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On embedding of level manifolds and sphere bundles

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1. Introduction

Let M be a compact, connected differentiable n-manifold of class C^∞, with $n>1$. Take a non-degenerate function f of class C^∞ on M whose existence is assured by [2] and [6]. In a neighborhood of a critical point P of f, we choose coordinates $x_1, \cdots, x_n$ so that the symmetric matrix

$$\left\| \frac{\partial^2 f}{\partial x_r \partial x_s} \right\|$$

is diagonal. By the index of the critical point P we mean the number of negative entries in the diagonal matrix. Let P^i_k $(k=1, 2, \cdots, n_i)$ be all the critical points of index i of f. Then f is called canonical if the following properties are satisfied:

1) $$f(P^i_1) = f(P^i_2) = \cdots = f(P^i_{n_i}) \quad (i = 0, \cdots, n)$$

2) $$\eta_0 < \eta_1 < \cdots < \eta_n$$

where $\eta_i = f(P^i_1)$.

In the present paper the existence of a canonical function will be established. As an application it is proved that a sphere bundle with fibre S^m and the base space M can be embedded into the$(2n+m+1)$-dimensional euclidean space R^{2n+m+1}.

2. Functions and vector fields

The following theorem is proved in [2] and [6].

THEOREM 2.1. Given any differentiable function $f: M \to R$, and given $\varepsilon>0$, there exists a differentiable function $g: M \to R$ such that

(1) g has at most finite critical points,

(2) at each critical point the determinant of the Jacobian matrix

$$\left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|$$

is not 0,

(3) $$|g(x) - f(x)| < \varepsilon, \quad \left| \frac{\partial g(x)}{\partial x_i} - \frac{\partial f(x)}{\partial x_i} \right| < \varepsilon$$

for all x in M and for all $1 \leqq i \leqq n$, where x_i $(i=1, \cdots, n)$ is a local coordinates system in a neighborhood of the critical point.

Since every differentiable manifold always admits a Riemannian metric we can introduce a metric $ds^2=g_{ij}(x)\,dx^i dx^j$ into M. In this paper a function of class C^∞ on M will be referred to simply as a function on M. Let f be a function on M satisfying the properties (1) and (2) of theorem 2.1. For a regular point P of f we put $f(P)=c$. Then the subvariety V_c defined by the equation $f(x)=c$ is a submanifold of M which is called level manifold. Let T_P^{n-1} be the tangent space of V_c at P and T_P^n the tangent space of M at P. Then we have direct sum

$$T_P^n = T_P^{n-1} \oplus T_P^1$$

where T_P^1 is normal to T_P^{n-1} with respect to the above Riemannian metric. Taking the unit vector in T_P^1 whose direction coincides with that of increasing of f, we have a vector field X on $M-\sum P_\nu$ where P_ν $(\nu=1, 2, \cdots)$ are all critical points of f.

If we take a suitable coordinates, f is represented in a neighborhood of a critical point as

$$f = c-x_1^2- \cdots -x_i^2+x_{i+1}^2+ \cdots +x_n^2. \tag{2,1}$$

Now we consider a vector field X in a neighborhood U of a critical point of f. From (2.1) we easily have

$$X(x) = -\left(\frac{x_1}{r}\frac{\partial}{\partial x_1}+ \cdots +\frac{x_i}{r}\frac{\partial}{\partial x_i}\right)+\left(\frac{x_{i+1}}{r}\frac{\partial}{\partial x_{i+1}}+ \cdots +\frac{x_n}{r}\frac{\partial}{\partial x_n}\right)$$

where $r=(x_1^2+ \cdots +x_n^2)^{1/2}$.

Let n_i be the number of the critical points of index i of f. Then the following theorem is well known (see [3]):

THEOREM 2.1. $\sum_{i=0}^{n}(-1)^i n_i=$the Euler number of M.

3. Level manifolds

LEMMA 3.1. The level manifold V_c defined by $f=c$ is diffeomorphic with the level manifold $V_{c'}$ defined by $f=c'$ if any critical value of f does not exist in the closed interval $[c, c']$.

Proof. Let $x_1, \cdots, x_n$ be local coordinates in a neighborhood W of P on V_c, and let X be the vector field induced from f. Write $X=\sum f^i\ (\partial/\partial x^i)$. Then f^i are differentiable functions since X is differentiable. We consider a system of linear differential equations

$$\frac{d\varphi_i}{dt} = f_i(\varphi_1(t), \cdots, \varphi_n(t)),\ i = 1, \cdots n, \tag{3.1}$$

for the unknown function $\varphi_1(t), \cdots, \varphi_n(t)$ of one variable t. By a fundamental existence theorem these equations have a unique set of solutions $\varphi_1(t;x), \cdots,$

$\varphi_n(t;x)$, defined for $|t|<\varepsilon$ and $|x_i|<\delta$, which satisfy the initial conditions

$$\varphi_i(0, x) = x_i .$$

The set of curves $\varphi(t;x)$ is then defined on $U=\{(x_1, \cdots, x_n)|x_i|<\delta\}$ by $\varphi(t;x)=(\varphi_1(t;x), \cdots, \varphi_n(t;x))$. From the construction, it is obvious that $\varphi(t;x)$ induce X in U. The uniqueness of these curves $\varphi(t;x)$ which induce X is clear from the fact $\varphi(t;x)$ must satisfy (3.1). It is also clear from (3.1) that any one of these curves does not intersect the others.

Put $V_{c,c'}=\{P|c\leqq f(P)\leqq c'\}$. Let $\cup U_j$ be a covering of $V_{c,c}$ and write ε_j for the above ε corresponding to U_j. By the compactness of $V_{c,c'}$ we can take $\varepsilon_0=\min_j \varepsilon_j>0$ as a common ε for all U_j.

