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Some general properties of Markov processes

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The main purpose of this paper is to study the conservative property and the recurrence of Markov processes on a separable locally compact space, following W. Feller's idea of sojourn sets (Feller [2], [3]) and combining them with Green measures.

In §1 we shall give the definition of Markov processes and introduce several notions useful for later considerations. Here we took much from the lecture of Professor H. P. McKean at Kyoto University in 1957-8. Next, we shall derive some results from the hypothesis (H. 1) concerning Green operators which we have introduced at the beginning of §2; we shall impose this hypothesis throughout the subsequent sections. §3 is concerned with the conservative property of Markov processes. Here we shall establish a theorem which characterizes the conservative property using both sojourn sets and Green measures, and we shall derive Feller's theorem concerning purely discontinuous Markov processes (c f. Feller [2; Theorem 7]) as its special case.

In §4 we shall characterize the recurrence either by Green measures or by sojourn sets, putting a new assumption (H. 2) on the continuity of harmonic measures. These results generalize some of the results by J. L. Doob [1] and by W. Feller [3]. In the last section we shall show some applications of the results of §4.

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§1. Definitions of Markov processes and fundamental notions.

Let E be a separable locally compact space. Adding an extra point ∞ to E as an isolated one, we shall get a separable locally compact space $\bar{E}=E+\infty$.

We denote a measurable function (sample path) from $[0,+\infty]$ into $\bar{E}$ by w and its position at time t by w_t or $x_t(w)$. Next, let W be the totality of the w's which satisfy the following conditions:

(W. 1) Put

(1. 1) $$\begin{aligned} \sigma_\infty(w) &= \inf(t; x_t(w)=\infty) && \text{if } x_t=\infty \text{ for some } t\geqq 0, \\ &= \infty && \text{otherwise.} \end{aligned}$$

Then $x_t(w)=\infty$ holds for every $t\geqq\sigma_\infty(w)$. Especially we shall define $x_\infty(w)=\infty$.

$(W. 2)$ $x_t(w)$ is right continuous and, for $t<\sigma_\infty$, its possible discontinuity is of the first kind.[1]

The condition $(W. 1)$ shows that every path has no return from ∞ to E.

Now, given an open or closed A of $\bar{E}$, the *passage time* σ_A for A is defined by

$$\begin{aligned}(1.\ 2) \qquad \sigma_A(w) &= \inf(t;\ x_t(w)\in A) \qquad \text{if } x_t(w)\in A \text{ for some } t\geqq 0,\\ &=\infty \qquad\qquad\qquad\qquad \text{otherwise.}\end{aligned}$$

Further we denote by $\tilde{\mathfrak{B}}$, the smallest Borel field containing the sets $(w; x_t(w)\in A)$, where t is an arbitrary fixed time and A is any Borel set of $\bar{E}$. Then we see from $(W. 2)$ that, if A is open, σ_A is a measurable function from $[W, \tilde{\mathfrak{B}}]$ into $[0, +\infty]$. But we are not sure that, if A is closed, σ_A is measurable with respect to $\tilde{\mathfrak{B}}$. Hence we need extend $\tilde{\mathfrak{B}}$ to the Borel field $\mathfrak{B}$ generated by $\tilde{\mathfrak{B}}$ and by the sets $[w;\ \sigma_F(w)>t]$, where F is any closed set of $\bar{E}$ and t runs over $[0, +\infty]$. It is clear that every passage time σ_A is a measurable function (random time) from $[W, \mathfrak{B}]$ into $[0, +\infty]$. Next, given a random time σ, we shall define the *stopped path* w_σ^- and the *shifted path* w_σ^+ as follows:

$$(1.\ 3) \qquad (w_\sigma^-)_t = x_{min(\sigma, t)}(w), \quad (w_\sigma^+)_t = x_{\sigma+t}(w).$$

It is easily shown from the definitions of W and $\mathfrak{B}$ that $w_\sigma^\pm\in W$, $(w;\ w_\sigma^\pm\in B)\in\mathfrak{B}$ and consequently $\mathfrak{B}_\sigma\subset\mathfrak{B}$, where B is any set belonging to $\mathfrak{B}$ and $\mathfrak{B}_\sigma$ is the totality of the sets $(w;\ w_\sigma^-\in B)$, i.e. the Borel field generated by the sets $(w;\ (w_\sigma^-)_t\in A)$ and $(w;\ \sigma_F(w_\sigma^-)<t)$.

We shall now introduce the notion of a Markov time.

DEFINITION 1. 1. A random time $\sigma(w)$ is called *a Markov time* if $(w;\ \sigma(w)\geqq t)\in\mathfrak{B}_t$ for every $t\geqq 0$.

We shall mention two lemmas necessary for later considerations without proof.[2]

LEMMA 1. 1. *Every Markov time $\sigma(w)$ is measurable with respect to* $\mathfrak{B}_{\sigma+} = \bigcap_{\varepsilon>0}\mathfrak{B}_{\varepsilon+\sigma}$.

LEMMA 1. 2. *Every passage time $\sigma_A(w)$ is a Markov time. Especially, if A is closed, σ_A is not only measurable with respect to $\mathfrak{B}_{\sigma_A+}$, but also to $\mathfrak{B}_{\sigma_A}$.*

Next we denote by P a system $\{P_x(\cdot);\ x\in\bar{E}\}$ of probability measures on $(W, \mathfrak{B})$ which satisfy the following conditions:

$(P. 1)$ $P_x(B)$ is measurable as a function of x for every fixed $B\in\mathfrak{B}$.

$(P. 2)$ $P_x(x_0(w)=x)=1$ for every $x\in\bar{E}$.

$(P. 3)$ $\mathfrak{B}$ coincides with $\tilde{\mathfrak{B}}$ up to P_x-probability 0 for each x.[3]

1) We don't assume the existence of the left limit $\lim_{t\uparrow\sigma_\infty} x_t(w)$.

2) cf. K. Itô and H. P. McKean [6].

3) This means that the completion of $\tilde{\mathfrak{B}}$ with respect to P_x includes $\mathfrak{B}$. For example, this is true under the hopothesis (A) in Hunt [4].

(P. 4) (MARKOV PROPERTY) For every $x \in \bar{E}$, $t \geqq 0$ and any bounded Borel function $f(w)$ on $(W, \mathfrak{B})$,

(1. 4) $E_x(f(w_t^+) | \mathfrak{B}_t) = E_{x_t}(f(w))$ with P_x-probability 1.

A combination $(W, \mathfrak{B}, P)$ (or simply x_t) is called *a Markov process on E*.

Finally, we shall define several notions concerning Markov processes on E.

DEFINITION 1.2. We consider two points x and y of $\bar{E}$. Then if $P_x(\sigma_V < +\infty) > 0$ for every open set $V \ni y$, y is said to be *accessible from* x and we use the notation $x \rightarrow y$. If $x \rightarrow y$ and $y \rightarrow x$, we say that x *and* y *have communication.*

DEFINITION 1.3. If P_x (w; $\sigma_\infty < +\infty) = 0$, then we say that *the process starting at x (or briefly x) is conservative on E.*

DEFINITION 1.4. x is called a *recurrent point* if $P_x\{\sigma_V(w^+_{\sigma_{U^c}}) < +\infty \mid \sigma_{U^c} < +\infty\} = 1$ holds for any open sets U and $V (\bar{V} \subset U)$ containing x.

