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On P-components of normal ideals in a semigroup

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Recently Professor Keizo Asano has shown to the author that P-components of normal ideals in rings can be characterized as follows:

Let G be a Brandt's groupoid of normal ideals in a ring R with a unit element, P a set of prime-spots, and $\mathfrak{a}_P$ the P-component of $\mathfrak{a}$ in G. Then the mapping φ_P: $\mathfrak{a}\rightarrow\varphi_P(\mathfrak{a})=\mathfrak{a}_P$ gives a homomorphism form G into the multiplicative semigroup $\mathfrak{S}$ consisting of all submodules of R, and it satisfies $\varphi_P(\mathfrak{a})\supseteq\bar{\mathfrak{a}}$. Conversely, let φ be a mapping from G into $\mathfrak{S}$ which satisfies $\varphi(\mathfrak{a})\supseteq\mathfrak{a}$ and $\varphi(\mathfrak{a}\mathfrak{b})=\varphi(\mathfrak{a})\varphi(\mathfrak{b})$ ($\mathfrak{a}\mathfrak{b}$: proper multiplication). Then φ coincides with some φ_P. Hence the set Φ of all φ satisfying $\varphi(\mathfrak{a})\supseteq\mathfrak{a}$ and $\varphi(\mathfrak{a}\mathfrak{b})=\varphi(\mathfrak{a})\varphi(\mathfrak{b})$ forms an atomic Boolean algebra under $\varphi\leq\psi$, where $\varphi\leq\psi$ means $\varphi(\mathfrak{a})\subseteq\psi(\mathfrak{a})$ for all $\mathfrak{a}$ in G.

In the present paper we shall generalize the above facts for the case of semigroups.

1. Let $\mathfrak{o}$ be an order of a (noncommutative) semigroup S with an identity 1. A subset $\mathfrak{a}$ of S is called a left s-$\mathfrak{o}$-ideal if (1) $\mathfrak{o}\mathfrak{a}\subseteq\mathfrak{a}$, (2) $\mathfrak{a}$ contains a regular element[1] and (3) $\mathfrak{a}\lambda\subseteq\mathfrak{o}$ for a suitable regular element λ in S. Right s-$\mathfrak{o}'$-ideals are defined in a similar fashion, where $\mathfrak{o}'$ denotes an order of S. If $\mathfrak{a}$ is a left s-$\mathfrak{o}$-ideal and a right s-$\mathfrak{o}'$-ideal, then $\mathfrak{a}$ is called an s-$\mathfrak{o}$-$\mathfrak{o}'$-ideal. An s-$\mathfrak{o}$-$\mathfrak{o}$-ideal is called an s-$\mathfrak{o}$-ideal.

Let $\{\mathfrak{o}^i, \mathfrak{o}^k, \cdots\}$ be a system of orders which are equivalent[2] to a fixed order $\mathfrak{o}$ of S. Then any two orders in the system are equivalent to each other. The product $\mathfrak{a}^{ik}\mathfrak{b}^{jl}$ of an s-$\mathfrak{o}^i$-$\mathfrak{o}^k$-ideal $\mathfrak{a}$ and an s-$\mathfrak{o}^j$-$\mathfrak{o}^l$-ideal $\mathfrak{b}$ is called proper if $k=j$. If $\mathfrak{o}^i$, $\mathfrak{o}^k$, $\cdots$ are maximal orders, then the set G of all v-ideals,[3] defined on this system of the orders, forms the Brandt's groupoid with respect to $\mathfrak{a}\circ\mathfrak{b}=(\mathfrak{a}\mathfrak{b})^*$, where $\mathfrak{a}\mathfrak{b}$ is proper.

We shall now impose that

1. $\mathfrak{o}^i$, $\mathfrak{o}^k$, $\cdots$ are maximal orders of S.
2. A fixed order $\mathfrak{o}^i$ is regular[4].

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1) An element of S is called regular if it satisfies both right and left cancelation laws.
2) Two subsets M, N of S are called equivalent if there exist regular elements λ, μ, λ', μ' in S such that $\lambda M\mu\subseteq N$ and $\lambda' N\mu'\subseteq M$. Two orders are called equivalent if they are equivalent as subsets of S. See [1]. [2] and [3].
3) An s-ideal $\mathfrak{a}$ is called a v-ideal if $\mathfrak{a}^*=\mathfrak{a}^{-1-1}=\mathfrak{a}$. See [3].
4) An order $\mathfrak{o}$ of S is called regular when, for any x in S, there exist two regular elements α and β in $\mathfrak{o}$ such that $x\mathfrak{o}\alpha\subseteq\mathfrak{o}$ and $\beta\mathfrak{o}x\subseteq\mathfrak{o}$. See [1], [2] and [3].

3. Ascending chain condition (A.C.C.) holds for integral two-sided v-$\mathfrak{o}^i$-ideals for a fixed order $\mathfrak{o}^i$.

