak{r}_o$ is the closed subsystem of $\mathfrak{r}$ defined by (1.6), i.e., $\mathfrak{r}_o = \{\alpha \in \mathfrak{r};\ \alpha \mid \mathfrak{h}_o^- = 0\}$.

PROPOSITION 2.1. *Let a σ-system of roots $\mathfrak{r}$ be normal, then $\mathfrak{r}^-$ is a root system in $(\mathfrak{h}_o^-)^*$.*

Proof. We identify $(\mathfrak{h}_o^-)^*$ with a subspace of $\mathfrak{h}_o{}^*$ which is the annihilator of $\mathfrak{h}_o^+$. Let us use the following notation:

$$\mathfrak{r}_\psi = \{\alpha \in \mathfrak{r};\ \alpha \mid \mathfrak{h}_o^- = \psi\}$$

for each $\psi \in \mathfrak{r}^-$. Let $\psi \in \mathfrak{r}^-$ and $\alpha \in \mathfrak{r}_\psi$. By the condition ($\nu$) only the following three cases are possible:

case a) $\alpha = \sigma\alpha = \psi$, then $\langle \alpha, \alpha \rangle = \langle \psi, \psi \rangle$;

case b) $\alpha \neq \sigma\alpha$ and $\langle \alpha, \sigma\alpha \rangle = 0$, then $\psi = (\alpha + \sigma\alpha)/2$

and $\langle\psi, \psi\rangle = \langle\alpha, \alpha\rangle/2 = \langle\sigma\alpha, \sigma\alpha\rangle/2$;

case c) $\alpha\neq\sigma\alpha$ and $\langle\alpha, \sigma\alpha\rangle < 0$, then $\psi = (\alpha+\sigma\alpha)/2$ and $\langle\psi, \psi\rangle = \langle\alpha, \alpha\rangle/4 = \langle\sigma\alpha, \sigma\alpha\rangle/4$.

Let $\lambda\in\mathfrak{r}^-$ and $\beta\in\mathfrak{r}_\lambda$.

In case a): $a_{\lambda,\psi} = 2\langle\lambda, \psi\rangle/\langle\psi, \psi\rangle = 2\langle\beta, \alpha\rangle/\langle\alpha, \alpha\rangle$
$= a_{\beta,\alpha}$, an integer.

Since $\beta - a_{\beta,\alpha}\alpha\in\mathfrak{r}$,

$$(\beta - a_{\beta,\alpha}\alpha)\,|\,\mathfrak{h}_o^- = \lambda - a_{\lambda,\psi}\psi\in\mathfrak{r}^-.$$

In case b):
$$\begin{aligned} a_{\lambda,\psi} &= 2\langle\lambda, \psi\rangle/\langle\psi, \psi\rangle = \langle\lambda, \alpha+\sigma\alpha\rangle/\langle\psi, \psi\rangle \\ &= \langle\beta, \alpha+\sigma\alpha\rangle/\langle\psi, \psi\rangle \\ &= 2\langle\beta, \alpha\rangle/\langle\alpha, \alpha\rangle + 2\langle\beta, \sigma\alpha\rangle/\langle\sigma\alpha, \sigma\alpha\rangle \\ &= a_{\beta,\alpha} + a_{\beta,\sigma\alpha}, \text{ an integer.}\end{aligned}$$

Put $\gamma = \beta - a_{\beta,\alpha}\alpha\in\mathfrak{r}$. Now
$$\begin{aligned} a_{\gamma,\sigma\alpha} &= 2\langle\beta - a_{\beta,\alpha}\alpha, \sigma\alpha\rangle/\langle\sigma\alpha, \sigma\alpha\rangle \\ &= 2\langle\beta, \sigma\alpha\rangle/\langle\sigma\alpha, \sigma\alpha\rangle = a_{\beta,\sigma\alpha}\end{aligned}$$
since $\langle\alpha, \sigma\alpha\rangle = 0$. Consequently
$$\delta = \gamma - a_{\gamma,\sigma\alpha}\sigma\alpha = \beta - a_{\beta,\alpha}\alpha - a_{\beta,\sigma\alpha}\sigma\alpha\in\mathfrak{r},$$
and
$$\delta\,|\,\mathfrak{h}_o^- = \lambda - a_{\beta,\alpha}\psi - a_{\beta,\sigma\alpha}\psi = \lambda - a_{\lambda,\psi}\psi\in\mathfrak{r}^-.$$

In case c):
$$\begin{aligned} a_{\lambda,\psi} &= 2\langle\lambda, \psi\rangle/\langle\psi, \psi\rangle = \langle\alpha+\sigma\alpha, \lambda\rangle/\langle\psi, \psi\rangle \\ &= 2(2\langle\alpha, \beta\rangle/\langle\alpha, \alpha\rangle + 2\langle\sigma\alpha, \beta\rangle/\langle\sigma\alpha, \sigma\alpha\rangle) \\ &= 2(a_{\beta,\alpha} + a_{\beta,\sigma\alpha}), \text{ an integer.}\end{aligned}$$

Put $\beta' = \beta - a_{\beta,\alpha}\alpha\in\mathfrak{r}$. Remarking that
$$a_{\alpha,\sigma\alpha} = 2\langle\alpha, \sigma\alpha\rangle/\langle\sigma\alpha, \sigma\alpha\rangle = -1,$$
we see that
$$a_{\beta',\sigma\alpha} = 2\langle\beta', \sigma\alpha\rangle/\langle\sigma\alpha, \sigma\alpha\rangle = a_{\beta,\sigma\alpha} + a_{\beta,\alpha}.$$

Next, put $\beta'' = \beta' - a_{\beta',\sigma\alpha}\sigma\alpha\in\mathfrak{r}$, then
$$\begin{aligned} a_{\beta'',\alpha} &= a_{\beta',\alpha} + a_{\beta',\sigma\alpha} \\ &= a_{\beta,\alpha} - 2a_{\beta,\alpha} + a_{\beta,\sigma\alpha} + a_{\beta,\alpha} = a_{\beta,\sigma\alpha}.\end{aligned}$$

Finally put $\beta''' = \beta'' - a_{\beta'',\alpha}\alpha\in\mathfrak{r}$, then
$$\begin{aligned} \beta'''\,|\,\mathfrak{h}_o^- &= \lambda - (a_{\beta,\alpha} + a_{\beta,\sigma\alpha} + a_{\beta'',\alpha})\psi \\ &= \lambda - 2(a_{\beta,\alpha} + a_{\beta,\sigma\alpha})\,\psi \\ &= \lambda - a_{\lambda,\psi}\psi\in\mathfrak{r}^-.\end{aligned}$$
q.e.d.

When $\mathfrak{r}$ is a normal σ-system of roots, $\mathfrak{r}^-$ is called the *restricted root system* with respect to $\mathfrak{h}_o^-$. Further when $\mathfrak{r}$ is normally extendable such that $\sigma\neq\tau^*$, then the root system $\mathfrak{r}^-$ in $(\mathfrak{h}_o^-)^*$ is usually called the root system of the corresponding infinitesimal symmetric pair $(\mathfrak{g}_\tau, \mathfrak{k}_{\tilde\sigma}, \tilde\sigma\,|\,\mathfrak{g}_\tau)$ (or $(\mathfrak{g}_{\tilde\sigma}, \mathfrak{k}_{\tilde\sigma}, \tau\,|\,\mathfrak{g}_{\tilde\sigma})$) with respect to the Cartan subalgebra $\sqrt{-1}\,\mathfrak{h}_o^-$ of $\sqrt{-1}\,\mathfrak{p}_{\tilde\sigma}$ (or $\mathfrak{h}_o^-$ of $\mathfrak{p}_{\tilde\sigma}$).

2.5. For any root system $\mathfrak{r}$, a subset $\mathfrak{r}'\subset\mathfrak{r}$ is called a closed subsystem if it satisfies that i) if $\alpha\in\mathfrak{r}'$ then $-\alpha\in\mathfrak{r}'$, and that ii) if $\alpha, \beta\in\mathfrak{r}'$ and $\alpha+\beta\in\mathfrak{r}$ then $\alpha+\beta\in\mathfrak{r}'$, Closed subsystems are also root systems as is easily seen.

If a root system $\mathfrak{r}$ is decomposable as a disjoint union of two subsets $\mathfrak{r}'$ and $\mathfrak{r}''$ such that $\langle \mathfrak{r}', \mathfrak{r}'' \rangle = 0$, then $\mathfrak{r}'$ (and $\mathfrak{r}''$) is called a factor of $\mathfrak{r}$. Every factor is a closed subsystem. A root system $\mathfrak{r}$ is called *irreducible* if every factor of r coincides with $\mathfrak{r}$ or is void. Every root system is decomposable into a disjoint union of non-voidal irreducible factors; then each factor is called a *component* of $\mathfrak{r}$. Any irreducible proper root system is isomorphic to a root system of a complex simple Lie algebra. About the length of roots of an irreducible proper root system $\mathfrak{r}$ we have the following two cases as is well known:

a) every root of $\mathfrak{r}$ has the same length, (in this case $\mathfrak{r}$ is called "simply-laced" according to a terminology of R. Bott);

b) $\mathfrak{r}$ is decomposed as a disjoint union of two non-voidal subsets $\mathfrak{r}_1$ and $\mathfrak{r}_2$ such that every root of $\mathfrak{r}_i$ (i=1 or 2) has the same length and that any root of $\mathfrak{r}_1$ is shorter than roots of $\mathfrak{r}_2$ (in this case $\mathfrak{r}$ is called "doubly-laced").

In case b) we have further two possibilities that the ratio of squares of length of roots of $\mathfrak{r}_1$ to that of $\mathfrak{r}_2$ is 1:2 or 1:3. We call $\mathfrak{r}$ to be "doubly-laced of type (1:2)" or "of type (1:3)" respectively. Every doubly-laced root system of type (1:3) is isomorphic to that of $\mathbf{G}_2$.

If a factor $\mathfrak{r}'$ of a σ-system of roots $\mathfrak{r}$ is invariant by σ, then $\mathfrak{r}'$ is called a *σ-factor*. A σ-system of roots $\mathfrak{r}$ is called *σ-irreducible* if every σ-factor of $\mathfrak{r}$ coincides with $\mathfrak{r}$ or is void. Every σ-irreducible σ-system of roots consists whether of a single component or of two isomorphic components. In the latter case σ permutes the two components. A non-voidal σ-irreducible σ-factor of a σ-system of roots is called a *σ-component*.

Let a σ-system of roots $\mathfrak{r}$ be such that the associated involution $\tilde{\sigma}$ of $\mathfrak{h}_0$ is extendable to an anti-involution of $\mathfrak{g}_C$. Then $\mathfrak{r}$ is σ-irreducible if and only if the corresponding real form $\mathfrak{g}_{\tilde{\sigma}}$ is simple, or equivalently the corresponding symmetric space is irreducible.

Let $\mathfrak{r}$ be a root system and $\mathfrak{r} \supset \mathfrak{r}'$ a subset. $\mathfrak{r}'$ is called to be *connected* if it cannot be decomposed into two mutually orthogonal parts. Thus any two elements α, β of $\mathfrak{r}$ is connected if $\langle \alpha, \beta \rangle \neq 0$. Every connected subset is contained in a single component.

Similarly we define the notion of *σ-connected* subset of a σ-system of roots. If a σ-invariant subset $\mathfrak{r}'$ of a σ-system of roots $\mathfrak{r}$ cannot be decomposed into two mutually orthogonal parts which are σ-invariant, then we say that $\mathfrak{r}'$ is σ-connected. Every σ-connected subset is contained in a σ-component.

2.6. Throughout this and the next Nos., σ-systems of roots $\mathfrak{r}$ are assumed to be normal. For $\psi \in \mathfrak{r}^-$, we put

$$\mathfrak{r}_\psi = \{\alpha \in \mathfrak{r};\ \alpha \mid \mathfrak{h}_o^- = \psi\}.$$

The number of elements of $\mathfrak{r}_\psi$ is called the multiplicity of ψ, and is denoted by $m(\psi)$. The elements of $\mathfrak{r}$ are called the roots of $\mathfrak{r}$ *associated to* the restricted root $\psi \in \mathfrak{r}^-$. We shall discuss some relations between multiplicities of restricted roots $\psi \in \mathfrak{r}^-$, and the inner products of roots of $\mathfrak{r}$ associated to ψ, which are especially useful for the classification of symmetric pairs of rank 1 (§**4**).

We identify $(\mathfrak{h}_o^+)^*$ and $(\mathfrak{h}_o^-)^*$ with the metric subspaces of $\mathfrak{h}_o^*$ which are the annihilators of $\mathfrak{h}_o^-$ and of $\mathfrak{h}_o^+$ respectively in the natural way. Then we have the orthogonal decomposition

$$\mathfrak{h}_o{}^* = (\mathfrak{h}_o^+)^* + (\mathfrak{h}_o^-)^*.$$

We use the notations $\hat{\alpha}$ and $\tilde{\alpha}$ for $\alpha \in \mathfrak{h}_o{}^*$ to denote each components of α, i.e., $\alpha = \hat{\alpha} + \tilde{\alpha}$, $\hat{\alpha} \in (\mathfrak{h}_o^-)^*$ and $\tilde{\alpha} \in (\mathfrak{h}_o^-)^*$.

PROPOSITION 2.2. *For $\psi \in \mathfrak{r}^-$, $\mathfrak{r}_\psi \ni \psi$ if and only if $m(\psi)$ is odd.*

Proof. If $\alpha \in \mathfrak{r}_\psi$, then $\sigma\alpha \in \mathfrak{r}_\psi$. Further, if $\alpha \in \mathfrak{r}_\psi - \{\psi\}$, then $\alpha \neq \sigma\alpha$. Hence $\mathfrak{r}_\psi - \{\psi\}$ is a disjoint union of pairs $(\alpha, \sigma\alpha)$, $\alpha \in \mathfrak{r} - \{\psi\}$, and has an even number of elements. q.e.d.

PROPOSITION 2.3. *If $\psi \in \mathfrak{r}^-$ has an odd multiplicity, then $2\psi \notin \mathfrak{r}^-$.*

Proof. Suppose that $2\psi \in \mathfrak{r}^-$, and let $\alpha \in \mathfrak{r}_{2\psi}$. Then $\tilde{\alpha} = 2\psi$.
Now

$$\langle \alpha, \psi \rangle = \langle \hat{\alpha} + \tilde{\alpha}, \psi \rangle = \langle 2\psi, \psi \rangle > 0.$$

Hence

$$(*) \qquad \langle \alpha, \alpha \rangle / \langle \psi, \psi \rangle = 1, 2 \text{ or } 3$$

by 1°) of No.2.1 since $\mathfrak{r}$ is a proper root system and $\langle \alpha, \alpha \rangle \geqq \langle \psi, \psi \rangle$. On the other hand

$$\langle \alpha, \alpha \rangle = \langle \hat{\alpha}, \hat{\alpha} \rangle + \langle \tilde{\alpha}, \tilde{\alpha} \rangle \geqq \langle \tilde{\alpha}, \tilde{\alpha} \rangle = 4\langle \psi, \psi \rangle,$$

contradicting to (*). q.e.d.

PROPOSITION 2.4. *If $\psi \in \mathfrak{r}^-$ has an even multiplicity and $2\psi \in \mathfrak{r}^-$, then 2ψ has an odd multiplicity.*

Proof. Since ψ has an even multiplicity, we have an element $\alpha \in \mathfrak{r}_\psi$ such that $\alpha \neq \sigma\alpha$. If 2ψ has an even multiplicity, then there exists an element $\beta \in \mathfrak{r}_{2\psi}$ such that $\beta \neq \sigma\beta$. Since $\mathfrak{r}$ is normal, $\langle \beta, \sigma\beta \rangle \leqq 0$. If $\langle \beta, \sigma\beta \rangle < 0$, then $\mathfrak{r} \ni \beta + \sigma\beta = 4\psi$, whence $\mathfrak{r}^- \ni 4\psi$, contradicting to 2°) of No.2.1. Hence $\langle \beta, \sigma\beta \rangle = 0$. Then

$$\langle \beta, \beta \rangle = 2\langle \tilde{\beta}, \tilde{\beta} \rangle = 2\langle 2\psi, 2\psi \rangle = 8\langle \psi, \psi \rangle.$$

On the other hand, by the assumption that 2ψ has an even multiplicity and Prop. 2.2 we see that

$$2\psi = \alpha + \sigma\alpha \notin \mathfrak{r}.$$

Therefore $\langle \alpha, \sigma\alpha \rangle = 0$, and $\langle \alpha, \alpha \rangle = 2\langle \psi, \psi \rangle$. Consequently

$$\langle \beta, \beta \rangle = 4\langle \alpha, \alpha \rangle,$$

which contradicts to the proper-ness of $\mathfrak{r}$. q.e.d.