DEFINITION 1. 5. x is called a *trap* if

$$P_x\{x_t(w) = x \textit{ for every } t \geqq 0\} = 1.$$

According to this terminology, ∞ is a trap.

DEFINITION 1.6. An open or closed set S containing x is called a *sojourn set with the center x*, if

$$P_x(\sigma_{S^c} = \sigma_\infty) > 0,\text{[1)]}$$

or equivalently, if

$$P_x(\sigma_{E-S} = \infty) > 0\,.$$

§2 Hypothesis (H. 1) and its results.

First we introduce several notations;

$$P(t, x, \cdot) = P_x(x_t \in \cdot),$$

$$P_A(t, x, \cdot) = P_x(x_t \in \cdot, \sigma_{A^c} > t) \qquad \text{for any open or closed set } A,$$

$$T_t f(x) = E_x(f(x_t)) = \int_E f(y) P(t, x, dy)\,,$$

$$G_\alpha f(x) = E_x\Big(\int_0^\infty e^{-\alpha t} f(x_t) dt\Big) = \int_0^\infty e^{-\alpha t} T_t f(x) dt \quad \text{for } \alpha > 0\,.$$

Here $f(x)$ is a Borel function on $\bar{E}$.

We now denote by $\mathfrak{C}$ the totality of functions which are continuous, bounded on E and equal to 0 at ∞. In the sequel we always assume that our process x_t satisfies the next hypothesis:

(H. 1) $\qquad G_\alpha$ maps $\mathfrak{C}$ into $\mathfrak{C}$ for every $\alpha > 0$.

1) The complement of a set is always considered with respect to $\bar{E}$.

This hypothesis is a little weaker than the continuity condition concerning T_t which is usually assumed[1]:

(H. 1)′ $\qquad T_t$ maps $\mathfrak{C}$ into $\mathfrak{C}$ for every $t \geqq 0$.

We now prove some theorems under (H. 1).

THEOREM 2. 1. (STRONG MARKOV PROPERTY) *If x_t satisfies* (H. 1), *then for every Markov time σ and for any bounded Borel function $f(w)$ on* $(W, \mathfrak{B})$,

$$E_x(f(w_\sigma^+) \mid \mathfrak{B}_{\sigma+}) = E_{x_\sigma}(f(w)) \tag{2. 1}$$

holds with P_x-probability 1.[2]

Proof. Since ∞ is a trap according to (P. 3), it suffices to show that for every $f \epsilon \mathfrak{C}$ and any $B \epsilon \mathfrak{B}_{\sigma+}$,

$$E_x(f(x_{\sigma+t});\ B) = E_t(E_{x_\sigma}(f(x_t));\ B).^{3)} \tag{2. 2}$$

Making use of (H. 1) and performing the same calculation as K. Itô [5; p. 15], we see that for any $B \epsilon \mathfrak{B}_{\sigma+}$

$$E_x\Big(\int_0^\infty e^{-\alpha t} f(x_{\sigma+t}) dt;\ B\Big) = E_x\Big(E_{x_\sigma}\Big(\int_0^\infty e^{-\alpha t} f(x_t) dt\Big); B\Big). \tag{2. 3}$$

Hence putting

$$g(t) = E_x(f(x_{\sigma+t});\ B),\quad h(t) = E_x(E_{x_\sigma}(f(x_t));\ B), \tag{2. 4}$$

we have

$$\int_0^\infty e^{-\alpha t} g(t) dt = \int_0^\infty e^{-\alpha t} h(t) dt\,.$$

According to the right continuity of the path, $g(t)$ and $h(t)$ are right continuous, so that it follows from the uniqueness of Laplace transform that for every $t \geqq 0$

$$g(t) = h(t),$$

which is what we wanted to show.

THEOREM 2. 2. *The accessible relation is transitive. Strictly speaking, if $x \to y$ and $y \to z$, then $x \to z$.*

To prove this we shall first show

LEMMA 2. 1. *Suppose that $x \to y$. Then given any open set $V \ni y$, there exist a $k > 0$ and an open set $U \ni x$ such that for every $\xi \epsilon \bar{U}$*[4]

$$E_\xi(e^{-\sigma_V}) \geqq k\,. \tag{2. 5}$$

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1) G. Hunt [4, p. 360] has proved that (H. 1)′ is equivalent to (H. 1) under some conditions.

2) K. Ito [5] proved this fact under (H, 1)′.

3) $E_x(f(w);\ B) = \int_B f(w)\ P_x(dw)$ for $B \epsilon \mathfrak{B}$ and a Borel function $f(w)$ on $(W, \mathfrak{B})$.

4) $\bar{U}$ means the closure of U, i. e. the smallest closed set containing U.

Proof. We choose an open set $V'\ni y$ whose closure is contained in V. Since $x\rightarrow y$, we have

$$E_{\xi}(e^{-\sigma_{V'}})>0. \tag{2. 6}$$

According to Urysohn's Lemma there exists a continuous function $f\geqq 0$ which is equal to 1 on $\overline{V}'$ and to 0 on V^c. Now we shall show that

$$E_x(f(x_{t_0}))>0 \text{ for some } t_0\geqq 0. \tag{2. 7}$$

For this purpose we assume that $E_x(f(x_t))=0$ for every $t\geqq 0$. Then, since f is non-negative and equals to 1 on $\overline{V}'$,

$$P_x(x_t\in\overline{V}')=0 \quad \text{for every} \quad t\geqq 0.$$

Hence it follows from the right continuity of the path that

$$P_x(x_t\in\overline{V}' \textit{ for some } t\geqq 0)=0,$$

which implies $P_x(\sigma_{V'}<+\infty)=0$. This contradicts with (2. 6). Thus (2. 7) has been proved. Hence again using the right continuity of the path, we have

$$E_x(f(x_t))>0 \quad \text{for every} \quad t\in(t_0,t_0+\varepsilon), \tag{2. 8}$$

where ε is a certain positive constant. From this we get

$$E_x\left(\int_0^{\infty} e^{-t}f(x_t)dt\right)=\int_0^{\infty} e^{-t}E_x(f(x_t))dt>0. \tag{2. 9}$$

Therefore making use of (H. 1), there exist a $k>0$ and an open set $U\ni x$ such that for every $\xi\in\overline{U}$

$$E_{\xi}\left(\int_0^{\infty} e^{-t}f(x_t)dt\right)\geqq k. \tag{2. 10}$$

On the other hand since $f=0$ on V^c, we get for any $\xi\in E$

$$E_{\xi}\left(\int_0^{\infty} e^{-t}f(x_t)dt\right)\leqq E_{\xi}\left(\int_{\sigma_V}^{\infty} e^{-t}dt\right)=E_{\xi}(e^{-\sigma_V}). \tag{2. 11}$$

Thus the lemma was completely proved.

Proof of THEOREM 2.2. We take an arbitrary open set $V\ni z$. Since $y\rightarrow z$, replacing x by y and y by z in the previous lemma, we have

$$E_{\eta}(e^{-\sigma_V})\geqq k \quad \text{for every} \quad \eta\in\overline{U}, \tag{1. 12}$$

where U is an open set containing y.

