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A generalization of a theorem of W. Hurewicz

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W. Hurewictz proved the following theorem for separable metric spaces R and S.

If f is a closed continuous mapping of R onto S such that for each point q of S the inverse image $f^{-1}(q)$ consists of at most $m+1$ points, then $\dim S \leqq \dim R+m$[1].

This theorem was extended by K. Morita to $\operatorname{ind}\dim R$ and $\dim S$ of normal spaces R and S[2].

The purpose of this brief note is to generalize Hurewicz's theorem as follows.

THEOREM. *If f is a closed continuous mapping of a normal space R onto a perfectly normal space S such that for each point q of S the boundary $B(f^{-1}(q))$ of $f^{-1}(q)$ consists of at most $m+1$ points ($m \geqq 0$), then*

$$\operatorname{ind}\dim S \leqq \operatorname{ind}\dim R+m\,.$$

Proof. We assume $\operatorname{ind}\dim R \leqq n$ and shall carry out the proof of $\operatorname{ind}\dim S \leqq n+m$ by induction with respect to $n \geqq -1$ and $m \geqq 0$.

1. This proposition is clearly valid for $n=-1$ and for every $m \geqq 0$.

2. Let us show the validity of this theorem for every $n > -1$ and for $m=0$. Assume G_1 and G_2 are arbitrary closed sets of S such that $G_1 \frown G_2=\phi$. Then $F_1=f^{-1}(G_1)$ and $F_2=f^{-1}(G_2)$ are disjoint closed sets of R. Hence we have, from $\operatorname{ind}\dim R \leqq n$, an open set U satisfying $F_1 \subseteq U \subseteq \bar{U} \subseteq F_2^c$[3], $\operatorname{ind}\dim(\bar{U}-U) \leqq n-1$. Since f is a closed mapping, $V=\{f(U^c)\}^c$ is an open set of S and it satisfies $G_1 \subseteq V \subseteq \bar{V} \subseteq G_2^c$. For $f^{-1}(G_1)=F_1 \subseteq U$ implies $G_1 \subseteq V$. $q \in G_2$ implies $f^{-1}(q) \subseteq F_2 \subseteq (\bar{U})^c$, and hence $(f(\bar{U}))^c=Q$ is an open nbd (=neighborhood) of q satisfying $Q \frown V=\phi$, proving $q \notin \bar{V}$ and consequently $\bar{V} \subseteq G_2^c$.

Letting $f(\bar{U}-U)=H$, we have a closed set H. Let q be an arbitrary point of $(\bar{V}-V)-H$; then $f^{-1}(q) \frown U \neq \phi$, $f^{-1}(q) \frown (\bar{U})^c \neq \phi$, $f^{-1}(q) \frown (\bar{U}-U)=\phi$. For $f^{-1}(q) \frown U =\phi$ implies $\{f(R-f^{-1}(q))\}^c=Q \ni q$, $Q \frown V=\phi$, i.e. $q \notin \bar{V}$. $f^{-1}(q) \frown (\bar{U})^c=\phi$ implies $q \in V$. The both cases are impossible. $f^{-1}(q) \frown (\bar{U}-U)=\phi$ is obvious.

We put $f^{-1}(q)_1=f^{-1}(q) \frown U$, $f^{-1}(q)_2=f^{-1}(q) \frown (\bar{U})^c$. Then we can show $B(f^{-1}(q)) \frown f^{-1}(q)_1 \neq \phi$. To show this we assume the contrary. Then $f^{-1}(q)_1$ is open,

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1) W. Hurewicz, Ein Theorem der Dimensionstheorie, Ann. Math., 31 (1930). We denote by $\dim R$ Lebesgue's dimension of R.