Under the assumption of the above proposition, for any $\alpha \in \mathfrak{r}_\psi$

$$2\psi = \alpha + \sigma\alpha \in \mathfrak{r}.$$

Here assume that $\langle \alpha, \sigma\alpha \rangle = 0$, then $\langle \alpha + \sigma\alpha, \alpha + \sigma\alpha \rangle = 2\langle \alpha, \alpha \rangle$ and

$$a_{\alpha+\sigma\alpha,\alpha} = 2,$$

whence $\mathfrak{r} \ni \alpha + \sigma\alpha - a_{\alpha+\sigma\alpha,\alpha}\alpha = \sigma\alpha - \alpha$, contradicting to the condition (ν) of $\mathfrak{r}$. Since the condition (ν) implies that $\langle \alpha, \sigma\alpha \rangle \leqq 0$, we have a conclusion that

$$(**) \qquad \langle \alpha, \sigma\alpha \rangle < 0.$$

Contrarily (**) implies that $2\psi = \alpha + \sigma\alpha \in \mathfrak{r}_{2\psi}$ and $2\psi \in \mathfrak{r}^-$.

Therefore we obtain the following

PROPOSITION 2.5. *When $\psi \in \mathfrak{r}^-$ have an even multiplicity,* i) *$2\psi \in \mathfrak{r}^-$ if and only* *if there exists an element* $\alpha \in \mathfrak{r}_\psi$ *such that* $\langle \alpha, \sigma\alpha \rangle < 0$, *and in this case* $\langle \beta, \sigma\beta \rangle < 0$ *for all* $\beta \in \mathfrak{r}_\psi$; ii) $2\psi \notin \mathfrak{r}^-$ *if and only if there exists an element* $\alpha \in \mathfrak{r}_\psi$ *such that* $\langle \alpha, \sigma\alpha \rangle = 0$, *and in this case* $\langle \beta, \sigma\beta \rangle = 0$ *for all* $\beta \in \mathfrak{r}_\psi$.

PROPOSITION 2.6. *Let* $\psi \in \mathfrak{r}^-$ *have an odd multiplicity, then* $\langle \alpha, \sigma\alpha \rangle = 0$ *for all* $\alpha \in \mathfrak{r}_\psi - \{\psi\}$.

Proof. If $\alpha \in \mathfrak{r}_\psi - \{\psi\}$ satisfy that $\langle \alpha, \sigma\alpha \rangle < 0$, then $\alpha + \sigma\alpha \in \mathfrak{r}$ and $2\psi = \alpha + \sigma\alpha \in \mathfrak{r}^-$ contradicting to Prop.2.3. q.e.d.

PROPOSITION 2.7. *Let* $\psi \in \mathfrak{r}^-$ *be such that* $2\psi \notin \mathfrak{r}^-$, *and* $\alpha, \beta \in \mathfrak{r}_\psi$ *satisfy that* $\alpha \neq \beta$ *and* $\alpha \neq \sigma\beta$, *then* $\langle \alpha, \beta \rangle > 0$.

Proof. $\alpha \neq \sigma\alpha$ or $\beta \neq \sigma\beta$ since $\alpha \neq \beta$. We may assume that $\beta \neq \sigma\beta$. Then

$$\langle \alpha, \beta \rangle + \langle \alpha, \sigma\beta \rangle = \langle \tilde{\alpha}, \beta + \sigma\beta \rangle = \langle \psi, 2\psi \rangle = 2\langle \psi, \psi \rangle > 0.$$

By Props.2.5 and 2.6 we see that $\langle \beta, \sigma\beta \rangle = 0$, whence

$$\langle \beta, \beta \rangle = \langle \sigma\beta, \sigma\beta \rangle = 2\langle \psi, \psi \rangle,$$

which implies that

$$(***) \qquad a_{\alpha,\beta} + a_{\alpha,\sigma\beta} = 2.$$

i) In case $\alpha \neq \sigma\alpha$: $\langle \alpha, \sigma\alpha \rangle = 0$ by Props.2.5 and 2.6, whence

$$\langle \alpha, \alpha \rangle = 2\langle \psi, \psi \rangle = \langle \beta, \beta \rangle,$$

which implies that

$$a_{\alpha,\beta} = 0 \text{ or } \pm 1, \quad a_{\alpha,\sigma\beta} = 0 \text{ or } \pm 1$$

by 3°) of No.2.1, which, combined with (***), implies that

$$a_{\alpha,\beta} = a_{\alpha,\sigma\beta} = 1, \text{ especially } \quad \langle \alpha, \beta \rangle > 0.$$

ii) In case $\alpha = \sigma\alpha$: $\alpha = \psi$, and

$$a_{\alpha,\beta} = 2\langle \psi, \beta \rangle / 2\langle \psi, \psi \rangle = 2\langle \psi, \psi \rangle / 2\langle \psi, \psi \rangle = 1,$$

in particular, $\langle \alpha, \beta \rangle > 0$. q.e.d.

PROPOSITION 2.8. *Let* $\psi \in \mathfrak{r}^-$ *be such that* $2\psi \in \mathfrak{r}^-$. *Then for any* $\alpha, \beta \in \mathfrak{r}_\psi$ *satisfying that* $\alpha \neq \beta$ *and* $\alpha \neq \sigma\beta$ *we have that* $\langle \alpha, \beta \rangle \geqq 0$, *and that* $\langle \alpha, \beta \rangle > 0$ *if and only if* $\langle \alpha, \sigma\beta \rangle = 0$.

Proof. $m(\psi)$ is even by Prop.2.3. And

$$\langle \alpha, \beta + \sigma\beta \rangle = \langle \tilde{\alpha}, 2\psi \rangle = 2\langle \psi, \psi \rangle > 0.$$

By Prop. 2.5.i), we see that $\langle \alpha, \sigma\alpha \rangle < 0$ and $\langle \beta, \sigma\beta \rangle < 0$, whence

$$\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = \langle \sigma\beta, \sigma\beta \rangle = 4\langle \psi, \psi \rangle.$$

Therefore

$$a_{\alpha,\beta} + a_{\alpha,\sigma\beta} = 2\langle \alpha, \beta + \sigma\beta \rangle / 4\langle \psi, \psi \rangle = 1,$$

which implies that $\langle \alpha, \beta \rangle = 0$ and $\langle \alpha, \sigma\beta \rangle > 0$, or that $\langle \alpha, \beta \rangle > 0$ and $\langle \alpha, \sigma\beta \rangle = 0$.

2.7. Let $\mathfrak{r}$ be a normal σ-system of roots as in the above No.

PROPOSITION 2.9. *Let* $\psi \in \mathfrak{r}^-$ *have an add multiplicity, then* $\mathfrak{r}_\psi$ *is connected. When* $m(\psi) > 1$, *let* $\mathfrak{r}_1$ *be the component of* $\mathfrak{r}$ *containing* $\mathfrak{r}_\psi$, *then* $\mathfrak{r}_1$ *is doubly-laced of type* (1:2).

Proof. $\psi \in \mathfrak{r}_\psi$ and $\langle \psi, \alpha \rangle = \langle \psi, \psi \rangle > 0$ for any $\alpha \in \mathfrak{r}_\psi$. Hence $\mathfrak{r}_\psi$ is connected. When $m(\psi) > 1$, for any $\alpha \in \mathfrak{r}_\psi - \{\psi\}$ $\langle \alpha, \sigma\alpha \rangle = 0$ by Prop. 2.6, whence $\langle \alpha, \alpha \rangle = 2\langle \psi, \psi \rangle$. q.e.d.

COROLLARY 2.10. *Let* $\mathfrak{r}$ *be irreducible and simply-laced or doubly-laced of type* (1:3).

If $m(\psi)$ is odd for $\psi\in\mathfrak{r}^-$, then $m(\psi)=1$.

PROPOSITION 2.11. *Let $\psi\in\mathfrak{r}^-$ be such that $2\psi\in\mathfrak{r}^-$, then $\mathfrak{r}_\psi$ is connected.*

Proof. Let $\alpha, \beta\in\mathfrak{r}_\psi$ be such that $\alpha\neq\beta$. If $\beta=\sigma\alpha$, then $\langle\alpha, \beta\rangle<0$ by Prop. 2.5. If $\beta\neq\sigma\alpha$, then $\langle\alpha, \beta\rangle>0$ or $\langle\alpha, \beta\rangle=0$ by Prop.2.8; in case $\langle\alpha, \beta\rangle=0$, $\langle\alpha, \sigma\beta\rangle>0$ and $\langle\sigma\beta, \beta\rangle<0$. q.e.d.

PROPOSITION 2.12. *Let $\psi\in\mathfrak{r}^-$ have even multiplicity and $2\psi\notin\mathfrak{r}^-$. If $m(\psi)>2$, then $\mathfrak{r}_\psi$ is connected.*

Proof. By assumptions, for any $\alpha\in\mathfrak{r}_\psi$, $\mathfrak{r}_\psi-\{\alpha, \sigma\alpha\}\neq\phi$. For any $\beta\in\mathfrak{r}_\psi-\{\alpha, \sigma\alpha\}$ we see that

$$\langle\alpha, \beta\rangle>0 \text{ and } \langle\beta, \sigma\alpha\rangle>0$$

by Prop.2.7, which implies that $\{\alpha, \beta, \sigma\alpha\}$ is connected. q.e.d.

When $m(\psi)=2$ and $2\psi\notin\mathfrak{r}^-$ for $\psi\in\mathfrak{r}^-$, then

$$\mathfrak{r}_\psi=\{\alpha\}\cup\{\sigma\alpha\} \text{ and } \langle\alpha, \sigma\alpha\rangle=0,$$

whence, by Props.2.9, 2.11 and 2.12, we have the

PROPOSITION 2.13. *For any $\psi\in\mathfrak{r}^-$, $\mathfrak{r}_\psi$ consists at most of two connected components; $\mathfrak{r}_\psi$ consists exactly of two components if and only if $m(\psi)=2$ and $2\psi\notin\mathfrak{r}^-$.*

2.8. Let $\mathfrak{r}$ be a σ-system of roots. Satake [7] defined the notion of a σ-fundamental system of roots which is useful also for our abstractly defined σ-system of roots $\mathfrak{r}$. A linear order in $\mathfrak{r}$ satisfying the following condition:

(σ) *if $\alpha\in\mathfrak{r}-\mathfrak{r}_0$ and $\alpha>0$, then $\sigma\alpha>0$,*

is called a *σ-order*. Put $\dim\mathfrak{h}_0=l$ ($=$ rank of $\mathfrak{r}$) and $\dim\mathfrak{h}_0^-=p$. One of the typical way to obtain a σ-order is to take a lexicographic order relative to a base $\{H_1,\ldots,H_l\}$ of $\mathfrak{h}_0$ such that $\{H_1,\ldots, H_p\}$ forms a base of $\mathfrak{h}_0^-$. A fundamental system of $\mathfrak{r}$ with respect to a σ-order is called a *σ-fundamental system.* If Δ is a σ-fundamental system of $\mathfrak{r}$, then $\Delta_0=\Delta\cap\mathfrak{r}_0$ is a fundamental system of $\mathfrak{r}_0$. Denoting by l_0 the rank of $\mathfrak{r}_0$, let

$$\Delta=\{\alpha_1,\ldots, \alpha_{l-l_0}, \alpha_{l-l_0+1},\ldots, \alpha_l\}$$

be a σ-fundamental system of $\mathfrak{r}$ such that $\Delta_0=\{\alpha_{l-l_0+1},\ldots, \alpha_l\}$. The Lemma 1 of [7] is applicable for $\mathfrak{r}$, and we get an involutive permutation $\bar{\sigma}$ of indices $\{1,\ldots\ldots, l-l_0\}$ such that

(2.1) $\quad \sigma(\alpha_i)=\alpha_{\bar{\sigma}(i)}+\sum_{j=l-l_0+1}^{l} c_j^{(i)}\alpha_j,\ c_j^{(i)}\geqq 0$ for $1\leqq i\leqq l-l_0$.

Let Δ^- be the set of distinct elements of $(\mathfrak{h}_0^-)^*$ obtained by restricting the elements of $\Delta-\Delta_0$ to $\mathfrak{h}_0^-$. As easily seen Δ^- forms a linear base of $(\mathfrak{h}_0^-)^*$ such that every element of $\mathfrak{r}^-$ is a linear combination of elements of Δ^- with integers of the same signs as coefficients. In particular, when $\mathfrak{r}$ is a normal σ-system of roots and Δ is a σ-fundamental system of $\mathfrak{r}$, then Δ^- is a fundamental system of $\mathfrak{r}^-$, in case of which Δ^- is called a *restricted fundamental system* of $\mathfrak{r}^-$ according to a definition of [7].

To describe a σ-fundamental system Δ we use the figure due to Satake [7], called *Satake figure*, which is defined as follows: take a Schläfli figure of Δ; every root of Δ_0 is denoted by black circle ● and every root of $\Delta-\Delta_0$ is denoted by white circle ○; if $\bar{\sigma}(i)=j$ such that $i\neq j$ for $1\leqq i\leqq l-l_0$, then simple roots α_i and α_j are connected by a curved arrow.

2.9. The propositions of [7] which are related only to the properties of root systems, are all applicable for our normal σ-systems of roots. For the proof of the following statements we refer to [7].

In this No. $\mathfrak{r}$ denotes a normal σ-system of roots. Let W, W_o and W^- denote respectively the Weyl groups of $\mathfrak{r}$, $\mathfrak{r}_o$ and $\mathfrak{r}^-$. Let W_σ be the subgroup of W consisting of all $s\in W$ commutative with σ. W_o is a normal subgroup of W_σ. For any $s\in W_\sigma$, $s\,\mathfrak{h}_o^- = \mathfrak{h}_o^-$. Hence $s\,|\,\mathfrak{h}_o^-$ is a linear transformation of $\mathfrak{h}_o^-$. Then $s\,|\,\mathfrak{h}_o^-\in W^-$. In this way we get a natural homomorphism

$$\varrho : W_\sigma \longrightarrow W^-.$$

(2.2) *ϱ is surjective with W_o as the kernel.*

Let Δ be a σ-fundamental system of $\mathfrak{r}$ and $s\in W_\sigma$, then $s\Delta$ is also a σ-fundamental system.

(2.3) *W_σ permutes transitively the σ-fundamental systems of $\mathfrak{r}$.*

Let $\mathfrak{r}_1$ and $\mathfrak{r}_2$ be two σ-systems of roots with involutions σ_1 and σ_2. We say that $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are *σ-isomorphic* if there is an isomorphism φ: $\mathfrak{r}_1 \cong \mathfrak{r}_2$ up to a homography such that $\varphi\sigma_1 = \sigma_2\varphi$. Correspondingly we define the notion of a σ-isomorphism of two σ-fundamental systems.

(2.4) *σ-fundamental systems of a normal σ-system of roots are σ-isomorphic to each other. For two normal σ-systems of roots, they are σ-isomorphic to each other if and only if their σ-fundamental systems are so.*

2.10. THEOREM 2.14. *Let $\mathfrak{r}$ be a normally extendable σ-system of roots in $\mathfrak{h}_o^*$ of a complex semi-simple Lie algebra $\mathfrak{g}_C$. The real forms corresponding to anti-involutions extending $\tilde{\sigma}$ such as to be normally related to $(\mathfrak{g}_\tau, \mathfrak{h}_C)$ are conjugate to each other by inner automorphisms of $\mathfrak{g}_C$ commuting with τ.*

Proof. Let σ_1 and σ_2 be two extensions satisfying the conditions of the Theorem. Put

$$\sigma_1 E_\alpha = \varrho_\alpha E_{\sigma\alpha},\ \sigma_2 E_\alpha = \varrho'_\alpha E_{\sigma\alpha}$$

for all $\alpha\in\mathfrak{r}$. $\sigma_1\sigma_2$ is an automophism of $\mathfrak{g}_C$ such that $\sigma_1\sigma_2\,|\,\mathfrak{h}_C =$ identity map, and that

$$\sigma_1\sigma_2 E_\alpha = \overline{\varrho'_\alpha}\varrho_{\sigma\alpha} E_\alpha$$

for $\alpha\in\mathfrak{r}$. By (1.2) and (1.5) we see that

$$\varrho_\alpha = \varrho_{\sigma\alpha} \text{ and } \varrho'_\alpha = \varrho'_{\sigma\alpha}.$$

Put

$$\sigma_1\sigma_2 E_\alpha = \eta_\alpha E_\alpha,$$

then

$$(*1)\qquad \begin{aligned} \eta_\alpha &= \eta_{\sigma\alpha} = \overline{\varrho'_\alpha}\varrho_\alpha = \overline{\varrho'_{\sigma\alpha}}\varrho_{\sigma\alpha} && \text{for } \alpha\in\mathfrak{r}, \\ \eta_\alpha &= 1 && \text{for } \alpha\in\mathfrak{r}_o. \end{aligned}$$

Let $\psi\in\mathfrak{r}^-$. By Props. 2.7 and 2.8 we see that, for any two α, $\beta\in\mathfrak{r}_\psi$ such that $\alpha\neq\beta$ and $\alpha\neq\sigma\beta$, $\alpha-\beta\in\mathfrak{r}_o$ or $\alpha-\sigma\beta\in\mathfrak{r}_o$. On the other hand $\eta_\alpha\eta_\beta = \eta_{\alpha+\beta}$ if α, β, $\alpha+\beta\in\mathfrak{r}$. These and (*1) imply that

(*2) $\eta_\alpha = \eta_\beta$ for any two α, $\beta\in\mathfrak{r}_\psi$ $(\psi\in\mathfrak{r}^-)$.