We now put $\sigma'_V(w)=\sigma_U(w)+\sigma_V(w^+_{\sigma_U})$. Since it is evident that $\sigma_V\leqq\sigma'_V$, we have

$$E_x(e^{-\sigma_V})\geqq E_x(e^{-\sigma'_V}) \tag{2. 13}$$

$$=E_x\{e^{-\sigma_U-\sigma_V(w^+_{\sigma_U})}\}$$

$$=E_x\{e^{-\sigma_U}E_{x_{\sigma_U}}(e^{-\sigma_V})\}$$

$$\geqq kE_x(e^{-\sigma_U})>0 .$$

Here we have used the fact that $x_{\sigma_U}\in\overline{U}$ and that $E_x(e^{-\sigma_U})>0$.

Finally, we shall give a lemma which is useful for §4.

LEMMA 2. 2. *If x is not a trap, there exist a $k>0$ and an open set $U\ni x$ such that*

$$E_\xi(\sigma_{U^c})\leqq k \quad \text{for every} \quad \xi\in\overline{U}.$$

This lemma is essentially due to E. B. Dynkin. The proof of K. Itô [5] under the assumption (H. 1)′ is also available to our case under (H. 1).

§3. Conservative property.

In this section we assume also (H. 1).

THEOREM 3. 1. *The following three conditions are equivalent to each other.*

(1) *The process starting from $x\in E$ is conservative, that is, $P(t,x,E)=1$ for every $t\geqq 0$.*

(2) *For every sojourn set S with the center x,*

$$E_x(\sigma_{S^c})=\int_0^\infty P_S(t,x,S)dt=\infty. \tag{3. 1}$$

(3) *For every open sojourn set S with the center x,*

$$E_x\left(\int_0^\infty \chi_S(x_t)dt\right)=\int_0^\infty P(t,x,S)dt=\infty,^{1)} \tag{3. 2}$$

where χ_S is the indicator function of S.

Proof. First, it is easily shown that the condition (1) impies the condition (2). In fact, if (1) holds, then $P_x(\sigma_\infty=\infty)=1$. Hence by the definition of sojourn sets, $P_x(\sigma_{S^c}=\infty)>0$. From this we get

$$E_x(\sigma_{S^c})=\infty .$$

Next it is evident that the second condition implies the third condition. Finally, we shall show that the third condition implies the first condition. For this purpose, suppose that the condition (1) does not hold, namely that $E_x(e^{-\sigma_\infty})>k$ for some $k>0$.

Let χ_E be the indicator function of E. Then we have

1) $G(x,\cdot)=\int_0^\infty P(t,x,\cdot)dt$ is called *Green measure*, which may take $+\infty$.

$$E_x\left(\int_0^\infty e^{-t}\chi_E\ (x_t)dt\right)=E_x\left(\int_0^{\sigma_\infty} e^{-t}dt\right)=1-E_x(e^{-\sigma_\infty}).$$

But, since χ_E belongs to $\mathfrak{C}$, $E_x(e^{-\sigma_\infty})$ is continuous in x. Hence we see that $U=\{y;\ y\in E,\ E_y(e^{-\sigma_\infty})>k\}$ is an open set containing x. Thus it is enough to show that (i) U is a sojourn set with the center x and that (ii)

$$\int_0^\infty P(t,x,U)dt<+\infty\ . \tag{3. 3}$$

(i) Suppose that U is not a sojourn set with the center x, that is, $P_x(\sigma_{U^c}=\sigma_\infty)=0$. Then we have

$$P_x(\sigma_{U^c}=\sigma_{E-U}<+\infty)=1\,,$$

so that noting that, U is open and that every sample path is right continuous, it is easily shown that

$$x_{\sigma_{U^c}}=x_{\sigma_{E-U}}\in E-U \quad \text{with} \quad P_x\text{-probability 1.}$$

Hence from the definition of U we get

$$E_{x_{\sigma_{U^c}}}(e^{-\sigma_\infty})\leqq k \quad \text{with} \quad P_x\text{-probability 1.}$$

We now calculate $E_x(e^{-\sigma_\infty})$.

$$k<E_x(e^{-\sigma_\infty})=E_x\{e^{-\sigma_{U^c}(w)-\sigma_\infty(w^+_{\sigma_{U^c}})}\}$$

$$=E_x\{e^{-\sigma_{U^c}(w)}E_{x_{\sigma_{U^c}}}(e^{-\sigma_\infty(w)})\}\leqq k\,.$$

This is a contradiction.

(ii) We have for any y of U

$$k<E_y(e^{-\sigma_\infty})\leqq P_y(\sigma_\infty\leqq t)+e^{-t},$$

so that, for a large t_0

$$0<k'=k-e^{-t_0}<P_y(\sigma_\infty\leqq t_0)$$

holds whenever $y\in U$. Therefore we get for every $y\in U$

$$P(t_0,y,E)=1-P_y(\sigma_\infty\leqq t_0)<1-k'.$$

Hence it is evident that a Borel set $A=\{y;\ y\in E,\ P(t_0,y,E)<1-k'\}$ contains U. Thus it is enough to show that

$$\int_0^\infty P(t,x,A)dt<+\infty. \tag{3. 4}$$

To do this, we shall prove that for any $n\geqq 0$

$$S_n(t)=\sum_{i=0}^{n}P(it_0+t,x,A)\leqq\sum_{i=0}^{n}(1-k')^i \tag{3. 5}$$

holds independently of $t\geqq 0$. Noting that ∞ is a trap, we have by the definition of A

$$P((n+1)t_0+t,x,E)=\int_A P(nt_0+t,x,dy)P(t_0,y,E)+\int_{E-A} P(nt_0+t,x,dy)P(t_0,y,E)$$
$$\leqq(1-k')P(nt_0+t,x,A)+P(nt_0+t,x,E-A).$$

Consequently,

$$P((n+1)t_0+t,x,E)+P(nt_0+t,x,A)$$
$$\leqq(1-k')P(nt_0+t,x,A)+P(nt_0+t,x,E).$$

Repeating the same calculation as above, we get

$$S_{n+1}(t)\leqq P((n+1)t_0+t,x,E)+\sum_{i=1}^{n}P(it_0+t,x,A)$$
$$\leqq(1-k')\sum_{i=0}^{n}P(it_0+t,x,A)+P(t_0,x,E)$$
$$\leqq(1-k')S_n(t)+1.$$

Since $S_0(t)\leqq 1$, (3. 5) is obtained by induction. Now making use of (3. 5) to calculate the left side of (3. 4), we have

$$\int_0^\infty P(t,x,A)dt=\int_0^{t_0}\sum_{i=0}^{\infty}P(it_0+t,x,A)dt$$
$$\leqq\frac{t_0}{k'}<+\infty.$$

This completes the proof of Theorem 3. 1.

THEOREM 3. 2. *If x is conservative on E and if $x\rightarrow y$, then y is also conservative on E.*

Proof. Suppose that y is not conservative on E. Then since $E_\eta(e^{-\sigma_\infty})$ is continuous with respect to η, it is greater than some positive k on the closure of an open set V containing y.