Now we put

$$P_k = \varphi(\varepsilon_j ; P_{k-1}) , \quad k = 1, 2, \cdots ,$$

where P_0 is an arbitrary point of V_c. Then it is clear that the curve $\psi(t;p)$ defined by

$$\psi(t;P) = \varphi(t-\varepsilon k;P_k) \quad \text{for} \quad \varepsilon k\leqq t\leqq \varepsilon(k+1)$$

satisfies (3.1), and hence it has the vector field $X|\psi$ which is the restriction of X on the curve ψ. From (3.1) we have

$$\frac{d}{dt}f(\psi(t;P)) = \sum \frac{\partial f}{\partial x_k}\frac{dx_k}{dt} = \sum \frac{\partial f}{\partial x_k}\cdot f_k(\psi(t;p)) = Xf$$

and hence

$$f(P_k)-f(P_0) = \int_0^{k\varepsilon} Xf \geqq k\varepsilon\eta$$

where $\eta=\min_{P\in V_{c,c'}} f(P)$. Since $V_{c\ c'}$ is compact and $f(x)>0$ for an arbitrary point x of $V_{c,c'}$, it follows that $\eta>0$. Taking k so that $k\geqq(c'-c)/\varepsilon\eta$, we have $f(P_k)\geqq c'$. Hence $\psi(t;p)$ can arrive at the point P' of V. By the correspondence $P\to P'$, we have a deffeomorphism of V_c to $V_{c'}$. Thus the lemma is proved.

Now we shall consider the differences of topological structure between V_c and $V_{c'}$ in the case critical points of f exist in $V_c, V_{c'}$.

LEMMA 3.2. If these exists only one critical point of f in $V_{c,c'}$, then the difference between the Euler number of V_c and that of $V_{c'}$ is ± 2 or 0 according as $\dim M$ is odd or even.

Proof. We may suppose without loss of generality that the critical point of f is $x=0$ and that f is written as $-x_1^2-\cdots-x_i^2+x_{i+1}^2+\cdots+x_n^2$ in a neighborhood of $x=0$. Denote by $B(\eta)$ the ball defined by $x_1^2+\cdots+x_n^2\leqq\eta$ and put $S_\eta=\partial B(\eta)$. Then $V_{-\varepsilon}\frown S(\eta)$ is written as

$$\begin{aligned} -x_1^2-\cdots-x_i^2+x_{i+1}^2+\cdots+x_n^2 &= -\varepsilon \\ x_1^2+\cdots+x_i^2+x_{i+1}^2+\cdots+x_n^2 &= \eta , \end{aligned}$$

or

$$x_1^2 + \cdots + x_i^2 = \tfrac{1}{2}(\varepsilon + \eta)$$
$$x_{i+1}^2 + \cdots + x_n^2 = \tfrac{1}{2}(-\varepsilon + \eta) .$$

Hence $V_{-\varepsilon} \cap S(\eta)$ is diffeomorphic to $S^{i-1} \times S^{n-i-l}$:

$$V_{-\varepsilon} \cap S(\eta) \simeq S^{i-1} \times S^{n-i-r} .$$

Since the set of $(x_{i+1}, \cdots, x_n)$ for all η is digeomorphic to $(n-i)$-dimensional ball which we shall denote by B^{n-i}, we have

$$V_{-\varepsilon} \cap B(\eta) \simeq S^{i-1} \times B^{n-i} .$$

Similarly we have

$$V_{\varepsilon} \cap B(\eta) \simeq B^{i} \times S^{n-i-1} .$$

Let g be a function on S^{i-1} which has just one critical point of index 0 and just one caitical point of index $i-1$. Let h be a function in R^{n-i} given by $h = y_1^2 + \cdots + y_{n-i}^2$ where y is the coordinate system in R^{n-i}. Put

$$F = g + h .$$

Then F is a function in $S^{i-1} \times R^{n-1}$ and it is obvious that F has only two critical points of index 0 and index $i-1$. Take a number c so that $c > \max_{P \in S^{i-1}} g(P)$. Then the subset F^c of points $P \in S^{i-1} \times R^{n-i}$ at which $F(P) \leqq c$ is diffeomorphic with $S^{i-1} \times B^{n-i}$. This is shown as follows. For an arbitrary point $P \in S^{i-1}$, $F \leqq c$ implies $y_1^2 + \cdots + y_{n-i}^4 \leqq c - g(P)$. Hence $F^c \simeq S^i \times B^{n-i}$. Let σ be a diffeomorphism of $V_{-\varepsilon} \cap B_\eta$ to F^c. Then we have a function $F \cdot \sigma$ which is defined in $V_{-\varepsilon} - B_\eta$ and constant on $\partial(V_{-\varepsilon} - B_\eta)$. Obviously $F \cdot \sigma$ has two critical points of index 0 and $i-1$. Now we extend to over $V_{-\varepsilon}$ and denote it by F_1.

Similarly let g' be a function on S^{n-i-1} which has only two critical points of index 0 and index $n-i-1$, and let h be a function in R^i such as $h' = y_1^4 + \cdots + y_i^2$. Put

$$F' = g' + h' .$$

Then the subset F'^c of points $P \in R^i \times S^{n-i-1}$ at which $F'(P) \leqq c$ is diffeomorphic with $B^i \times S^{n-i-1}$. Let σ' be a diffeomorphism of $V_\varepsilon \;\, B_\eta$ to F'^c. We have a function $F' \sigma'$ which is defined in $V_\varepsilon \cap B_\eta$ and constant on $\partial(V_\varepsilon \;\, B_\eta)$, and which has just one critical point of index 0 and just one crictical point of index $n-i-1$. Since $V_{-\varepsilon} - (V_{-\varepsilon} - B_\eta) \simeq V_\varepsilon - (V_\varepsilon - B_\eta)$, $F_1 = \text{const}$ on $(V_{-\varepsilon} - B_\eta)$, and $F'\sigma' = \text{const}$ on $\partial(V_\varepsilon - B_\eta)$. Consequently we can extend $F'\sigma'$ to a function F_2 on V_ε so that $n_k = \bar{n}_k$ where n_k or $\bar{n}_k$ is the number of critical points of index k of F_1 in $V_{-\varepsilon} - B_\eta$ or F_2 in $V_\varepsilon - B_\eta$. From theorem 2.1 we have

$$\text{the Euler number of } V_{-\varepsilon} = \sum (-1)^k n_k + 1 + (-1)^{i-1} ,$$
$$\text{the Euler number of } V_{\varepsilon} = \sum (-1)^k \bar{n}_k + 1 + (-1)^{n-i-1} .$$

Thus the difference between these numbers is $(-1)^{i-1}-(-1)^{n-i-1}=(-1)^{i-1}(1-(-1)^n)$, and the lemma is proved.

4. Ortho-f-arcs

Let Ω_0 be the set of critical points of f on M. Then by (3.1) the trajectories orthogonal to the level manifolds of f are well-defined in $M-\Omega_0$. These trajectories is celled ortho-f-arcs on M. From now on we suppose that the direction of trajectories accords with that of increasing of f.