Let Δ be a σ-fundamental system of $\mathfrak{r}$ and $\Delta^- = \{\lambda_1,\ldots\ldots,\lambda_p\}$ be the associated restricted fundamental system of $\mathfrak{r}^-$. Since $\lambda_1, \ldots, \lambda_p$ are linearly independent, we can choose an $H \in \sqrt{-1}\,\mathfrak{h}_o^-$ so that $\lambda_i(H)$ take arbitrarily given pure imaginary values. In particular we can choose $H \in \sqrt{-1}\,\mathfrak{h}_o^-$ so that

(*3) $\quad e^{\lambda_i(H)} = \eta_\alpha \quad$ for any $\alpha \in \mathfrak{r}_{\lambda_i}$, $1 \leqq i \leqq p$,

because of (*2). "$H \in \sqrt{-1}\,\mathfrak{h}_o^-$" implies that "$\gamma(H) = 0$ for all $\gamma \in \mathfrak{r}_o$", whence

(*4) $\quad e^{\gamma(H)} = 1 = \eta_\gamma \quad$ for all $\gamma \in \mathfrak{r}_o$.

Since $\Delta \subset \mathfrak{r}_o \cup \mathrm{pr}^{-1}(\Delta^-)$, where pr: $\mathfrak{h}_o^* \to (\mathfrak{h}_o^-)^*$ is the restriction, we see that

(*5) $\quad e^{\alpha(H)} = \eta_\alpha \quad$ for all $\alpha \in \Delta$,

by (*3) and (*4). Then (*5) and the fact that $\sigma_1\sigma_2 \mid \mathfrak{h}_C =$ the identity map imply that

(*6) $\quad e^{\beta(H)} = \eta_\beta \quad$ for all $\beta \in \mathfrak{r}$.

Now, put $\xi = \mathrm{Exp}(\mathrm{Ad}(H/2))$, which is an inner automorphism of $\mathfrak{g}_C$ such that $\xi\tau = \tau\xi$. And

$$\begin{aligned}\xi\sigma_2\xi^{-1}E_\alpha &= \xi\sigma_2(e^{-\alpha(H/2)}E_\alpha) = \xi(e^{\alpha(H/2)}\varrho'_\alpha E_{\sigma\alpha}) \\ &= \varrho'_\alpha e^{\alpha(H/2 - \bar{\sigma}H/2)}E_\alpha = \varrho'_\alpha e^{\alpha(H)}E_\alpha = \varrho'_\alpha \eta_\alpha E_\alpha = \varrho_\alpha E_\alpha\end{aligned}$$

for all $\alpha \in \mathfrak{r}$. Hence $\xi\sigma_2\xi^{-1} = \sigma_1$. q.e.d.

This Theorem and (2.4) imply immediately the

COROLLARY 2.15. *Let two real forms $\mathfrak{g}_{\sigma_1}$ and $\mathfrak{g}_{\sigma_2}$ of a complex semi-simple Lie algebra $\mathfrak{g}_C$ be normally related to $(\mathfrak{g}_\tau, \mathfrak{h}_C)$. Then, $\mathfrak{g}_{\sigma_1} \cong \mathfrak{g}_{\sigma_2}$ if and only if their σ-fundamental systems are σ-isomorphic.*

Thus the classification problem of real simple Lie algebras is reduced to the classification of their σ-irreducible σ-fundamental systems of roots.

§3. A reduction to the case of restricted rank 1.

3.1. Let $\mathfrak{g}_C$ be a complex semi-simple Lie algebra, $\mathfrak{h}_C$ a Cartan subalgebra of $\mathfrak{g}_C$, $\mathfrak{r}$ the system of (non-zero) roots of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$ and $\{E_\alpha, \alpha \in \mathfrak{r}\}$ a Weyl base of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$, and $\mathfrak{g}_\tau$ be a compact form of $\mathfrak{g}_C$ such that $\tau\mathfrak{h}_C = \mathfrak{h}_C$ and $\tau E_\alpha = E_{-\alpha}$ for all $\alpha \in \mathfrak{r}$. Now we quote the following well-known theorem:

(3.1) *For any automorphism φ of $\mathfrak{r}$, there exists an automorphism ψ of $\mathfrak{g}_C$ such that $\psi\mathfrak{h}_C = \mathfrak{h}_C$, $\psi\tau = \tau\psi$ and that $\psi^* = \varphi$ as a linear map*: $\mathfrak{h}_C^* \to \mathfrak{h}_C^*$, i.e., $(\varphi\alpha)(H) = \alpha(\psi(H))$ for all $\alpha \in \mathfrak{h}_C^*$ and $H \in \mathfrak{h}_C$.

The automorphism ψ of (3.1) satisfies that

$$\psi\mathfrak{g}_\alpha = \mathfrak{g}_{\varphi^{-1}\alpha} \qquad \text{for } \alpha \in \mathfrak{r}.$$

Put

$$\psi E_\alpha = \xi_\alpha E_{\varphi^{-1}\alpha},$$

then the condition that $\psi\tau = \tau\psi$ is equivalent to saying that

(3.2) $\quad |\,\xi_\alpha\,| = 1 \quad$ *for all* $\alpha \in \mathfrak{r}$.

Let Δ be a fundamental system of $\mathfrak{r}$. The proof of the following statement is also contained in the usual proof of (3.1).

(3.3) *The automorphism ψ of* (3.1) *can be chosen to take arbitrary pre-assigned values ξ_α satisfying* (3.2) *for $\alpha \in \Delta$; by these values ξ_α for $\alpha \in \Delta$, ψ is determined uniquely.*

Now let $\mathfrak{r}$ be a σ-system of roots in $\mathfrak{h}_o{}^*$, and Δ be a σ-fundamental system of $\mathfrak{r}$. Apply (3.1) and (3.3) to $\sigma\tau^*$ with the assignment that $\xi_\alpha = 1$ for all $\alpha \in \Delta_o$ (ξ_α is arbitrary for $\alpha \in \Delta - \Delta_o$). Then we obtain an automorphism ψ of $\mathfrak{g}_C$ satisfying (3.1) such that $\psi^* = \sigma\tau^*$. Since Δ_o is a fundamental system of $\mathfrak{r}_o$ and $\xi_\alpha\xi_\beta = \xi_{\alpha+\beta}$ for $\alpha, \beta, \alpha+\beta \in \mathfrak{r}_o$, we see that $\xi_\alpha = 1$ for all $\alpha \in \mathfrak{r}_o$. If we put $\eta = \psi\tau$, then η is an anti-automorphism of $\mathfrak{g}_C$ (i.e., an automorphism of $\mathfrak{g}_C$ as a Lie algebra over $\mathbf{R}$ and an anti-linear map of $\mathfrak{g}_C$ as a vector space over $\mathbf{C}$) such that $\eta^* = \sigma$, and that, if we put $\eta E_\alpha = \varrho_\alpha E_{\sigma\alpha}$ for $\alpha \in \mathfrak{r}$, $\varrho_\alpha = 1$ for all $\alpha \in \mathfrak{r}_o$. Namely we obtained the

Proposition 3.1. *Let $\mathfrak{r}$ be a σ-system of roots in $\mathfrak{h}_o{}^*$. There exists an anti-automorphism η of $\mathfrak{g}_C$ such that $\eta\mathfrak{h}_C = \mathfrak{h}_C$, $\eta \mid \mathfrak{h}_C = \tilde{\sigma}$ and $\eta\tau = \tau\eta$, and that, if we put $\eta E_\alpha = \varrho_\alpha E_{\sigma\alpha}$ for all $\alpha \in \mathfrak{r}$, then $\varrho_\alpha = 1$ for all $\alpha \in \mathfrak{r}_o$.*

The next question is to seek conditions whether the anti-automorphism η of Prop. 3.1 can be chosen to be involutive or not.

3.2. Let $\mathfrak{r}$ be a σ-system of roots in $\mathfrak{h}_o{}^*$. When $\mathrm{rank}(\mathfrak{r}^-) = p$, we say that $\mathfrak{r}$ is of *restricted rank p*. For each $\lambda \in \mathfrak{r}^-$, let $\tilde{\mathfrak{r}}_\lambda$ denote the union of $\mathfrak{r}_{m\lambda}$, $m \in \mathbf{R}$, such that $m\lambda \in \mathfrak{r}^-$. Clearly $\mathfrak{r}_o \cup \tilde{\mathfrak{r}}_\lambda$ is closed and σ-invariant in $\mathfrak{r}$.

Lemma 3.2 *$\tilde{\mathfrak{r}}_\lambda$ is σ-connected.*

Proof. Let $\alpha \in \mathfrak{r}_\lambda$ and $\beta \in \mathfrak{r}_{m\lambda}$ ($m\lambda \in \mathfrak{r}^-$), then

$$\langle \alpha+\sigma\alpha, \beta+\sigma\beta \rangle = \langle 2\lambda, 2m\lambda \rangle = 4m\langle \lambda, \lambda \rangle > 0.$$

Hence $\langle \alpha, \beta \rangle \neq 0$ or $\langle \alpha, \sigma\beta \rangle \neq 0$. q.e.d.

Let $\bar{\mathfrak{r}}_\lambda$ denote the σ-component of $\mathfrak{r}_o \cup \tilde{\mathfrak{r}}_\lambda$ containing $\tilde{\mathfrak{r}}_\lambda$, and $\mathfrak{h}_\lambda$ be a subspace of $\mathfrak{h}_o$ generated by all H_α such that $\alpha \in \bar{\mathfrak{r}}_\lambda$. Then $\bar{\mathfrak{r}}_\lambda$ is a σ-irreducible σ-system of roots in $\mathfrak{h}_\lambda{}^*$ (considered as a subspace of $\mathfrak{h}_o{}^*$) with the induced involution $\sigma_\lambda = \sigma \mid \mathfrak{h}_\lambda^*$. Put $\mathfrak{h}_\lambda^- = \mathfrak{h}_o^- \cap \mathfrak{h}_\lambda$ and $\mathfrak{h}_\lambda^+ = \mathfrak{h}_o^+ \cap \mathfrak{h}_\lambda$, then $\mathfrak{h}_\lambda^* = (\mathfrak{h}_\lambda^+)^* \oplus (\mathfrak{h}_\lambda^+)^*$ and $(\mathfrak{h}_\lambda^-)^*$ (or $(\mathfrak{h}_\lambda^+)^*$) is the eigenspace of the value $+1$ (or -1) of σ_λ. Since $(\mathfrak{h}_\lambda^-)^*$ is of dimension 1, generated by λ as is easily seen, we have the

Lemma 3.3. *$\bar{\mathfrak{r}}_\lambda$ is a σ-irreducible σ-system of roots of restricted rank* 1.

Take a σ-order in $\mathfrak{r}$. Let Δ be the σ-fundamental system of $\mathfrak{r}$ relative to the σ-order, and Δ_o, Δ^- be fundamental systems of $\mathfrak{r}_o$, $\mathfrak{r}^-$ respectively defined as in No. 2.8. This σ-order induces a σ-order in $\bar{\mathfrak{r}}_\lambda$ for each $\lambda \in \mathfrak{r}^-$. Let Δ^λ, Δ_o^λ and $(\Delta^\lambda)^-$ be the corresponding fundamental systems of $\bar{\mathfrak{r}}_\lambda$, $(\bar{\mathfrak{r}}_\lambda)_o$ and $(\bar{\mathfrak{r}}_\lambda)^-$. If $\lambda \in \Delta^-$, then $(\Delta^\lambda)^- = \{\lambda\}$.

Proposition 3.4. *Let $\lambda \in \Delta^-$, then $\Delta^\lambda = \Delta \cap \bar{\mathfrak{r}}_\lambda$ and $\Delta_o^\lambda = \Delta_o \cap \bar{\mathfrak{r}}_\lambda$.*

Proof. It is sufficient to prove that every simple root of Δ^λ is a simple root of Δ. Assume that a root $\gamma \in \Delta^\lambda$ is non-simple for the original σ-order of $\mathfrak{r}$. Then there exists $\alpha, \beta \in \mathfrak{r}$ such that $\alpha > 0$ and $\beta > 0$ and that $\gamma = \alpha + \beta$.

Put $\alpha \mid \mathfrak{h}_o^- = \mu$ and $\beta \mid \mathfrak{h}_o^- = \nu$, then $\mu, \nu \in \mathfrak{r}^-$ and $\mu \geqq 0$, $\nu \geqq 0$. We have two cases: $\gamma \mid \mathfrak{h}_o^- = 0$ or λ.

i) In case $\gamma \mid \mathfrak{h}_o^- = 0$: $\mu + \nu = 0$, $\mu \geqq 0$ and $\nu \geqq 0$. Hence $\mu = \nu = 0$. And $\alpha, \beta \in \mathfrak{r}_o$. As is easily seen, $\langle \alpha, \gamma \rangle \neq 0$ or $\langle \beta, \gamma \rangle \neq 0$. If $\langle \alpha, \gamma \rangle \neq 0$, then α is connected with $\bar{\mathfrak{r}}_\lambda$ and contained in $\mathfrak{r}_o$. Hence $\alpha \in \bar{\mathfrak{r}}_\lambda$. Since $\bar{\mathfrak{r}}_\lambda$ is closed in $\mathfrak{r}$, $\beta \in \bar{\mathfrak{r}}_\lambda$. Similarly "$\langle \beta, \gamma \rangle \neq 0$" implies also that $\alpha, \beta \in \bar{\mathfrak{r}}_\lambda$. A contradiction.

ii) In case $\gamma \mid \mathfrak{h}_0^- = \lambda$: $\mu+\nu = \lambda$, $\mu \geqq 0$ and $\nu \geqq 0$. Since λ is simple for Δ^-, $\mu = 0$ or $\nu = 0$. We may consider as $\mu = 0$, then $\alpha \in \mathfrak{r}_0$ and $\beta \in \mathfrak{r}_\lambda$. Hence $\beta \in \bar{\mathfrak{r}}_\lambda$, and $\alpha \in \bar{\mathfrak{r}}_\lambda$ by the closed-ness of $\bar{\mathfrak{r}}_\lambda$ in $\mathfrak{r}$, which contradicts to the assumption that $\gamma \in \Delta^\lambda$. q.e.d.

By this proposition we see easily the

PROPOSITION 3.5. *When we are given a σ-fundamental system Δ of $\mathfrak{r}$, then Δ^λ consists of* $\mathrm{pr}^{-1}(\lambda) \cap \Delta$ *plus all elements of Δ_0 which are Δ_0-connected with* $\mathrm{pr}^{-1}(\lambda) \cap \Delta$ *for each $\lambda \in \Delta^-$, where* $\mathrm{pr}: \Delta - \Delta_0 \to \Delta^-$ *is the restriction map.*

For the definition of "Δ_0-connected," *cf.*, [7], No.1.3., p.81.

3.3. Let $\mathfrak{r}$ be a σ-system of roots in $\mathfrak{h}_0{}^*$. Using the notations of the above No. and choosing a σ-order in $\mathfrak{r}$, let $\mathfrak{g}_{\lambda C}$ denote the semi-simple part of the centralizer of the plane

$$p_\lambda = \{H \in \mathfrak{h}_C;\ \alpha(H) = 0 \quad \text{for all } \alpha \in \bar{\mathfrak{r}}_\lambda\}$$

in $\mathfrak{g}_C$ for each $\lambda \in \Delta^-$, i.e.,

$$\mathfrak{g}_{\lambda C} = (\mathfrak{h}_\lambda)_C + \sum_{\alpha \in \bar{\mathfrak{r}}_\lambda} \mathfrak{g}_\alpha.$$

$(\mathfrak{h}_\lambda)_C$ is a Cartan subalgebra of $\mathfrak{g}_{\lambda C}$, $\bar{\mathfrak{r}}_\lambda$ the root system of $\mathfrak{g}_{\lambda C}$ with respect to $(\mathfrak{h}_\lambda)_C$, $\{E_\alpha, \alpha \in \bar{\mathfrak{r}}_\lambda\}$ a Weyl base of $\mathfrak{g}_{\lambda C}$ relative to $(\mathfrak{h}_\lambda)_C$. $\mathfrak{g}_{\lambda C}$ is invariant under τ. Put $\tau \mid \mathfrak{g}_{\lambda C} = \tau_\lambda$ for $\lambda \in \Delta^-$. Then $\mathfrak{g}_{\tau_\lambda} = \mathfrak{g}_{\lambda C} \cap \mathfrak{g}_\tau$ is a compact form of $\mathfrak{g}_{\lambda C}$ such that $\tau_\lambda(\mathfrak{h}_\lambda)_C = (\mathfrak{h}_\lambda)_C$ and $\tau_\lambda E_\alpha = E_{-\alpha}$ for $\alpha \in \bar{\mathfrak{r}}_\lambda$. Further we put $\tilde{\sigma} \mid \mathfrak{h}_\lambda = \tilde{\sigma}_\lambda$.