Then it may be seen that every order $\mathfrak{o}^k$ in the system is regular, and the A.C.C. holds for integral two-sided v-$\mathfrak{o}^k$-ideals. Moreover it is verified that the A.C.C. holds for v-$\mathfrak{o}^i$-$\mathfrak{o}^k$-ideals which are contained in any fixed s-$\mathfrak{o}^i$-$\mathfrak{o}^k$-ideal. Using this fact we can prove that there exist, for any s-$\mathfrak{o}^i$-$\mathfrak{o}^k$-ideal $\mathfrak{a}$, a finite number of elements $c_1, \cdots, c_n$ in $\mathfrak{a}$ such that $\mathfrak{a}^*$ is generated by $c_1, \cdots, c_n$. That is, $\mathfrak{a}^*=[c_1, \cdots, c_n] =(\bigcup_{\nu=1}^{n}\mathfrak{o}^i c_\nu \mathfrak{o}^k)^*$.

A subset A of S is called an $\mathfrak{o}^i$-$\mathfrak{o}^k$-set if $\mathfrak{o}^i A\mathfrak{o}^k\subseteq A$ and A contains a regular element of S. For any $\mathfrak{o}^i$-$\mathfrak{o}^k$-set A we define a closure operation as the set-theoretical sum of all v-$\mathfrak{o}^i$-$\mathfrak{o}^k$-ideals generated by a finite number of elements in A, i.e.

$$\bar{A}^{(ik)}=\bigcup_{a_\nu\in A}(\mathfrak{o}^i a_1\mathfrak{o}^k\cup\cdots\cup\mathfrak{o}^i a_n\mathfrak{o}^k)^*.$$

LEMMA 1. *If an* $\mathfrak{o}^i$-$\mathfrak{o}^k$-*set* A *is an* $\mathfrak{o}^j$-$\mathfrak{o}^l$-*set, then* $\bar{A}^{(ik)}=\bar{A}^{(jl)}$.

Proof. Let x be any element in $\bar{A}^{(jl)}$. Then there exists an v-$\mathfrak{o}^j$-$\mathfrak{o}^l$-ideal $\mathfrak{c}=[c_1, \cdots, c_m]$, $c_\nu\in A$, which contains x. Since $\mathfrak{a}=\mathfrak{o}^i(\mathfrak{o}^j c_1\mathfrak{o}^l\cup\cdots\cup\mathfrak{o}^j c_m\mathfrak{o}^l)\mathfrak{o}^k\subseteq\mathfrak{o}^i A\mathfrak{o}^k=A$, and $\mathfrak{a}^*=[a_1, \cdots, a_n]$, $a_\nu\in\mathfrak{a}\subseteq A$, we obtain $x\in\mathfrak{c}\subseteq\mathfrak{a}^*\subseteq\bar{A}^{(ik)}$, i.e. $\bar{A}^{(jl)}\subseteq\bar{A}^{(ik)}$. Similarly $\bar{A}^{(ik)}\subseteq\bar{A}^{(jl)}$. Therefore we have $\bar{A}^{(ik)}=\bar{A}^{(jl)}$ *q.e.d.*

Now we define $\bar{A}$ by $\bar{A}=\bar{A}^{(ik)}=\bar{A}^{(jl)}$. Then the operation $A\to\bar{A}$ has the following properties:

1) $A\subseteq\bar{A}$,

2) $\bar{\bar{A}}=\bar{A}$,

3) If A, B are $\mathfrak{o}^i$-$\mathfrak{o}^k$-sets then $A\subseteq B$ implies $\bar{A}\subseteq\bar{B}$,

4) If A and B are $\mathfrak{o}^i$-$\mathfrak{o}^k$-and $\mathfrak{o}^k$-$\mathfrak{o}^l$-sets respectively, then $\bar{A}\bar{B}\subseteq\overline{AB}$.

An $\mathfrak{o}^i$-$\mathfrak{o}^k$-set A is called closed if $\bar{A}=A$. For s-ideals, the closure operation coincides with the $*$-operation: $\bar{\mathfrak{a}}=\mathfrak{a}^*$. Hence $\mathfrak{a}$ is a closed ideal if it is a v-ideal.

LEMMA 2. *Let* A, B *be* $\mathfrak{o}^i$-$\mathfrak{o}^k$-, $\mathfrak{o}^i$-$\mathfrak{o}^l$-*sets respectively, and* M *a subset of* S. *If* $AM\subseteq B$ *and* $A\lambda\subseteq B$ *for a regular element* λ, *then* $\bar{A}M\subseteq\bar{B}$. *Particularly, if* $\mathfrak{a}$, $\mathfrak{b}$ *are* s-$\mathfrak{o}^i$-$\mathfrak{o}^k$-, s-$\mathfrak{o}^i$-$\mathfrak{o}^l$-*ideals respectively, then* $\mathfrak{a}M\subseteq\mathfrak{b}$ *implies* $\mathfrak{a}^*M\subseteq\mathfrak{b}^*$.