For every critical point P of f we choose coordinates $x_1, \cdots, x_n$ in a neighborhood U_P of P so that

$$f = c_0 - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2 .$$

Furthermore in M we can introduce such a Riemannian metric as $ds^2 = \sum_j dx_j^2$ in U_P. Under the above metric we have

LEMMA 4.1. In the above neighborhood of a critical point P, denote by L the set of all ortho-f-arcs stretched into P and by L' the set of all ortho-f-arcs issueing out from P. Then $L \cup P \simeq B^i$ and $L' \cup P \simeq B^{n-i}$.

Proof. Since

$$f = c_0 - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2 ,$$

the vector field induced from f is written as

$$X = \sum_j \varepsilon_j \frac{x_i}{r} \frac{\partial}{\partial x_j}, \quad r = (x_1^2 + \cdots + x_n^2)^{1/2}$$

where $\varepsilon_j = -1$ for $1 \leqq j \leqq i$ and $\varepsilon_j = 1$ for $i < j \leqq n$. Hence every ortho-f-arc satisfies the differential equations

$$\frac{dx_j(t)}{dt} = c(t)\,\varepsilon_j x_j(t), \quad 1 \leqq j \leqq n ,$$

and the solution of these equations is

$$x_j(t) = c_j \exp \varepsilon_j c(t)$$

where t is an arbitrary common parameter on all ortho-f-arcs. Therefore if we put $c_{i+1} = \cdots = c_n = 0$ and make $c(t) \to \infty$, it follows that $L \cup P = \{x \mid x_{i+1} = \cdots = x_n = 0\}$. Similarly putting $c_1 = \cdots = c_i = 0$ and making $c(t) \to -\infty$, we see that $L' \cup P = \{x \mid x_1 = \cdots = x_i = 0\}$.

LEMMA 4.2. Denote by L_c^i the set of all points which are on the ortho-f-arcs stretched into the critical point P, and which satisfies $f(P) \geqq f(x) > c$, then L_c^i is diffeomophic with i-dimensional ball $B^i \subset M$ if and only if there is no critical point in L_c^i.

Proof. Each ortho-f-arc φ of L_c^i satisfies the system of linear differential equations

$$\frac{d\varphi_i}{dt} = f_j(\varphi_1(t), \cdots, \varphi_n(t)), \quad i = 1, \cdots, n. \tag{4.1}$$

By the fundamental existence theorem the unique set of solutions $\varphi_i(t; x)$ with initial conditions $\varphi_j(0, x) = x$ are differentiable with respect to t and x. For $|x| < \delta$ where $\delta > 0$ is sufficiently small, by using lemma 4.1 we have $L_c^i \quad B_\delta \simeq B^i$. Hence we can uniquely represent $x(|x| \geqq \delta)$ by using t and $x_1, \cdots, x_i$ satisfying $x_1^2 + \cdots + x_n^2 = \delta$. Thus L_c^i is diffeomorphic with a i-dimensional ball.

REMARK. Denote by $L_{c'}^{\prime n-j}$ the set of all points x which lie on the ortho-f-arcs issuing from a critical point Q of index j and satisfy $c' > f(x) \geqq f(Q)$. If L_c^i and $\partial L_{c'}^{n-j}$ are in a general position, we have

$$\dim (L_c^i \cap \partial L_{c'}^{n-j}) = i - j - 1.$$

Hence we may suppose that on L_c^i there is not any critical point of index j, $j \geqq i$.

5. Existence of canonical functions

THEOREM 5.1. There esists a function f with the following properties.

1) For all critical points P_j^i $(j = 1, 2, \cdots n_i)$ of index i and for all i, $0 \leqq i \leqq n$

$$f(P_1^i) = f(P_2^i) = \cdots = f(P_{n_i}^i),$$

2)
$$f(P_1^0) < f(P_1^1) < \cdots < f(P_{n_i}^i).$$

We shall call a function to be canonical if it satisfies 1) and 2) in theorem 5.1.

Proof. We arrange these critical points in a sequence $P_1, P_2, \cdots$ so that the index of $P_\mu \leqq$ the index of $P_{\mu+1}$. Now we shall prove it by the induction for μ. By certain coordinates in a neighborhood of P_0 f is written as

$$f = a + x_1^2 + \cdots + x_n^2.$$

Take a function g which satisfies the following conditions:

1) $g(r) = \alpha$ for $0 \leqq r \leqq \delta$ and $g(r) = 0$ for $r \geqq 2\delta$

2) $0 \leqq g(r) \leqq \alpha$, $g'(r) > 0$ and $\alpha + f(P_0) < \min_{Q \in M} f(Q)$

where $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, and δ is a sufficiently small positive number.

Putting

$$\tilde{f} = f + g$$

we have

$$\tilde{f}(P_0) = f(P_0) + g(P_0) = f(P_0) + \alpha$$

and

$$\frac{\partial \tilde{f}}{\partial r} = 2r + g'(r) > 0 \quad \text{for} \quad 0 < r \leqq 2\delta.$$

Hence $\tilde{f}$ has the same critical points as f and $\tilde{f}(P_0) < \tilde{f}(P_\mu)$, $\mu \geqq 1$.

Now we assume that f satisfies the following conditions.

1) If the index of P_μ = the index of P_ν and $\mu, \nu \leqq k$, then $f(P_\mu) = f(P_\nu)$.

2) If the index of $P_\mu <$ the index of P_ν and $\mu \leqq k$, then $f(P_\mu) < f(P_\nu)$ for all ν.

We shall show that we can modify f so that the critical points are unchangeable and the conditions 1) and 2) are satisfied for P_μ, $\mu \leqq k+1$.

Let L_{k+1} be the set of all ortho-f-arcs stretched into the critical point P_{k+1}. By the remark we may assume that $P_\mu \notin L_{k+1}$, $\mu \geqq k+1$. By lemma 4.2. L_{k+1} is diffeomorphic with a ball $B^{k'}$, k' = the index of P_{k+1}. Since along an arbitrary ortho-f-arc the values of f increase in a monotone, $V_{c_0} \cap L_{k+1}$ is diffeomorphic to $S^{k'}$.