THEOREM 3.6. *Let $\mathfrak{r}$ be a σ-system of root in $\mathfrak{h}_0{}^*$. Using the above notations, $\mathfrak{r}$ is normally extendable with respect to $(\mathfrak{g}_\tau, \mathfrak{h}_C)$ if and only if $\bar{\mathfrak{r}}_\lambda$ with σ_λ is normally extendable with respect to $(\mathfrak{g}_{\tau_\lambda}, (\mathfrak{h}_\lambda)_C)$ for each $\lambda \in \Delta^-$.*

Proof. "only if" part is clear.

Assume that each $\tilde{\sigma}_\lambda$, $\lambda \in \Delta^-$, is extended to an anti-involution of $\mathfrak{g}_{\lambda C}$ such that it is normally related to $(\mathfrak{g}_{\tau_\lambda}, (\mathfrak{h}_\lambda)_C)$. Put

$$\Delta'^\lambda = \Delta^\lambda - \Delta_0^\lambda \qquad \text{for } \lambda \in \Delta^-.$$

Then, Δ is decomposed into a disjoint union

$$\Delta = \Delta_0 \cup (\bigcup_{\Delta^- \ni \lambda} \Delta'^\lambda)$$

by a reason of Prop. 3.4. Next we put

$$\tilde{\sigma}_\lambda E_\alpha = \varrho_\alpha^\lambda E_{\sigma\alpha} \qquad \text{for all } \alpha \in \bar{\mathfrak{r}}_\lambda,\ \lambda \in \Delta^-.$$

Then

$$\varrho_\alpha^\lambda = 1 \qquad \text{for all } \alpha \in \Delta_0^\lambda,\ \lambda \in \Delta^-.$$

Hence, if we define $\tilde{\sigma} E_\alpha = \varrho_\alpha E_{\sigma\alpha}$ by putting $\varrho_\alpha = 1$ for $\alpha \in \Delta_0$ and $\varrho_\alpha = \varrho_\alpha^\lambda$ for $\alpha \in \Delta'^\lambda$, then ϱ_α is defined for all $\alpha \in \Delta$ and $\tilde{\sigma}$ is extended uniquely to an anti-automorphism of $\mathfrak{g}_C$ by (3.3) and Prop. 3.1, which coincides with $\tilde{\sigma}_\lambda$ on $\mathfrak{g}_{\lambda C}$ for all $\lambda \in \Delta^-$. Now

$$\tilde{\sigma}\tilde{\sigma} E_\alpha = E_\alpha \qquad \text{for all } \alpha \in \Delta,$$

by our definition and the assumption that $\tilde{\sigma}_\lambda \tilde{\sigma}_\lambda = 1$ for all $\lambda \in \Delta^-$.
Therefore

$$\tilde{\sigma}\tilde{\sigma} = \text{the identity automorphism of } \mathfrak{g}_C$$

by the uniqueness of (3.3). q.e.d.

This theorem reduces our problem to the classification of normally extendable

σ-systems of roots of restricted rank 1.

By Prop.3.5 and Theo.3.6 we see easily the

PROPOSITION 3.7. *Let Δ be a σ-fundamental system of a normally extendable σ-system of roots. Let $\lambda \in \Delta^-$ and Δ' be $\Delta - \Delta^\lambda$ plus all elements of Δ_0 which are Δ_0-connected to an element of $\Delta - \Delta^\lambda$. Then Δ' is also a σ-fundamental system of a normally extendable σ-system of roots.*

This proposition will be used frequently in §5.

REMARK. In case $\sigma = \tau^*$, $\mathfrak{r} = \mathfrak{r}_0$ and $\Delta = \Delta_0$. Hence $\Delta^- = \phi$ and the condition "normally extendable" of Theo. 3.6 is trivially satisfied. Actually we have that

$$\tilde{\sigma} = \tau.$$

Hence $\mathfrak{g}_{\tilde{\sigma}} = \mathfrak{g}_\tau$ and the corresponding symmetric space is reduced to "a point".

This trivial case will be omitted out of our subsequent discussions, i.e., hereafter $\sigma \neq \tau^*$ always and $\mathfrak{g}_{\tilde{\sigma}}$ is non-compact if σ is normally extendable.

§4. Classification (Case of rank 1).

4.1 First we classify σ-fundamental systems of σ-irreducible normal σ-systems of roots of restricted rank 1. Let $\mathfrak{r}$ be such a σ-system of roots. Choosing a σ-order in $\mathfrak{r}$, put $\Delta^- = \{\lambda\}$. By discussions of §2 we have the following three cases:

i) $m(\lambda)$ is odd, then $\mathfrak{r}^- = \{-\lambda, \lambda\}$:

ii) $m(\lambda)$ is even and $2\lambda \notin \mathfrak{r}^-$, then $\mathfrak{r}^- = \{-\lambda, \lambda\}$;

iii) $m(\lambda)$ is even and $2\lambda \in \mathfrak{r}^-$, then $\mathfrak{r}^- = \{-2\lambda, -\lambda, \lambda, 2\lambda\}$.

As a σ-order in $\mathfrak{r}$ we use a lexicographic order with respect to a linear base $\{H_1, \dots, H_l\}$ of $\mathfrak{h}_0$ such that $H_1 \in \mathfrak{h}_0^-$ throughout this paragraph. The convenience of usage of this order is that $\mathfrak{r}_{-\lambda} < \mathfrak{r}_0 < \mathfrak{r}_\lambda$ in cases i) and ii), and $\mathfrak{r}_{-2\lambda} < \mathfrak{r}_{-\lambda} < \mathfrak{r}_0 < \mathfrak{r}_\lambda < \mathfrak{r}_{2\lambda}$ in case iii).

The next lemma will be used sometimes in the sequel.

LEMMA 4.1. *Every root γ of $\mathfrak{r}_0$ is connected with some root of* $\mathfrak{r}_\lambda \cup \mathfrak{r}_{2\lambda}$ (where $\mathfrak{r}_{2\lambda} = \phi$ in cases i) and ii)).

Proof. Since $\mathfrak{r}$ is σ-rreducible and hence σ-connected, there exists a chain $\{\psi_1, \psi_2, \dots, \psi_n\}$ in $\mathfrak{r}_0$ which connects γ to some root $\alpha' \in \mathfrak{r}_\lambda \cup \mathfrak{r}_{2\lambda}$, i.e., $\langle \gamma, \psi_1 \rangle \neq 0$, $\langle \psi_i, \psi_{i+1} \rangle \neq 0$ for $1 \leq i \leq n-1$ and $\langle \psi_n, \alpha' \rangle \neq 0$. Assume that $n>0$, and put $\gamma = \psi_0$. If $\langle \psi_{n-1}, \alpha' \rangle \neq 0$, then the length n of this chain can be reduced by 1. If $\langle \psi_{n-1}, \alpha' \rangle = 0$, we put $\alpha'' = \alpha' - a_{\alpha', \psi_n} \psi_n$, then $\alpha'' \in \mathfrak{r}_\lambda \cup \mathfrak{r}_{2\lambda}$ and $\langle \psi_{n-1}, \alpha'' \rangle \neq 0$. Thus $\{\psi_1, \dots, \psi_{n-1}\}$ is a chain in $\mathfrak{r}_0$ to connect γ to $\alpha'' \in \mathfrak{r}_\lambda \cup \mathfrak{r}_{2\lambda}$, and the length of the chain is reduced by 1. Continue this process up to $n=0$, then the Lemma is proved.

4.2. *Case* i). Put $m = m(\lambda)$ and $m = 2m' - 1$. Let the roots of

(4.2.1) $\qquad \mathfrak{r}_\lambda = \{\alpha_1, \dots, \alpha_{m'}, \dots, \alpha_m\}$

be arranged in the increasing order with respect to the given σ-order, i.e., $\alpha_i < \alpha_j$ if $i<j$. Since $\alpha_i + \sigma\alpha_i = \alpha_j + \sigma\alpha_j = 2\lambda$ for $\alpha_i, \alpha_j \in \mathfrak{r}_\lambda$,

(4.2.2) $\qquad \alpha_i < \alpha_j$ *if and only if* $\sigma\alpha_i > \sigma\alpha_j$.

In particular,

(4.2.3) $\qquad \sigma\alpha_i = \alpha_{m-i+1} \qquad$ for $1 \leqq i \leqq m$,
$\qquad\qquad \sigma\alpha_{m'} = \alpha_{m'} = \lambda$.

By Prop.2.6 we see that

(4.2.4) $\qquad \langle\alpha_i, \alpha_i\rangle = 2\langle\lambda, \lambda\rangle \qquad$ for $i \neq m'$,
$\qquad\qquad \langle\alpha_{m'}, \alpha_{m'}\rangle = \langle\lambda, \lambda\rangle$,

and by Prop.2.7 we see that

(4.2.5) $\qquad \langle\alpha_i, \alpha_j\rangle > 0 \qquad$ if $i+j \neq m+1$;

in particular,

(4.2.6) $\qquad \gamma_i = \alpha_{i+1} - \alpha_i \in \mathfrak{r}_o^+ \qquad$ for $1 \leqq i < m$

($\mathfrak{r}_o^+$ denotes the set of positive roots of $\mathfrak{r}_o$). And

(4.2.7) $\qquad \langle\gamma_i, \gamma_i\rangle = 2\langle\lambda, \lambda\rangle \qquad$ if $1 \leqq i < m'-1$,
$\qquad\qquad \langle\gamma_{m'-1}, \gamma_{m'-1}\rangle = \langle\lambda, \lambda\rangle$.

Further, from (4.2.3) and (4.2.6) we see that

(4.2.8) $\qquad \gamma_{m'+i} = \gamma_{m'-i-1} \qquad$ for $0 \leqq i < m'-1$.

By the property of the used σ-order stated at the middle of the above No., we see that

(4.2.9) $\qquad \alpha_{m-i+1}$ *is the* i-th *root from the highest root for* $1 \leqq i \leqq m$.

Then (4.2.6), (4.2.8) and (4.2.9) prove that

(4.2.10) $\qquad \gamma_1, \ldots, \gamma_{m'-1}$ *are simple roots.*

On the other hand.

(4.2.11) $\qquad \alpha_1$ *is a simple root,*

since it is the lowest root of $\mathfrak{r}_\lambda$. By (4.2.4)-(4.2.7) we see easily that $\{\alpha_1, \gamma_1, \ldots, \gamma_{m'-1}\}$ form a fundamental system of roots of type $B_{m'}$ if $m' \geqq 2$. Clearly every root of $\mathfrak{r}_\lambda$ is expressed as a linear combination of $\alpha_1, \gamma_1, \ldots \gamma_{m'-1}$, e.g.,

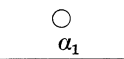
$$\alpha_m = \alpha_1 + 2(\gamma_1 + \cdots + \gamma_{m'-1}).$$

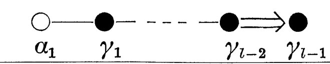
Let $\gamma \in \mathfrak{r}_o^+$; by Lemma 4.1 there exist $\alpha_i, \alpha_j \in \mathfrak{r}_\lambda$ such that $\gamma + \alpha_i = \alpha_j$. Then

$$\gamma = \gamma_i + \cdots + \gamma_{i+j-1},$$

i.e., it is expressed as a linear combination of $\gamma_1, \ldots, \gamma_{m'-1}$ if $m' \geqq 2$. Thus we obtain the

PROPOSITION 4.2. *Normal σ-systems of roots of restricted rank* 1 *of case* i) *are classified by their σ-fundamental systems described by Satake figures as follows.*

A_1i)  *with* $\sigma\alpha_1 = \alpha_1$ *and* $m(\lambda) = 1$,

B_li)  *for all* $l \geqq 2$

with $\sigma\alpha_1 = \alpha_1 + 2(\gamma_1 + \cdots + \gamma_{l-1})$ *and* $m(\lambda) = 2l-1$, where the arrow $\Longrightarrow$ directs from the longer root to the shorter one.

As is easily seen, thus defined σ are involutive automorphisms of $\mathfrak{r}$ and the σ-systems of roots defined by the above σ-fundamental systems are actually normal.

4.3. *Case* ii). Put $m(\lambda) = m = 2m'$. By Prop. 2.5. ii) all roots of $\mathfrak{r}_\lambda$ have the same length and

(4.3.1) $\langle \alpha, \alpha \rangle = 2\langle \lambda, \lambda \rangle$ for all $\alpha \in \mathfrak{r}_\lambda$.

Let the roots of $\mathfrak{r}_\lambda = \{\alpha_1, \dots, \alpha_m\}$ be arranged in the inc.easing order with respect to the given σ-order, and discuss of them in a parallel way to No. 4.2 using Prop. 2.7. Then first we obtain:

(4.3.2) $\sigma\alpha_i = \alpha_{m-i+1}$ for $1 \leqq i \leqq m$,

(4.3.3) $\langle \alpha_i, \alpha_j \rangle > 0$ if $i+j \neq m+1$.

Here we put

(4.3.4) $$\begin{aligned} \alpha_{i+1} - \alpha_i &= \gamma_i \quad \text{for } 1 \leqq i < m', \\ \alpha_{m'+1} - \alpha_{m'-1} &= \gamma_{m'}, \end{aligned}$$

then

(4.3.5) $\gamma_i \in \mathfrak{r}_o^+$ and $\langle \gamma_i, \gamma_i \rangle = 2\langle \lambda, \lambda \rangle$ for $1 \leqq i \leqq m'$.

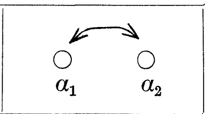
Now all differences $\alpha_j - \alpha_i$ $(j > i)$ are expressed as a linear combination of $\gamma_1, \dots, \gamma_{m'}$; in particular

$$\alpha_m = \sigma\alpha_1 = \alpha_1 + 2(\gamma_1 + \cdots\cdots + \gamma_{m'-2}) + \gamma_{m'-1} + \gamma_{m'}.$$

Next, every element of $\mathfrak{r}_o^+$ is expressed as a linear combination of $\gamma_1, \dots, \gamma_{m'}$ by a reasoning of use of Lemma 4.1. Finally, discussing $\langle \gamma_i, \gamma_j \rangle$ and $\langle \alpha_1, \gamma_j \rangle$ by (4.3.3)-(4.3.4), we see that $\{\alpha_1, \gamma_1, \dots, \gamma_{m'}\}$ form the fundamental system with respect to the given σ-oider if $m' > 1$. They are of type $D_{m'+1}$ if $m' \geqq 3$ and of type A_3 if $m' = 2$. If $m' = 1$, then $\mathfrak{r}_o = \phi$ and $\mathfrak{r}_\lambda = \{\alpha_1, \sigma\alpha_1\}$ is not connected. Thus we obtain a classification of σ-fundamental systems of case ii). It is easy to see thus obtained σ are involutive automorphisms of $\mathfrak{r}$ and that the σ-systems of roots defined by these σ-fundamental systems are all normal.

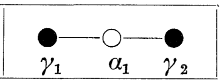
PROPOSITION 4.3. *Normal σ-sytems of roots of restricted rank* 1 *of case* ii) *are classified by their σ-fundamental systems described by Satake figures as follows:*

$A_1 \times A_1$ ii)



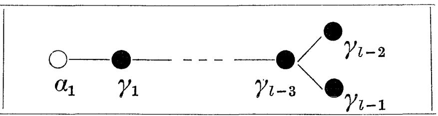
with $\sigma\alpha_1 = \alpha_2$ *and* $m(\lambda) = 2$,

A_3 ii)



with $\sigma\alpha_1 = \alpha_1 + \gamma_1 + \gamma_2$ *and* $m(\lambda) = 4$,

D_l ii)



for all $l \geqq 4$

with $\sigma\alpha_1 = \alpha_1 + 2(\gamma_1 + \dots + \gamma_{l-3}) + \gamma_{l-2} + \gamma_{l-1}$ *and* $m(\lambda) = 2l - 2$.

