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Semi cubical theory on homotopy classification

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If a topological space X has vanishing homotopy groups π_i for $i \neq n$ and $\pi_n = \Pi$, it is a well known result of Eilenberg and MacLane [4] that there exists a one to one correspondence between semi simplicial maps of semi simplicial complex K

$$K \longrightarrow K(\Pi, n)$$

and cocycles of $Z^n(K;\ \Pi)$, where $K(\Pi, n)$ is regarded as a minimal complex of $S(X)$.

Generally, let a topological space X have vanishing homotopy groups π_i for $i \neq (n, q, q', \cdots, q^{(m)})$ $(1 < n < q < q' \cdots < q^{(m)})$ and $\pi_n = \Pi$, $\pi_q = G$, $\pi_{q'} = G'$, $\cdots$, $\pi_{q^{(m)}} = G^{(m)}$. According to the Mathematical Reviews 18 (1957), it is reported that M. M. Postnikov classified semi simplicial maps

$$K \longrightarrow \mathfrak{P}(K(\Pi, n), k, N, k', N', \cdots, k^{(m)}, N^{(m)})$$

by defining sequences $(x_n, x_q, x'_q, \cdots, x_{q^{(m)}})$ each of which consists of one cocycle x_n and cochains $x_q, x'_q, \cdots, x_{q^{(m)}}$, where $\mathfrak{P}$ is regarded as the minimal complex of $S(X)$.

In this paper, we shall construct a complex $\mathfrak{P}$ as a semi cubical complex and discuss the chain homotopies between these maps. Our main purpose is to construct these homotopies explicitly and as an application we shall explain some classification theorems in the cases where the Eilenberg-MacLane invariant and Postnikov invariants associated to $\mathfrak{P}$ are additive in a sense [4].

1. Semi cubical complex

Let X be a topological space, consider the usual cubical singular complex $Q(X)$ associated with X, in which the r-cube is a function $\sigma(t_1, \ldots, t_r) \in X$ defined for $0 \leq t_i \leq 1$ and continuous in the topology of the cartesian product of the variables. If $r = 0$, then σ is interpreted as a single point of X. The front and aft faces $F_i^0\sigma$ and $F_i^1\sigma (i = 1, \ldots, r)$ of σ are defined as $(r-1)$-cubes

$$F_i^0\sigma(t_1, \ldots, t_{r-1}) = \sigma(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{r-1}),$$

$$F_i^1\sigma(t_1, \ldots, t_{r-1}) = \sigma(t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_{r-1}).$$

The i^{th} degenerate cube $D_i\sigma$ of σ is defined as $(r+1)$-cube

$$D_i\sigma(t_1, \ldots, t_{r+1}) = \sigma(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{r+1}).$$

Observation of the formal properties of F_i^0, F_i^1 and D_i presented above leads to the following [3]

DEFINITION. A *semi cubical complex* (S. Q. complex) $K=\bigcup_r K_r$ is a sequence of disjoint sets K_r, $r=0, 1, 2, \ldots$ (r is called as the dimension of the element of K_r) together with mappings

$$F_i^0=F_{i,r}^0: K_r \to K_{r-1} \qquad i=1, \ldots, r \qquad r>0,$$
$$F_i^1=F_{i,r}^1: K_r \to K_{r-1} \qquad i=1, \ldots, r \qquad r>0,$$
$$D_i=D_{i,r}: K_r \to K_{r+1} \qquad i=1, \ldots, r+1 \quad r\geq 0,$$

subject to the following identities

$$F_i^0F_j^0=F_{j-1}^0F_i^0, \quad F_i^1F_j^1=F_{j-1}^1F_i^1 \quad i<j,$$
$$F_i^0F_j^1=F_{j-1}^1F_i^0, \quad F_i^1F_j^0=F_{j-1}^0F_i^1 \quad i<j,$$
$$(1.\ 1) \qquad D_iD_j=D_{j+1}D_i \qquad i\leq j,$$
$$F_i^0D_j=D_{j-1}F_i^0, \quad F_i^1D_j=D_{j-1}F_i^1 \quad i<j,$$
$$F_j^0D_j=F_j^1D_j=I \qquad (I=\text{identity}),$$
$$F_i^0D_j=D_jF_{i-1}^0, \quad F_i^1D_j=D_jF_{i-1}^1 \quad i>j.$$

The integral chain complex $C(K)$ of S. Q. complex K is defined by letting $C_r(K)$ be the free abelian group generated by K_r and setting, for $\sigma \in K_r$, $r>0$,

$$(1.\ 2) \qquad \partial\sigma=\textstyle\sum_{i=1}^r(-1)^i(F_i^0\sigma-F_i^1\sigma)$$

and $\partial\sigma=0$ if $r=0$.

It follows from (1. 1) that $\partial\partial=0$. The chain and cochain complexes with any coefficients are defined accordingly, and give rise to the homology and cohomology groups of K.

If K and L are $S.\,Q.$ complexes, an *S. Q. map* $T: K \longrightarrow L$ is a sequence of mappings

$$T_r: K_r \longrightarrow L_r$$

such that $F_i^0T_r=T_{r-1}F_i^0$, $F_i^1T_r=T_{r-1}F_i^1$ and $D_iT_r=T_{r+1}D_i$ for any i. This map T is then also a chain transformation of $C(K)$ into $C(L)$.

If K and L are $S.\,Q.$ complexes, then the *cartesian product* $K\times L$ is the $S.\,Q.$ complex defined by

$$(K\times L)_r=K_r\times L_r$$

$$F_i^0(a\times b)=F_i^0a\times F_i^0b, \qquad F_i^1(a\times b)=F_i^1a\times F_i^1b,$$
$$D_i(a\times b)=D_ia\times D_ib, \qquad \text{for } a \in K_r,\ b \in L_r.$$

And, the *tensor product* $C(K)\otimes C(L)$ is the chain complex with $(K\otimes L)_r$

$=\sum_{p+q=r}K_p\otimes L_q$ and with boundary operator defined by

(1. 3) $$\partial(a_p\otimes b_q)=\partial a_p\otimes b_q+(-1)^p a_p\otimes\partial b_q \quad \text{for } a_p\in K_p,\ b_q\in L_q.$$

2. The complexes $K(\Pi, n)$, $F(\Pi, n)$.

For each integer $r\geq 0$ let $I^r=I\times\cdots\times I$ be the r-fold cartesian product of the unit interval I. We shall introduce the special mappings

$$\varepsilon_i^0: I^{r-1}\to I^r,\quad \varepsilon_i^1: I^{r-1}\to I^r,\quad \eta_i: I^{r+1}\to I^r$$

defined as

$$\varepsilon_i^0(t_1, \dots, t_{r-1})=(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{r-1}) \quad 1\leq i\leq r,$$
$$\varepsilon_i^1(t_1, \dots, t_{r-1})=(t_1, \dots, t_{i-1}, 1, t_i, \dots, t_{r-1}) \quad 1\leq i\leq r,$$
$$\eta_i(t_1, \dots, t_{r+1})=(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{r+1}) \quad 1\leq i\leq r+1.$$

The composites of pairs of these mappings satisfy the identities

(2. 1)
$$\begin{array}{lll}
\varepsilon_j^0\varepsilon_i^0=\varepsilon_i^0\varepsilon_{j-1}^0, & \varepsilon_j^1\varepsilon_i^1=\varepsilon_i^1\varepsilon_{j-1}^1 & i<j,\\
\varepsilon_j^1\varepsilon_i^0=\varepsilon_i^0\varepsilon_{j-1}^1, & \varepsilon_j^0\varepsilon_i^1=\varepsilon_i^1\varepsilon_{j-1}^0 & i<j,\\
\eta_j\eta_i=\eta_i\eta_{j+1}, & & i\leq j,\\
\eta_j\varepsilon_i^0=\varepsilon_i^0\eta_{j-1}, & \eta_j\varepsilon_i^1=\varepsilon_i^1\eta_{j-1} & i<j,\\
\eta_j\varepsilon_j^0=\eta_j\varepsilon_j^1=\iota & & (\iota=\text{identity}),\\
\eta_j\varepsilon_i^0=\varepsilon_{i-1}^0\eta_j, & \eta_j\varepsilon_i^1=\varepsilon_{i-1}^1\eta_j & i>j.
\end{array}$$

In the following we shall denote ε_i as the mapping ε_i^0 or as the mapping ε_i^1 if no confusion would occur.