Let Q be an arbitrary point close to L_{k+1} and let Q' be the intersecting point of V_{c_0} and the ortho-f-arc passing through Q. Consider on V_{c_0} the metric induced from M. Then on V_{c_0} we can draw the unique geodesic which passes through Q' and is orthogonal to $V_c \cap L_{k+1}$. Denote by Q'' the intersecting point of this geodesic and $V_{c_0}\ L_{k+1}$ and by $r(Q)$ the geodesic distance on V_{c_0} between Q' and Q''. Since $r(Q) \to 0$ $(Q \to Q_0 \in L_{k+1} \setminus L'_{k+1}$ where L'_{k+1} is the set of all ortho-f-arcs issuing from the point $P_{k+1})$, we may consider that $r(Q_0) = 0$, $Q_0 \in L_{k+1} \setminus L'_{k+1}$. Then we have. $r(Q) = 0$ if and only if $Q \in L_{k+1} \setminus L'_{k+1} \cup P_{k+1}$.

Denote by $L_{k+1}(c)$ the set of all points Q on L_{k+1}, which satisfies $f(P_{k+1}) \geqq f(Q) > c$.

a) In the case the index of P_k = the index of P_{k+1}.

Define a function g such that

$$\begin{aligned} g(t) &= 0 \quad \text{for} \quad t \leqq f(P_k) - 2\varepsilon\,, \\ &= f(P_k) - f(P_{k+1}) \quad \text{for} \quad t \geqq f(P_{k+1}) - \delta\,, \\ g'(t) &> -1 \quad \text{for all} \quad t\,, \end{aligned}$$

where ε is a sufficiently small positive number so that on $L_{k+1}(c)$, $c = f(P_k) - 2\varepsilon$, there is not any critical point except P_{k+1}.

Furthermore define a function $h(t)$ such that

$$\begin{aligned} h(t) &= 1 \quad \text{for} \quad t \leqq \delta \\ &= 0 \quad \text{for} \quad t \geqq 2\delta \end{aligned}$$

where δ is a small positive number.

Put

$$\tilde{f} = f(Q) + g(f(Q))\, h(r(Q))\,.$$

Since $r(P_{k+1}) = 0$, from 1) and 2) we have

$$\tilde{f}(P_{k+1}) = f(P_{k+1}) + f(P_k) - f(P_{k+1}) = f(P_k)\,.$$

Consider a vector X_Q at Q, which is orthogonal to the level manifold $V_{f(Q)}$ defined by $f = f(Q)$. Then we have

$$X_Q \tilde{f} = X_Q f + g'(f)(X_Q f) h$$
$$= X_Q f(1+g'(f)h) .$$

Since $X_Q f \neq 0$ for $Q \in U(L_{k+1}(c))$ where $U(L_{k+1}(c))$ is a sufficiently small neighborhood of $L_{k+1}(c)$, it follows that

$$X_Q \tilde{f} \neq 0 \quad \text{for} \quad Q \neq P .$$

In a neighborhood of P we have

$$\tilde{f}(Q) = f(Q) + \text{const} ,$$

and hence $\tilde{f}$ has the same critical points as f.

b) In the case the index of $P_k <$ the index of P_{k+1}.

Define a function g such that

$$g(t) = 0 \quad \text{for} \quad t \leqq f(P_k)$$
$$= f(P_k) - f(P_{k+1}) + 2\varepsilon \quad \text{for} \quad t \geqq f(P_{k+1}) - \varepsilon$$
$$g'(t) > -1$$

where $f(P_k) + 2\varepsilon = \min_{\mu \geqq k+1} f(P_\mu)$.

Put

$$\hat{f}(Q) = f(Q) + g(f(Q)) h(r(Q)) .$$

Then in the same way as a) we see that $\hat{f}$ satisfies 1) and 2) for $\mu \leqq k+1$.

Thus the theorem is proved.

6. Regular embedding

LEMMA 6.1. Let

$$f_i(x_1, \cdots, x_n) = 0 \quad (i = 1, \cdots, n+1) \tag{6.1}$$

(f are polynomials) be a set of non-homogeneous equations with indeterminate coefficients and let

$$\bar{f}_i(x_1, \cdots, x_n) = 0 \quad (i = 1, \cdots, n+1) \tag{6.2}$$

be the equations obtained from (6.1) by a given specialization of the coefficients in (6.1). Then a necessary condition for the existence of a solution of the equations (6.2) is $T(\bar{a}) = 0$ where T is a certain polynomial in the indeterminate coefficients $(a_1, a_2, \cdots, a_\nu)$ of f_i and $(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_\nu)$ is the given specialization of $(a_1, a_2, \cdots, a_\nu)$.

Proof. Make f_i homogeneous by introducing a new indeterminate x_0 and replacing x_k/x_0 for x_k. Then there is a non-zero resultant form for $n+1$ equations in $n+1$ unknowns such that

$$T(a_1 \cdots a_\nu) x_0^\tau \equiv \sum_1^r A_i(x_0 \cdots x_n) f_i(x_0 \cdots x_n)$$

for a suitable integer τ, the A_j being polynomials in $x_0 \cdots x_n$ with coefficient $K[a_1, \cdots, a_\nu]$. Here let $\bar{x}$ be one of the solutions of (6.2), and we have

$$T(\bar{a}_1 \cdots \bar{a}_\nu) = 0.$$

LEMMA 6.2. Let

$$f_i(x_1 \cdots x_n) = f_i(y_1 \cdots y_n) \quad (i = 1, \cdots, 2n+1) \tag{6.3}$$

be a set of non-homogeneous equations with indeterminate coefficients and let

$$\bar{f}_i(x_i \cdots x_n) = \bar{f}_i(y_1 \cdots y_n) \quad (i = 1, \cdots, 2n+1) \tag{6.4}$$

be the equations obtained from (6.3) by a given specialization of the coefficient in (6.3). Then a necessary condition for the existence of a solution of (6.4) such as $x \neq y$ is $R(\bar{a})=0$ where R is some polynomial in the indeterminate coefficients $(a_1 \cdots a_\mu)$ of f_i and $(\bar{a}_1 \cdots \bar{a}_\mu)$ is the specialization of $(a_1 \cdots a_\mu)$.