We now put $\sigma'_\infty=\sigma_V+\sigma_\infty(w^+_{\sigma_V})$. Noting that $\sigma'_\infty\geqq\sigma_\infty$, we have

$$E_x(e^{-\sigma_\infty})\geqq E_x(e^{-\sigma'_\infty}) \tag{3. 6}$$
$$=E_x(e^{-\sigma_V}E_{x_{\sigma_V}}(e^{-\sigma_\infty}))$$
$$\geqq k\cdot E_x(e^{-\sigma_V})>0.$$

To get the last inequality, we have used the relation $x\rightarrow y$. Therefore it turns out that x is not conservative on E, contrary to the assumption.

Next we apply Theorem 3.1 to the Markov processes whose sample path has only jumps with probability 1. Then we can obtain Feller's theorem concerning so called purely discontinuous Markov processes. But here, for short, we consider the case when E is a denumerable space, i.e. $E=\{1, 2, 3\cdots\cdots\}$. In this case our processes defined in §1 automatically have the following properties.

1. Every path w is a right continuous step function for $t<\sigma_\infty$.

2. Since the hypothesis (H. 1) is trivially satisfied, the strong Markov property holds for every Markov time (c f. Theorem 2. 1).

3. Define the first jumping time by $\sigma_1(w)=\sup\{t: x_0(w)=x_s(w)$ *for every* $s\leqq t\}$. Since $\sigma_1(w)$ is a Markov time, it follows from 2 that

$$P_x(\sigma_1(w)\geqq t)=e^{-p(x)t}, \tag{3. 7}$$

where $p(x)$ is a non-negative number which cannot take $+\infty$ (from the right continuity of path and (P. 2)). If $p(x)=0$, x is a trap.

4. If x is not a trap, put

$$\Pi(x,y)=P_x(x_{\sigma_1}=y). \tag{3. 8}$$

If x is a trap, put

$$\Pi(x,y)=\delta_{x,y}\text{[1)}$$

Then

$$\Pi=\left(\Pi(x,y),\ \begin{matrix} x\downarrow 1.\ 2.\ \cdots\cdots,\ \infty \\ y\rightarrow 1.\ 2.\ \cdots\cdots,\ \infty\end{matrix}\right)$$

is a strictly stochastic matrix on $\bar{E}$.

Further we denote by $\Pi^n(x,y)$ the element of the matrix $\Pi^n=\underbrace{\Pi\cdot\Pi\cdots\cdots\Pi}_{n}$ ($n\geqq 1$). Π^0 is, by definition, the identity matrix on $\bar{E}$.

Finally we denote by Π_A the restriction of Π to a set A which is defined by

$$\Pi_A(x,y)=\Pi(x,y) \quad \text{if } x,y\in A, \tag{3. 9}$$
$$=0 \qquad \text{otherwise.}$$

We understand $\Pi_A^n(x,y)$, Π_A^n in the same way as for Π.

We shall now characterize the conservative property by means of p and Π. These quantities, however, are determined by the generator of the process, so that we can say that the conservative property is characteaized by the generator.

THEOREM 3. 3. (W. FELLER) *The following three conditions are equivalent to each other.*

(1) *The process starting at x is conservative, that is,*

$$P(t,x,E)=1 \quad \text{for every} \quad t\geqq 0.$$

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1) $\delta_{x,y}=\begin{cases}1 & \text{if } x=y,\\ 0 & \text{otherwise.}\end{cases}$

(2)

$$\sum_{y\in E}\Pi^n(x,y)=1 \qquad (n=0,1,2,\cdots\cdots), \tag{3.10}$$

and

$$\sum_{n=0}^{\infty}\sum_{y\in A}\Pi_A^n(x,y)\frac{1}{p(y)}=\infty, \tag{3.11}$$

for A satisfying

$$\lim_{n\to\infty}\sum_{y\in A}\Pi_A^n(x,y)>0. \tag{3.12}$$

(3) (3.10) *holds and*

$$\sum_{n=0}^{\infty}\sum_{y\in A}\Pi^n(x,y)\frac{1}{p(y)}=\infty, \tag{3.13}$$

for A satisfying (3.12).

To prove this we shall prepare two lemmas.

LEMMA 3.1. *Suppose that* (3.10) *holds. Then A is a sojourn set with the center x if and only if* (3.12) *holds.*

Proof. It is enough to show that

$$\lim_{n\to\infty}\sum_{y\in A}\Pi_A^n(x,y)=P_x(\sigma_{A^c}=\sigma_\infty).$$

For this we define jumping times:

$$\begin{aligned}
\sigma_0(w) &\equiv 0 && \text{(0-th jumping time)},\\
\sigma_1(w) &= \text{first jumping time},\\
\sigma_2(w) &= \sigma_1(w)+\sigma_1(w^+_{\sigma_1}) && \text{(second jumping time)},\\
&\vdots\\
\sigma_n(w) &= \sigma_{n-1}(w)+\sigma_1(w^+_{\sigma_{n-1}}) && \text{(n-th jumping time)}.
\end{aligned}$$

Here if $\sigma_k=\infty$, we put $\sigma_{k+1}=\sigma_{k+2}\cdots=\infty$.

First we shall prove that the condition (3.10) implies

$$P_x(\sigma_n<\infty,\ x_{\sigma_n}=\infty)=0. \tag{3.14}$$

In fact, if (3.14) is not true, we have

$$\begin{aligned}
0&<P_x(\sigma_n<\infty,\ x_{\sigma_n}=\infty)\\
&=P_x(x_{\sigma_1}\in E,\ x_{\sigma_2}\in E,\ \cdots,\ x_{\sigma_{n-1}}\in E,\ x_{\sigma_n}=\infty)\\
&=\sum_{y\in E}\Pi(x,y)P_y(x_{\sigma_1}\in E,\ \cdots,\ x_{\sigma_{n-2}}\in E,\ x_{\sigma_{n-1}}=\infty)\\
&=\Pi^n(x,\infty).
\end{aligned}$$

This contradicts (3.10).

Next it follows from (3.14) that if $\sigma_{n-1}<\infty$ and if $\sigma_n=\infty$, then $x_{\sigma_{n-1}}$ are traps with P_x-probability 1. Hence we have

$$(3.15)\qquad P_x(\sigma_{A^c}=\sigma_\infty)=\lim_{n\to\infty}\{\sum_{k=1}^{n}P_x(x_{\sigma_i}\epsilon A,\ i<k;\ \sigma_{k-1}<\infty,\ \sigma_k=\infty)$$
$$+P_x(x_{\sigma_i}\epsilon A,\ i\leqq n;\ \sigma_n<\infty)\}.$$

To calculate the right side of the above equality we shall denote by T the totality of traps contained in A. Then we have

$$P_x(x_{\sigma_i}\epsilon A,\ i<k;\ \sigma_{k-1}<\infty,\ \sigma_k=\infty)$$
$$=\sum_{z\in T}\sum_{y\in A-T}\Pi_A^{k-2}(x,y)\Pi(y,z)$$
$$=\sum_{z\in T}(\sum_{y\in A}\Pi_A^{k-2}(x,y)\Pi(y,z)-\sum_{y\in T}\Pi_A^{k-2}(x,y)\Pi(y,z))$$
$$=\sum_{z\in T}(\Pi_A^{k-1}(x,z)-\Pi_A^{k-2}(x,z)),$$
$$P_x(x_{\sigma_i}\epsilon A,i\leqq n;\sigma_n<\infty)=\sum_{y\in A-T}\Pi_A^{n-1}(x,y).$$

Therefore we have

$$\sum_{k=1}^{n}P_x(x_{\sigma_i}\epsilon A,\ i<k;\ \sigma_{k-1}<\infty,\sigma_k=\infty)+P_x(x_{\sigma_i}\epsilon A,\ i\leqq n;\ \sigma_n<\infty)$$
$$=\sum_{k=1}^{n}\sum_{z\in T}(\Pi_A^{k-1}(x,z)-\Pi_A^{k-2}(x,z))+\sum_{y\in A-T}\Pi_A^{n-1}(x,y)$$
$$=\sum_{y\in T}\Pi_A^{n-1}(x,y)+\sum_{y\ni A-T}\Pi_A^{n-1}(x,y)$$
$$=\sum_{A}\Pi_A^{n-1}(x,y).$$

Thus Lemma 3.1 was proved.