For each integer $r\geq 0$ the *standard complex* $Q(I^r)$ is the $S.$ $Q.$ complex defined as follows. An n-cube of $Q(I^r)$ is a map $\alpha: I^n\to I^r$ which is a composite of mappings presented above, and can be written uniquely as a form

$$\alpha=\varepsilon_{i_1}\dots\varepsilon_{i_s}\eta_{j_1}\dots\eta_{j_t}$$

$r\geq i_1>\dots>i_s\geq 1$, $1\leq j_1<\dots<j_t\leq n$, $r-s+t=n$, [4]. In the following we shall denote α, β, ... as such a map. For each map β: $I^p\to I^n$ the β-face of α is the p-cube of $Q(I^r)$ defined as the composite map $\alpha\beta$. Especially we define

$$F_i^1(\alpha)=\alpha\varepsilon_i^1,\quad F_i^0(\alpha)=\alpha\varepsilon_i^0,\quad D_i(\alpha)=\alpha\eta_i.$$

Denote by $C^n(I^r;\Pi)$ the (multiplicative) group of n-cochains σ on $Q(I^r)$ with coefficients in the (multiplicative) abelian group Π. We assume that these cochains are normalized in the sense that they vanish on all degenerate cubes of $Q(I^r)$. (The map α is called as a degenerate cube if $t>0$ in above formula.) We could equivalently regard $C^n(I^r;\Pi)$ as the group of n-cochains σ, coefficients in Π, for the complex $Q_N(I^r)$ in which all the n-cubes are $\alpha=\varepsilon_{i_1}\dots\varepsilon_{i_s}$.

Any map $\beta: I^p \to I^r$ induces an $S. Q.$ map $\beta_\#: Q(I^p) \to Q(I^r)$ as $\beta_\#(\alpha) = \beta\alpha$ for $\alpha \in Q(I^r)$ and hence a homomorphism

$$\beta^*: C^n(I^r; \Pi) \to C^n(I^p; \Pi).$$

In particular, the maps ε_i^0, ε_i^1 and η_i yield operators

$$\varepsilon_i^{0*} = F_i^0: C^n(I^r; \Pi) \to C^n(I^{r-1}; \Pi),$$
$$\varepsilon_i^{1*} = F_i^1: C^n(I^r; \Pi) \to C^n(I^{r-1}; \Pi),$$
$$\eta_i^* = D_i: C^n(I^r; \Pi) \to C^n(I^{r+1}; \Pi),$$

satisfying the identities (1. 1). (It follows from (2. 1) obviously.)

The $S. Q.$ complex $F(\Pi, n)$ is now defined as the complex $F(\Pi, n) = \cup_r F_r(\Pi, n)$ in which $F_r(\Pi, n)$ is the integral group ring of $C^n(I^r; \Pi)$, and the F_i^0, F_i^1, D_i homomorphisms are those induced above. An element σ of $C^n(I^r; \Pi)$ will be called an r-cube of $F(\Pi, n)$. Clearly the r-cubes are free generators of the additive group $F_r(\Pi, n)$.

The $S. Q.$ complex $K(\Pi, n)$ is defined as the subcomplex $K(\Pi, n) = \cup_r K_r(\Pi, n)$ of $F(\Pi, n)$ in which $K_r(\Pi, n)$ is the integral group ring of $Z^n(I^r; \Pi)$. An element σ of $Z^n(I^r; \Pi)$ will be called as an r-cube of $K(\Pi, n)$, and for any map $\beta: I^{n+1} \to I^r$

$$(\Delta\sigma)(\beta) = \sigma\textstyle\sum_{i=1}^{n+1}(-1)^i(F_i^0 - F_i^1)\beta$$
$$= \sigma\textstyle\sum_{i=1}^{n+1}(-1)^i(\beta\varepsilon_i^0 - \beta\varepsilon_i^1) = 0,$$

where $\Delta: C^n(I^r; \Pi) \to C^{n+1}(I^r; \Pi)$ is the coboundary homomorphism.

3. Suspension operators

For any r-cube σ of $F(\Pi, n)$ we define an $(r+1)$-cube $S(\sigma)$ of $F(\Pi, n+1)$ as follows

$$S(\sigma)(\alpha) = \sigma(\eta_1\alpha\varepsilon_1) \qquad \text{in the case (1),}$$
$$(3.\ 1) \qquad = \text{unit element of } \Pi \qquad \text{in the case (2),}$$
$$= (\Delta\sigma)(\eta_1\alpha) \qquad \text{in the case (3);}$$

Case (1): α is written as $\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n}}: I^{n+1} \to I^{r+1}$, where $r+1 \geq i_1 > \cdots > i_{r-n} > 1$.

In this case $\eta_1\alpha\varepsilon_1 = \varepsilon_{i_1-1}\cdots\varepsilon_{i_{r-n}-1}: I^n \to I^r$. It is convenient to denote $\eta_1\alpha\varepsilon_1$ as $\bar{\bar{\alpha}}$ in the following.

Case (2): α is written as $\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n-1}}\varepsilon_1^0: I^{n+1} \to I^{r+1}$, where $r+1 \geq i_1 > \cdots > i_{r-n-1} > 1$, or α is a degenerate cube.

Case (3): α is written as $\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n-1}}\varepsilon_1^1: I^{n+1} \to I^{r+1}$, where $r+1 \geq i_1 > \cdots > i_{r-n-1} > 1$.

In this case $\eta_1\alpha = \varepsilon_{i_1-1}\cdots\varepsilon_{i_{r-n-1}-1}: I^{n+1} \to I^r$.

We shall prove in the following that $S(\sigma)$ is an element of $K_{r+1}(\Pi, n+1)$.

Case (1): If $\alpha=\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n-1}}: I^{n+2}\to I^{r+1}$, $r+1\geq i_1>\cdots>i_{r-n-1}>1$, then

$$\begin{aligned}(\Delta S(\sigma))(\alpha)&=S(\sigma)\textstyle\sum_{i=1}^{n+2}(-1)^i(\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n-1}}\varepsilon_i^0-\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n-1}}\varepsilon_i^1)\\&=(\Delta\sigma)(\overline{\alpha})+\textstyle\sum_{i=2}^{n+2}(-1)^i\sigma(\overline{\alpha\varepsilon_i^0}-\overline{\alpha\varepsilon_i^1})\\&=(\Delta\sigma)(\overline{\alpha})-\textstyle\sum_{i=1}^{n+1}(-1)^i\sigma(\overline{\alpha}\varepsilon_i^0-\overline{\alpha}\varepsilon_i^1)=0.\end{aligned}$$

Case (2): If $\alpha=\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n-2}}\varepsilon_1^0: I^{n+2}\to I^{r+1}$, $r+1\geq i_1>\cdots>i_{r-n-2}>1$, then

$$\begin{aligned}(\Delta S(\sigma))(\alpha)&=S(\sigma)\textstyle\sum_{i=1}^{n+1}(-1)^i(\varepsilon_{i_1}\cdots\varepsilon_1^0\varepsilon_i^0-\varepsilon_{i_1}\cdots\varepsilon_1^0\varepsilon_i^1)\\&=S(\sigma)\textstyle\sum_{i=1}^{n+1}(-1)^i(\varepsilon_{i_1}\cdots\varepsilon_{i+1}^0\varepsilon_1^0-\varepsilon_{i_1}\cdots\varepsilon_{i+1}^1\varepsilon_1^0)=0.\end{aligned}$$

Case (2′): If $\alpha=\varepsilon_{i_1}\cdots\varepsilon_{i_s}\eta_{j_1}\cdots\eta_{j_t}$ and $t>0$, then

$$(\Delta S(\sigma))(\alpha)=S(\sigma)\textstyle\sum_{i=1}^{n+2}(-1)^i(\alpha\varepsilon_i^0-\alpha\varepsilon_i^1)=0, \qquad \text{if } t\geq 2.$$

$$\begin{aligned}(\Delta S(\sigma))(\alpha)&=S(\sigma)\textstyle\sum_{i=1}^{n+2}(-1)^i(\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n}}\eta_j\varepsilon_i^0-\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n}}\eta_j\varepsilon_i^1)\\&=S(\sigma)\textstyle\sum_{i<j}(-1)^i(\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n}}\varepsilon_i^0\eta_{j-1}-\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n}}\varepsilon_i^1\eta_{j-1})\\&\quad+(-1)^jS(\sigma)(\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n}}-\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n}})\\&\quad+\textstyle\sum_{i>j}(-1)^iS(\sigma)(\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n}}\varepsilon_{i-1}^0\eta_j-\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n}}\varepsilon_{i-1}^1\eta_j)=0, \quad \text{if } t=1.\end{aligned}$$

Case (3): If $\alpha=\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n-2}}\varepsilon_1^1: I^{n+2}\to I^{r+1}$, $r+1\geq i_1>\cdots>i_{r-n-2}>1$, then

$$\begin{aligned}(\Delta S(\sigma))(\alpha)&=S(\sigma)\textstyle\sum_{i=1}^{n+2}(-1)^i(\varepsilon_{i_1}\cdots\varepsilon_1^1\varepsilon_i^0-\varepsilon_{i_1}\cdots\varepsilon_1^1\varepsilon_i^1)\\&=S(\sigma)\textstyle\sum_{i=1}^{n+2}(-1)^i(\varepsilon_{i_1}\cdots\varepsilon_{i+1}^0\varepsilon_1^1-\varepsilon_{i_1}\cdots\varepsilon_{i+1}^1\varepsilon_1^1)\\&=(\Delta\sigma)\textstyle\sum_{i=1}^{n+2}(-1)^i(\varepsilon_{i_1-1}\cdots\varepsilon_i^0-\varepsilon_{i_1-1}\cdots\varepsilon_i^1)\\&=(\Delta\sigma)\textstyle\sum_{i=1}^{n+2}(-1)^i(\eta_1\alpha\cdot\varepsilon_i^0-\eta_1\alpha\cdot\varepsilon_i^1)\\&=(\Delta\Delta\sigma)(\eta_1\alpha)=0.\end{aligned}$$

We shall prove in the following that

(3. 2) $\quad F_i(S\sigma)=S(F_{i-1}\sigma) \qquad$ for $i\geq 2$, $\sigma\in F_r(\Pi, n)$,

(3. 3) $\quad \partial S(\sigma)=-S\partial(\sigma) \qquad$ for $\sigma\in K_r(\Pi, n)$.