LEMMA 3. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *be* v-$\mathfrak{o}^i$-$\mathfrak{o}^k$-, v-$\mathfrak{o}^i$-$\mathfrak{o}^l$-*ideals respectively, and* M *a subset of* S. *If* $\mathfrak{a}M\subseteq\mathfrak{b}$ *then* $(\mathfrak{c}\cdot\mathfrak{a})M\subseteq\mathfrak{c}\cdot\mathfrak{b}$ *for any* v-$\mathfrak{o}^j$-$\mathfrak{o}^i$-*ideal* $\mathfrak{c}$. *If particularly* $\mathfrak{a}x\subseteq\mathfrak{b}$ *then* $x\in\mathfrak{a}^{-1}\cdot\mathfrak{b}$.

The proofs of the above two lemmas are similarly obtained as in [3, §5].

From now on 'ideals' will always mean v-ideals and 'c-ideals' closed ideals, respectively.

Let $p=p^{ii}$ be a prime ideal. The set $\boldsymbol{p}$ of all (prime) ideals which are conjunctive to p is called a prime spot of S. Let $p'=p^{kk}$ be a prime ideal conjunctive to $p=p^{ii}$, then $(\mathfrak{a}^{ik}\cup\mathfrak{p})^*=\mathfrak{o}^i$ implies $(\mathfrak{a}^{ik}\cup\mathfrak{p}')^*=\mathfrak{o}^k$ and conversely. In such a case $\mathfrak{a}=\mathfrak{a}^{ik}$ is called coprime to $\boldsymbol{p}$. Let P be any set of prime spots in S. Then $\mathfrak{a}$ is called coprime to P when $\mathfrak{a}$ is coprime to all prime spots in P.

We now define, analogously to the case of rings[5], the P-component $\mathfrak{a}_P=\mathfrak{a}_P{}^{ik}$ of an ideal $\mathfrak{a}=\mathfrak{a}^{ik}$ as the set of all elements x in S such that $\mathfrak{n}x\subseteq\mathfrak{a}$ for a suitable integral ideal $\mathfrak{n}=\mathfrak{n}^{ii}$ which is coprime to P. If $\mathfrak{n}=\mathfrak{n}^{kk}$ is conjunctive to $\mathfrak{n}$, then by Lemma 2 $\mathfrak{n}x\subseteq\mathfrak{a}$ implies $x\mathfrak{n}'\subseteq\mathfrak{a}$ and conversely. Hence $\mathfrak{a}_P$ is defined symmetrically with respect to the left and the right orders of $\mathfrak{a}$, and represented as the set-theoretical sum of all $\mathfrak{n}^{-1}\cdot\mathfrak{a}=\mathfrak{a}\cdot\mathfrak{n}'^{-1}$ with $\mathfrak{n}(\mathfrak{n}')$ coprime to P.

Let P be any set of prime spots in S. Then $\{\mathfrak{o}_P^i, \mathfrak{o}_P^k, \ldots\}$ forms an another system of orders (equivalent to one another) of S. Our main object is the closed $\mathfrak{o}_P^i$-$\mathfrak{o}_P^k$-ideals. In the following $\mathfrak{n}, \mathfrak{n}', \ldots$ will denote ideals which are coprime to P.

Lemma 4. *Let $\mathfrak{a}=\mathfrak{a}^{ik}$ be an ideal. Then $\mathfrak{a}_P$ forms a closed (c-) $\mathfrak{o}_P^i$-$\mathfrak{o}_P^k$-ideal, and*

$$\mathfrak{a}_P=\overline{\mathfrak{o}_P^i\mathfrak{a}}=\overline{\mathfrak{a}\mathfrak{o}_P^k}=\overline{\mathfrak{o}_P^i\mathfrak{a}\mathfrak{o}_P^k}.$$

Proof. This is similarly obtained as in [3, §5].

Lemma 5. *$\mathfrak{o}_P^i, \mathfrak{o}_P^k, \ldots$ form a system of regular orders equivalent to one another.*

Proof. Regularity was proved in [3, §5]. Equivalency is evident by Lemma 4.

Lemma 6. *If $\mathfrak{A}$ is an s-$\mathfrak{o}_P^i$-$\mathfrak{o}_P^k$-ideal, then $\overline{\mathfrak{A}}$ is a c-$\mathfrak{o}_P^i$-$\mathfrak{o}_P^k$-ideal.*

Lemma 7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be any two ideals Then*

$$((\mathfrak{a}\mathfrak{b})^*)_P=\overline{\mathfrak{a}_P\mathfrak{b}_P}.$$

The proofs of the above two lemmas are similarly obtained as in [3, §5].