4.4. *Case* iii). Put $m(\lambda) = 2m'$ and $m(2\lambda) = 2m'' - 1$, and let the roots of

(4.4.1) $\mathfrak{r}_\lambda = \{\alpha_1, \dots, \alpha_{2m'}\}$, $\mathfrak{r}_{2\lambda} = \{\beta_1, \dots, \beta_{m''}, \dots, \beta_{2m''-1}\}$

be arranged in the increasing order with respect to the given σ-order. By Prop. 2.13 $\mathfrak{r}_\lambda$ and $\mathfrak{r}_{2\lambda}$ are connected. Hence $\mathfrak{r}$ must be connected since it is σ-irreducible. As in No. 4.2 we see that

(4.4.2) $$\begin{aligned} \sigma\alpha_i &= \alpha_{2m'+1-i} \quad \text{for } 1 \leqq i \leqq 2m', \\ \sigma\beta_j &= \beta_{2m''-j} \quad \text{for } 1 \leqq j \leqq 2m'' - 1. \end{aligned}$$

By Prop. 2.5. i) and Prop. 2.6 we see that

$$(4.4.3)\qquad \begin{aligned} \langle \alpha_i,\ \sigma\alpha_i\rangle &< 0 \qquad \text{for all } i, \\ \langle \beta_j,\ \sigma\beta_j\rangle &= 0 \qquad \text{for } j\neq m'', \end{aligned}$$

$$(4.4.4)\qquad 2\lambda = \beta_{m''} = \alpha_i+\alpha_{2m'+1-i} \qquad \text{for } 1\leqq i \leqq m'.$$

Then we see that

$$(4.4.5)\qquad \begin{aligned} \langle \alpha_i,\ \alpha_i\rangle = \langle \beta_{m''},\ \beta_{m''}\rangle &= 4\langle \lambda,\ \lambda\rangle \qquad \text{for all } i, \\ \langle \beta_j,\ \beta_j\rangle &= 8\langle \lambda,\ \lambda\rangle \qquad \text{for } j\neq m''. \end{aligned}$$

LEMMA 4.4. *In case $m'' = 1$ $\mathfrak{r}$ is simply-laced, and in case $m''>1$ $\mathfrak{r}$ is doubly-laced of type* (2:1).

Proof. The assertion for the case $m''>1$ is clear from (4.4.5). In case $m'' = 1$, assume that $\mathfrak{r}$ is not simply-laced. Since the roots of $\mathfrak{r}_\lambda \cup \mathfrak{r}_{2\lambda}$ have the same length, there must exist a root $\gamma\in\mathfrak{r}_o$ such that $\langle\gamma,\gamma\rangle \neq \langle\alpha_1,\alpha_1\rangle$. Then, by Lemma 4.1 there exists a root $\alpha\in\mathfrak{r}_\lambda\cup\mathfrak{r}_{2\lambda}$ such that $\langle\gamma,\alpha\rangle\neq 0$. We may assume that $\langle\gamma,\alpha\rangle<0$ by replacing γ by $-\gamma$ if necessary. Now put

$$\delta = \gamma - a_{\gamma,\alpha}\alpha \in \mathfrak{r}_\lambda\cup\mathfrak{r}_{2\lambda},$$

then $\langle\delta,\delta\rangle = \langle\gamma,\gamma\rangle \neq \langle\alpha_1,\alpha_1\rangle$ which contradicts to the fact that all roots of $\mathfrak{r}_\lambda\cup\mathfrak{r}_{2\lambda}$ have the same length. q.e.d.

Now the case iii) is further divided into two cases; *case* a) $m''=1$, and *case* b) $m''>1$.

4.4.a. *Case* iiia). β_1 is the highest root, and α_1 is a simple root since it is the lowest root of $\mathfrak{r}_\lambda\cup\mathfrak{r}_{2\lambda}$. Let n be the coefficient of α_1 in the expression of β_1 as a linear combination of simple roots. Since $\beta_1|\mathfrak{h}_o^- = 2\lambda$ and $\alpha_1|\mathfrak{h}_o^- = \lambda$, $n=1$ or 2. And, in case $n=2$ all simple roots other than α_1 must belong to $\mathfrak{r}_o$; in case $n=1$ only one simple root differing from α_1 belongs to $\mathfrak{r}_\lambda$ and all other simple roots must belong to $\mathfrak{r}_o$.

Further we note that

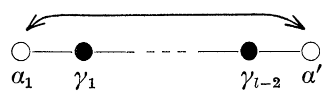
$$(4.4.6)\qquad \langle\beta_1,\ \alpha_1\rangle>0$$

since $\beta_1 = \alpha_1+\sigma\alpha_1$. This determines the possible simple roots which can be α_1 in the given Schläfli figure. Simple roots of $\mathfrak{r}_o$ will be denoted by γ_i.

In the present case $\mathfrak{r}$ is simply-laced by Lemma 4.4, hence of type A_l, D_l or E_l.

Type A_l. By (4.4.6) α_1 must be the one of the external roots in the Schläfli figure of A_l. Then $n=1$. Let α' be another simple root of $\mathfrak{r}_\lambda$, then $\langle\beta_1,\alpha'\rangle>0$ since $\beta_1=\alpha'+\sigma\alpha'$. Hence α' must be the another external root than α_1 in the Schläfli figure of A_l. Therefore the possible σ-fundamental system is determined uniquely. It is described by Satake figure as follows:

$A_l$iii)



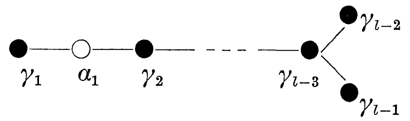
for every $l \geqq 2$

with $\sigma\alpha_1 = \alpha'+(\gamma_1+\dots+\gamma_{l-2})$ *and* $m(\lambda)=2(l-1)$, $m(2\lambda) = 1$.

It is easy to see that each one of the above σ-fundamental systems determines a normal σ-system of roots.

Type $\mathbf{D}_l$. By (4.4.6) α_1 is determined uniquely in the Schläfli figure of D_l and $n=2$. Hence the possible σ-fundamental system is determined uniquely, which is described by

$D_l iii)$



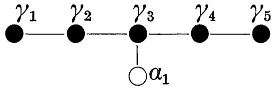
for every $l\geqq 4$

with $\sigma\alpha_1 = \gamma_1+\alpha_1+2(\gamma_2+\cdots\cdots+\gamma_{l-3})+\gamma_{l-2}+\gamma_{l-1}$ *and* $m(\lambda) = 2(2l-4)$.

As is easily seen each one of the above σ-fundamental systems atcually makes $\mathfrak{r}$ a normal σ-system of roots.

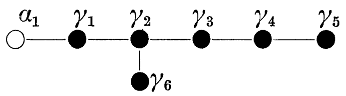
Type $\mathbf{E}_l$. Discuss parallelly to the above ones, then the possible σ-fundamental system is determined uniquely for each $l=6$, 7 and 8, which is described as follows:

$E_6 iii)$



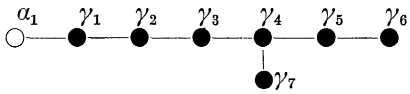
with $\sigma\alpha_1 = \alpha_1+\gamma_1+2\gamma_2+3\gamma_3+2\gamma_4+\gamma_5$,

$E_7 iii)$



with $\sigma\alpha_1 = \alpha_1+3\gamma_1+4\gamma_2+3\gamma_3+2\gamma_4+\gamma_5+2\gamma_6$,

$E_8 iii)$



with $\sigma\alpha_1 = \alpha_1+3\gamma_1+4\gamma_2+5\gamma_3+6\gamma_4+4\gamma_5+2\gamma_6+3\gamma_7$.

These σ-fundamental systems gives normal σ-systems of roots as is easily checked.

4.4.b. *Case* iii b). $\mathfrak{r}$ is doubly-laced of type (2:1) by Lemma 4.4, hence is of type B_l, C_l or F_4. $\beta_{2m''-1}$ is the highest root and $\beta_{2m''-i}$ is the i-th root from the highest root for $1\leqq i\leqq m''$ with respect to the given σ-order. For roots of $\mathfrak{r}_{2\lambda}$ we can apply Prop. 2.7, and we see that

(4.4.7) $\quad \langle\beta_{2m''-i}, \beta_{2m''-j}\rangle > 0 \quad$ for $1\leqq i<j\leqq m''$.

Hence

(4.4.8) $\quad \gamma_i = \beta_{2m''-i} - \beta_{2m''-i-1}$ $(\in\mathfrak{r}_0^+)$ *is a simple root for* $1\leqq i< m''$.

Since $\beta_{2m''-i}$, $1\leqq i<m''$, are long roots and $\beta_{m''}$ is a short root, by (4.4.7) we see that

(4.4.9) $\quad \gamma_1,\dots,\gamma_{m''-2}$ *are long roots, and* $\gamma_{m''-1}$ *is a short root.*

Furthermore,

(4.4.10) $\quad \alpha_1$ *is a short simple root,*

since it is the lowest root of $\mathfrak{r}_\lambda$; and

(4.4.11) $\quad \langle\alpha_1, \beta_{m''}\rangle > 0$

since $\beta_{m''} = \alpha_1+\sigma\alpha_1$.

Type $\mathbf{B}_l$. By the above generality α_1 and $\gamma_{m''-1}$ are different short simple roots, and every fundamental system of roots of type B_l contains only one short root. Hence there does not exist any normal σ-system of roots of type B_l belonging to the considered case.

Type C_l. The Schläfli figure of type C_l is described as

$$\underset{\varphi_1}{\circ}-\underset{\varphi_2}{\circ}-\cdots-\underset{\varphi_{l-1}}{\circ}\Leftarrow\underset{\varphi_l}{\circ}.$$

The highest root

$$\beta_{2m''-1} = 2(\varphi_1+\dots+\varphi_{l-1})+\varphi_l.$$

By (4.4.7)-(4.4.8) $\langle\beta_{2m''-1}, \gamma_1\rangle > 0$ and γ_1 is simple. Hence γ_1 is uniquely determined as $\gamma_1 = \varphi_1$. Here we note that φ_1 is a short root, then by (4.4.9) $m''-1 = 1$, i. e., only "$m''=2$" is possible. Hence

$$\beta_{m''} = \beta_{2m''-2} = \varphi_1+2(\varphi_2+\dots+\varphi_{l-1})+\varphi_l.$$

Now, by (4.4.11) α_1 is uniquely determined as $\alpha_1 = \varphi_2$. Since the coefficient of α_1 is 2 in the above expression of $\beta_{m''}$ as a linear combination of simple roots, the possible σ-fundamental system is uniquely determined, which is described as follows:

$C_l$iii)	$\underset{\gamma_1}{\bullet}-\underset{\alpha_1}{\circ}-\underset{\gamma_2}{\bullet}-\cdots-\underset{\gamma_{l-2}}{\bullet}\Leftarrow\underset{\gamma_{l-1}}{\bullet}$

for every $l \geqq 3$

with $\sigma\alpha_1 = \gamma_1+\alpha_1+2(\gamma_2+\dots+\gamma_{l-2})+\gamma_{l-1}$, $m(\lambda) = 4(l-2)$ *and* $m(2\lambda) = 3$.

This figure gives certainly a normal σ-system of roots for every $l \geqq 3$ as is easily seen.

Type F_4. The Sclăfli figure of type F_4 is described as

$$\underset{\varphi_1}{\circ}-\underset{\varphi_2}{\circ}\Rightarrow\underset{\varphi_3}{\circ}-\underset{\varphi_4}{\circ}$$

with the highest root

$$\beta_{2m''-1} = 2\varphi_1+3\varphi_2+4\varphi_3+2\varphi_4.$$

By (4.4.7)-(4.4.8) $\langle\beta_{2m''-1}, \gamma_1\rangle > 0$ and γ_1 is simple. Hence $\gamma_1 = \varphi_1$, a long root. Next

$$\beta_{2m''-2} = \beta_{2m''-1}-\varphi_1 = \varphi_1+3\varphi_2+4\varphi_3+2\varphi_4,$$

and, by (4.4.7)-(4.4.8) $\langle\beta_{2m''-2}, \gamma_2\rangle > 0$ and γ_2 is simple. Hence $\gamma_2 = \varphi_2$, a long root. Then

$$\beta_{2m''-3} = \beta_{2m''-1}-\varphi_2 = \varphi_1+2\varphi_2+4\varphi_3+2\varphi_4.$$

By (4.4.7)-(4.4.8) $\langle\beta_{2m''-3}, \gamma_3\rangle > 0$ and γ_3 is simple. Hence $\gamma_3 = \varphi_3$, a short root.

Now, by (4.4.9) $m''-1 = 3$, i.e., only "$m''=4$" is possible. And

$$\beta_{m''} = \beta_{2m''-4} = \varphi_1+2\varphi_2+3\varphi_3+2\varphi_4.$$

Then, by (4.4.11) α_1 is uniquely determined as $\alpha_1 = \varphi_4$. And the possible σ-fundamental system is determined uniquely, which is described by Satake figure as follows:

F_4iii)	$\underset{\gamma_1}{\bullet}-\underset{\gamma_2}{\bullet}\Rightarrow\underset{\gamma_3}{\bullet}-\underset{\alpha_1}{\circ}$

with $\sigma\alpha_1 = \gamma_1+2\gamma_2+3\gamma_3+\alpha_1$,
$m(\lambda) = 8$ *and* $m(2\lambda) = 7$.

This figure determines certainly a normal σ-system of roots as is easily checked.

Summarizing the discussions of No. 4.4 we obtain the following

PROPOSITION 4.5. *Normal σ-systems of roots of restricted rank* 1 *of case iii) are classified as follows: $A_l$iii) for $l\geqq 2$, $D_l$iii) for $l\geqq 4$, $E_l$iii) for $l=6$, 7 and 8, $C_l$iii) for $l\geqq 3$, and F_4iii).*

4.5. In Nos. 4.2, 4.3 and 4.4 we classified all normal σ-systems of roots of restricted rank 1 by their σ-fundamental systems. Now we shall determine whether they are normally extendable or not. For this we need two lemmas (Lemmas 4.6 and 4.7).

First we quote two propositions about the structure constants $N_{\alpha,\beta}$ (α, $\beta\in\mathfrak{r}$) of a complex semi-simple Lie algebra with respect to a Weyl base $\{E_\alpha,\ \alpha\in\mathfrak{r}\}$ which are well known since H. Weyl. We use the convention that $N_{\alpha,\beta}=0$ if $\alpha+\beta\neq 0$ and $\notin\mathfrak{r}$.

(4.5.1) *Let α, β, γ be non-zero roots such that $\alpha+\beta+\gamma=0$, then*

$$N_{\alpha,\beta}=N_{\beta,\gamma}=N_{\gamma,\alpha}.$$

(4.5.2) *Let α, β, γ and δ be non-zero roots such that $\alpha+\beta+\gamma+\delta=0$, $\beta+\gamma\neq 0$, $\gamma+\delta\neq 0$ and $\delta+\beta\neq 0$, then*

$$N_{\alpha,\beta}N_{\gamma,\delta}+N_{\alpha,\gamma}N_{\delta,\beta}+N_{\alpha,\delta}N_{\beta,\gamma}=0.$$

Let $\mathfrak{r}$ be a normal σ-system of roots of restricted rank 1, η be any anti-automorphism of $\mathfrak{g}_C$ extending $\tilde{\sigma}$ in the sense of Prop. 3.1.

Put

$$\eta E_\alpha=\varrho_\alpha E_{\sigma\alpha} \qquad \text{for each } \alpha\in\mathfrak{r}.$$

Then $\varrho_\alpha=1$ for any $\alpha\in\mathfrak{r}_0$.