If α is some type in the case (1) or (2), (3. 2) follows from the facts;

$$F_i(S\sigma)(\alpha)=(S\sigma)(\overline{\varepsilon_i\alpha}),\ \varepsilon_i\alpha=\varepsilon_{i-1}\overline{\alpha},$$

and if α is a degenerate cube, $\varepsilon_i\alpha$ also degenearte.

If $\alpha=\varepsilon_{i_1}\cdots\varepsilon_{i_{r-n-2}}\varepsilon_1^1: I^{n+1}\to I^r$ and $r\geq i_1>\cdots>i_{r-n-2}>1$, then

$$\begin{aligned}F_i(S\sigma)(\alpha)&=(S\sigma)(\varepsilon_{i_1+1}\cdots\varepsilon_{i_p+1}\varepsilon_i\varepsilon_{i_{p+1}}\cdots\varepsilon_1^1)\\&=(\Delta\sigma)(\varepsilon_{i_1}\cdots\varepsilon_{i_p}\varepsilon_{i-1}\varepsilon_{i_{p+1}-1}\cdots\varepsilon_{i_{r-n-2}-1})\\&=(\Delta\sigma)(\varepsilon_{i-1}\varepsilon_{i_1-1}\cdots\varepsilon_{i_{r-n-2}-1})\\&=(\Delta\sigma)(\varepsilon_{i-1}\eta_1\alpha), \qquad \text{if } i_p\geq i>i_{p+1},\end{aligned}$$

and

$$\begin{aligned} S(F_{i-1}\sigma)(\alpha) &= \Delta(F_{i-1}\sigma)(\eta_1\alpha) \\ &= (F_{i-1}\sigma)\textstyle\sum_{j=1}^{n+1}(-1)^j(F_j^0-F_j^1)(\eta_1\alpha) \\ &= \sigma\textstyle\sum_{j=1}^{n+1}(-1)^j(\varepsilon_{i-1}\eta_1\alpha\varepsilon_j^0-\varepsilon_{i-1}\eta_1\alpha\varepsilon_j^1) \\ &= (\Delta\sigma)(\varepsilon_{i-1}\eta_1\alpha). \end{aligned}$$

If $\sigma\in K_r(\Pi, n)$, $\Delta\sigma=0$ therefore (3. 3) is the immediate consequence of (3. 2), considering

$$F_1^0(S\sigma)(\alpha)=F_1^1(S\sigma)(\alpha)=\text{unit element of } \Pi$$

for any map α: $I^{n+1}\rightarrow I^r$.

4. Minimal complexes

Two singular r-cubes σ_0 and σ_1 in a topologicial space X are called *compatible* if their faces coincide: $F_i^0\sigma_0=F_i^0\sigma_1$, $F_i^1\sigma_0=F_i^1\sigma_1$ for $1\leq i\leq r$. If in addition σ_0 and σ_1 are members of a continuous one parameter family σ_t, $0\leq t\leq 1$, of singular r-cubes, all of which are compatible, we say that σ_0 and σ_1 are *homotopic.* For $r=0$ any two cubes are compatible, and if X is assumed to be arcwise connected, they are also homotopic.

A subcomplex $M=M(X)$ of $Q(X)$ will be called *minimal* provided:

(4. 1) *For each r the collapsed r-cube σ: $I^r\rightarrow x_0$ is in M.*

(4. 2) *If σ is a singular r-cube all of whose faces are in M, then M contains a unique singular r-cube σ' compatible with and homotopic to σ.*

The existence of such a subcomplex is obvious [1] and we have:

If the homotopy groups $\pi_i(X)$ vanish for $i<n$ then the minimal subcomplex consists of some singular cubes such that all faces of σ of dimensions $>n$ are collapsed. Consider an r-cube σ in M. Let α: $I^n\rightarrow I^r$ is any n-cube in $Q(I^r)$, α-face $\sigma\alpha$ of σ determines an element

$$\kappa(\sigma)(\alpha)=a(\sigma\alpha)\in\pi_n(X).$$

The function $\sigma\rightarrow\kappa(\sigma)$ defined above becomes an $S.$ $Q.$ map

$$\kappa:\ M\rightarrow K(\pi_n(X),\ n). \tag{4. 3}$$

If the homotopy groups $\pi_i(X)$ vanish for $i<n$ and $n<i<q$ $(1<n<q)$, then there is an $S.$ $Q.$ map

$$\bar{\kappa}:\ K(\pi_n(X),\ n)\rightarrow M \tag{4. 4}$$

satisfying the following conditions :

(4. 4. 1) *For each r-cube ϕ of $K(\pi_n(X), n)$, where $r<q$, there is a unique r-cube*

$\bar{\kappa}(\phi)$ in M such that $\kappa(\bar{\kappa}(\phi))=\phi$.

(4.4.2.) *For each q-cube ϕ of $K(\pi_n(X), n)$ there is at least one q-cube σ of M such that $\kappa(\sigma)=\phi$. Any two such cubes are compatible. One of these σ will be selected and denoted by $\bar{\kappa}(\phi)$. Thus $\kappa(\bar{\kappa}(\phi))=\phi$. For the collased q-cube ϕ, we choose $\bar{\kappa}(\phi)$ to be the collapsed q-cube in M.*

(4.4.3) *For each $(q+1)$-cube ϕ of $K(\pi_n(X), n)$, if we define a map $f_\phi:(I^{q+1})^{\cdot}\to X$ such that $f_\phi\varepsilon_i=\bar{\kappa}(\phi\varepsilon_i)$, then we have*

$$k(\phi)=c(f_\phi)\in\pi_q(X).$$

The function $\phi\to k(\phi)$ defined above depends upon the selection in (4.4.2) and becomes a cocycle

$$k\in Z^{q+1}(K(\pi_n(X), n); \pi_q(X)).$$

The cohomology class of this cocycle is a topological invariant (well known as the Eilenberg-MacLane invariant [2]).

5. Postnikov constructions

Let X be a topological space, and the homotopy groups $\pi_i(X)$ of X vanish without $i=n, q, q', \ldots, q^{(m)} (1<n<q<\ldots<q^{(m)})$. We shall denote $\pi_n(X)=\Pi$, $\pi_q(X)=G$, $\pi_{q'}(X)=G', \ldots, \pi_{q^{(m)}}(X)=G^{(m)}$ in the following.

Consider an r-cube σ in $M=M(X)$. Let $\beta: I^q\to I^r$ be any q-cube in $Q(I^r)$. We have r-cube $\kappa(\sigma)$ of $K(\Pi, n)$ as above, and r-cube $\psi=\psi(\sigma)$ of $F(G, q)$ defined as

$$\psi(\beta)=d(\bar{\kappa}(\kappa(\sigma\beta)), \sigma\beta)\in\pi_q(X)=G.$$

Above pair $(\kappa(\sigma), \psi(\sigma))$ for any r-cube σ satisfies the condition

$$k(\kappa(\sigma\gamma))=\Delta\psi(\sigma\gamma) \quad \text{in} \quad Z^{q+1}(I^{q+1}; G)$$

where $\gamma: I^{q+1}\to I^r$ is any $(q+1)$-cube in $Q(I^r)$ and $\Delta: C^q(I^{q+1}; G)\to C^{q+1}(I^{q+1}; G)$ is the coboundary homomorphism.

The *S. Q.* complex $N=\mathfrak{P}(K(\Pi, n), G, q, k)$ (simply denoted by $K(\Pi, n, G, q, k)$ in the following) is now defined as the subcomplex of the cartesian product $K(\Pi, n)\times F(G, q)$ in which r-cube is a pair (ϕ, ψ) subject to the condition:

(5. 1) $\quad k(\phi\gamma)=\Delta\psi(\gamma)$ *for any $(q+1)$-cube γ of $Q(I^r)$.*

By the above definitions, we have obviously

$$K_i(\Pi, n, G, q, k)\cong K_i(\Pi, n) \quad \text{for } i<q,$$
$$K(\Pi, n, G, q, k)\supset K(G, q),$$

and $K(\Pi, n, G, q, k)$ is $(q+1)$-trivial (If σ and σ' are compatible r-cubes in $K(\Pi, n, G, q, k)$ and $r\geq q+1$ then $\sigma=\sigma'$).