Lemma 8. *Let $\mathfrak{A}$ be an s-$\mathfrak{o}_P^i$-$\mathfrak{o}_P^k$-ideal. Then*

$$\mathfrak{A}\subseteq\mathfrak{o}_P^i \leftrightarrows \mathfrak{A}\subseteq((\mathfrak{o}^k\mathfrak{o}^i)^{-1})_P \leftrightarrows \mathfrak{A}\subseteq\mathfrak{o}_P^k.$$

Proof. Suppose that $\mathfrak{A}\subseteq\mathfrak{o}_P^i$. Then $\overline{\mathfrak{o}_P^k\mathfrak{o}_P^i}\mathfrak{A}\overline{\mathfrak{o}_P^k\mathfrak{o}_P^i}\subseteq\overline{\mathfrak{o}_P^k\mathfrak{o}_P^i\mathfrak{A}\mathfrak{o}_P^k\mathfrak{o}_P^i}\subseteq\overline{\mathfrak{o}_P^k\mathfrak{A}\mathfrak{o}_P^i}\subseteq\overline{\mathfrak{o}_P^k\mathfrak{o}_P^i\mathfrak{o}_P^i}=\overline{\mathfrak{o}_P^k\mathfrak{o}_P^i}$. This implies $\mathfrak{A}\subseteq\overline{\mathfrak{o}_P^k\mathfrak{o}_P^i}^{-1}=[((\mathfrak{o}^k\mathfrak{o}^i)^*)_P]^{-1}=((\mathfrak{o}^k\mathfrak{o}^i)^{*-1})_P{}^{6)}=((\mathfrak{o}^k\mathfrak{o}^i)^{-1})_P$. Hence $\mathfrak{A}\subseteq\mathfrak{o}_P^i$ implies $\mathfrak{A}\subseteq((\mathfrak{o}^k\mathfrak{o}^i)^{-1})_P$. The converse is evident. The other part is similarly obtained.

Theorem 1. *Let $\mathfrak{A}$ be an s-$\mathfrak{o}_P^i$-$\mathfrak{o}_P^k$-ideal contained in $\mathfrak{o}_P^i$. Then*

$$\mathfrak{a}=\mathfrak{A}\cap(\mathfrak{o}^k\mathfrak{o}^i)^{-1}$$

is an s-$\mathfrak{o}^i$-$\mathfrak{o}^k$-ideal, and

$$\overline{\mathfrak{A}}=(\mathfrak{a}^*)_P.$$

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5) See [1] and [2].

6) Let $\mathfrak{a}$ be any ideal. Then the mapping $\mathfrak{a}\rightarrow\mathfrak{a}_P$ gives a groupoid-homomorphism from G of all v-ideals onto the P-components of all ideals in G. Hence $(\mathfrak{a}^{-1})_P=(\mathfrak{a}_P)^{-1}$.

Proof. $\mathfrak{a}$ is evidently s-$\mathfrak{o}^i$-$\mathfrak{o}^k$-ideal. Since $\mathfrak{a}\subseteq\mathfrak{A}$, $\mathfrak{A}$ contains $\mathfrak{n}^{-1}\mathfrak{a}$ for all $\mathfrak{n}=\mathfrak{n}^{ii}$. Hence $\overline{\mathfrak{A}}\supseteq\mathfrak{n}^{-1}\cdot\mathfrak{a}$ for all $\mathfrak{n}$, hence $\overline{\mathfrak{A}}\supseteq(\mathfrak{a}^*)_P$. Suppose that $a\in\overline{\mathfrak{A}}$. Then there exist $a_\nu\in\mathfrak{A}(\nu=1, \dots, n)$ such that $a\in[a_1, \dots, a_n]$. Since $\mathfrak{A}\subset\mathfrak{o}^i_P$, by Lemma 8 $a_\nu\in(\mathfrak{o}^k\mathfrak{o}^i)^{-1}_P$. Hence there exists $\mathfrak{n}=\mathfrak{n}^{ii}$ such that $\mathfrak{n}a_\nu\subseteq(\mathfrak{o}^k\mathfrak{o}^i)^{-1}$ $(\nu=1, \dots, n)$. On the other hand, $\mathfrak{n}a_\nu\subseteq\mathfrak{o}^i\mathfrak{A}=\mathfrak{A}$. We obtain $\mathfrak{n}a_\nu\subseteq\mathfrak{A}\frown(\mathfrak{o}^k\mathfrak{o}^i)^{-1}=\mathfrak{a}$, $a_\nu\in\mathfrak{n}^{-1}\cdot\mathfrak{a}^*(\nu=1, \dots, n)$. Hence $a\in[a_1, \dots, a_n]\subseteq\mathfrak{n}^{-1}\cdot\mathfrak{a}^*\subseteq(\mathfrak{a}^*)_P$, as desired.