Proof. Put

$$y_k = x_k + x'_k \quad (k = 1, \cdots, n)$$

and suppose $x'_1 \neq 0$ then we have

$$(f(x+x')-f(x))/x'_1 = 0 \quad (i = 1, \cdots, 2n+1).$$

Here we can consider that $x_1 \cdots x_n\ 1/x'_1\ x'_2 \cdots x'_n$ are unknowns. Let R be the resultant form for $(f_i(x+x')-f_i(x))/x'_1$ and from lemma 6.1 we have immediately

$$R(\bar{a}) = 0.$$

Let f'_ν, f''_ν $(\nu=1 \cdots m)$ be arbitrary differentiable functions on a compact manifold K and $\{U_r\}$ be a covering of K and $w^r_1 \cdots w^r_\mu$ be local coordinates of U_r. Define $d_K(f', f'')$ as the following:

$$d_K(f', f'') = \max_{P \in K} \left\{ \sum_\nu |f'_\nu(p) - f''_\nu(P) + \sum_{\mu, \nu, r} \left| \frac{\partial f'_\nu}{\partial w^r_\mu}(P) - \frac{\partial f''_\nu}{\partial w^r_\mu}(P) \right| \right\}.$$

LEMMA 6.3. If by $y_\nu = f'_\nu$ a compact manifold K is regularly embedded into R^m then there exists a positive number such that K is always regularly embedded into R^m by $y_\nu = f''_\nu$ only if $d(f', f'') < \eta$.

Proof. By the hypothesis there exists $\delta > 0$ as follows.

1) $$\min_{P \in K,\, Q \in K - B\delta(P)} |f'(P) - f'(Q)| > \eta' > 0$$

where $B_\delta(P)$ is a geodesic ball with radius δ having P as its center.

2) For some $s(0 \leqq s \leqq m-\mu)$

$$\left| \frac{\partial(f'_1 \cdots f'_{\mu-1}\ f'_{\mu+s}}{\partial(w^{(o)}_1 \cdots w^{(r)}_\mu)} \right| > \eta'' > 0$$

where $w^{(r)}_1 \cdots w^{(r)}_\mu$ are the coordinates of U_r.

From

$$\min_{P\in K,\, Q\in K-B\delta(P)} |f''(P)-f''(Q)| > \min_{P\in K,\, Q\in K-B\delta(P)} |f'(P)-f'(Q)|$$
$$- \max_{P\in K} |f'(P)-f''(P)| - \max_{Q\in K} |f'(Q)-f''(Q)|$$
$$(\max |f'(P)-f''(P)| + \max|f'(Q)-f''(Q)| \to 0 \quad (\eta \to 0)),$$

and

$$\left|\frac{\partial(f_1''\cdots f_{\mu-1}'' f_{\mu+s}'')}{\partial(w_1\cdots w_\mu)}\right| \geqq \left|\frac{\partial(f_1' f_{\mu-1}' f_{\mu+s}')}{\partial(w\cdots w_\mu)}\right|$$
$$-\left|\frac{\partial(f_1''\cdots f_{\mu-1}'' f_{\mu+s}'')}{\partial(w_1\cdots w_\mu)} - \frac{\partial(f_1'\cdots f_{\mu-1}' f_{\mu+s}')}{\partial(w_1\cdots w_\mu)}\right|$$
$$\left(\left|\frac{\partial(f_1''\cdots f_{\mu-1}'' f_{\mu+s}'')}{\partial(w_1\cdots w_\mu)} - \frac{\partial(f\cdots f_{\mu+1}' f_{\mu+s}'')}{\partial(w_1\cdots w_\mu)}\right| \to 0 \quad (\eta\to 0)\right),$$

If η is sufficiently small, it follows immediately that

$$\min_{P\in K,\, Q\in K-B\delta(P)} |f'(P)-f''(Q)| > \frac{\eta'}{2},$$
$$\left|\frac{\partial(f_1''\cdots f_{\mu-1}'' f_{\mu+s}'')}{\partial(w_1\cdots w_\mu)}\right| > \frac{\eta''}{2}.$$

Hence the lemma is proved.

7. Regular embedding of level manifold

From now on for the sake of simplicity we write

$$(x_1^k, \cdots, x_i^k) = y^k, \quad (x_{i+1}^k, \cdots, x_n^k) = z^k,$$
$$(u_1^k, \cdots, u_i^k) = u^k, \quad (v_1^k, \cdots, v_{n-i}^k) = v^k,$$
$$\sum_{j=1}^{h} (x_j^k)^2 = (x^k)^2 = (y^k)^2 + (z^k)^2,$$
$$\sum_{p=1}^{i} (u_p^k)^2 = (u^k)^2, \quad \sum_{b=1}^{n-i} (v_q^k)^1 = (v^k)^2$$

Choose coordinates x^k in o neighborhood of a critical point P_k of index i so that f is represented as

$$f = -c^k - (y^k)^2 + (z^k)^2.$$

Here we assume $c^k=0$ and $V_{-\varepsilon}$ is written as

$$V_{-\varepsilon} | -(y^k)^2 + (z^k)^2) = -\varepsilon.$$

Put

$$G_k(\varepsilon) = \{(y^k, z^k) | (y^k)^2 = \delta, \quad z^k = 0\},$$
$$H_k(\delta) = \{(u^k, v^k) | (u^k)^2 < 1, \quad (v^k)^2 = \delta^2\},$$

and identify (u^k, v^k) and (y^k, z^k) by

(7.1) $$y^k = u^k(\delta^2 + \varepsilon/(u^k)^2)^{1/2}$$
$$z^k = |u^k| v^k \quad (|u^k|^2 = (u^k)^2).$$

Then from $(V_{-\varepsilon}-\sum_k G_k(\varepsilon)) \smile \sum_k H_k(\delta)$ we have a manifold denoted as

$$V_{-\varepsilon,\delta} = (V_{-\varepsilon}-\sum_k G_k) \underset{\sigma_\varepsilon}{\cup} \sum H_k(\delta)$$

where σ_ε represents the identifying (7.1).

LEMMA 7.1. $(V_{-\varepsilon}-\sum_k G_k(\varepsilon)) \underset{\sigma_\varepsilon}{\cup} \sum H_k)\delta) \simeq (V_{-\varepsilon'}-\sum G_k(\varepsilon')) \underset{\sigma_{\varepsilon'}}{\cup} \sum H_k(\delta)$.