Generally, if $M(X)\cong N^{(i-1)}$ in dimensions less than $q^{(i)}$, and $N^{(i-1)}$ is $q^{(i)}$-trivial, there is a cocycle

$$k^{(i)} \in Z^{q^{(i)}+1}(N^{(i-1)}; G^{(i)})$$

associated to a pair of $S.Q.$ maps $(\kappa', \bar{\kappa}')$ satsifying the following conditions:

(5. 2) $$\kappa': M \to N^{(i-1)},$$

(5. 3) $$\bar{\kappa}': N^{(i-1)} \to M,$$

(5. 3. 1) $\kappa'\bar{\kappa}'(\phi)=\phi$ *for r-cube* $\phi \in N^{(i-1)}$, $r\leq q^{(i)}$,

(5. 3. 2) *For each* $(q^{(i)}+1)$*-cube* ϕ *of* $N^{(i-1)}$, *if we define a map* f_{ϕ}: $(I^{q^{(i)}+1})^{\cdot}\to X$ *such that* $f_{\phi}\varepsilon_i=\bar{\kappa}'(\phi\varepsilon_i)$, *then we have*

$$k^{(i)}(\phi)=c(f_{\phi}) \in \pi_q(i)(X)=G^{(i)}.$$

Consider an r-cube σ in M. Let $\beta: I^{q^{(i)}}\to I^r$ be any $q^{(i)}$-cube in $Q(I^r)$, we have r-cube $\psi=\psi(\sigma)$ of $F(G^{(i)}, q^{(i)})$ defined as

$$\psi(\beta)=d(\bar{\kappa}'(\kappa'(\sigma\beta)), \sigma\beta) \in \pi_{q^{(i)}}(X)=G^{(i)}.$$

Above pair $(\kappa'(\sigma), \psi(\sigma))$ for any r-cube σ satisfies the condition

$$k^{(i)}(\kappa'(\sigma\gamma))=\Delta\psi(\sigma\gamma) \quad \text{in } Z^{q^{(i)}+1}(I^{q^{(i)}+1}: G^{(i)}),$$

where γ: $I^{q^{(i)}+1}\to I^r$ is any $(q^{(i)}+1)$-cube in $Q(I^r)$.

The $S.Q.$ complex $N^{(i)}=\mathfrak{P}(N^{(i-1)}, G^{(i)}, q^{(i)}, k^{(i)})$ is now defined as the subcomplex of the cartesian product $N^{(i-1)}\times F(G^{(i)}, q^{(i)})$ in which r-cube is a pair (ϕ, ψ) subject to the condition:

(5. 4) $k^{(i)}(\phi\gamma)=\Delta\psi(\gamma)$ *for any* $(q^{(i)}+1)$*-cube* γ *of* $Q(I^r)$.

These constructions make an $S.Q.$ complex

$$\mathfrak{P}(K(\Pi, n), k, N, k', N', \dots, k^{(m)}, N^{(m)})$$

inductively, and obviously this complex isomorphic to $M(X)$.

6. Representations of $S.Q.$ maps

We wish to classify $S.Q.$ maps of an $S.Q.$ complex K

(6. 1) $$T: K\to \mathfrak{P}(K(\Pi, n), k, N, k', N', \dots, k^{(m)}, N^{(m)}).$$

Such a map determines a sequence

(6. 2) $$x_n, x_q, x_{q'}, \dots, x_{q^{(m)}}$$

of a cocycle and cochains

$$x_n=T^{\#}b_n \in Z^n(K; \Pi)$$

$$x_q=T^{\#}b_q \in C^q(K; G)$$

$$x_{q'}=T^{\#}b_{q'} \in C^{q'}(K; G'),$$

....

$$x_{q^{(m)}}=T^{\#}b_{q^{(m)}} \in C^{q^{(m)}}(K; G^{(m)}).$$

In the above formula b_n is the basic cocycle in $Z^n(N^{(m)}; \Pi)\cong Z^n(\Pi, n; \Pi)$ and $b_{q^{(i)}}$ is the basic cochain in $C^{q^{(i)}}(N^{(m)}; G^{(i)})\cong C^{q^{(i)}}(N^{(i)}; G^{(i)})=C^{q^{(i)}}(\mathfrak{P}(N^{(i-1)}, G^{(i)}, k^{(i)}); G^{(i)})$ defined by

$$b_n(\phi)=\phi(\varepsilon^n) \qquad \text{for any } n\text{-cube } \phi \text{ of } K(\Pi, n),$$

$$b_{q^{(i)}}(\phi, \psi)=\psi(\varepsilon^{q^{(i)}}) \quad \text{for any } q^{(i)}\text{-cube } (\phi, \psi) \text{ of } N^{(i)}$$

where ε^n: $I^n \to I^n$ and $\varepsilon^{q^{(i)}}$: $I^{q^{(i)}} \to I^{q^{(i)}}$ are the identity maps, and $q^{(0)}=q$, $N^{(0)}=N=K(\Pi, n, G, q, k)$.

It is well known that there is a one to one correspondence between $S. Q.$ maps

$$T: K \to K(\Pi, n)$$

and cocycles $x_n \in Z^n(K; \Pi)$ [4]. It is convenient to make a one to one correspondence between $S. Q.$ maps

$$T: K \to F(\Pi, n)$$

and cochains $x_n \in C^n(K; \Pi)$ by defining $x_n = T^{\#}b_n$ for the basic cochain $b_n \in C^n(F(\Pi, n); \Pi)$, and

$$T(\sigma)(\alpha)=x_n(\sigma\alpha) \qquad \text{for any } n\text{-cube } \alpha \text{ of } Q(I^{\dim \sigma}).$$

Generally, there is a one to one correspondence between $S. Q.$ maps (6. 1) and sequence (6. 2) satisfying the conditions

$$(6.\ 3) \qquad k^{(i)}T(x_n, x_q, \ldots, x_{q^{(i-1)}})=\delta x_{q^{(i)}} \qquad i=0, 1, \ldots, m,$$

where $T(x_n, x_q, \ldots, x_{q^{(i-1)}})$: $K \to N^{(i-1)}$ is the $S.Q.$ map which corresponds inductively to the subsequence $(x_n, x_q, \ldots, x_{q^{(i-1)}})$ of (6. 2).

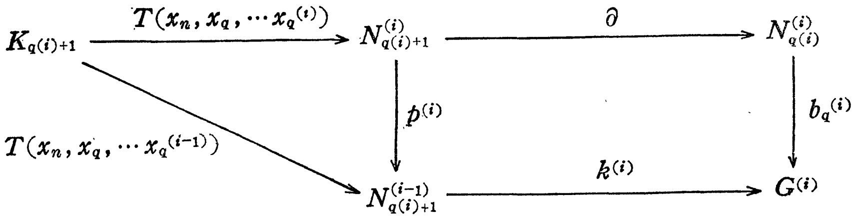
The map T corresponding in this fashion to the sequence $(x_n, x_q, x_{q'}, \ldots, x_{q^{(m)}})$ will be denoted by $T(x_n, x_q, x_{q'}, \ldots, x_{q^{(m)}})$. Then, this map is characterized as that $S. Q.$ map for which

$$T(x_n, x_q, x_{q'}, \ldots, x_{q^{(m)}})\sigma=(\phi, \psi) \quad \text{for an } r\text{-cube } \sigma \text{ of } K,$$

where $\phi=T(x_n, x_q, x_{q'}, \ldots, x_{q^{(m-1)}})\sigma \in N_r^{(m-1)}$ is the r-cube defined by induction and ψ is the r-cube of $F(G^{(m)}, q^{(m)})$ defined by

$$\psi(\beta)=x_{q^{(m)}}(\sigma\beta) \quad \text{for any } q^{(m)}\text{-cube } \beta \text{ of } Q(I^r).$$

Consider the diagram:



then (6. 3) is an immediate consequence of the commutativity of this diagram. Namely

$$b_{q^{(i)}}\partial T(x_n, x_q, \ldots, x_{q^{(i)}})=\delta x_{q^{(i)}}$$

and

(6. 4) $$k^{(i)}p^{(i)}=b_{q^{(i)}}\partial.$$

In the above diagram $p^{(i)}$ is the projection defined by

$$p^{(i)}(\phi, \psi)=\phi \qquad \text{for any cube } (\phi,\psi) \in N^{(i)}_{q(i)+1},$$

and if $i=0$, (6. 4) is rewritten as

$$kp=b_q\partial$$

where $p: N^{(0)}_{q+1} \to N^{(-1)}_{q+1}$, namely

$$p: K_{q+1}(\Pi, n, G, q, k) \to K_{q+1}(\Pi, n).$$

These formulas (6. 4) may be proved directly from (5. 1), (5. 4), for instance

$$\begin{aligned} kp(\phi, \psi) &= k(\phi)=\Delta\psi(\varepsilon^{q+1})=\psi\textstyle\sum_{i=1}^{q+1}(-1)^i(F_i^0-F_i^1)(\varepsilon^{q+1}) \\ &= \psi\textstyle\sum_{i=1}^{q+1}(-1)^i(\varepsilon_i^0-\varepsilon_i^1)=\sum_{i=1}^{q+1}(-1)^i\{\psi(\varepsilon_i^0)-\psi(\varepsilon_i^1)\} \\ &= \textstyle\sum_{i=1}^{q+1}(-1)^i\{F_i^0\psi(\varepsilon^q)-F_i^1\psi(\varepsilon^q)\}=b_q\partial(\phi, \psi). \end{aligned}$$

7. Chain homotopies

We make reference to chain homotopies between special $S.Q.$ maps of an $S.Q.$ complex K.