Next we shall introduce several notations:

$$P=\begin{pmatrix}p(1) & & 0\\ & p(2) & \\ 0 & & \ddots\end{pmatrix},\quad I=\text{identity matrix},$$

$$P_A(t)=\left(P_A(t,x,y),\ \begin{matrix}x\downarrow 1,2,3,\cdots,\infty\\ y\to 1,2,3,\cdots,\infty\end{matrix}\right),$$

$$G_A(\alpha)=\left(G_A(\alpha,x,y),\ \begin{matrix}x\downarrow 1,2,3,\cdots,\infty\\ y\to 1,2,3,\cdots,\infty\end{matrix}\right),$$

where $$G_A(\alpha,x,y)=\int_0^\infty e^{-\alpha t}P_A(t,x,y)dt.$$

LEMMA 3.2 *For any set A and $\alpha\geqq 0$,*

$$(3.16)\qquad G_A(\alpha)=\sum_{n=0}^{\infty}((\alpha I+P)^{-1}P\cdot\Pi_A)^n\cdot(\alpha I+P)^{-1}.$$

If $\alpha=0$, the both sides of (3.16) *may take* $+\infty$.

Proof. Define

$$P_A^{(n)}(t,x,y)=P_x(x_t=y,\ \sigma_n\leqq t<\sigma_{n+1}),$$

then we can see that it is equal to

$$\left(\int_0^{t_1}e^{-Pt_1}P\Pi_A dt_1\int_0^{t-t_1}e^{-Pt_2}P\Pi_A dt_2\cdots\int_0^{t-t_1-t_2-\cdots-t_{n-1}}e^{-Pt_n}dt_n\right)(x,y).$$

Hence we have

$$G_A^{(n)}(\alpha,x,y)\equiv\int_0^\infty e^{-\alpha t}P_A^{(n)}(t,x,y)dt=((\alpha I+P)^{-1}P\Pi_A)^n(\alpha I+P)^{-1}(x,y).$$

Noting that $P_A(t,x,y)=\sum_{n=0}^{\infty}P_A^{(n)}(t,x,y)$, (3.16) is obtained immediately.

Proof of THEOREM 3.3. Summing up Theorem 3.1, Lemma 3.1 and Lemma 3.2, it is evident that the condition (3) implies the condition (1). Hence it suffices to show that the condition (1) implies (3.10). But this is directly derived by the definition of Π.

§4. Recurrence

In this section we shall assume the following hypothesis (H. 2) besides (H. 1):

(H. 2) Given any closed set F for every $f\in\mathfrak{C}$,

$$E_x(f(x_{\sigma_F}))=\int_A f(y)h_F(x,dy) \tag{4.1}$$

belongs to $\mathfrak{C}$.

Here $h_F(x,\cdot)$ is the distribution of x_{σ_F} which we call the harmonic measure over F ineuced by the process x_t, since this measure is exactly the function-theoretical harmonic measure in case x_t is the two-dimensional Brownian motion. Using this notation, we have

$$P_x(\sigma_F<+\infty)=h_F(x,F). \tag{4.2}$$

THEOREM 4.1. *x is a recurrent point if and only if*

$$\int_0^\infty P(t,x,U)dt=\infty \tag{4.3}$$

for every open set U containing x.

Proof. (i) First we shall show that if x is a recurrent point, then (4.3) will hold. Suppose in the contrary that for some open set $U\ni x$

$$\int_0^\infty P(t,x,U)dt<\infty. \tag{4.4}$$

The condition (4. 4) implies $E_x(\sigma_{U^c})<\infty$, because

$$\int_0^\infty P(t,x,U)dt=E_x\Big(\int_0^\infty \chi_U(x_t)dt\Big)$$

$$\leqq E_x\Big(\int_0^{\sigma_{U^c}} \chi_U(x_t)dt\Big)=E_x(\sigma_{U^c}),$$

where χ_U is the indicator function of U.

Let f be a continuous function which equals 1 at x, 0 on U^c and lies between 0 and 1 elesewhere. Then we have

$$\lim_{\alpha\to 0} G_\alpha f(x)=G_{0+}f(x)\leqq\int_0^\infty P(t,x,U)dt<+\infty.$$

Therefore, given any small $\varepsilon>0$, there exists an α_0 such that

$$G_{\alpha_0}f(x)>G_{0+}f(x)-\varepsilon.$$

Hence using (H. 1), we get

$$G_{\alpha_0}f(\xi)>G_{+0}f(x)-\varepsilon,$$

whenever ξ runs over a certain open set V containing x such that $\overline{V}\subset U$. But since $G_\alpha f(\xi)$ is monotone non-increasing as a function of α for any fixed ξ, we have

(4.5) $$G_\alpha f(\xi)\geqq G_{+0}f(x)-\varepsilon \quad \text{for every } \xi\in\overline{V} \text{ and for every } 0\leqq\alpha\leqq\alpha_0.$$

On the other hand it follows from the definition of recurrence and from $E_x(\sigma_{U^c})<\infty$ that

(4. 6) $$P_x(\sigma_{U^c}+\sigma_V(w^+_{\sigma_{U^c}})<+\infty)=1.$$

For short we denote σ_{U^c} by σ_1 and $\sigma_{U^c}+\sigma_V(w^+_{\sigma_{U^c}})$ by σ_2, repectively. Then according to (4. 6),

(4. 7) $$\lim_{\alpha\to 0} E_x(e^{-\alpha\sigma_2})=P_x(\sigma_2<+\infty)=1.$$

We now calculate $G_\alpha f(x)$ for $0<\alpha\leqq\alpha_0$.

$$G_{+0}f(x)\geqq G_\alpha f(x)=E_x\Big(\int_0^\infty e^{-\alpha t}f(x_t)dt\Big)$$

$$\geqq E_x\Big(\int_0^{\sigma_1} e^{-\alpha t}f(x_t)dt\Big)+E_x(e^{-\alpha\sigma_2}G_\alpha f(x_{\sigma_2}))$$

$$\geqq E_x\Big(\int_0^{\sigma_1} e^{-\alpha t}f(x_t)dt\Big)+(G_{+0}f(x)-\varepsilon)E_x(e^{-\alpha\sigma_2}).$$

Hence we have

$$(4.\ 8)\qquad G_{+0}f(x)\geqq\frac{E_x\left(\int_0^{\sigma_1}e^{-\alpha t}f(x_t)dt\right)-\varepsilon\cdot E_x(e^{-\alpha\sigma_2})^{1)}}{1-E_x(e^{-\alpha\sigma_2})}.$$

As α tends to 0, the right side of (4. 8) goes to ∞. This is a contradiction.