Proof. It is sufficient for the purpose to prove the lemma when $|\varepsilon-\varepsilon'|$ is sufficiently small. If we correspond $(u^k, v^k) \in V_{-\varepsilon,\delta}$ to $(u^k, v^k) \in V_{-\varepsilon',\delta}$, from (7.1) it induces the correspondence between $(V_{-\varepsilon}-\sum G_k(\varepsilon))$ $B_k(2\delta^2+\varepsilon)^{1/2}$ and $(V_{-\varepsilon'}-\sum G_k(\varepsilon'))$ $B_k(2\delta^2+\varepsilon')^{1/2}$ such as

$$\begin{aligned} &x'^k = x^k + h^k(\varepsilon, \varepsilon', x^k) \quad (x \in \sigma_\varepsilon H_k(\delta),\ x' \in \sigma_{\varepsilon'} H_k(\delta)), \\ &h^k,\ \partial h^k_j/\partial x^k_i \to 0 \quad (|\varepsilon-\varepsilon'| \to 0) \end{aligned} \tag{7.2}$$

where $h^k=(h^k_1, \cdots, h^k_n)$. On the other hand at every point $x \in V_{-\varepsilon}-\sum \bar{B}_k(\delta^2/2+\varepsilon)^{1/2}$ $(G_k(\varepsilon) \subset B_k(\delta^2/2+\varepsilon)^{1/2})$ we draw the geodesic orthogonal to $V_{-\varepsilon}$. Let x' be the intersecting point of the geodesic and $V_{-\varepsilon'}$. Then the correspondence $x \to x'$ is written in $B_k(2\delta^2+\varepsilon')^{1/2}$ as

$$\begin{aligned} &x'^k = x^k + g^k(\varepsilon, \varepsilon', x^k), \\ &g^k,\ \partial g^h_j/\partial x^k_i \to 0 \quad (|\varepsilon-\varepsilon'| \to 0), \end{aligned} \tag{7.3}$$

where $g_k = (g^k_1, \cdots, g^k_n)$.

Define a function $\varphi(x^k)$ such as

$$\varphi(x^k) = \begin{cases} 0 & (x^k)^2 \geqq 2\delta^2+\varepsilon, \\ 1 & (x^k)^2 \leqq \delta^2/2+\varepsilon. \end{cases}$$

Let $r(a, b)$ be the geodesic distance (with respect to the metric on $V_{-\varepsilon'}$ induced from M) on $V_{-\varepsilon'}$ between a and b $(a, b \in V_{-\varepsilon'})$. Moreover let x''^k be the point on the geodesic passing through x^k+g^k and x^k+h^k $(x^k+g^k,\ x^k+h^k \in V_{-\varepsilon'})$, which satisfies

$$\frac{d(x^k+g^k,\ x''^k)}{d(x^k+g^k,\ x^k+h^k)} = \varphi(x^k)$$

where x''^k is between x^k+g^k and x^k+h^k on the geodesic. Then we have immediately

$$x''^k_j = x^k_j + g^k_j + \psi^k_j(x^k)(k^k_j - g^k_j) \tag{7.4}$$

where $\psi^k_j=0$ for $(x^k)^2 \geqq 2\delta^2+\varepsilon$ and $\psi^k_j=1$ $(x^k)^2 \leqq \delta^2/2+\varepsilon$. Now we define the correspondence between $V_{\varepsilon,\delta}$ and $V_{-\varepsilon',\delta}$ as follows.

$$\begin{gathered} V_{-\varepsilon,\delta} \ni (u^k, v^k) \to (u^k, v^k) \in V_{-\varepsilon',\delta} \\ V_{-\varepsilon} \frown (B_k(2\delta^2+\varepsilon)^{1/2} - \bar{B}_k(\delta^2/2+\varepsilon)^{1/2}) \ni x^k \\ \to x''^k \in V_{-\varepsilon'}. \end{gathered}$$

From (7.2), (7.3) and (7.4) this correspondence is $1-1$ when $|\varepsilon'-\varepsilon|$ is sufficiently small and it induces $V_{-\varepsilon,\delta}\simeq V_{-\varepsilon',\delta}$.

LEMMA. 7.2. $$V_\varepsilon \simeq V_{-\varepsilon,\delta}.$$

Proof. From lemma 7.1 it is sufficient to prove $V_{\varepsilon',\delta^2}\simeq V_{-\varepsilon',\delta}$ for small $\varepsilon'>0$. An arbitrary point $x^k\in V_{\varepsilon'\delta^2}\cap\bar{B}_k(2\delta^2+\varepsilon'\delta^2)^{1/2}$ is written as

$$\begin{aligned} y^k &= u^k\delta \quad ((u^k)^2\leqq 1)\,, \\ z^k &= v^k((u^k)^2+\varepsilon')^{1/2} \quad ((v^k)^2=\delta^2)\,. \end{aligned} \tag{7.5}$$

Hence from (7.1) the correspondence $V_{\varepsilon'\delta^2}\ni x=(y(u,v),\ z(u,v))\to x'=(u,v)\in V_{-\varepsilon',\delta}$ is written in $B_k(2\delta^2+\delta^2\varepsilon')^{1/2}-\bar{B}_k(\delta^2/2+\varepsilon'\delta^2)^{1/2}$ as

$$\begin{aligned} x'_k &= x^k+h^k(\varepsilon',x_k)\,, \\ h,\ \partial h_j^k/\partial x_i^k &\to 0 \quad (\varepsilon'\to 0)\,. \end{aligned}$$

Define a function $\varphi(x^k)$ such as

$$\varphi(x^k)=\begin{cases} 0 & (x^k)^2\geqq\delta(1+\varepsilon')^{1/2} \\ 1 & (x^k)^2\leqq\delta(\tfrac{1}{2}+\varepsilon')^{1/2}\,, \end{cases}$$

and by the same way as that in the proof of lemma 7.1 we easily $V_{\varepsilon'\delta^2}\simeq V_{-\varepsilon',\delta}$.

THEOREM 7.1. If a level manifold V_c in M^n is regularly embedded into R^{n+i-1} and in $V_{c,c'}$ there exists no other critical point than index i, then $V_{c'}$ can be regularly embedded into R^{n+i}.

Proof. By the way similar to the proof of the theorem (5.1) we may suppose that for all critical points P_k $(k=1,2,\cdots)$ $f(P_k)=0$. Hence we have $V_\varepsilon\simeq V_c$ and $V_{-\varepsilon}\simeq V_{c'}$. Put

$$\begin{aligned} y_k &= \alpha^k((\beta^k)^2+\varepsilon)^{1/2} \quad (\alpha^k)^2=1 \\ z^k &= \beta^k \qquad\qquad (\beta^k)^2<1 \end{aligned} \tag{7.6}$$

where $\alpha^k=(\alpha_1^k\cdots\alpha_i^k)$ and $\beta^k=(\beta_1^k\cdots\beta_{n-i}^k)$.