Let $w \in C^{n-1}(K; \Pi)$, $\delta w \in Z^n(K; \Pi)$ induces the map

$$T(\delta w): K \to K(\Pi, n).$$

LEMMA (7. 1). $T(\delta w)=T(0)$.

PROOF. Put

$$\begin{aligned} &\overline{w}(I\otimes\sigma_{n-1})=w(\sigma_{n-1}) && \text{for } I\otimes\sigma_{n-1} \in (I\otimes K)_n, \\ &\overline{w}(1\otimes\sigma_n)=\delta w(\delta_n) && \text{for } 1\otimes\sigma_n \in (I\otimes K)_n, \\ &\overline{w}(0\otimes\sigma_n)=0 && \text{for } 0\otimes\sigma_n \in (I\otimes K)_n, \end{aligned}$$

then

$$\delta\overline{w}(I\otimes\sigma_n)=\overline{w}(\partial I\otimes\sigma_n)-\overline{w}(I\otimes\partial\sigma_n)$$
$$=\overline{w}(1\otimes\sigma_n-0\otimes\sigma_n)-\overline{w}(I\otimes\partial\sigma_n)=\delta w(\sigma_n)-w(\partial\sigma_n)=0,$$
$$\delta\overline{w}(1\otimes\sigma_{n+1})=\overline{w}(1\otimes\partial\sigma_{n+1})=\delta w(\partial\sigma_{n+1})=0,$$
$$\delta\overline{w}(0\otimes\sigma_{n+1})=\overline{w}(0\otimes\partial\sigma_{n+1})=0.$$

Therefore, we have a cocycle $\overline{w}\in Z^n(I\otimes\dot{K};\Pi)$ and also an $S.Q.$ map
$$T(\overline{w}): I\otimes K\to K(\Pi,n).$$

Let I: $K\to I\otimes K$ be a map defined by
$$I(\sigma_r)=I\otimes\sigma_r\,.$$

Then we have a chain homotopy
$$T(\overline{w})\cdot I: C_r(K)\to C_{r+1}(K(\Pi,n))$$
between the maps $T(\delta w)$ and $T(0)$. The proof is complete.

The commutativity of the following diagram is obvious:

$$\begin{array}{ccc} C_r(K) & \xrightarrow{\quad T(w)\quad} & C_r(F(\Pi,n)) \\ \downarrow I & & \downarrow S \\ C_{r+1}(I\otimes K) & \xrightarrow{\quad T(\overline{w})\quad} & C_{r+1}(K(\Pi,n)). \end{array}$$

In the following, we shall denote $T(\overline{w})\cdot I=S\cdot T(w)$ simply by $D(w)$.

Let $v\in C^{q-1}(K;G)$ and $c\in C^{q^{(i)}-1}(K;G^{(i)})$, $\delta v\in Z^q(K;G)$ and $\delta c\in Z^{q^{(i)}}(K;G^{(i)})$ induce the $S.Q.$ maps
$$T(0,\delta v): K\to K(G,q)\subset K(\Pi,n,G,q,k)$$
$$T(0,\dots,0,\delta c): K\to K(G^{(i)},q^{(i)})\subset N^{(i)}$$
respectively.

LEMMA (7. 2).
$$T(0,\delta v)\cong T(0,0),$$
$$T(0,\dots,0,\delta c)\cong T(0,\dots,0,0).$$

Proof of this lemma is similar to above. Namely we may define cocycles $\overline{v}\in Z^q(I\otimes K;G)$, $\overline{c}\in Z^{q^{(i)}}(I\otimes K;G^{(i)})$ such that the commutativity of the following diagrams is satisfied

$$\begin{array}{ccc} C_r(K) & \xrightarrow{\quad T(v)\quad} & C_r(F(G,q)) \\ \downarrow I & & \downarrow S \\ C_{r+1}(I\otimes K) & \xrightarrow{\quad T(\overline{v})\quad} & C_{r+1}(K(G,q)), \end{array}$$

$$\begin{array}{ccc} C_r(K) & \xrightarrow{\quad T(c)\quad} & C_r(F(G^{(i)},q^{(i)})) \\ \downarrow I & & \downarrow S \\ C_{r+1}(I\otimes K) & \xrightarrow{\quad T(\overline{c})\quad} & C_{r+1}(K(G^{(i)},q^{(i)})). \end{array}$$

In the following, we shall denote $T(\bar{v})\cdot I=S\cdot T(v)$ and $T(\bar{c})\cdot I=S\cdot T(c)$ simply by $D(v)$, $D(c)$ respectively.

We have considered the constructions of chain homotopies; conversely, we consider now the representations of chain homotopies.

LEMMA (7. 3). *Any chain homotopy*

$$D: C_r(K)\rightarrow C_{r+1}(N^{(i)})$$

between S.Q. maps $T(x_n, x_q, \dots, x_{q^{(i)}})$ *and* $T(x'_n, x'_q, \dots, x'_{q^{(i)}})$ *may be regarded as the combination* $T\cdot I$, *where*

$$T=T(y_n, y_q, \dots, y_{q^{(i)}})$$

is an S. Q. map of $I\otimes K$ *in* $N^{(i)}$

$$\begin{array}{ccc} C_r(K) & \xrightarrow{\quad D \quad} & \\ \downarrow I & & \\ C_{r+1}(I\otimes K) & \xrightarrow{\quad T \quad} & C_{r+1}(N^{(i)}) \end{array}$$

PROOF.

Case (7. 3. 1); $i=-1$, namely $N^{(-1)}=K(\Pi, n)$.

We define a n-cocycle $y_n\in Z^n(I\otimes K; \Pi)$ by

$$y_n(1\otimes\sigma_n)=b_n(F_1^1(D(\sigma_n)))=x_n(\sigma_n) \quad \text{for } \sigma_n\in K_n,$$

$$y_n(0\otimes\sigma_n)=b_n(F_1^0(D(\sigma_n)))=x'_n(\sigma_n) \quad \text{for } \sigma_n\in K_n,$$

$$y_n(I\otimes\sigma_{n-1})=b_n(D(\sigma_{n-1})) \quad \text{for } \sigma_{n-1}\in K_{n-1}.$$

It is obvious $D=T(y_n)\cdot I$.

Case (7. 3. 2); $i=0$, namely $N^{(0)}=N=K(\Pi, n, G, q, k)$.

We define y_n as (7. 3. 1) and define a q-cochain $y_q\in C^q(I\otimes K; G)$ by

$$y_q(1\otimes\sigma_q)=b_q(F_1^1(D(\sigma_q)))=x_q(\sigma_q) \quad \text{for } \sigma_q\in K_q,$$

$$y_q(0\otimes\sigma_q)=b_q(F_1^0(D(\sigma_q)))=x'_q(\sigma_q) \quad \text{for } \sigma_q\in K_q,$$

$$y_q(I\otimes\sigma_{q-1})=b_q(D(\sigma_{q-1})) \quad \text{for } \sigma_{q-1}\in K_{q-1}.$$

Then

$$\delta y_q(1\otimes\sigma_{q+1})=\delta x_q(\sigma_{q+1})=kT(x_n)(\sigma_{q+1})=kT(y_n)(1\otimes\sigma_{q+1}),$$

$$\delta y_q(0\otimes\sigma_{q+1})=\delta x'_q(\sigma_{q+1})=kT(x'_n)(\sigma_{q+1})=kT(y_n)(0\otimes\sigma_{q+1}),$$

$$\delta y_q(I\otimes\sigma_q)=y_q(1\otimes\sigma_q-0\otimes\sigma_q-I\otimes\partial\sigma_q)=x(\sigma_q)-x'_q(\sigma_q)-b_q(D(\partial\sigma_q))$$
$$=x_q(\sigma_q)-x'_q(\sigma_q)+b_q((\partial D-T(x_n, x_q)+T(x'_n, x'_q))\sigma_q)$$

$$=b_q\partial D(\sigma_q)+x_q(\sigma_q)-x'_q(\sigma_q)-x_q(\sigma_q)+x'_q(\sigma_q)$$
$$=(\delta b_q)(D(\sigma_q))=kT(b_n)(D(\sigma_q))$$
$$=kT(y_n)(I\otimes\sigma_q).$$

Case (7. 3. 3): $i>0$.

We may prove similarly as above. For instance we define a $q^{(i)}$-cochain $y_{q^{(i)}}\in C^{q^{(i)}}(I\otimes K; G^{(i)})$ by

$$y_{q^{(i)}}(1\otimes\sigma_{q^{(i)}})=b_{q^{(i)}}(F_1^1(D(\sigma_{q^{(i)}})))=x_{q^{(i)}}(\sigma_{q^{(i)}}) \quad \text{for } \sigma_{q^{(i)}}\in K_{q^{(i)}},$$
$$y_{q^{(i)}}(D\otimes\sigma_{q^{(i)}})=b_{q^{(i)}}(F_1^0(D(\sigma_{q^{(i)}})))=x'_{q^{(i)}}(\sigma_{q^{(i)}}) \quad \text{for } \sigma_{q^{(i)}}\in K_{q^{(i)}},$$
$$y_{q^{(i)}}(I\otimes\sigma_{q^{(i)}-1})=b_{q^{(i)}}(D(\sigma_{q^{(i)}-1})) \quad \text{for } \sigma_{q^{(i)}-1}\in K_{q^{(i)}-1}.$$

8. Classifiaction of $S.Q.$ maps

For our purpose we shall define the internal products of $S.Q.$ complexes $F(\Pi,n)$, $K(\Pi,n)$, $K(\Pi,n,G,q,k)$, ..., $N^{(i)}$, $\cdots$ as follows:

(8. 1. 1) $(\psi\circ\psi')(\alpha)=\psi(\alpha)+\psi'(\alpha)$ for any n-cube α of $Q(I^r)$ for r-cubes ψ,ψ' of $F(\Pi,n)$.