LEMMA 4.6. *Let $\alpha\in\mathfrak{r}_\lambda$ and γ, $\delta\in\mathfrak{r}_0$ such that $\{\alpha+\gamma,\ \alpha+\delta\}\subset\mathfrak{r}$, $\gamma+\delta\neq 0$, $N_{\gamma,\delta}=0$ and $\sigma\alpha=\alpha+\gamma+\delta$. Then*

$$\bar{\varrho}_\alpha\varrho_{\sigma\alpha}=1.$$

LEMMA 4.7. *Let $\alpha\in\mathfrak{r}_\lambda$ and γ, δ, $\varepsilon\in\mathfrak{r}_0$ such that $\sigma\alpha=\alpha+\gamma+\delta+\varepsilon$, $\{\alpha+\gamma,\ \alpha+\delta,\ \alpha+\varepsilon,\ \sigma\alpha-\gamma,\ \sigma\alpha-\delta,\ \sigma\alpha-\varepsilon\}\subset\mathfrak{r}$, $\gamma+\delta\neq 0$, $\delta+\varepsilon\neq 0$, $\gamma+\varepsilon\neq 0$* and *$N_{\gamma,\delta}=N_{\delta,\varepsilon}=N_{\gamma,\varepsilon}=0$. Then*

$$\bar{\varrho}_\alpha\varrho_{\sigma\alpha}=-1.$$

Proof of Lemma 4.6. Apply η on both sides of

$$[[E_\alpha,\ E_\gamma],\ E_\delta]=N_{\alpha,\gamma}N_{\alpha+\gamma,\delta}E_{\sigma\alpha}.$$

And compare the coefficients of both sides remarking that $N_{\alpha,\beta}=N_{-\alpha,-\beta}\in\mathbf{R}$, $|\varrho_\alpha|=1$, $\varrho_\gamma=\varrho_\delta=1$. Then we obtain

$$\bar{\varrho}_\alpha\varrho_{\sigma\alpha}=(N_{\alpha+\gamma+\delta,-\gamma}N_{\alpha+\delta,-\delta})/(N_{\alpha,\gamma}N_{\alpha+\gamma,\delta}). \tag{4.i}$$

Here, apply (4.5.2) to the 4-ple $\{\alpha+\gamma+\delta,\ -\alpha,\ -\gamma,\ -\delta\}$, then we see that

$$N_{\alpha+\gamma+\delta,-\gamma}N_{-\delta,-\alpha}+N_{\alpha+\gamma+\delta,-\delta}N_{-\alpha,-\gamma}=0.$$

Therefore

$$N_{\alpha+\gamma+\delta,-\gamma}/N_{\alpha,\gamma}=N_{\alpha+\gamma+\delta,-\delta}/N_{\alpha,\delta}. \tag{4.ii}$$

Next, apply (4.5.1) to triples $\{\alpha+\gamma+\delta,\ -\delta,\ -\alpha-\gamma\}$ and $\{\alpha+\delta,\ -\delta,\ -\alpha\}$. Then we see that

$$N_{\alpha+\gamma+\delta,-\delta}=N_{-\delta,-\alpha-\gamma}=-N_{\alpha+\gamma,\delta}, \tag{4.iii}$$

and that

$$N_{\alpha+\delta,-\delta}=N_{-\delta,-\alpha}=-N_{\alpha,\delta}. \tag{4.iv}$$

From (4.i)-(4.iv) we conclude that

$$\bar{\varrho}_\alpha\varrho_{\sigma\alpha}=1. \qquad \text{q.e.d.}$$

Proof of Lemma 4.7. Apply η on both sides of

$$[[[E_\alpha,\ E_\gamma],\ E_\delta],\ E_\varepsilon]=N_{\alpha,\gamma}N_{\alpha+\gamma,\delta}N_{\alpha+\gamma+\delta,\varepsilon}E_{\sigma\alpha},$$

and compare the coefficients of both sides. Then we obtain

(4.v) $\bar{\varrho}_\alpha \varrho_{\sigma\alpha} = (N_{\alpha+\gamma+\delta+\varepsilon,-\gamma} N_{\alpha+\delta+\varepsilon,-\delta} N_{\alpha+\varepsilon,-\varepsilon}) / (N_{\alpha,\gamma} N_{\alpha+\gamma,\delta} N_{\alpha+\gamma+\delta,\varepsilon})$.

Apply (4.5.2) to the 4-ple $\{\alpha+\gamma+\delta+\varepsilon,\ -\alpha-\delta,\ -\gamma,\ -\varepsilon\}$, then we see that

$$N_{\alpha+\gamma+\delta+\varepsilon,-\gamma} N_{-\varepsilon,-\alpha-\delta} + N_{\alpha+\gamma+\delta+\varepsilon,-\varepsilon} N_{-\alpha-\delta,-\gamma} = 0,$$

hence

(4.vi) $N_{\alpha+\gamma+\delta+\varepsilon,-\gamma} / N_{\alpha+\delta,\gamma} = N_{\alpha+\gamma+\delta+\varepsilon,-\varepsilon} / N_{\alpha+\delta,\varepsilon}$.

Next apply (4.5.1) to the triple $\{\alpha+\gamma+\delta+\varepsilon,\ -\varepsilon,\ -\alpha-\gamma-\delta\}$, then we see that

(4.vi′) $N_{\alpha+\gamma+\delta+\varepsilon,-\varepsilon} = N_{-\varepsilon,-\alpha-\gamma-\delta} = -N_{\alpha+\gamma+\delta,\varepsilon}$.

From (4.vi) and (4.vi′) we obtain

(4.vii) $N_{\alpha+\gamma+\delta+\varepsilon,-\gamma} / N_{\alpha+\gamma+\delta,\varepsilon} = -N_{\alpha+\delta,\gamma} / N_{\alpha+\delta,\varepsilon}$.

Similarly, applying (4.5.2) to the 4-ple $\{\alpha+\delta+\varepsilon,\ -\alpha,\ -\delta,\ -\varepsilon\}$ and then applying (4.5.1) to the triple $\{\alpha+\delta+\varepsilon,\ -\varepsilon,\ -\alpha-\delta\}$, we see that

(4.viii) $N_{\alpha+\delta+\varepsilon,-\delta} / N_{\alpha,\delta} = -N_{\alpha+\delta,\varepsilon} / N_{\alpha,\varepsilon}$.

Further, apply (4.5.1) to the triple $\{\alpha+\varepsilon,\ -\varepsilon,\ -\alpha\}$, then we obtain

(4.ix) $N_{\alpha+\varepsilon,-\varepsilon} = N_{-\varepsilon,-\alpha} = -N_{\alpha,\varepsilon}$.

Finally, since $[E_\gamma, E_\delta] = 0$ we have the equality

$$[[E_\alpha, E_\gamma], E_\delta] = [[E_\alpha, E_\delta], E_\gamma],$$

which implies that

(4.x) $N_{\alpha,\delta} / (N_{\alpha,\gamma} N_{\alpha+\gamma,\delta}) = 1 / N_{\alpha+\delta,\gamma}$.

Multiply (4.vii)-(4.x) side by side, and then compare with (4.v). Then we conclude that

$$\bar{\varrho}_\alpha \varrho_{\sigma\alpha} = -1. \qquad \text{q.e.d.}$$

4.6 THEOREM 4.8. a) *The following normal σ-systems of roots of restricted rank* 1: $A_1 i)$, $B_l i)$, $A_1 \times A_1 ii)$, $A_3 ii)$, $D_l ii)$, $A_l iii)$, $C_l iii)$ *and* $F_4 iii)$, *are normally extendable.* b) *The following ones: $D_l iii)$ and $E_l iii)$* ($6 \leqq l \leqq 8$), *are not normally extendable.*

Proof. For the σ-fundamental system Δ of the normal σ-system of roots $\mathfrak{r}$ of restricted rank 1 of each type, choose an anti-automorphism η satisfying Prop. 3.1 such that ϱ_{α_1} ($|\varrho_{\alpha_1}| = 1$) is arbitrary, and that $\varrho_{\alpha_2} = \varrho_{\alpha_1}$ for $A_1 \times A_1 ii)$, $\varrho_{\alpha'} = (N_{\alpha',\gamma} / N_{\sigma\alpha',-\gamma}) \varrho_{\alpha_1}$ for $A_l iii)$ by putting $\gamma = \gamma_1 + \dots + \gamma_{l-2}$.

First we see easily that

$$\begin{aligned} \varrho_{\sigma\alpha_1} \overline{\varrho_{\alpha_1}} &= 1 && \text{for } A_1 i), \\ \varrho_{\sigma\alpha_1} \overline{\varrho_{\alpha_1}} &= \varrho_{\sigma\alpha_2} \overline{\varrho_{\sigma\alpha_2}} = 1 && \text{for } A_1 \times A_1 ii), \\ \varrho_{\sigma\alpha_1} \overline{\varrho_{\alpha_1}} &= \varrho_{\sigma\alpha'} \overline{\varrho_{\alpha'}} = 1 && \text{for } A_l iii). \end{aligned}$$

Further, applying Lemma 4.6 for $\alpha = \alpha_1$, we see that

$$\varrho_{\sigma\alpha_1} \overline{\varrho_{\alpha_1}} = 1$$

by putting $\gamma = \delta = \gamma_1 + \dots + \gamma_{l-1}$ for $B_l i)$, $\gamma = \gamma_1$ and $\delta = \gamma_2$ for $A_3 ii)$, $\gamma = \gamma_1 + \dots + \gamma_{l-3} + \gamma_{l-2}$ and $\delta = \gamma_1 + \dots + \gamma_{l-3} + \gamma_{l-1}$ for $D_l ii)$, $\gamma = \gamma_1$ and $\delta = 2(\gamma_2 + \dots + \gamma_{l-2}) + \gamma_{l-1}$ for $C_l iii)$, $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ and $\delta = \gamma_2 + 2\gamma_3$ for $F_4 iii)$. Hence

$$\eta\eta E_\beta = E_\beta$$

for every simple root $\beta \in \Delta$ of each former type. Therefore

$$\eta\eta = \text{the identity automorphism of } \mathfrak{g}_C$$

for each former type by the uniqueness of (3.3).

Next apply Lemm 4.7 for Δ of $D_l iii)$ by putting $\alpha=\alpha_1$, $\gamma=\gamma_1$, $\delta=\gamma_2+\dots+\gamma_{l-3}+\gamma_{l-2}$ and $\varepsilon=\gamma_2+\dots+\gamma_{l-3}+\gamma_{l-1}$; for $E_6 iii)$ by putting $\alpha=\alpha_1$, $\gamma=\gamma_3$, $\delta=\gamma_2+\gamma_3+\gamma_4$ and $\varepsilon=\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5$; for $E_7 iii)$ by putting $\alpha=\alpha_1$, $\gamma=\gamma_1+\gamma_2+\gamma_3$, $\delta=\gamma_1+\gamma_2+\gamma_6$ and $\varepsilon=\gamma_1+2\gamma_2+2\gamma_3+2\gamma_4+\gamma_5+\gamma_6$; for $E_8 iii)$ by putting $\alpha=\alpha_1$, $\gamma=\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5$, $\delta=\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_7$ and $\varepsilon=\gamma_1+2\gamma_2+3\gamma_3+4\gamma_4+3\gamma_5+2\gamma_6+2\gamma_7$. Then we see that

$$\varrho_{\sigma\alpha_1}\overline{\varrho_{\alpha_1}}=-1$$

for each latter type. Hence η cannot be involutive for any choice of ϱ_{α_1} for each latter type.

By the above Theorem we obtain a classification of irreducible infinitesimal symmetric pairs of rank 1 by their σ-fundamental systems.

§5. Classification (General case).

5.1 In this paragraph we classify σ-fundamental systems Δ of σ-irreducible normally extendable σ-systems of roots. This will give us a classification of infinitesimal irreducible symmetric pairs *via* Prop. 1.2 and Cor. 2.15. Let $\mathfrak{r}$ be such a σ-system of roots. $\mathfrak{r}$ consists at most of two components.

If $\mathfrak{r}$ consists exactly of two components $\mathfrak{r}_1$ and $\mathfrak{r}_2$, then $\sigma\mathfrak{r}_1=\mathfrak{r}_2$, and a σ-fundamental system consists of two connected-ness components Δ_1 and Δ_2 such that Δ_i forms a fundamental system of $\mathfrak{r}_i$ for each $i=1, 2$, and $\sigma\Delta_1=\Delta_2$. Hence $\Delta^-\cong\Delta_i$, and Δ^λ is of type $A_1\times A_1 ii)$ for each $\lambda\in\Delta^-$ by a notation of No.3.2. Therefore $\mathfrak{r}$ is normally extendable by Theorems 3.6 and 4.8. This is the case that the corresponding compact symmetric space is the space of a compact simple Lie group of the same type as Δ^-.

In subsequent Nos. we discuss the cases of $\mathfrak{r}$ being connected, and hence of $\mathfrak{g}_\sigma$ being simple.

Theorem 3.6 is the key theorem for our classification, by which only σ-fundamental systems of normally extendable ones are possible as Δ^λ, $\lambda\in\Delta^-$.

In the sequel we use the following arguments frequently: under some assumptions about a σ-fundamental system Δ, $\sigma\varphi_i$ and $\sigma\varphi_j$ are determined for two simple roots φ_i, φ_j of Δ in such a way that

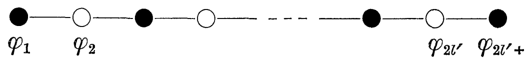
$$\langle\sigma\varphi_i, \sigma\varphi_j\rangle\neq\langle\varphi_i, \varphi_j\rangle,$$

which contradicts to "σ is isometric"; hence there exists no σ-fundamental system which is normally extendable and satisfies the given assumptions. This type of argument is called an "isometry argument" for the sake of simplicity.

To describe the types of real simple Lie algebras $\mathfrak{g}_{\tilde{\sigma}}$ we use the usual notation due to E.Cartan.

5.2. Type $\mathbf{A}_l$. Schläfli figure of Δ: $\underset{\varphi_1}{\circ}\!-\!\!-\!\underset{\varphi_2}{\circ}\!-\!\!-\cdots-\!\!-\!\underset{\varphi_{l-1}}{\circ}\!-\!\!-\!\underset{\varphi_l}{\circ}$.
Since $\Delta^\lambda\subset\Delta$ for each $\lambda\in\Delta^-$, Δ^λ must be of types $A_1 i)$, $A_3 ii)$, $A_m iii)$ or $A_1\times A_1 ii)$. Our discussion is divided in four cases.

a) *The case that at least one external root of Δ belongs to Δ_0.* We may assume that $\varphi_1\in\Delta_0$. Δ^λ, $\lambda\in\Delta^-$, containing φ_1 must be of type $A_3 ii)$ since in the remaining possible three types every external root does not belong to Δ_o; then $\varphi_3 \in \Delta_o$, and $\varphi_4 \notin \Delta_o$ by Prop. 3.5. Next Δ^λ, $\lambda \in \Delta^-$, containing φ_4 must be of type $A_3 ii)$ by the same reason as above since it contains φ_3 by Prop. 3.5, and $\varphi_5 \in \Delta_o$, $\varphi_6 \notin \Delta_o$; the same arguments continue iteratedly; finally l must be odd, and we obtain a normally extendable σ-fundamental system



In particular: $l = 2l'+1$ $(l' \geqq 1)$, $\varphi_{2i+1} \in \Delta_o$ $(0 \leqq i \leqq l')$, $\varphi_{2i} \notin \Delta_o$ $(1 \leqq i \leqq l')$, and

$$\sigma\varphi_{2i} = \varphi_{2i-1} + \varphi_{2i} + \varphi_{2i+1} \qquad (1 \leqq i \leqq l').$$

The corresponding real simple Lie algebra $\mathfrak{g}_{\tilde{\sigma}}$ is of type **AII.**

In the remaining cases two external roots of Δ belong to $\Delta - \Delta_o$. We put $\varphi_1 \mid \mathfrak{h}_o^- = \lambda_1$, i.e., $\varphi_1 \in \Delta^{\lambda_1}$.

b) *The case that Δ^{λ_1} is of type $A_m iii)$.* If $m < l$, then $\varphi_{m+1} \notin \Delta_o$ by Prop. 3.5. Since

$$0 > \langle \varphi_{m+1}, \varphi_m \rangle = \langle \sigma\varphi_{m+1}, \sigma\varphi_m \rangle = \langle \sigma\varphi_{m+1}, \varphi_1 + \cdots\cdot + \varphi_{m-1} \rangle,$$

there exists an $i \leqq m-1$ such that

$$\langle \sigma\varphi_{m+1}, \varphi_i \rangle < 0,$$

which becomes impossible after discussing possible types of Δ^λ such that $\Delta^\lambda \ni \varphi_{m+1}$ Hence $l = m$, and $\Delta = \Delta^{\lambda_1}$ with $l \geqq 2$. The corresponding $\mathfrak{g}_{\tilde{\sigma}}$ is of type **AIV.**

c) *The case that Δ^{λ_1} is of type $A_1 \times A_1 ii)$.* Put $\Delta^{\lambda_1} = \{\varphi_1, \varphi_m\}$, then $m > 2$. $\{\varphi_2, \varphi_{m-1}\} \subset \Delta - \Delta_o$ (and $\varphi_{m+1} \in \Delta - \Delta_o$ if $m < l$) by Prop. 3.5. Here

$$\langle \sigma\varphi_{m-1}, \varphi_1 \rangle < 0,$$

and, if $m < l$, then

$$\langle \sigma\varphi_{m+1}, \varphi_1 \rangle < 0$$

similarly as in the above case. Putting $\varphi_{m-1} \mid \mathfrak{h}_o^- = \lambda'$ and $\varphi_{m+1} \mid \mathfrak{h}_o^- = \lambda''$, we see from the above formula that

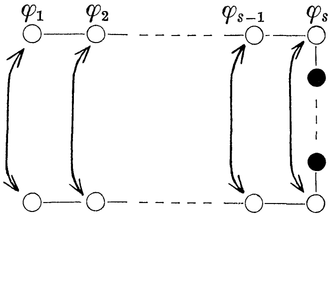
$$\lambda' = \varphi_2 \mid \mathfrak{h}_o^- = \lambda''$$

by checking the possible types of $\Delta^{\lambda'}$ and of $\Delta^{\lambda''}$. But this is impossible. Hence it must be that $m = l$.