Then (7.6) induces $-y^2+z^2=-\varepsilon$.

Let F be a regular embedding map of $V_{-\varepsilon}$ into R^{n+i-1}. Then for $B_k((2+\varepsilon)^{1/2})$ $F(V_{-\varepsilon})$ is represented by (α^k,β^k) as

$$X^k=F^k(\alpha^k,\beta^k)\,,\quad F^k=(F_1^k,\cdots,F_{n+i-1}^k)\,,$$

where $X^k=(X_1^k\cdots X_{n+i-1}^k)$ is the coordinates of R^{n+i-1}. Define maps $f^k(u,v)$ from $0<|u^k|<1$, $|v^k|\leqq 1$ into R^{n+i} as the following:

$$f^k(u^k,v^k)=(F^k(u^k/|u^k|,\ |u^k|v^k),\ \exp(1/((u^k)^2-1)) \quad |u^k|<1.$$

To simplify the notations for a while we abbreviate index k. From

$$f(u,v)=f(u',v') \quad (|u|,\ |u'|<1)$$

we have

$$u/|u| = u'/|u'|, \quad |u|v = |u'|v', \quad |u| = |u'|$$

and hence we have

$$u = u', \quad v = v'.$$

Put

$$\alpha_1 = \frac{u_1}{|u|}, \cdots, \alpha_i = \frac{u_i}{|u|}, \quad \beta_1 = v_1, \cdots, \beta_{n-i} = v_{n-i}.$$

If $u_t \neq 0$ $(1 \leqq t \leqq i)$ we can use $|u|, \alpha_1, \cdots, \hat{\alpha}_t, \cdots, \alpha_i, \beta_1, \cdots, \beta_{n-i}$ as local coordinates in $\{(u, v) | u^2 < 1, v^2 \leqq 1\}$. Then we see

$$\frac{\partial(f_1 \cdots f_{n-1}\ f_{n+s-1}\ f_{n+i})}{\partial(\alpha_1 \cdots \hat{\alpha}_t \cdots \alpha_i\ \beta_1 \cdots \beta_{n-i} | u|)} = -\frac{2|u|}{(u^2-1)^2} e^{\frac{1}{u^2-1}} \frac{\partial(F_1 \cdots F_{n-1}\ F_{n+s-1})}{\partial(\alpha_1 \cdots \hat{\alpha}_t \cdots \alpha_i\ \beta_1 \cdots \beta_{n-i})}.$$

Hence the Jacobian of the right hand is not zero for some s. Thus by f the set $K = \{u, v) | \frac{1}{3} \leqq |u| < 1, |v| \leqq 1\}$ is regularly embedded into R^{n+i}, and hence for K there exists η in lemma 6.3.

Let $h_\nu(u, v)$ be polynomials and put

$$\bar{f}(u, v) = f(u, v) + \varphi(u)(h(u, v) - f(u, v))$$
$$(h = (h_1 \cdots h_{n+i}))$$

where

$$\varphi(u) = 1 \quad |u| \leqq \frac{1}{3}$$
$$= 0 \quad |u| \geqq \frac{2}{3}.$$

Now consider the equations

$$(7.9) \qquad \left.\frac{\partial(h_1 \cdots h_{n-1}\ h_{n+s-1})}{\partial(u_1 \cdots u_i\ v_1 \cdots v_{n-i})}\right]_{v=0} = 0 \quad (s = 1, \cdots, i+1)$$

with unknowns $u_1, \cdots, u_i$. By lemma 1 there exist polynomials h_ν such that $d_K(\bar{f}, f) < \eta$ and (7.9) has no solution. Hence for a given $u|u| \leqq \frac{1}{3})$ there exist $s = s(u)$ and $\delta(u)$

$$\frac{\partial(h_1 \cdots h_{n-1}\ h_{n+s-1})}{\partial(u_1 \cdots u_i\ v_1 \cdots v_{n-i})} \neq 0 \quad \text{for} \quad |v| \leqq \delta(u).$$

Put $\delta = \min\limits_{|u| \leqq \frac{1}{3}} \delta(u)$ and we easily see $\delta > 0$. Furthermore from lemma (6.2) we can take h_ν so that the equations $h_\nu(u, 0) = h_\nu(u', 0)$ $(\nu = 1, \cdots, n+i;\ i < n)$ have no solution and $\min\limits_{u, |u-u'| > \delta} |h(u, 0) - h(u', 0)| > 0$. Hence if v, v' is sufficiently small we also have $\min\limits_{u, |u-u'| > \delta} |h(u, v) - h(u', v')| > 0$. Hence by h $\{(u, v) | |u| \leqq \frac{1}{3}, |v| = \delta\}$ is regularly embedded where δ is sufficiently small.

Now we shall show that $\{(u, v) | |u| < 1, |v| = \delta\}$ is regularly embedded by $\bar{f}(u, v)$ into R^{n+i}. It is clear for $|u| \leqq \frac{1}{3}$ or $\frac{2}{3} \leqq |u| < 1$ from $\bar{f} = h$ or $\bar{f} = f$. It is clear for $\frac{1}{3} \leqq |u| \leqq \frac{2}{3}$ by lemma 6.3 since $d_{K'}(f, \bar{f}) \leqq \eta$ where $K' = \{(u, v) | \frac{1}{3} \leqq u \leqq \frac{2}{3}, |v| = \delta\} \subset K$. Hence it is proved.