COROLLARY. *Let $\mathfrak{A}$ be a c-$\mathfrak{o}^i_P$-$\mathfrak{o}^k_P$-ideal contained in $\mathfrak{o}^i_P$. Then there exists an $\mathfrak{o}^i$-$\mathfrak{o}^k$-ideal $\mathfrak{a}$ such that $\mathfrak{A}=\mathfrak{a}_P$.*

REMARK. Let $\mathfrak{A}$ be an s-$\mathfrak{o}^i_P$-$\mathfrak{o}^k_P$-ideal contained in $\mathfrak{o}^i_P$. Then evidently $\mathfrak{a}_1=\mathfrak{A}\frown\mathfrak{o}^i\mathfrak{o}^k$ is an s-$\mathfrak{o}^i$-$\mathfrak{o}^k$-ideal and it is proved that $\overline{\mathfrak{A}}=(\mathfrak{a}_1^*)_P$. Hence by Theorem 1, we obtain $\overline{\mathfrak{A}}=(\mathfrak{a}^*)_P=(\mathfrak{a}_1^*)_P=(\mathfrak{a}_i^*)_P=(\mathfrak{a}_k^*)_P$, where $\mathfrak{a}_i=\mathfrak{A}\frown\mathfrak{o}^i$, $\mathfrak{a}_k=\mathfrak{A}\frown\mathfrak{o}^k$.

THEOREM 2. *The c-$\mathfrak{o}^i_P$-$\mathfrak{o}^k_P$-ideals $\mathfrak{A}$, $\mathfrak{B}$, ... form a groupoid G_P with respect to the product $\mathfrak{A}\cdot\mathfrak{B}=\overline{\mathfrak{A}\mathfrak{B}}$, where $\mathfrak{A}$ is a c-$\mathfrak{o}^i_P$-$\mathfrak{o}^k_P$-ideal and $\mathfrak{B}$ a c-$\mathfrak{o}^k_P$-$\mathfrak{o}^l_P$-ideal. G_P is homomorphic to G of all ideals as groupoids: $G\simeq G_P$.*

Proof. The mapping $\mathfrak{a}\to\mathfrak{a}_p(\mathfrak{a}\in G)$ is a homomorphism of G into G_P. If $\mathfrak{A}$ is a c-$\mathfrak{o}^i_P$-$\mathfrak{o}^k_P$-ideal contained in $\mathfrak{o}^i_P$, then there exists $\mathfrak{a}=\mathfrak{a}^{ik}\in G$ such that $\mathfrak{A}=\mathfrak{a}_P$. If $\mathfrak{C}$ is a c-$\mathfrak{o}^i_P$-$\mathfrak{o}^k_P$-ideal not contained in $\mathfrak{o}^i_P$, then there exists $\mathfrak{a}_P(\mathfrak{a}=\mathfrak{a}^{ii})$ such that $\mathfrak{a}_P\cdot\mathfrak{C}\subset\mathfrak{o}^i_P$. Hence by the above corollary there exists $\mathfrak{b}=\mathfrak{b}^{ik}$ in G such that $\mathfrak{a}_P\cdot\mathfrak{C}=\mathfrak{b}_P$. Hence $\mathfrak{C}=\mathfrak{a}_P^{-1}\cdot\mathfrak{b}_P=(\mathfrak{a}^{-1}\cdot\mathfrak{b})_P$, and the proof is complete.

THEOREM 3. *Let $\mathfrak{a}=\mathfrak{a}^{ik}$ be an integral ideal in G. Then the following conditions are equivalent.*

1. $\mathfrak{a}$ *is coprime to P.*
2. $\mathfrak{a}(\mathfrak{o}^i\mathfrak{o}^k)^{-1}$ *is coprime to P.*

2′. $(\mathfrak{o}^i\mathfrak{o}^k)^{-1}\mathfrak{a}$ *is coprime to P.*

3. $\mathfrak{a}_P=\mathfrak{o}^i_P$.

3′. $\mathfrak{a}_P=\mathfrak{o}^k_P$.

Proof. $1\to2$: $\mathfrak{a}\cdot(\mathfrak{o}^i\mathfrak{o}^k)^{-1}=\mathfrak{c}$ is evidently a two-sided $\mathfrak{o}^i$-ideal contained in $\mathfrak{a}$. Suppose that $\mathfrak{a}$ is coprime to P. Then $(\mathfrak{a}\cup\mathfrak{p})^*=\mathfrak{o}^i$ for any $\mathfrak{p}=\mathfrak{p}^{ii}$ in $\boldsymbol{p}$ $(\boldsymbol{p}\in P)$. Hence we have $(\mathfrak{c}\cup\mathfrak{p})^*=(\mathfrak{a}\cup\mathfrak{p}\cdot(\mathfrak{o}^i\mathfrak{o}^k)^*)\cdot(\mathfrak{o}^i\mathfrak{o}^k)^{-1}=(\mathfrak{a}\cup\mathfrak{p}\mathfrak{o}^i\mathfrak{o}^k)^*\cdot(\mathfrak{o}^i\mathfrak{o}^k)^{-1}=((\mathfrak{a}\cup\mathfrak{p})\mathfrak{o}^k)^*\cdot(\mathfrak{o}^i\mathfrak{o}^k)^{-1}=((\mathfrak{a}\cup\mathfrak{p})^*\mathfrak{o}^k)^*\cdot(\mathfrak{o}^i\mathfrak{o}^k)^{-1}=(\mathfrak{o}^i\mathfrak{o}^k)^*\cdot(\mathfrak{o}^i\mathfrak{o}^k)^{-1}=\mathfrak{o}^i$. $2\to3$: From $\mathfrak{c}_P=\mathfrak{o}^i_P\supseteq\mathfrak{a}_P\supseteq\mathfrak{c}_P$, we obtain $\mathfrak{a}_P=\mathfrak{o}^i_P$. $3\to1$: If $\mathfrak{a}_P=\mathfrak{o}^i_P$, then $\mathfrak{a}_P\ni1$. Hence there exists $\mathfrak{n}=\mathfrak{n}^{ii}$ such that $1\in\mathfrak{n}^{-1}\cdot\mathfrak{a}$. Hence $\mathfrak{n}\subseteq\mathfrak{a}$. Thus $\mathfrak{a}$ is coprime to P. Similarly we obtain $1\to2'\to3'\to1$.