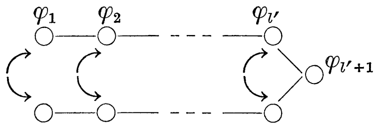
Now $\Delta - \Delta^{\lambda_1}$ is a normally extendable σ-fundamental system by Prop. 3.7, and

$$\varphi_2 \mid \mathfrak{h}_o^- = \varphi_{l-1} \mid \mathfrak{h}_o^- = \lambda'.$$

If $l = 3$, then $\Delta^{\lambda'} = \{\varphi_2\}$ is of type $A_1 i)$. If $l > 3$, then $\Delta^{\lambda'}$ is of type $A_{l-2} iii)$ by b) or $A_1 \times A_1 ii)$ with $l > 4$. In case $\Delta^{\lambda'}$ being of type $A_1 \times A_1 ii)$ we can iterate the same discussion as above. Continue the same discussions iteratedly. Finally we obtain the following σ-fundamental systems normally extendable:



for $2 \leqq s \leqq l/2$,



in case $l = 2l'+1$.

The corresponding $\mathfrak{g}_{\tilde{\sigma}}$ are of type **AIII.**

d) *The case that* Δ^{λ_1} *is of type* A_1i). $\Delta^{\lambda_1} = \{\varphi_1\}$ and $\sigma\varphi_1 = \varphi_1$. Hence $\varphi_2 \notin \Delta_0$ by Prop. 3.5, and

$$\langle \varphi_1, \sigma\varphi_2 \rangle < 0.$$

Since Δ^{λ_2}, $\lambda_2 = \varphi_2 \,|\, \mathfrak{h}_o^-$, is of type $A_1i)$, $A_1 \times A_1ii)$ or $A_miii)$, the above formula implies immediately that

$$\Delta^{\lambda_2} = \{\varphi_2\} \text{ and } \sigma\varphi_2 = \varphi_2.$$

Similarly we see that $\sigma\varphi_3 = \varphi_3$, and so on; thus we obtain a normally extendable σ-fundamental system

$$\Delta : \ \bigcirc\!\!-\!\!\bigcirc\!-\cdots-\!\bigcirc\!\!-\!\!\bigcirc\ .$$

The corresponding $\mathfrak{g}_{\tilde{\sigma}}$ is of type **AI** (normal form of $\mathfrak{g}_C$).

5.3. Type B$_l$. Schläfli figure of Δ: $\underset{}{\overset{\varphi_l}{\bigcirc}}\!-\cdots-\!\overset{\varphi_2}{\bigcirc}\!\Longrightarrow\!\overset{\varphi_1}{\bigcirc}$.

Δ^λ, $\lambda \in \Delta^-$, can be only of types $A_1i)$, $A_1 \times A_1ii)$, $A_3ii)$, $A_miii)$ or $B_mi)$. Our discussion is divided in two cases.

a) *The case* $\varphi_1 \notin \Delta_o$ Put $\varphi_1 \,|\, \mathfrak{h}_o^- = \lambda_1$. Δ^{λ_1} must be of type $A_1i)$ as is easily seen. Then $\Delta - \Delta^{\lambda_1}$ is of type A_{l-1}, and is a normally extendable σ-fundamental system by Prop. 3.7. Since

$$\langle \sigma\varphi_2, \varphi_1 \rangle = \langle \varphi_2, \varphi_1 \rangle < 0,$$

we see easily that Δ^λ, containing φ_2, is of type $A_1i)$ and $\sigma\varphi_2 = \varphi_2$. Then, as in No. 5.3.d), we see that

$$\sigma\varphi_i = \varphi_i$$

for all $i \leqq l$. Thus we have a normally extendable σ-fundamental system

$$\Delta : \ \bigcirc\!\!-\!\!\bigcirc\!-\cdots-\!\bigcirc\!\Longrightarrow\!\bigcirc\ .$$

The corresponding $\mathfrak{g}_{\tilde{\sigma}}$ is the normal form of $\mathfrak{g}_C$, and is a special case of type **BI.**

b) *The case* $\varphi_1 \in \Delta_o$. There exists an $m < l$ such that $\{\varphi_1, \dots, \varphi_m\} \subset \Delta_o$ and $\varphi_{m+1} \notin \Delta_o$. Then, putting $\varphi_{m+1} \,|\, \mathfrak{h}_o^- = \lambda_1$, Δ^{λ_1} must be of type $B_{m+1}i)$. Now discuss parallelly to the above case, then we see immediately that

$$\sigma\varphi_j = \varphi_j \qquad \text{for } m+1 < j \leqq l.$$

Thus we have normally extendable σ-fundamental systems

$$\Delta : \ \underset{\varphi_l}{\bigcirc}\!-\!\bigcirc\!-\cdots-\!\underset{\varphi_{m+1}}{\bigcirc}\!-\!\underset{\varphi_m}{\bullet}\!-\cdots-\!\underset{\varphi_2}{\bullet}\!\Longrightarrow\!\underset{\varphi_1}{\bullet} \qquad \text{for } 1 \leqq m < l.$$

Corresponding $\mathfrak{g}_{\tilde{\sigma}}$ are of type **BI** for $m < l-1$, and of type **BII** for $m = l-1$.

5.4 Type C$_l$. Schläfli figure of Δ: $\overset{\varphi_l}{\bigcirc}\!-\cdots-\!\overset{\varphi_2}{\bigcirc}\!\Longleftarrow\!\overset{\varphi_1}{\bigcirc}$

Δ^λ, $\lambda \in \Delta^-$, can be only of types $A_1i)$, $A_1 \times A_1ii)$, $A_3ii)$, $A_miii)$, $B_2i)$ or $C_miii)$. The discussion is divided in three cases.

a) *The case* $\varphi_1 \in \Delta_o$. There is an $m \geqq 2$ such that $\{\varphi_1, \dots, \varphi_{m-1}\} \subset \Delta_o$ and $\varphi_m \notin \Delta_o$. Then, putting $\varphi_m \,|\, \mathfrak{h}_o^- = \lambda_1$, Δ^{λ_1} must be of type $C_{m+1}i)$. Hence, $m \leqq l-1$,

$\Delta^{\lambda_1} = \{\varphi_1, \dots, \varphi_{m+1}\}$, $\varphi_{m+1} \in \Delta_o$ and $\varphi_{m+2} \notin \Delta_o$. Now $\{\varphi_{m+1}, \varphi_{m+2}, \dots, \varphi_l\}$ is a σ-fundamental system of type A_{l-m} such that $\varphi_{m+1} \in \Delta_o$, which is normally extendable by Prop. 3.7. Therefore the remaining discussions are reduced to the case of No. 5.2. a). For each $\lambda \in \Delta^-$ suct that $\lambda \neq \lambda_1$, Δ^λ is of type A$_3$ii). Thus we obtain a normally extendable σ-fundamental system

Δ: ●—○—●— · · · —○—●—○—●— - - - - —●⇐●
φ_l $\qquad\qquad$ φ_{m+2} φ_{m+1} φ_m $\qquad\qquad$ φ_1

for $2 \leqq m \leqq l-1$ such that $l-m$ is odd. The corresponding $\mathfrak{g}_{\bar\sigma}$ is of type **CII.**

In the remaining cases $\varphi_1 \notin \Delta_o$. Putting $\varphi_1 | \mathfrak{h}_o^- = \lambda_1$, Δ^{λ_1} must be of type $B_2i)$ or $A_1i)$.

b) *The case Δ^{λ_1} being of type $B_2i)$.* Then $\Delta^{\lambda_1} = \{\varphi_1, \varphi_2\}$, $\varphi_2 \in \Delta_o$ and $\varphi_3 \notin \Delta_o$. $\{\varphi_2, \dots, \varphi_l\}$ form a σ-fundamental system of type A_{l-1} which is normally extendable by Prop. 3.7. Since $\varphi_2 \in \Delta_o$, the remaining discussions are reduced to the case of No. 5.2.a). Δ^λ, $\lambda \in \Delta^- - \{\lambda_1\}$, are all of type $A_3ii)$. Thus we obtain a normally extendable σ-fundamental system

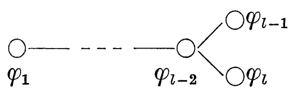
Δ: ●—○—●— - - - - —○—●⇐○

for l even. The corresponding $\mathfrak{g}_{\bar\sigma}$ is a special case of type **CII.**

c) *The case Δ^{λ_1} being of type $A_1i)$.* By a discussion similar as in No. 5.3.a), we obtain a normally extendable σ-fundamental sytem

Δ: ○— - - - —○⇐○ .

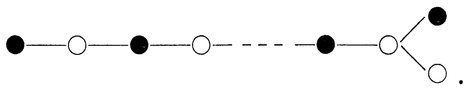
The corresponding $\mathfrak{g}_{\bar\sigma}$ is of type **CI** (normal form of $\mathfrak{g}_c$).

5.5 Type D$_l$. Schläfli figure of Δ: 

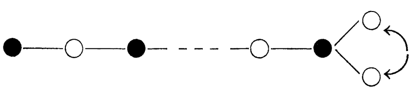
Δ^λ, $\lambda \in \Delta^-$, can be only of types $A_1i)$, $A_1 \times A_1ii)$, $A_3ii)$, $A_miii)$ or $D_mii)$. The discussion is divided in three cases.

a) *The case $\varphi_1 \in \Delta_o$.* When $l > 4$, the possible type of Δ^λ containing φ_1 is only $A_3ii)$. Hence the same discussion as in No. 5.2.a) continue until $\sigma\varphi_{l-3}$ is determined.

If l is even, then $\varphi_{l-2} \notin \Delta_o$. Put $\varphi_{l-2} | \mathfrak{h}_o^- = \lambda'$. Since $\varphi_{l-3} \in \Delta_o$, $\Delta^{\lambda'}$ must be of type $A_3ii)$. Therefore, $\varphi_{l-1} \in \Delta_o$ and $\varphi_l \notin \Delta_o$, or $\varphi_{l-1} \notin \Delta_o$ and $\varphi_l \in \Delta_o$. Thus we obtain a normally extendable σ-fundamentsl system

Δ:  .

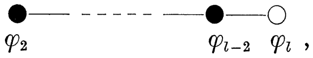
If l is odd, then $\varphi_{l-2} \in \Delta_o$. And $\{\varphi_{l-1}, \varphi_{l-2}, \varphi_l\}$ is a σ-fundamental system of type A_3 which is normally extendable by Prop. 3.7. Therefore it must be of restricted rank 1 and of type $A_3iii)$ as is easily checked. We obtain a normally extendable σ-fundamental system

Δ: 

The corresponding $\mathfrak{g}_{\bar{\sigma}}$ are of type **DIII** for the above two σ-fundamental systems.

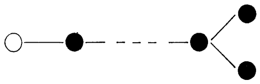
In the remaining two cases $\varphi_1 \notin \Delta_0$. Put $\varphi_1 | \mathfrak{h}_0^- = \lambda_1$.

b) *The case* $\varphi_2 \in \Delta_0$. Δ^{λ_1} must of type $A_m iii)$ or $D_l ii)$. If Δ^{λ_1} is of type A_m $iii)$, then $\Delta^{\lambda_1} = \{\varphi_1, \dots, \varphi_m\}$. First we see that $m \leqq l-2$, since otherwise $m = l-1$ and Δ^{λ} containing φ_l has by Prop. 3.5 the following Satake figure:



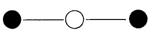
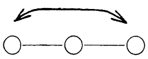
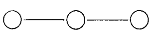
,

to which there corresponds no normal σ-fundamental system of roots of restricted rank 1 by discussions of §4. Now $\varphi_{m+1} \notin \Delta_0$, and applying an "isometry argument" to the pair $(\varphi_m, \varphi_{m+1})$ we arrive to a contradiction. Hence Δ^{λ_1} can not be of type $A_m iii)$, and must be of typc $D_l ii)$. Thus we see that

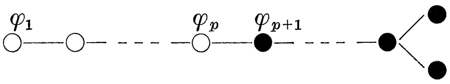
$\Delta = \Delta^{\lambda_1}$: 

The corresponding $\mathfrak{g}_{\bar{\sigma}}$ is of type **DII.**

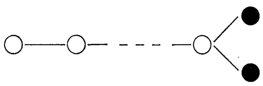
c) *The case* $\varphi_2 \notin \Delta_0$. Δ^{λ_1} must be of type $A_1 i)$ or $A_1 \times A_1 ii)$. If Δ^{λ_1} is of type $A_1 \times A_1 ii)$ such that $\Delta^{\lambda_1} = \{\varphi_1, \varphi_m\}$, then discussing $\langle \sigma\varphi_{m-1}, \varphi_1 \rangle$ and $\langle \sigma\varphi_{m+1}, \varphi_1 \rangle$ we arrive readily to a contradiction. Hence Δ^{λ_1} must be of type $A_1 i)$. Then, $\Delta - \{\varphi_1\}$ is by Prop.3.7 a normally extendable σ-fundamental system with $\varphi_2 \notin \Delta_0$ and of type D_{l-1} if $l > 4$, or of type A_3 if $l = 4$. In case $l = 4$, by the classification of No. 5.2, the Satake figure of $\Delta - \{\varphi_1\}$ must be

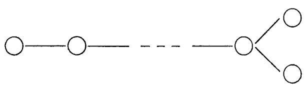
 ,  or  .

In case $l > 4$, the discussions of b) and c) can be applied again to $\Delta - \{\varphi_1\}$. Continue these arguments iteratedly, then finally we obtain the following normally extendable σ-fundamental systems:

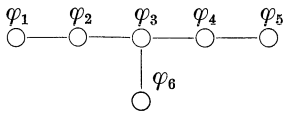


for $2 \leqq p < l-2$,

 ,  ,

 .

The corresponding $\mathfrak{g}_{\bar{\sigma}}$ are of type **DI.**

5.6 Type $\mathbf{E_6}$. Schläfli figure of Δ: 

.

Δ^{λ}, $\lambda \in \Delta^-$, can be only of types $A_1 i)$, $A_1 \times A_1 ii)$, $A_3 ii)$, or $D_m ii)$.

Assume that $\varphi_1 \in \Delta_0$, then Δ^{λ} containing φ_1 must be of type $A_3 ii)$; hence $\varphi_3 \in \Delta_0$ and $\{\varphi_2, \varphi_4, \varphi_6\} \subset \Delta - \Delta_0$ by Prop. 3.5. Then, putting $\varphi_6 | \mathfrak{h}_0^- = \lambda'$, $\Delta^{\lambda'}$ must be of type $A_3 iii)$ such that $\sigma\varphi_6 = \varphi_3 + \varphi_4$, and $\varphi_5 \notin \Delta_0$; finally, putting $\varphi_5 | \mathfrak{h}_0^- = \lambda''$,

$\varDelta^{\lambda''} = \{\varphi_5\}$ must be of type $A_1 i$). Now, applying an "isometry argument" to the pair (φ_4, φ_5) we arrive easily to a contradiction. Therefore we see that

(5.6.1) $$\varphi_1 \notin \varDelta_o.$$

Similarly we see that

(5.6.2) $$\varphi_5 \notin \varDelta_o.$$

Here we put $\varphi_1 | \mathfrak{h}_o^- = \lambda_1$. Subsequent discussions are divided in two cases according as $\varphi_2 \in \varDelta_o$ or not.

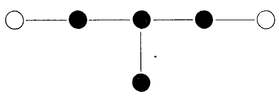
a) *The case* $\varphi_2 \in \varDelta_o$. If we assume that $\varphi_3 \notin \varDelta_o$, then $\varDelta^{\lambda_1}$ must be of type $A_3 iii$), $\{\varphi_4, \varphi_6\} \subset \varDelta - \varDelta_o$, and $\sigma\varphi_6$ is a linear combination of φ_4, φ_5 and φ_6. Now an "isometry argument" of the pair (φ_3, φ_6) leads to a contradiction. Hence

$$\varphi_3 \in \varDelta_o.$$

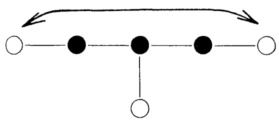
Next, if we assume that $\varphi_4 \notin \varDelta_o$, then $\varDelta^{\lambda_1}$ must be of type $A_4 iii$), and $\varphi_6 \notin \varDelta_o$. Then we must have a $\lambda \in \varDelta^-$, such that the figure of $\varDelta^\lambda$ is ○—●—● by Prop. 3.5, which is impossible. Hence

$$\varphi_4 \in \varDelta_o.$$

Thus we obtain the following two normally extendable σ-fundamental systems according as $\varphi_6 \in \varDelta_o$ or not.



and



.