(8. 1. 2) $(\phi\circ\phi')(\alpha)=\phi(\alpha)+\phi'(\alpha)$ for any n-cube α of $Q(I^r)$ and for r-cubes ϕ,ϕ' of $K(\Pi,n)$.

(8. 1. 3) $$(\phi,\psi)\circ(\iota,\psi)=(\phi,\psi\circ\psi')$$
for r-cubes (ϕ,ψ), (ι,ψ') of $K(\Pi,n,G,q,k)$, where ι is the collapsed r-cube of $K(\Pi,n)$, then (ι,ψ') may be regarded as an r-cube of $K(G,q)$.

(8. 1. 4) $$(\phi,\psi)\circ(\iota,\psi')=(\phi,\psi\circ\psi')$$
for r-cubes (ϕ,ψ), (ι,ψ') of $N^{(i)}=\mathfrak{P}(N^{(i-1)}, G^{(i)}, q^{(i)}, k^{(i)})$, where ι is the collapsed r-cube of $N^{(i-1)}$, then (ι,ψ') may be regarded as an r-cube of $K(G^{(i)},q^{(i)})$.

THEOREM (8. 2). *The cocycles x_n, $x'_n\in Z^n(K;\Pi)$ are cohomologous if and only if the maps $T(x_n)$, $T(x'_n)$ are chain homotopic.* (*Theorem* 5. 2 *of* III [4]).

PROOF. Since $b_n\in Z^n(K(\Pi, n);\ \Pi)$ is a cocycle, $T(x_n)\simeq T(x'_n)$ implies that $x_n=b_nT(x_n)$ and $x'_n=b_nT(x'_n)$ are cohomologous. Conversely, assume that $x_n-x'_n=\delta w$, for some $w\in C^{n-1}(K;\Pi)$, and construct a chain homotopy $D(w)$ as in the proof of (7. 1). On the other hand, we define a map

$$D'=D_1T(x'_n): C_r(K)\to C_{r+1}(K(\Pi, n)),$$

then we have a chain homotopy

(8. 3) $$E=D'\circ D(w): C_r(K)\to C_{r+1}(K(\Pi, n)),$$

where $\circ$ is the internal product defined in (8.1.2). Namely;

$$(\partial E+E\partial)\sigma=\partial(D'(\sigma)\circ D(w)(\sigma))+(D'\circ D(w))(\partial\sigma)$$
$$=\sum_{i=1}^{r+1}(-1)^i(F_i^0D'(\sigma)\circ F_i^0D(w)(\sigma)-F_i^1D'(\sigma)\circ F_i^1D(w)(\sigma))$$
$$+\sum_{i=1}^{r}(-1)^i((D'\circ D(w))F_i^0\sigma-(D'\circ D(w))F_i^1\sigma)$$
$$=F_1^1D_1T(x_n')(\sigma)\circ F_1^1D(w)(\sigma)-F_1^0D_1T(x_n')(\sigma)\circ F_1^0D(w)(\sigma)$$
$$+\sum_{i=2}^{r+1}(-1)^i(F_i^0D'(\sigma)\circ F_i^0D(w)(\sigma)-F_i^1D'(\sigma)\circ F_i^1D(w)(\sigma))$$
$$+\sum_{i=1}^{r}(-1)^i(D'(F_i^0\sigma)\circ D(w)(F_i^0\sigma)-D'(F_i^1\sigma)\circ D(w)(F_i^1\sigma))$$
$$=T(x_n')(\sigma)\circ T(\delta w)(\sigma)-T(x_n')(\sigma)\circ T(0)(\sigma)$$
$$=T(x_n)(\sigma)-T(x_n')(\sigma),$$

since $F_{i+1}D'(\sigma)\circ F_{i+1}D(w)(\sigma)=F_{i+1}D_1T(x_n')(\sigma)\circ F_{i+1}ST(w)(\sigma)$

$$=D_1F_iT(x_n')(\sigma)\circ SF_iT(w)(\sigma)$$
$$=D_1T(x_n')(F_i\sigma)\circ ST(w)(F_i\sigma)$$
$$=D'(F_i\sigma)\circ D(w)(F_i\sigma).$$

We next consider two *S. Q.* maps

$$T(x_n, x_q),\ T(x_n', x_q'): K\rightarrow N=K(\Pi, n, G, q, k).$$

If x_n and x_n' are cohomologous, there exists a chain homotopy

$$E: T(x_n)\simeq T(x_n').$$

Note that in this case $x_q-x_q'-kE$ is a cocycle, because

$$\delta(x_q-x_q'-kE)=kT(x_n)-kT(x_n')-kE\partial$$
$$=k(E\partial+\partial E)-kE\partial=0,$$

since k is a cocycle.

THEOREM (8. 4) *S. Q. maps $T(x_n, x_q)$ and $T(x_n', x_q')$ are chain homotopic if and only if x_n and x_n' are cohomologous and $x_q-x_q'-kE$ is cohomologous zero for arbitrary chain homotopy $E: T(x_n)\simeq T(x_n')$.*

PROOF. Let $E': C_r(K)\rightarrow C_{r+1}(N)$ be a chain homotopy between $T(x_n, x_q)$ and $T(x_n', x_q')$; i. e.

$$\partial E'+E'\partial=T(x_n, x_q)-T(x_n', x_q'),$$

it is obvious that $x_n=b_nT(x_n, x_q)$ and $x_n'=b_nT(x_n', x_q')$ are cohomologous, and

$$pE': C_r(K)\rightarrow C_{r+1}(K(\Pi, n))$$

gives a chain homotopy $E: T(x_n)\simeq(x_n')$.

Then,

$$x_q-x_q'-kE=b_qT(x_n, x_q)-b_qT(x_n', x_q')-kE$$
$$=b_q(\partial E'+E'\partial)-kpE'$$
$$=\delta(b_qE')$$

since $b_q\partial = kp$ (6.4).

Conversely, let $E: T(x_n) \cong T(x'_n)$ be a chain homotopy and $x_q - x'_q - kE = \delta v$ for $v \in C^{q-1}(K; G)$. We first determine a cocycle $y_n \in Z^n(I \otimes K; \Pi)$ as in Lemma (7.3.1), and define a cochain $\bar{x}'_q \in C^q(I \otimes K; G)$ as follows;

$$\begin{aligned} &\bar{x}'_q(1 \otimes \sigma_q) = (x'_q + kE)(\sigma_q) && \text{for } \sigma_q \in K_q, \\ &\bar{x}'_q(0 \otimes \sigma_q) = x'_q(\sigma_q) && \text{for } \sigma_q \in K_q, \\ &\bar{x}'_q(I \otimes \sigma_{q-1}) = 0 && \text{for } \sigma_{q-1} \in K_{q-1}, \end{aligned}$$

then we have a pair $(y_n, \bar{x}'_q)$, and

$$\begin{aligned} \delta\bar{x}'_q(1 \otimes \sigma_{q+1}) &= \bar{x}'_q(1 \otimes \partial\sigma_{q+1}) = (x'_q + kE)(\partial\sigma_{q+1}) \\ &= (x_q - \delta v)(\partial\sigma_{q+1}) = x_q(\partial\sigma_{q+1}) \\ &= \delta x_q(\sigma_{q+1}) = kT(x_n)(\sigma_{q+1}) \\ &= kT(y_n)(1 \otimes \sigma_{q+1}), \end{aligned}$$

$$\begin{aligned} \delta\bar{x}'_q(0 \otimes \sigma_{q+1}) &= \bar{x}'_q(0 \otimes \partial\sigma_{q+1}) = x'_q(\partial\sigma_{q+1}) = \delta x'_q(\sigma_{q+1}) \\ &= kT(x'_n)(\sigma_{q+1}) = kT(y_n)(0 \otimes \sigma_{q+1}), \end{aligned}$$

$$\begin{aligned} \delta\bar{x}'_q(I \otimes \sigma_q) &= \bar{x}'_q(1 \otimes \sigma_q - 0 \otimes \sigma_q - I \otimes \partial\sigma_q) \\ &= (x'_q + kE)(\sigma_q) - x'_q(\sigma_q) = kE(\sigma_q) \\ &= kT(y_n)(I \otimes \sigma_q), \end{aligned}$$

therefore $T(y_n, \bar{x}'_q)$ is regarded as an $S. Q.$ map

$$I \otimes K \rightarrow N = K(\Pi, n, G, q, k).$$

Finally we have

$$D = T(y_n, \bar{x}'_q) \cdot I : C_r(K) \rightarrow C_{r+1}(N)$$

with the following property

$$\partial D + D\partial = T(x_n, x'_q + kE) - T(x'_n, x'_q).$$

The remainder of our proof is due to Lemma (7.2) and (8.1.3). Namely, we define a map

$$D'_{(0)} = D_1 T(x_n, x'_q + kE) : C_r(K) \rightarrow C_{r+1}(N),$$

then we have a chain homotopy

$$D'_{(0)} \circ D(v) : T(x_n, x_q) \cong T(x_n, x'_q + kE).$$

Inductively we shall consider two $S. Q.$ maps

$$T(x_n, \ldots, x_{q(i-1)}, x_{q(i)}), T(x'_n, \ldots, x'_{q(i-1)}, x_{q(i)}) : K \rightarrow N^{(i)} = \mathfrak{P}(N^{(i-1)}, G^{(i)}, q^{(i)}, k^{(i)}).$$

Let the following conditions be satisfied:

(8. 5. 1) x_n and x'_n are cohomologous,

(8. 5. 2) $x_q - x'_q - kE$ is cohomologous zero,

....