The groupoid-homomorphism $G\simeq G_P$ in Theorem 2 is characterized by the following

THEOREM 4. *Let G be a groupoid of v-ideals defined on the system $\{\mathfrak{o}^i, \mathfrak{o}^k, \dots\}$, and $\mathfrak{M}$ the set of all closed $\mathfrak{o}^i$-$\mathfrak{o}^k$-sets*[7] *of S. Then the mapping φ_P: $\mathfrak{a}\to\varphi_P(\mathfrak{a})=\mathfrak{a}_P$*

7) If S is a ring, then $\mathfrak{M}$ coincides with the set of all $\mathfrak{o}^i$-$\mathfrak{o}^k$-modules, each of which contains a regular element. See [3, §5].

from G into $\mathfrak{M}$ *satisfies* $\varphi_P(\mathfrak{a})\supseteq\mathfrak{a}$ *and* $\varphi_P(\mathfrak{a}\mathfrak{b})=\varphi_P(\mathfrak{a})\cdot\varphi_P(\mathfrak{b})$, $\mathfrak{a}=\mathfrak{a}^{ik}$, $\mathfrak{b}=\mathfrak{b}^{kl}$. *Conversely, if a mapping* φ *from G into* $\mathfrak{M}$ *satisfies*

1) $\varphi(\mathfrak{a})\supseteq\mathfrak{a}$,

2) $\varphi(\mathfrak{a}\mathfrak{b})=\varphi(\mathfrak{a})\varphi(\mathfrak{b})$,

then φ *coincides with some* φ_P. *Hence the set* Φ *of all* φ *satisfying* 1) *and* 2) *forms an atomic Boolean algebra under* $\varphi\leq\psi$, *where* $\varphi\leq\psi$ *means* $\varphi(\mathfrak{a})\subseteq\psi(\mathfrak{a})$ *for all* $\mathfrak{a}$ *in G. Moreover,* $\mathfrak{a}_P$ *coincides with the set-theoretical sum of all inverse image of* $\varphi_p(\mathfrak{a})$.

Proof. The first part is easy. We now prove the latter part. Since $\varphi(\mathfrak{o}^i)\supseteq\mathfrak{o}^i$ and $\varphi(\mathfrak{o}^i)\varphi(\mathfrak{o}^i)=\varphi(\mathfrak{o}^i)$, $\varphi(\mathfrak{o}^i)$ forms a closed $\mathfrak{o}^i$-semigroup[8]. Hence there exists a set $\mathfrak{P}_i$ of prime $\mathfrak{o}^i$-ideals such that $\varphi(\mathfrak{o}^i)=\mathfrak{o}^i_{P_i}$[9]. We shall now use $P(i)$ to denote the set of all prime spots which contain $\mathfrak{p}=\mathfrak{p}^{ii}$ in $\mathfrak{P}_i$. Then $P(i)=P(k)$ for arbitrary indices i and k. Because if $P(k)\not\subseteq P(i)$, then we can take $\mathfrak{p}'$ such that $\mathfrak{p}'\in\boldsymbol{p}'$, $\boldsymbol{p}'\in P(k)$, and $\mathfrak{p}_0=\mathfrak{c}^{-1}\cdot\mathfrak{p}'\cdot\mathfrak{c}(\mathfrak{c}=\mathfrak{c}^{ki})$ is not contained in every $\boldsymbol{p}$ in $P(i)$. Hence $\varphi(\mathfrak{o}_i)\not\subseteq\mathfrak{o}^i_{\mathfrak{p}_0}$. This implies $\mathfrak{p}_0^{-1}\subseteq\varphi(\mathfrak{o}^i)$.[10] Therefore $\mathfrak{p}'^{-1}=\mathfrak{c}\cdot\mathfrak{p}_0^{-1}\cdot\mathfrak{c}^{-1}\subseteq\overline{\varphi(\mathfrak{c})\varphi(\mathfrak{o}^i)\varphi(\mathfrak{c}^{-1})}\subseteq\varphi(\mathfrak{c}\cdot\mathfrak{o}^i\cdot\mathfrak{c}^{-1})=\varphi(\mathfrak{o}^k)=\mathfrak{o}^k_{\mathfrak{P}_k}\subseteq\mathfrak{o}^k_{\mathfrak{p}'}$, i. e. $\mathfrak{p}'^{-1}\subseteq\mathfrak{o}'_{\mathfrak{p}'}$. This is a contradiction. Hence $P(k)\subseteq P(i)$. Similarly $P(i)\subseteq P(k)$. Hence we obtain $P=P(i)=P(k)$, as desired.