(ii) Suppose that x is not a recurrent point. According to Lemma 2. 2 and the definition of recurrence, we can choose some constant $k>0$ and two open sets U, U' $(\bar{U}'\subset U)$ containing x such that

$$(4.\ 9)\qquad E_\xi(\sigma_{U^c})\leqq k \quad \text{for every} \quad \xi\in\bar{U}',$$

and that

$$(4.\ 10)\qquad P_x(\sigma_{U^c}+\sigma_{\bar{U}'}(w^+_{\sigma_{U^c}})<+\infty)=\int_{U^c}h_{U^c}(x,\,dy)h_{\bar{U}'}(y,\bar{U}')<1.$$

Using (H. 2), it is easily shown that the right side of (4. 10) is continuous in x, so that, for a certain $\varepsilon>0$ and for some open neighbourhood $V(\bar{V}\subset U')$ of x we have

$$(4.\ 11)\qquad P_\xi(\sigma_{U^c}+\sigma_{\bar{U}'}(w^+_{\sigma_{U^c}})<\infty)\leqq 1-\varepsilon,$$

whenever ξ runs over $\bar{V}$. Hence noting that

$$\sigma_{U^c}+\sigma_{\bar{U}'}(w^+_{\sigma_{U^c}})\leqq\sigma_{U^c}+\sigma_V(w^+_{\sigma_{U^c}}),$$

we get from (4. 11)

$$(4.\ 12)\qquad P_\xi(\sigma_{U^c}+\sigma_V(w^+_{\sigma_{U^c}})<+\infty)\leqq 1-\varepsilon \qquad \text{for every } \xi\in\bar{V}.$$

We now define

$$(4.\ 13)\qquad \begin{array}{ll} \sigma_0(w)\equiv 0,\ \sigma_1(w)=\sigma_{U^c}(w), & \sigma_2(w)=\sigma_1(w)+\sigma_V(w^+_{\sigma_1}), \\ \vdots & \vdots \\ \sigma_{2n-1}(w)=\sigma_{2n-2}(w)+\sigma_{U^c}(w^+_{\sigma_{2n-2}}), & \sigma_{2n}(w)=\sigma_{2n-1}(w)+\sigma_V(w^+_{\sigma_{2n-1}}); \end{array}$$

if $\sigma_i=\infty$, then we put $\sigma_{i+1}=\sigma_{i+2}=\cdots=\infty$.

According to (4. 12) we get

$$P_x(\sigma_{2n}<+\infty)\leqq(1-\varepsilon)^n.$$

Using this and (4. 9), we shall calculate $G_\alpha\chi_V(x)$.

$$G_{\alpha V}\chi(x)=E_x\left(\int_0^\infty e^{-\alpha t}\chi_V(x_t)dt\right)=\sum_{n\geqq 0}E_x\left(\int_{\sigma_{2n}}^{\sigma_{2n+1}}e^{-\alpha t}\chi_V(x_t)dt\right).^{2)}$$

1) The first term of the numerator is positive, decreasing as a function of α and independent of the ε. Hence we may assume that it exceeds ε for every $\alpha\leqq\alpha_0$.

2) Added in proof: Here we have used the fact that $\sigma_n\uparrow\sigma_\infty$ with P_x-probability 1, which we shall show in the proof of Lemma 4.2.

$$E_x\Big(\int_{\sigma_{2n}}^{\sigma_{2n+1}} e^{-\alpha t}\chi_V(x_t)dt\Big)\leqq E_x(e^{-\alpha\sigma_{2n}}E_{x_{\sigma_{2n}}}\Big(\int_0^{\sigma_1} e^{-\alpha t}\chi_V(x_t)dt\Big)\leqq k(1-\varepsilon)^n.$$

Hence we get

$$G_\alpha\chi_V(x)\leqq k\sum_{n\geqq 0}(1-\varepsilon)^n=\frac{k}{\varepsilon}<+\infty\,. \tag{4. 14}$$

But since the right side of (4. 14) is independent of α,

$$\int_0^\infty P(t,x,V)dt=G_{0+}\chi_V(x)=\lim_{\alpha\downarrow 0}\ G_\alpha\chi_V(x)\leqq\frac{k}{\varepsilon}<+\infty\,,$$

which is what we wanted to show.

COROLLARY. *Every recurrent point is conservative.*

Combining Theorem 3. 1 and Theorem 4. 1, our statement is evident.

Next we shall give two lemmas useful for the subsequent theorems.

LEMMA 4. 1. *If x and y have communication with probability* 1,[1] *both x and y are recurrent.*

Proof. It is enough to prove that x is recurrent. For this purpose, given any pair of open sets U and U' containing x such that $\bar{U}'\subset U$ and that $U\not\ni y$, we shall show

$$P_x\{\sigma_{U^c}+\sigma_{\bar{U}'}(w^+_{\sigma_{U^c}})<+\infty\}=1. \tag{4. 15}$$

Using the assumption $P_y(\sigma_{\bar{U}'}<+\infty)=1$ and (H. 2), given an arbitrary small $\varepsilon>0$, there exists an open set V containing y such that

$$P_\eta(\sigma_{\bar{U}'}<+\infty)\geqq 1-\varepsilon \text{ for every } \eta\in\bar{V}. \tag{4. 16}$$

With no loss of generality we may assume $\bar{U}\cap\bar{V}=\phi$. But since $P_x(\sigma_V<+\infty)=1$ by the assumption, we have

$$P_x\{\sigma_V+\sigma_{\bar{U}'}(w^+_{\sigma_V})<+\infty\}=E_x(P_{x\sigma_V}(\sigma_{\bar{U}'}<+\infty))\geqq 1-\varepsilon\,.$$

Hence noting that $\sigma_{U^c}+\sigma_{\bar{U}'}(w^+_{\sigma_{U^c}})\leqq\sigma_V+\sigma_{\bar{U}'}(w^+_{\sigma_V})$, we get

$$P_x(\sigma_{U^c}+\sigma_{\bar{U}'}(w^+_{\sigma_{U^c}})<+\infty)\geqq 1-\varepsilon,$$

which shows (4. 15). Thus Lemma 4. 1 was completely proved.

LEMMA 4. 2. *Suppose that x is a recurrent point which is not a trap and that U is an open set containing x whose closure is compact and $E_x(\sigma_{U^c})<+\infty$. Then given any open set V ($\bar{V}\subset U$, $V\ni x$), σ_n defined by* (4. 13) *is finite with P_x-probability* 1 *for every n, and as n goes to ∞, $\sigma_n\uparrow\infty$ with P_x-probability* 1.