(8. 5. 3) $x_{q^{(i-1)}} - x'_{q^{(i-1)}} - k^{(i-1)}E^{(i-1)}$ is cohomologous zero,

where $E^{(j)}: T(x_n, \ldots, x_{q^{(j-1)}}) \cong T(x'_n, \ldots, x'_{q^{(j-1)}})$ $j=0, 1, \ldots i-1$ are the chain homotopy defined inductively.

Then, there exists a chain homotopy

(8. 5. 4) $$E^{(i)}: T(x_n, \ldots, x_{q^{(i-1)}}) \cong T(x'_n, \ldots, x'_{q^{(i-1)}})$$

inductively. In this case note that $x_{q^{(i)}} - x'_{q^{(i)}} - k^{(i)}E^{(i)}$ is a cocycle, because

$$\begin{aligned}&\delta(x_{q^{(i)}} - x'_{q^{(i)}} - k^{(i)}E^{(i)})\\ &\quad = k^{(i)}T(x_n, \ldots, x_{q^{(i-1)}}) - k^{(i)}T(x'_n, \ldots, x'_{q^{(i-1)}}) - k^{(i)}E^{(i)}\partial\\ &\quad = k^{(i)}(E^{(i)}\partial + \partial E^{(i)}) - k^{(i)}E^{(i)}\partial = 0,\end{aligned}$$

since $k^{(i)}$ is a cocycle.

Theorem (8.6). *S. Q. maps* $T(x_n, \ldots, x_{q^{(i)}})$ *and* $T(x'_n, \ldots, x'_{q^{(i)}})$ *are chain homotopic if and only if* (8.5.1), (8.5.2),...(8.5.3) *are satisfied and* $x_{q^{(i)}} - x'_{q^{(i)}} - k^{(i)}E^{(i)}$ *is cohomologous zero for arbitrary chain homotopy* $E^{(i)}$ (8.5.4).

Proof. Let $E^{(i+1)}: C_r(K) \to C_{r+1}(N^{(i)})$ be a chain homotopy between $T(x_n, \ldots, x_{q^{(i)}})$ and $T(x'_n, \ldots, x'_{q^{(i)}})$; i.e.

$$\partial E^{(i+1)} + E^{(i+1)}\partial = T(x_n, \ldots, x_{q^{(i)}}) - T(x'_n, \ldots, x'_{q^{(i)}}),$$

then inductively (8.5.1), (8.5.2), $\cdots$, (8.5.3) are obvious since

$$p^{(i)}E^{(i+1)}: C_r(K) \to C_{r+1}(N^{(i-1)})$$

brings a chain homotopy $E^{(i)}: T(x_n, \ldots, x_{q^{(i-1)}}) \cong T(x'_n, \ldots x'_{q^{(i-1)}})$. And,

$$\begin{aligned}&x_{q^{(i)}} - x'_{q^{(i)}} - k^{(i)}E^{(i)}\\ &\quad = b_{q^{(i)}}T(x_n, \ldots, x_{q^{(i)}}) - b_{q^{(i)}}T(x'_n, \ldots, x'_{q^{(i)}}) - k^{(i)}E^{(i)}\\ &\quad = b_{q^{(i)}}(\partial E^{(i+1)} + E^{(i+1)}\partial) - k^{(i)}p^{(i)}E^{(i+1)}\\ &\quad = \delta(b_{q^{(i)}}E^{(i+1)})\end{aligned}$$

since $b_{q^{(i)}}\partial = k^{(i)}p^{(i)}$ (6. 4).

Conversely, let $E^{(i)}: T(x_n, \ldots, x_{q^{(i-1)}}) \cong T(x'_n, \ldots, x'_{q^{(i-1)}})$ is a chain homotopy and $x_{q^{(i)}} - x'_{q^{(i)}} - k^{(i)}E^{(i)} = \delta c$ for $c \in C^{q^{(i)}-1}(K; G^{(i)})$. We first determine a sequence

$$y_n, y_q, y_{q'}, \ldots, y_{q^{(i-1)}}$$

of a cocycle y_n and cochains $y, y_{q'}, \ldots, y_{q(i-1)}$ associated to the chain homotopy $E^{(i)}$ as in Lemma (7. 6), and define a cochain $\bar{x}'_{q(i)} \in C^{q^{(i)}}(I \otimes K; G^{(i)})$ as follows;

$$\bar{x}'_{q(i)}(1 \otimes \sigma_{q(i)}) = (x'_{q(i)} + k^{(i)}E^{(i)})(\sigma_{q(i)}) \quad \text{for} \quad \sigma_{q(i)} \in K_{q(i)},$$

$$\bar{x}'_{q(i)}(0 \otimes \sigma_{q(i)} = (x'_{q(i)}(\sigma_{q(i)}) \quad \text{for} \quad \sigma_{q(i)} \in K_{q(i)},$$

$$\bar{x}'^{q}{}_{(i)}(I \otimes \sigma_{q(i)-1}) = 0 \quad \text{for} \quad \sigma_{q(i)-1} \in K_{q(i)-1},$$

then we have a sequence

$$y_n, y_q, y'_q, \ldots, y_{q(i-1)}, \bar{x}'_{q(i)}.$$

Here

$$\begin{aligned}
\delta\bar{x}'_{q(i)}(1 \otimes \sigma_{q(i)+1}) &= \bar{x}'_{q(i)}(1 \otimes \partial\sigma_{q(i)+1}) = (x'_{q(i)} + k^{(i)}E^{(i)})(\partial\sigma_{q(i)+1}) \\
&= (x_{q(i)} - \delta c)(\partial\sigma_{q(i)+1}) = x_{q(i)}(\partial\sigma_{q(i)+1}) \\
&= \delta x_{q(i)}(\sigma_{q(i)+1}) = k^{(i)}T(x_n, \ldots, x_{q(i-1)})(\sigma_{q(i)+1}) \\
&= k^{(i)}T(y_n, \ldots, y_{q(i-1)})(1 \otimes \sigma_{q(i)+1}),
\end{aligned}$$

$$\begin{aligned}
\delta\bar{x}'_{q(i)}(0 \otimes \sigma_{q(i)+1}) &= \bar{x}'_{q(i)}(0 \otimes \partial\sigma_{q(i)+1}) = x'_{q(i)}(\partial\sigma_{q(i)+1}) \\
&= \delta x'_{q(i)}(\sigma_{q(i)+1}) = k^{(i)}T(x'_n, \ldots, x'_{q(i-1)})(\sigma_{q(i)}) \\
&= k^{(i)}T(y_n, \ldots, y_{q(i-1)})(0 \otimes \sigma_{q(i)+1}),
\end{aligned}$$

$$\begin{aligned}
\delta\bar{x}'_{q(i)}(I \otimes \sigma_{q(i)}) &= \bar{x}'_{q(i)}(1 \otimes \sigma_{q(i)}) - \bar{x}'_{q(i)}(0 \otimes \sigma_{q(i)}) - \bar{x}'_{q(i)}(I \otimes \partial\sigma_{q(i)}) \\
&= (x'_{q(i)} + k^{(i)}E^{(i)})(\sigma_{q(i)}) - x'_{q(i)}(\sigma_{q(i)}) \\
&= k^{(i)}E^{(i)}(\sigma_{q(i)}) = k^{(i)}T(y_n, \ldots, y_{q(i-1)})(I \otimes \sigma_{q(i)}),
\end{aligned}$$

then $(y_n, y_q, \ldots, y_{q(i-1)}, x'_{q(i)})$ satisfies the condition (6. 3), therefore $T(y_n, y_q, \ldots, y_{q(i-1)}, \bar{x}'_{q(i)})$ is regarded as an $S. Q.$ map

$$C_r(I \otimes K) \to C_r(N^{(i)}).$$

Finally we have

$$D = T(y_n, y_q, \ldots, y_{q(i-1)}, \bar{x}'_{q(i)}) \cdot I \; : C_r(K) \to C_{r+1}(N^{(i)})$$

with the following property

$$\partial D + D\partial = T(x_n, \ldots, x_{q(i-1)}, x'_{q(i)} + k^{(i)}E^{(i)}) - T(x'_n, \ldots, x'_{q(i-1)}, x'_{q(i)}).$$

The remainder of our proof is due to lemma (7. 2), (8. 1. 4). Namely, we define a map

$$D'_{(i)} = D_1 T(x_n, \ldots, x_{q(i-1)}, x'_{q(i)} + k^{(i)}E^{(i)}) \colon C_r(K) \to C_{r+1}(N^{(i)}),$$

then we have a required chain homotopy

$$D'_{(i)} \circ D(c) \colon T(x_n, \ldots, x_{q(i-1)}, x_{q(i)}) \cong T(x_n, \ldots, x_{q(i-1)}, x'_{q(i)} + k^{(i)}E^{(i)})$$

9. Applications to additive cases

It is obvious from the theorem (8. 3) that there is a one to one correspondence between chain homotopy classes of $S.\,Q.$ maps

$$T: K \to K(\Pi, n)$$

and cohomology classes of $H^n(K; \Pi)$.