Next we prove that $\varphi(\mathfrak{a})=\mathfrak{a}_P$. Since $\mathfrak{o}^i_P=\varphi(\mathfrak{o}^i)\subseteq\overline{\varphi(\mathfrak{o}^i)\mathfrak{a}\cdot\mathfrak{a}^{-1}}=\overline{\varphi(\mathfrak{o}^i)\mathfrak{a}\mathfrak{a}^{-1}}\subseteq\overline{\varphi(\mathfrak{o}^i)\varphi(\mathfrak{a})\mathfrak{a}^{-1}}=\overline{\varphi(\mathfrak{a})\mathfrak{a}^{-1}}\subseteq\overline{\varphi(\mathfrak{a})\varphi(\mathfrak{a}^{-1})}=\overline{\varphi(\mathfrak{a}\cdot\mathfrak{a}^{-1})}=\overline{\varphi(\mathfrak{o}^i)}=\varphi(\mathfrak{o}^i)$, we have $\mathfrak{o}^i_P=\overline{\varphi(\mathfrak{a})\mathfrak{a}^{-1}}$ and $\mathfrak{a}_P=\overline{\mathfrak{o}^i_P\mathfrak{a}}=\overline{\varphi_P(\mathfrak{a})\mathfrak{a}^{-1}\mathfrak{a}}=\overline{\varphi(\mathfrak{a})\mathfrak{o}^k}$. On the other hand, since $\varphi(\mathfrak{a})\subseteq\overline{\varphi(\mathfrak{a})\mathfrak{o}^k}\subseteq\overline{\varphi(\mathfrak{a})\varphi(\mathfrak{o}^k)}\subseteq\varphi(\mathfrak{a})$, we have $\varphi(\mathfrak{a})=\overline{\varphi(\mathfrak{a})\mathfrak{o}^k}$. Therefore we obtain $\varphi(\mathfrak{a})\mathfrak{o}=\mathfrak{a}_P=\varphi_P(\mathfrak{a})$.

Suppose that A is the inverse image of $\varphi_P(\mathfrak{a})$. If $\mathfrak{c}\in A$, then $\mathfrak{c}_P=\mathfrak{a}_P$. Hence $\mathfrak{c}\subseteq\mathfrak{a}_P$, $\bigcup_{\mathfrak{c}\in A}\mathfrak{c}\subseteq\mathfrak{a}_P$. Conversely let a be any element in $\mathfrak{a}_P$. Then there exists $\mathfrak{c}$ such that $a\in\mathfrak{c}=\mathfrak{n}^{-1}\cdot\mathfrak{a}$. Since $\mathfrak{c}_P=\mathfrak{n}_P^{-1}\cdot\mathfrak{a}_P=\mathfrak{o}^i_P\cdot\mathfrak{a}_P=\mathfrak{a}_P$, we have $\mathfrak{c}\in A$. Hence $\mathfrak{a}_P=\bigcup_{\mathfrak{c}\in A}\mathfrak{c}$. This completes the proof.

2. We now consider a lattice-formulation of P-components of two-sided $\mathfrak{o}$-ideals in a semigroup. Let L be a lattice-ordered group (l-group) with the ascending chain condition for integral elements,[11] and P a set of prime elements of L. A P-component of an element of L can be defined as follows:

Definition. *The ideal generated by* $\{ap^{-1};\ p\in P\}$ *is called a P-component of* $a\in L$. *Symbol*: $\varphi_P(a)$.

The object of this paragraph is to prove

Theoerm 5. *Let L be an l-group satisfying the ascending chain condition for integral elements. Then the mapping* $\varphi_P: a\to\varphi_P(a)$ *gives a homomorphism from*

8) See §5 in [3].

9) See §5 in [3]. If $\varphi(\mathfrak{o}^i)=S$, then we define $\varphi(\mathfrak{c}^i)=\mathfrak{o}_\phi$, where ϕ denotes the vacuous.

10) See §5 in [3].

11) An element x of L is called integral if x is contained in an identity of L. By the ascending chain condition for integral elements, L forms a commutative group.