Now consider $V_{-\varepsilon, \delta}$ and from (7.1) and (7.6) we have

$$u^k(x^k) = |z^k| y^k / |\delta y^k| \, , \quad v^k(x^k) = \delta z^k / |z^k|$$

and

$$\alpha^k(x^k) = y^k / |y^k| \, , \quad \beta^k(x^k) = z^k$$

which induce

$$\alpha^k(x^k) = u^k(x^k)/|u^k(x^k)| \, , \quad \beta^k(x^k) = |u^k(x^k)| v(x^k) \, . \tag{7.10}$$

Define a map of $V_{-\varepsilon, \delta}$ into R^{n+i} as follows:

$$V_{-\varepsilon,\delta} - \sum_k \{u^k, v^k) \ \ |u^k| < 1 \, , \ |v^k| = \delta\} \ni x^k \to (F(x^k), \, 0)$$

$$V_{-\varepsilon,\delta} \cap \sum_k \left\{ \{u^k, v^k) \ \ \frac{2}{3} < |u^k| < 1 \, , \ |v^k| = \delta \right\} \ni x^k$$

$$\to (F(x^k) \, , \ \exp 1/|u^k(x^k)|^2 - 1)$$

$$\sum \{(u^k, \, v^k) | \ \ |u^k| < 1, \ |v^k| = \delta\} \ni (u^k, \, v^k) \to \bar{f}(u^k, \, v^k) \, .$$

Since for $\frac{2}{3} < |u^k| < 1$ from (7.7), (7.8) and (7.10) it follows that

$$\begin{aligned} \bar{f}(u^k(x^k), \, v^k(x^k)) &= f(u^k(x^k) \, , \, v^k(x^k) \\ &= (F(u^k(x^k)/|u^k(x^k)|, \ |u^k(x)| v^k(x)) \, , \ \exp 1/(|u^k(x^k)|^2 - 1)) \\ &= (F(\alpha^k(x^k) \, , \, \beta^k(x^k)) \, , \ \exp 1/(|u^k(x^k)|^2 - 1)) \\ &= (F(x^k) \, , \ \exp 1/(|u^k(x)|^2 - 1)) \end{aligned}$$

the above definition is well defined.

It has already been proved that the above map embeds $V_{-\varepsilon\ \delta} - \sum_k \{(u^k, \, v^k) | \frac{2}{3} < |u^k| < 1, \ |v^k| = \delta\}$ and every $\{(u^k, v^k) | \, |u^k| < 1, \ |v^k| = \delta\}$ into R^{n+i}. It is necessary to show that the image of $\{(u^k, v^k) | \, |u^k| < \frac{2}{3}, \ |v^k| = \delta\}$ and the image of $\{(u^l, v^l) | \, |u^l| < \frac{2}{3}, \ |v^l| = \delta\}$ hove no intersection if $k \neq l$.

From (7.8) we have

$$|\bar{f}(u^k, v^k) - \bar{f}^l(u^l, v^l)| \geqq |f^k(u^k, v^k) - f^l(u^l, v^l)| - \eta^k - \eta^l$$

$$\eta^k = |f^k(u^k, v^k) - h^k(u^k, v^k)| \quad \text{and}$$

$$\eta^l = |f^l(u^l, v^l) - h^l(u^l, v^l)| \, .$$

Since there exists $r > 0$ such as

$$|f^k(u^k, v^k) - f^l(u^l, v^l)| > r \quad \text{for all } k, l \quad (k \neq l)$$

if we take h^k, h^l so that $|f^k - h^k| < \frac{r}{4}$ and $|f^l - h^l| < \frac{r}{4}$ we have

$$|\bar{f}^k(u^k, v^k) - \bar{f}^l(u^l, v^l)| > \frac{r}{2} \, .$$

Furthermore it is clear that $(V_{-\varepsilon\ \delta} - \sum \{(u^k, v^k) | \frac{2}{3} < |u^k| < 1, \ |v^k| = \delta\}) \cap \{(u^l, v^l) | \frac{2}{3} < |u^l| < 1, \ |v^l| = \delta\} = \phi$. Hence the theorem is proved.

Let f be a cononical function as in theorem 5.1. If $\eta_0 < c < \varepsilon_1$ it is obvious that $V_c | f=c$ is diffeomorphic with spheres and V_c is regularly embedded into R^n. Hence by using theorem 7.1 and the induction we have immediately

COROLLARY. Let f be a canonical function as in theorem 5.1. Thcn $V_c | f=c$ $(\eta_i < c < \eta_{i+1})$ is regularly embedded into R^{n+i}.

8. Embedding of sphere bundles

Let ζ be a sphere bundle consisting of $[E, M, \pi]$, where π is a map from E onto M^n and whose fibre and group are S^m and O^m where O^m is the m-dimensional orthogonal group. Consider $(m+1)$-plane $\bar{E}_p$ such as $\bar{E}_p \supset \pi^{-1}(P) \cap P$. Let $\bar{\pi}$ be the map $\bar{E}_p \to P$. Then we have $(m+1)$-plane bundle $\zeta = [\bar{E}, M, \bar{\pi}]$ associated with ζ and we can consider that $\bar{E} \supset M \cup E$. Introduce a Riemannian metric into ζ and denote by $\Upsilon(Q, \pi(Q))$ the geodesic distance on $\bar{E}_{\pi(Q)}$ between Q and $\pi(Q)$.

Let f be a function of M which satisfies 1) and 2) in theorem 5.1. Put

$$\bar{f}(Q) = f(\pi(Q)) + r^2(Q, \pi(Q)) .$$

Then f has the same critical points as f. Denote all the critical points of index i by P^i_k $(k=1, 2, \cdots)$. Then we can choose coordinates (x^k, y^k) in a neighborhood of P^i_k so that

$$\bar{f} = a^k - (x^k_1)^2 - \cdots - (x^k_i)^2 + (x^k_{i+1})^2 + \cdots + (x^k_n)^2 \\ + (y^k_1) + \cdots + (y^k_{m+1})^2 .$$

Hence $\bar{f}$ has the index i at P^i_k. Since $\bar{f}(P^i_k) = f(P^i_k)$, $\bar{f}$ satisfies 1) and 2) in theorem 5.1. Putting

$$c > \max_{P \in M} \bar{f}(P)$$

we have $c > \eta_n$. Since the maximum index of the critical points of $\bar{f}$ in $\bar{E}^{n+m+1}$ in n, by using corollary of theorem 7.1 we see that $T_c | \bar{f}=c$ is regularly embedded into R^{2n+m+1}.

For an arbitrary point $Q \in E$ we have $Q \in \bar{E}_{\pi(Q)}$. On $\bar{E}_{\pi(Q)}$ we consider ortho-$f | \bar{E}_{\pi(Q)}$-arc κ_Q passing through Q and $\pi(Q)$ where $f | \bar{E}_{\pi(Q)}$ is the restriction of f on $\bar{E}_{\pi(Q)}$. Since on κ_Q there exists a unique point Q' such as

$$r^2(Q', \pi(Q)) = c - f(\pi(Q)) > 0 ,$$

by $Q \to Q'$ we get the 1–1 correspondence between E and V_c, which induce $E \simeq V_c$. Hence we have

THEOREM 8.1. A sphere bundle with fibre S^m, group O^m and base space M^n can be regularly embedded in R^{2n+m+1}.

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