The former figure corresponds to $\mathfrak{g}_{\bar{\sigma}}$ of type **EVI**, and the latter to that of type **EIII.**

b) *The case* $\varphi_2 \notin \varDelta_o$. If $\varphi_4 \in \varDelta_o$, then by a discussion parallel to that of a) we must conclude that $\varphi_2 \in \varDelta_o$, which is impossible by our assumption. Therefore

$$\varphi_4 \notin \varDelta_o.$$

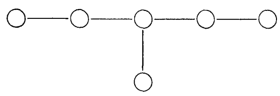
If we assume that $\varphi_3 \in \varDelta_o$, then, putting $\varphi_2 | \mathfrak{h}_o^- = \lambda_2$, $\varDelta^{\lambda_2}$ must be of type A_3 iii), and $\varphi_6 \notin \varDelta_o$. Then we must have a $\lambda \in \varDelta^-$ such that the figure of $\varDelta^\lambda$ is ○—● by Prop. 3.5, which is impossible. Hence

$$\varphi_3 \notin \varDelta_o.$$

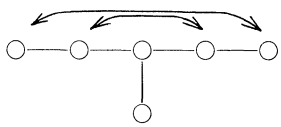
Further we see readily that

$$\varphi_6 \notin \varDelta_o.$$

Thus $\varDelta_o = \phi$, and $\sigma | \varDelta$ must be an involutive automorphism of $\varDelta$ so that we obtain the following two normally extendable σ-fundamental systems



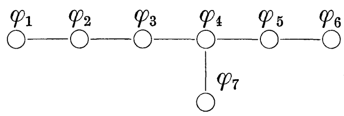
and



.

The former figure corresponds to $\mathfrak{g}_{\bar{\sigma}}$ of type **EI** (normal form of E_6), and the latter to that of type **EII.**

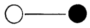
5.7. Type E$_7$. Schläfli figure of $\varDelta$:



.

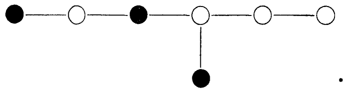
First, the same discussion as in the proof of (5.6.1) shows that

(5.7.1) $$\varphi_6 \notin \Delta_0.$$

a) *The case* $\varphi_1 \in \Delta_0$. Δ^λ containing φ_1 must be of type $A_3 ii)$; $\varphi_2 \notin \Delta_0$, $\varphi_3 \in \Delta_0$ and $\varphi_4 \notin \Delta_0$. Then Δ^λ containing φ_4 must be of type $A_3 ii)$, and $\varphi_5 \in \Delta_0$ or $\varphi_7 \in \Delta_0$. If we assume that $\varphi_5 \in \Delta_0$, then $\Delta^{\lambda'}$, $\varphi_6 | \mathfrak{h}_0^- = \lambda'$, must have a figure  by Prop. 3.5 which is impossible. Therefore

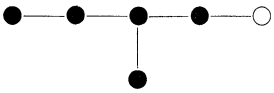
$$\varphi_5 \notin \Delta_0 \quad \text{and} \quad \varphi_7 \in \Delta_0 .$$

Thus we obtain a normally extendable σ-fundamental system

Δ: 

The corresponding $\mathfrak{g}_\sigma$ is of type **EVI.**

b) *The case* $\varphi_1 \notin \Delta_0$. Put $\varphi_1 | \mathfrak{h}_0^- = \lambda_1$. If we assume that $\varphi_2 \in \Delta_0$, then the possible type of Δ^{λ_1} is $A_m iii)$, $3 \leqq m \leqq 6$, or $D_6 ii)$. In case Δ^{λ_1} is of type $D_6 ii)$, $\Delta^{\lambda'}$, $\lambda' = \varphi_6 | \mathfrak{h}_0^-$, must have the figure



by Prop. 3.5, which is impossible. Similar arguments or "isometry arguments" show that all other possible types of Δ^{λ_1} do not occur, and hence we obtain

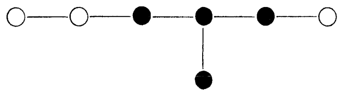
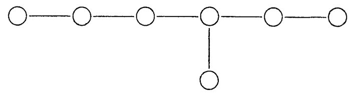
(5.7.2) $$\varphi_2 \notin \Delta_0.$$

Now Δ^{λ_1} must be of type $A_1 i)$ or $A_1 \times A_1 ii)$. Checking all possibilities of Δ^{λ_1} being of type $A_1 \times A_1 ii)$ by "isometry arguments", we see that

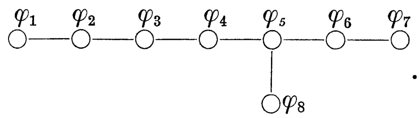
(5.7.3) Δ^{λ_1} is of type $A_1 i)$.

Then $\Delta - \{\varphi_1\}$ is a normally extendable σ-fundamental system by Prop. 3.7, and must have one of the four figures of No. 5.6, two of which become impossible after applying "isometry arguments" to the pair (φ_1, φ_2).

Thus we obtain the following two normally extendable σ-fundamental systems Δ:

 and 

The former corresponds to $\mathfrak{g}_\sigma$ of type **EVII,** and the latter to that of type **EV** (normal form of E_7).

5.8. Type E$_8$. Schläfli figure of Δ: 

In the same way as in the proof of (5.6.1), first we see that

(5.8.1) $$\varphi_1 \notin \Delta_0 \text{ and } \varphi_7 \notin \Delta_0.$$

Next, parallelly to the discussions of No. 5.7. b), we see that

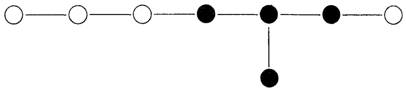
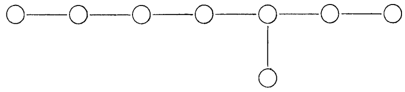
(5.8.2) $$\varphi_2 \notin \Delta_0 \text{ and } \varphi_3 \notin \Delta_0,$$

and that

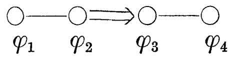
(5.8.3) Δ^{λ_1}, $\varphi_1 | \mathfrak{h}_0^- = \lambda_1$, is of type $A_1 i)$.

Then $\varDelta-\{\varphi_1\}$ is a normally extendable σ-fundamental system by Prop. 3.7, which must have one of the two figures of No. 5.7. b) since (5.8.2).

Thus we obtain the following two normally extendable σ-fundamental systems $\varDelta$:

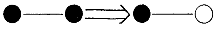
 and  .

The former corresponds to $\mathfrak{g}_{\bar{\sigma}}$ of type **EIX,** and the latter to that of type **EVIII** (normal form of E_8).

5.9. Type $\mathbf{F}_4$. Schläfli figure of $\varDelta$:  .

$\varDelta^\lambda$, $\lambda\in\varDelta^-$, can be only of types $A_1i)$, $A_2iii)$, $B_mi)$ ($m=2$ or 3), $C_3iii)$, or $F_4iii)$.

a) *In Case* $\varphi_1\in\varDelta_0$. Then only $F_4iii)$ is possiblle as a type of $\varDelta^\lambda$ as is easily checked; hence we obtained a normally extendable σ-fundamental system

$\varDelta=\varDelta^\lambda$: 

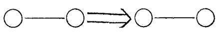
which corresponds to $\mathfrak{g}_{\bar{\sigma}}$ of type **FII.**

b) *In case* $\varphi\notin\varDelta_0$. If we assume that $\varphi_2\in\varDelta_0$, then $\varphi_3\in\varDelta_0$ since otherwise we have a figure ○—● for $\varDelta^\lambda$ containing φ_1 by Prop. 3.5, which is impossible. Then $\varphi_4\notin\varDelta_0$ since otherwise we obtain a figure ○—●⟹●—● which is impossible. Now $\varDelta^\lambda$ containing φ_4 has the figure ●⟹●—○ which is also impossible. Hence

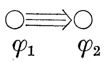
(5.9.1) $\qquad \varphi_2\notin\varDelta_0$.

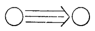
And $\varDelta^{\lambda_1}$, $\lambda_1=\varphi_1\,|\,\mathfrak{h}_0^-$, must be of type $A_1i)$ necessarily. Then, $\varDelta-\{\varphi_1\}$ form a normally extendable σ-fundamental system by Prop.3.7.

By the classification of No.5.4 and (5.9.1), $\varDelta-\{\varphi_1\}$ must be of type *CI.* Thus we obtain a normally extendable σ-fundamental system

$\varDelta$: 

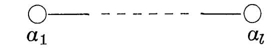
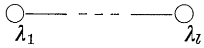
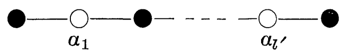
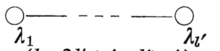
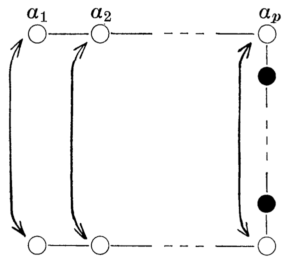
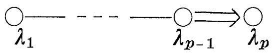
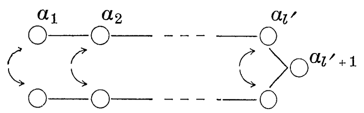
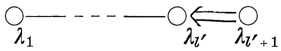
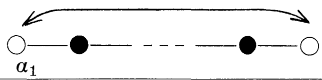
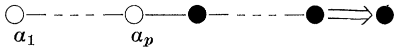
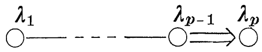
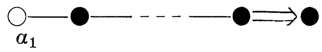
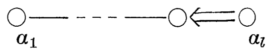
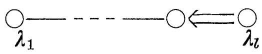
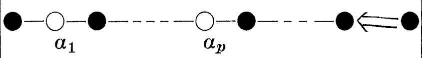
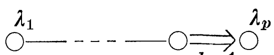
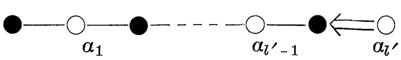
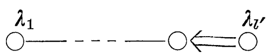
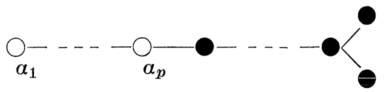
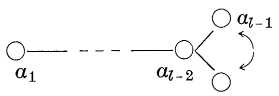
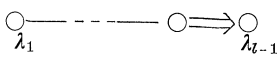
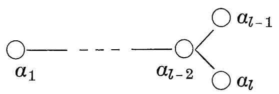
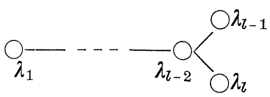
which corresponds to $\mathfrak{g}_{\bar{\sigma}}$ of type **FI** (normal form of F_4).

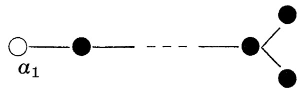
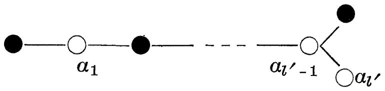
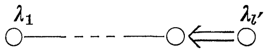
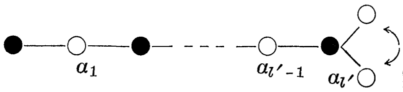
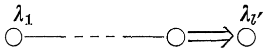
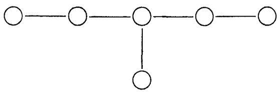
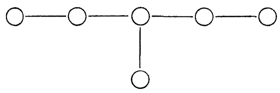
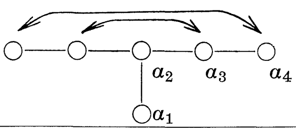
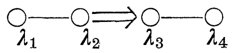
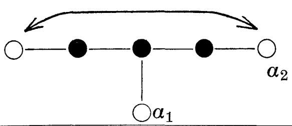
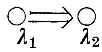
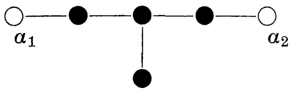
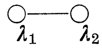
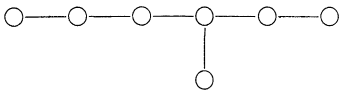
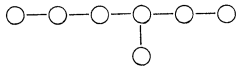
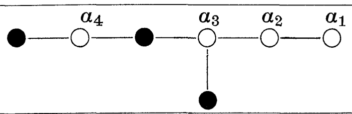
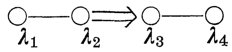
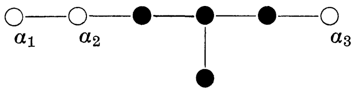
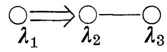
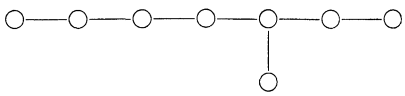
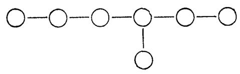
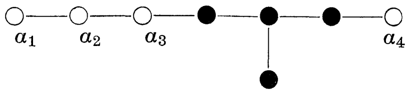
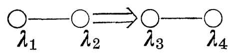
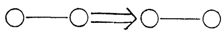
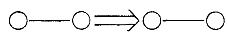
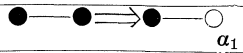
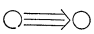
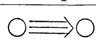
5.10. Type $\mathbf{G}_2$. Schläfli figure of $\varDelta$:  . Since every normal σ-system of roots of restricted rank 1 is simply-laced or doubly-laced of type (2:1), we see immediately that $\varphi_1\notin\varDelta_0$ and $\varphi_2\notin\varDelta_0$; and we obtain only one normally extendable σ-fundamental system with Satake figure.

 .

The corresponding $\mathfrak{g}_{\bar{\sigma}}$ is of type **G** (normal form of G_2).

5.11. Finally we give a table of σ-fundamental systems described by Satake figure, and multiplicities $m(\lambda)$ and $m(2\lambda)$ of simple roots $\lambda\in\varDelta^-$, for all irreducible symmetric spaces such that $\mathfrak{g}_\sigma$ are simple. Some simple roots of $\varDelta-\varDelta_0$ are denoted by letters α_i, and simple root of $\varDelta^-$ which is obtained as a restriction of α_i to $\mathfrak{h}_0^-$ is denoted by λ_i. l denotes the rank of $\varDelta$.

	Δ	Δ^-	$m(\lambda_i)$	$m(2\lambda_i)$
AI			1	0
AII		 $(l=2l'+1,\ l'\geqq 1)$	4	0
AIII		 $(2\leqq p\leqq \frac{l}{2})$	2 (for $i<p$) $2(l-2p+1)$ (for $i=p$)	0 1
		 $(l=2l'+1,\ l'\geqq 1)$	2 (for $i\leqq l'$) 1 $(i=l'+1)$	0 0
AIV			$2(l-1)$	1
BI		 $(l\geqq 2,\ 2\leqq p\leqq l)$	1 $(i<p)$ $2(l-p)+1$ $(i=p)$	0 0
BII			$2l-1$	0
CI			1	0
CII		 $(l\geqq 3,\ 1\leqq p\leqq \frac{l-1}{2})$	4 $(i<p)$ $4(l-2p)$ $(i=p)$	0 3
		 $(l=2l',\ l'\geqq 2)$	4 $(i<l')$ 3 $(i=l')$	0 0
DI		 $(l\geqq 4,\ 2\leqq p\leqq l-2)$	1 $(i<p)$ $2(l-p)$ $(i=p)$	0 0
			1 $(i<l-1)$ 2 $(i=l-1)$	0 0
			1	0

DII			$2(l-1)$	0
DIII		 $(l=2l',\ l'\geqq 2)$	4 $(i<l')$ 1 $(i=l')$	0 0
		 $(l=2l'+1,\ l'\geqq 2)$	4 $(i<l')$ 4 $(i=l')$	0 1
EI			1	0
EII			1 $(i=1, 2)$ 2 $(i=3, 4)$	0 0
EIII			6 $(i=1)$ 8 $(i=2)$	0 1
EIV			8	0
EV			1	0
EVI			1 $(i=1, 2)$ 4 $(i=3, 4)$	0 0
EVII			1 $(i=1)$ 8 $(i=2, 3)$	0 0
EVIII			1	0
EIX			1 $(i=1, 2)$ 8 $(i=3, 4)$	0 0
FI			1	0
FII			8	7
G			1	0

References

[1] R. BOTT and H. SAMELSON, Applications of the Morse theory to symmetric spaces, Amer, J. of Math., **80** (1958), 964-1029.

[2] E. CARTAN, Les groupes réels simples finis et continus, Ann. Éc. Normale Sup., 3-ième série, **31** (1914), 263-355.

[3] ———, Sur une classe remarquable d'espaces de Riemann, Bull. Soc. Math. France, **54** (1926), 214-264; **55** (1927), 114-134.

[4] ———, Sur certaines formes riemanniennes remarquables des géométries à groupe fondamental simple, Ann. Éc. Normale Sup., 3-ième série, **44** (1927), 345-467.

[5] ———, Groupes simples clos et ouverts et géométrie riemannienne, J. Math. pure et appl., **8** (1929), 1-33.

[6] F. GANTMACHER, On the classification of real simple Lie groups, Recueil Math., **5** (1939), 217-250.

[7] I. SATAKE, On representations and compactifications of symmetric Riemannian spaces, Ann. of Math., **71** (1960), 77-110.

[8] Seminaire "Sophus Lie", 1954-55, Théorie des algèbres de Lie, Topologie des groupes de Lie. (Mimeographed note)