---

1) Strictly speaking, for any open $U\ni x$ and for any open $V\ni y$

$$P_x(\sigma_V<+\infty)=P_y(\sigma_U<+\infty)=1.$$

Proof. (1) To prove the first statement it is enough to show that $\sigma_{2n}<+\infty$. From the definition of recurrence, $\sigma_2<+\infty$. Hence according to (H. 2), given an arbitrary small $\varepsilon>0$, $V'=\{\xi;\ \xi\in V,\ P_\xi(\sigma_2<+\infty)>1-\varepsilon\}$ is an open set containing x. Putting $\sigma_2'=\sigma_{U^c}+\sigma_{V'}(w^+_{\sigma_{U^c}})$, $\sigma_4'=\sigma_2'+\sigma_2(w^+_{\sigma_2'})$ and noting that $\sigma_4'\geqq\sigma_4$ and that $P_x(\sigma_2'<+\infty)=1$, we have

$$P_x(\sigma_4<+\infty)\geqq P_x(\sigma_4'<+\infty)=E_x(P_{x_{\sigma_2'}}(\sigma_2<+\infty))\geqq 1-\varepsilon,$$

which shows $P_x(\sigma_4<+\infty)=1$. By the same argument, we have $P_x(\sigma_{2n}<+\infty)=1$.

(2) Since $\sigma_n(w)$ is an increasing sequence of n for any fixed w, $\lim_{n\uparrow\infty}\sigma_n(w)=\sigma(w)$ is well defined for every w. To prove the second statement it is enough to show that $P_x(\sigma=\infty)=1$. In the contrary, if we assume that $P_x(\sigma<\infty)>0$, noting that $P_x(\sigma_\infty=\infty)=1$ (from Corollary to Theorem 4. 1) we have $P_x(\sigma<\sigma_\infty)>0$. Therefore, the set $\{w;\sigma(w)<\sigma_\infty(w)\}$ is not null. If $w\in\{w;\ \sigma(w)<\sigma_\infty(w)\}$, according to $(W.\ 2)$, $x_{\sigma-}(w)=\lim_{n\uparrow\infty}x_{\sigma_n}(w)$ exists. But from the definition of σ_{2n} and σ_{2n+1}, $x_{\sigma-}(w)$ has to belong both to $\bar{V}$ and to U^c. This is impossible.[1)]

THEOREM 4. 2. *If x is recurrent and if $x\to y$, y is also recurrent.*

Proof. It is enough to show that the condition of Lemma 4. 1 is satisfied.

(1) *y is accessible from x with P_x-probability* 1.

By the assumption, given any open set V containing y, $k=P_x(\sigma_V<+\infty)>0$. It remains only to show that $k=1$. Now we take U as in Lemma 4. 2 and given any small $\varepsilon>0$, choose an open set $U'(\bar{U}'\subset U)$ containing x which satisfies

$$P_\xi(\sigma_V<+\infty)>k-\varepsilon \quad \text{for every} \quad \xi\in\bar{U}'$$

Then by the second statement of Lemma 4. 2,[2)] there exists a certain constant γ independent of ε such that for sufficiently large $n>n_0$

$$P_x(\sigma_V<+\infty,\ \sigma_V<\sigma_{2n})=\gamma_n\geqq\gamma>0. \tag{4. 17}$$

Therefore, using the first statement of Lemma 4. 2, we have

$$\begin{aligned}k&=P_x(\sigma_V<+\infty)=P_x(\sigma_V<+\infty,\ \sigma_V<\sigma_{2n})+P_x(\sigma_V<+\infty,\ \sigma_V\geqq\sigma_{2n})\\ &\geqq\gamma_n+E_x(P_{x_{\sigma_{2n}}}(\sigma_V<+\infty);\ \sigma_{2n}\leqq\sigma_V)\\ &\geqq\gamma_n+(1-\gamma_n)(k-\varepsilon).\end{aligned}$$

Noting that $\gamma_n\geqq\gamma>0$, we obtain

$$k\geqq 1-\frac{1-\gamma_n}{\gamma_n}\cdot\varepsilon\geqq 1-\frac{1-\gamma}{\gamma}\cdot\varepsilon.$$

This shows that $k=1$.

1) This part of the proof was suggested by K. Itô.

2) Lemma 4.2 is applied, replacing V by U'.

(2) *x is accessible from y with P_y-probability* 1.

Suppose that for some open set $U' \ni x$

$$P_y(\sigma_{\bar{U}'} < +\infty) < \alpha < 1 .$$

Here we may further assume that $\bar{U}'$ is contained in a certain open set U for which $P_x(\sigma_{U^c} < +\infty) = 1$. Using (H. 2), we can take an open set $V \ni y$ such that $\bar{V} \frown \bar{U} = \phi$ and

$$P_\eta(\sigma_{\bar{U}'} < +\infty) \leqq \alpha \quad \text{for every } \eta \in \bar{V}.$$

Next we can see that for a small open $U''(\subset U')$ containg x

$$P_x\{\sigma_{\bar{V}} < \sigma_{U^c} + \sigma_{U''}(w^+_{\sigma_{U^c}})\} = \beta > 0 .\text{[1)]} \tag{4. 18}$$

On the other hand, noting that $\bar{V} \frown \bar{U} = \phi$ and $\bar{U}'' \subset \bar{U}'$, we have

$$P_\eta(\sigma_{\bar{U}''} < +\infty) \leqq \alpha \quad \text{for every } \eta \in \bar{V}. \tag{4. 19}$$

Now putting $\sigma_2 = \sigma_{U^c} + \sigma_{\bar{U}''}(w^+_{\sigma_{U^c}})$, we shall calculate the left side of (4. 18).

$$\begin{aligned}\beta = P_x(\sigma_{\bar{V}} < \sigma_2) &= P_x\{\sigma_2 = \sigma_{\bar{V}} + \sigma_{\bar{U}''}(w^+_{\sigma_{\bar{V}}}),\ \sigma_{\bar{V}} < \sigma_2\} \\ &= E_x(P_{x\sigma_{\bar{V}}}(\sigma_{\bar{U}''} < +\infty)\,;\ \sigma_{\bar{V}} < \sigma_2) \leqq \alpha \cdot \beta < \beta .\end{aligned}$$

This is a contradiction. Thus Theorem 4. 2 was completely proved.

Next we shall characterize recurrence by means of sojourn sets. For this purpose we shall define a special sojourn set.

DEFINITION 4. 1. A sojourn set S is said to be *minimal* if S contains no proper sojourn sets. Here we say that S' is a *proper sojourn set* of S if S' is a sojourn set and if $S' \subset S \not\subset S'$.

We now put $A_x = \{y\,;\, x \rightarrow y\}$. Then it is almost evident that A_x is a closed sojourn set and that $P_x(\sigma_{A_x^c} = \sigma_\infty) = 1$.

THEOREM 4. 3. *x is recurrent if and only if x is conservative and if A_x is minimal.*

Proof. (1) Supposing that x is recurrent, we shall show that $P_x(\sigma_\infty = \infty) = 1$ and that A_x is minimal. First, if x is a trap, our statement is trivial. Next, if x is a recurrent point which is not a trap, according to Corollary of Theorem 4. 1, x is conservative. Further as was shown in the proof of Theorem 4. 2, any two points which belong to A_x have communication probability-1. This shows that A_x has no proper sojourn sets.