Now, we assume that the Eilenberg MacLane invariant is additive, namely $\boldsymbol{k} \in H^{q+1}(K(\Pi, n); G)$ induces the additive internal operation $\boldsymbol{k} \vdash$ [4]. If the space X is a space of loops, it is well known that $\boldsymbol{k}$ is in the image of the suspension map

$$S: H^{q+2}(\Pi, n+1; G) \to H^{q+1}(\Pi, n; G)$$

therefore $\boldsymbol{k}$ is additive (Theorem 16. 2 of III [4]).

THEOREM (9. 1). *Chain homotopy classes of S. Q. maps*

$$T: K \to K(\Pi, n, G, q, \boldsymbol{k})$$

are calculated by elements of pairs of classes

$$[\textit{kernel } (k \vdash) \cap H^n(K; \Pi),\ H^q(K; G)/k \vdash H^{n-1}(K; \Pi)].$$

PROOF. We consider two chain homotopies

$$E_1, E_2: T(x_n) \cong T(x'_n)$$

induced by two chain homotopies between the chain homotopic maps $T(x_n, x_q)$, $T(x'_n, x'_q)$. Then, we have two cochains $w_1, w_2 \in C^{n-1}(K; \Pi)$ defined by

$$w_i = b_n E_i, \qquad i=1, 2.$$

It is obvious that E_i is represented as

$$E_i = D' \circ D(w_i).$$

Here

$$(x_q - x'_q - kE_1) - (x_q - x'_q - kE_2) = kE_2 - kE_1$$

and

$$\begin{aligned}\delta(kE_2 - kE_1) &= kE_2\partial - kE_1\partial \\ &= k(\partial E_1 - T(x_n) + T(x'_n)) - k(\partial E_2 - T(x_n) + T(x'_n)) \\ &= 0,\end{aligned}$$

then $kE_2 - kE_1$ is a q-cocycle of $Z^q(K; G)$.

On the other hand,

$$\delta(w_2 - w_1) = (x_n - x'_n) - (x_n - x'_n) = 0$$

then $w_2 - w_1 = u$ is an $(n-1)$-cocycle of $Z^{n-1}(K; \Pi)$.

Since k is additive, the cohomology class of kE_2-kE_1 is uniqely determind by $k(E_2\circ(E_1)^{-1})$, here

$$E_2\circ(E_1)^{-1}=(D'\circ D(w_2))\circ(D'\circ D(w_1))^{-1}$$
$$=D(w_2)\circ(D(w_1))^{-1}=D(w_2-w_1)$$
$$=D(u)=ST(u),$$

therefore the cohomology class of kE_2-kE_1 is the class determined by $(Sk)\vdash u$.

Conversely, if $x_q-x'_q-kE_1$ is in $B^q(K;G)\cup(Sk)\vdash Z^{n-1}(K;\Pi)$, we may represent $x_q-x'_q-kE_1=\delta v+kST(u)$ for some pair (v,u) of $v\in C^{q-1}(K;G)$ and $u\in Z^{n-1}(K;\Pi)$, then we have a chain homotopy

$$E_2=E_1\circ ST(u):\ T(x_n)\cong T(x'_n)$$

such that $x_q-x'_q-kE_2$ is cohomologous zero. The remainder of our proof is due to Theorem (8. 4).

Generally, we consider the Postnikov invariant $\boldsymbol{k}^{(i)}$, which is the cohomology class of $k^{(i)}\in Z^{q^{(i)}+1}(N^{(i-1)};G^{(i)})$. In this case there exists an injective $S.Q.$ map

$$j:\ K(G^{(i-1)},q^{(i-1)})\to N^{(i-1)}=\mathfrak{P}(N^{(i-2)},G^{(i-1)},q^{(i-1)},k^{(i-1)})$$

then $\boldsymbol{k}^{(i)}$ induces a cohomlogy class

$$j^*\boldsymbol{k}^{(i)}\in H^{q^{(i)}+1}(K(G^{(i-1)},q^{(i-1)});G^{(i)}).$$

Now, we assume that these invariants are additive. If the space X is a space of loops, it is well known that our assumption is satisfied.

THEOREM (9.2). *Consider an S.Q. map*

$$T(x_n,\ldots,x_{q^{(i-2)}}):\ K\to N^{(i-2)}$$

and its prolongations

$$T:\ K\to N^{(i)}=\mathfrak{P}(N^{(i-1)},G^{(i)},q^{(i)},k^{(i)}).$$

Then, chain homotopic classes of S.Q. maps T, $p^{(i)}$-image of which are chain homotopic mutually, are calculated by cosets of

$$H^{q^{(i)}}(K;G)/(j^*K^{(i)})\vdash H^{q^{(i-1)}-1}(K;G^{(i-1)}).$$

PROOF. We consider two chain homotopies

$$E_1^{(i)},E_2^{(i)}\ :T(x_n,\ldots,x_{q^{(i-2)}},x_{q^{(i-1)}})\cong T(x_n,\ldots,x_{q^{(i-2)}},x'_{q^{(i-1)}})$$

induced by two chain homotopies between the chain homotopic maps

$$T(x_n,\ldots,x_{q^{(i-2)}},x_{q^{(i-1)}},x_{q^{(i)}}),\ T(x_n,\ldots,x_{q^{(i-2)}},x_{q^{(i-1)}},x'_{q^{(i)}}).$$

Then we have two cochains $c_1, c_2 \in C^{q^{(i-1)}-1}(K; G^{(i-1)})$ defined by

$$c_j = b_{q^{(i-1)}} E_j^{(i)} \qquad j=1,2.$$

It is obvious that $E^{(i)}$ is represented as

$$E_j^{(i)} = (D_1 T(x_n, \ldots, x_{q^{(i-2)}}, x'_{q^{(i-1)}})) \circ D(c_j).$$

Here

$$(x_{q^{(i)}} - x'_{q^{(i)}} - k^{(i)} E_1^{(i)}) - (x_{q^{(i)}} - x'_{q^{(i)}} - k^{(i)} E_2^{(i)}) = k^{(i)} E_2^{(i)} - k^{(i)} E_1^{(i)},$$

and

$$\delta(k^{(i)} E_2^{(i)} - k^{(i)} E_1^{(i)}) = 0.$$

On the other hand

$$\partial(c_2 - c_1) = (x_{q^{(i-1)}} - x'_{q^{(i-1)}}) - (x_{q^{(i-1)}} - x'_{q^{(i-1)}}) = 0.$$

Since $k^{(i)}$ is additive, the cohomology class of $k^{(i)} E_2^{(i)} - k^{(i)} E_1^{(i)}$ is uniquely determined by $k^{(i)}(E_2^{(i)} \circ (E_1^{(i)})^{-1})$,

$$E_2^{(i)} \circ (E_1^{(i)})^{-1} = D(c_2) \circ (D(c_1))^{-1} = D(c_2 - c_1)$$

then, the cohomology class of $k^{(i)} E_2^{(i)} - k^{(i)} E_1^{(i)}$ is the class determined by $S(j^* k^{(i)}) \vdash (c_2 - c_1)$.

Conversely, if $x_{q^{(i)}} - x'_{q^{(i)}} - k^{(i)} E_1^{(i)}$ is in $B^{q^{(i)}}(K; G^{(i)}) \cup [S(j^* k^{(i)}) \vdash Z^{q^{(i-1)}-1}(K; G^{(i-1)})]$, we may represent $x_{q^{(i)}} - x_{q^{(i)}} - k^{(i)} E_1^{(i)} = \delta v + k^{(i)} j \cdot S \cdot T(u)$ for some pair (v, u) of $v \in C^{q^{(i)}-1}(K; G^{(i)})$ and $u \in Z^{q^{(i-1)}-1}(K; G^{(i-1)})$, then we have a chain homotopy

$$E_2^{(i)} = E_1^{(i)} \circ (S \cdot T(u)) : T(x_n, \ldots, x_{q^{(i-1)}}) \cong T(x_n, \ldots, x'_{q^{(i-1)}})$$

such that $x_{q^{(i)}} - x'_{q^{(i)}} - k^{(i)} E_2^{(i)}$ is cohomologous zero. The remainder of our proof is due to Theorem (8. 6).

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