L into the l-semigroup[12] $\mathfrak{J}$ *consisting of all ideals of L, and it satisfies* $\varphi_P(a)\ni a$. *Conversely, let* φ *be a mapping from L into* $\mathfrak{J}$ *which satisfies* $\varphi(a)\ni a$ *and* $\varphi(ab)=\varphi(a)\varphi(b)$. *Then* φ *coincides with some* φ_P. *Hence the set* Φ *of all group-homomorphisms* φ *from L into* $\mathfrak{J}$, *each of which satisfies* $\varphi(a)\ni a$ *for every element* $a\in L$, *forms an atomic Boolean algebra under an inclusion relation* $\leq$, *where* $\varphi\leq\psi$ *means* $\varphi(x)\subseteq\psi(x)$ *for all x in L.*

LEMMA. *Let* $J(\neq I)$ *be an m-ideal*[13] *of L. Then there exists a suitable set P of prime elements such that* $J=\varphi_P(e)$.

Proof of Lemma. Since $J\supseteq I$ and $J\neq I$, we can take a non-integral element c in J. Let $c^{-1}\cap e=p_1\cdots p_r$ be the factorization into prime elements p_i. Then since $p_i\geq c^{-1}\cap e$, we have $p_i^{-1}\leq c\cup e\in J$. Hence $p_i^{-1}\in J$. That is to say, $P=\{p;\ p^{-1}\in J\}$ is non-void. We now prove that $\varphi_P(e)=J$. Evidently $\varphi_P(e)$ is contained in J. Let $\varphi_P(e)\neq J$ and take an element $a\in J$ such that $a\notin\varphi_P(e)$. Then $a\cup e\notin\varphi_P(e)$. Let $(a\cup e)^{-1}=a^{-1}\cap e=\prod_{i=1}^{n}p_i$ be the factorization into prime elements p_i. Then $a\cup e=\prod_{i=1}^{n}p_i^{-1}$. If $p_1^{-1},\cdots,p_n^{-1}$ are contained in $\varphi_P(e)$, then $a\cup e\in\varphi_P(e)$ and $a\in\varphi_P(e)$, a contradiction. Hence there exists p_i^{-1} which is not contained in $\varphi_P(e)$. On the other hand, since $p_i^{-1}\leq a\cup e\in J$, we have $p_i^{-1}\in J$. This is a contradiction. Hence $J=\varphi_P(e)$, as desired.

Proof of Theorem. The first part of the theorem is easily obtained. We now prove the later part of the theorem. Evidently $\varphi(e)$ forms an m-ideal of L. If $\varphi(e)\neq I$, then by Lemma, $\varphi(e)=\varphi_P(e)$ for a suitable set P of prime elements of L. Since $\varphi(e)=a^{-1}a\varphi(e)\subseteq a^{-1}\varphi(a)\varphi(e)=a^{-1}\varphi(a)\subseteq\varphi(a^{-1})\varphi(a)=\varphi(a^{-1}a)=\varphi(e)$, we have $a^{-1}\varphi(a)=\varphi(e)=\varphi_P(e)$. Hence $\varphi(a)=a\varphi_P(e)=\varphi_P(a)$. Conversely, for any set P of prime elements, the mapping φ: $a\rightarrow\varphi_P(a)$ gives a group-homomorphism satisfying $\varphi(a)\ni a$. It is easily verified that $\varphi_P(e)=\varphi_Q(e)$ implies $P=Q$. Hence the mapping $\varphi\rightarrow P(\varphi=\varphi_P)$ is one-to-one between Φ and $\mathfrak{P}$, where $\mathfrak{P}$ is the set of all prime elements of L. If $\varphi_P(x)\subseteq\varphi_Q(x)$, then $P\subseteq Q$. Hence the mapping $\varphi\rightarrow P$ gives a lattice-isomorphism between Φ and $\mathfrak{P}$. Φ forms therefore an atomic Boolean algebra. This completes the proof.

REMARK. By the proof of Theorem 5, we obtain that Φ is lattice-isomorphic to the lattice $\mathfrak{L}$ of all the l-ideals[14] of L, and also to the lattice $\mathfrak{N}$ of all the m-ideals of L. Hence of course $\mathfrak{L}$ is lattice-isomorphic to $\mathfrak{N}$. In details this isomorphism is represented as follows:

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12) A multiplication $J\cdot J'$ of J and J' in $\mathfrak{J}$ is defined as an ideal generated by $\{xx';\ x\in J,\ x'\in J'\}$. Then $\mathfrak{J}$ forms an l-semigroup ([4]) under this multiplication and set-inclusion relation.

13) An ideal of an l-group is called an m-ideal when it forms a semigroup containing the identity e. Then the set I of all integral elements forms an m-ideal, any and m-ideal contains I.

14) Cf. [4] Chapter XIII.

$$N \to J = J(e, \{p^{-1};\ p \in N\}), \quad (N \in \mathfrak{L}),$$
$$L \to N = J \wedge J^*, \qquad (J \in \mathfrak{N}),$$

where J^* denotes the dual ideal of J, and $\wedge$ denotes the intersection.

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