---

1) Define σ_{2n} as in (4.13), for a pair of U and U'. Then if $\beta = 0$ for every open $U''(\subset U')$, by the consideration analogous to the first statement of Lemme 4.2, we have

$$P_x(\sigma_{\bar{V}} < \sigma_{2n}) = 0 \quad \text{for every } n > 0.$$

This contradicts the facts that $\sigma_{2n} \uparrow \infty$ and that $P_x(\sigma_V < +\infty) > 0$.

(2) If x is conservative, it follows from Theorem 3. 2 that all the points belonging to A_x are conservative. Hence if A_x has no proper sojourn sets, x and any fixed $y \in A$ have communication with probability 1. Therefore, according to Lemma 4. 1, x is recurrent.

THEOREM 4.4. *If a compact set K contains no recurrent points, then for every $x \in E$*

$$\int_0^\infty P(t, x, K) < +\infty. \tag{4. 20}$$

Proof. Let y be not recurrent. Then it follows from the proof of Theorem 4. 1 that there exists an open neighbourhood $U(y)$ of y such that

$$\int_0^\infty P(t, \eta, U(y))dt < k_y < +\infty \tag{4. 21}$$

holds independently of $\eta \in U(y)$.

We now take such open sets for every $y \in K$. By compactness of K, it is possible to cover K with a finite number of $U(y)$. We shall denote these sets by U_1, U_2, $\cdots$, U_n and calculate $\int_0^\infty P(t, x, U_i)dt$, using the passage time σ_i to U_i.

$$\int_0^\infty P(t, x, U_i)\, dt = E_x\Big(\int_0^\infty \chi_{U_i}(x_t)dt\Big) = E_x\Big(\int_{\sigma i}^\infty \chi_{U_i}(x_t)dt\Big)$$

$$= E_x\Big(E_{x_{\sigma_i}}\Big(\int_0^\infty \chi_{U_i}(x_t)dt\Big); \sigma_i < +\infty\Big) \leqq k_i < +\infty\,,$$

so that we have

$$\int_0^\infty P(t, x, K)dt \leqq \sum_{i=1}^n \int_0^\infty P(t, x, U_i)dt \leqq \sum_{i=1}^n k_i < +\infty\,.$$

This completes the proof.

COROLLARY. *Suppose that E is compact, that at least one point belonging to E is conservative and that any two points of E have commumication. Then every point of E is recurrent.*

§5. Applications of §4.

In this section we shall apply the results of the preceeding section to special cases.

I. Case of $E = \{1.2.3 \cdots\cdots\cdots\cdots\}$.

In this case all the processes defined in §1 trivially satisfy the hypotheses (H. 1) and (H. 2). Therefore the criterion of recurrence by Green measures

(Theorem 4. 1) is always true. Furthermore, since the topology in E is discrete, x is recurrent if and only if

(5. 1) $$\int_0^\infty P(t, x, x)dt=\infty .$$

First we shall consider the case in which x is not a trap. Then, putting $A=E$ and $\alpha=0$ in Lemma 3. 2, we have

(5. 2) $$\int_0^\infty P(t, x, x)dt=\left(\sum_{n=0}^\infty \Pi^n(x, x)\right)\cdot\frac{1}{p(x)}.$$

But $p(x)>0$, as x is not a trap. Therefore the condition (5.1) is equivalent to

(5. 3) $$\sum_{n=0}^\infty \Pi^n(x, x)=\infty .$$

Next, even if x is a trap, (5. 3) holds by the definition of Π. Thus we have

THEOREM 5. 1. *Let x_t be a Markov process on a denumerable state space. Then the point x is recurrent if and only if* (5. 3) *holds.*

This theorem shows that the recurrence property does not depend on $p(\cdot)$ which describes the speed of our process. Analogously, using only Π, we can easily reformulate the criterion of recurrence by sojourn sets (Theorem 4. 3) in the more concrete form, though the detail is omitted.

II. Diffusion processes with Brownian hitting probabilities.

Let E be n-dimensional Euclidean space. The process x_t on E defined in §1 is called a *diffusion process with Brownian hitting probabilities*[1] if the harmonic measures of x_t coincide with those of n-dimensional Brownian motion, i. e. with the n-dimensional classical harmonic measures. In this case it is easily shown that sample paths are continuous with probability 1. Further the continuity of classical harmonic functions implies that the following condition which is stronger than (H. 2) is satisfied:

(H. 2)′ $h_F(x, \cdot)$ maps any bounded Borel function f into $\mathfrak{C}$.

Next, it follows from the definition that any two points on E have communication and that the recurrence property of x_t is the same as that of n-dimensional Brownian motion. But since n-dimensional Brownian motion is recurrent[1] for $n\leqq 2$ and not recurrent for $n>2$, assuming that our process x_t satisfies (H. 1) we have

THEOREM 5. 2. *Let x_t be a diffusion process with Brownian hitting probabilities. Then,*

1) This terminology is due to K. Itô and H. P. McKean.

2) A Markov process x_t is said to be *recurrent* if every point of E is recurrent with respect to x_t. Analogously, the non-recurrence of a process is defined by that of every point in E.

(1) *if $n \leqq 2$, every point of E is recurrent and hence we have for every x and for every open set U* (not necessarily containing x)

$$\int_0^\infty P(t, x, U) dt = \infty; \tag{5. 4}$$

(2) *if $n > 2$, every point of E is not recurrent and hence we have for every x and for every compact set K*

$$\int_0^\infty P(t, x, K) dt < +\infty . \tag{5. 5}$$

III. Green measures of killed processes.

Let x_t be a Markov process which satisfies (H. 1) and (H. 2). Furthermore we assume that any two points of E have communication. Given an open set A, the *killed process x_t^0 on A* is defined as follows:

$$\begin{aligned} x_t^0 &= x_t && \text{if } t < \sigma_A, \\ &= \infty && \text{if } t \geqq \sigma_A . \end{aligned} \tag{5. 6}$$

Now we denote the transition probabilities and the Green measures of x_t^0 by $P^0(t, x, \cdot)$ and $G^0(x, \cdot)$, respectively; namely

$$P^0(t, x, \cdot) = P_x(x_t \epsilon \cdot, \ \sigma_A > t),$$

$$G^0(x, \cdot) = \int_0^\infty P^0(t, x, \cdot) dt.$$

Since it results from Corollary to Theorem 4. 1 and from our assumption that every point of $E - A$ is not recurrent with respect to x_t^0, it is expected that for every compact set $K \subset E - A$

$$G^0(x, K) < +\infty^{1)}. \tag{5. 7}$$

In fact, if x_t is not recurrent, our statement is clear by $G^0(x, K) \leqq G(x, K)$ and Theorem 4. 4. On the other hand, if x_t is recurrent, using the same technique as in Theorem 4. 1 and Lemma 4. 2, we see that, for every $x \epsilon E - A$ and for some open set U containing x, $G^0(\xi, U)$ is uniformly bounded whenever ξ runs over U. From this we can easily derive (5. 7) in the same way as in Theorem 4. 4. Thus we have

THEOREM 5. 3 *The Green measures of the killed process are finite for every compact sets.*

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1) We cannot derive this fact directly from Theorem 4. 4, because we are not sure that the killed process x_t^0 satisfies (H, 1), though (H. 2) is always satisfied.

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