2) K. Morita, On closed mapping and dimension, Proc. Japan Acad., 32, no. 3 (1956). $\operatorname{Ind}\dim \phi=-1$, $\operatorname{ind}\dim R \leqq n$ if and only if for any pair of a closed set F and an open set G with $F \subseteq G$ there exists an open set U such that $F \subseteq U \subseteq \bar{U} \subseteq G$, $\operatorname{ind}\dim(\bar{U}-U) \leqq n-1$.

3) We denote by F^c the complement set of F.

and hence $P=f^{-1}(q)_1 \cup (\bar{U})^c$ is an open set containing $f^{-1}(q)$. Since $P \cap U=f^{-1}(q)_1$, $Q=\{f(P^c)\}^c$ is an open nbd of q satisfying $Q \cap V=\phi$, which contradicts $q \in \bar{V}$. Therefore $B(f^{-1}(q)) \cap f^{-1}(q)_2=\phi$, i.e. $f^{-1}(q)_2$ is open.

i) In the case of $n=0$ we have $\bar{U}-U=\phi$, and hence $H=\phi$. Therefore for every point $q \in \bar{V}-V$ it follows from the openness of $f^{-1}(q)_2$ that $W=\{f(U \cup f^{-1}(q)_2)^c\}^c$ is an open nbd of q such that $W \cap V^c=\{q\}$. Hence $\bar{V}$ is open and closed, proving ind dim $S \leqq 0$.

ii) In the case of $n>0$ we assume the validity of this proposition for $n-1$. Since f is a continuous, closed mapping of $\bar{U}-U$ onto H, we have ind dim $H \leqq n-1$ from the assumption. We can choose, by the perfect normality of S, closed sets $H_k (k=1, 2 \cdots)$ such that $(\bar{V}-V)-H=\bigcup_{k=1}^{\infty} H_k$. It follows from $f^{-1}(H_k) \cap (\bar{U}-U)=\phi$ that $f^{-1}(H_k) \cap (\bar{U})^c=\cup\{f^{-1}(q)_2 \mid q \in H_k\}=E_k$ is a closed set of R. Since $f(E_k)=H_k$, f is a closed, continuous mapping of E_k onto H_k. Since $f^{-1}(q)_2$ for every point q of H_k is open, q is an isolated point of H_k. Hence the subspace H_k is a discrete space, and consequently ind dim $H_k \leqq 0$. Thus we can conclude, from the sum-theorem, ind dim $(\bar{V}-V)=$ ind dim $(H \cup (\bigcup_{k=1}^{\infty} H_k)) \leqq n-1$, i.e. ind dim $S \leqq n$.

3. Now we assume the validity of this proposition for the case where ind dim $R \leqq n-1$ and $B(f^{-1}(q))$ consists of at most $m+1$ points and for the case where ind dim $R \leqq n$ and $B(f^{-1}(q))$ consists of at most m points. Then we shall prove it for the case where ind dim $R \leqq n$ and $B(f^{-1}(q))$ consists of at most $m+1$ points.

Let G_1 and G_2 be arbitrary closed sets of S with $G_1 \cap G_2=\phi$; then we can define F_1, F_2. U, V and H in the same way as in the above proof 2. Since $\bar{U}-U$ is closed, f is a closed, continuous mapping of $\bar{U}-U$ onto H. Therefore ind dim $(\bar{U}-U) \leqq n-1$ combining with the inductive assumption implies ind dim $H \leqq n+m-1$. Furthermore we define H_k and E_k $(k=1, 2 \cdots)$ as in the above. Then f is a closed, continuous mapping of E_k onto H_k. It follows from $B(f^{-1}(q)) \cap f^{-1}(q)_1 \neq \phi$ that the boundary of $f^{-1}(q) \cap E_k$ in E_k consists of at most m points. Hence we have ind dim $H_k \leqq n+m-1$ from the inductive assumption. Thus we can conclude ind dim $(\bar{V}-V)=$ ind dim $(H \cup (\bigcup_{k=1}^{\infty} H_k)) \leqq n+m-1$, which completes the proof of ind dim $S \leqq n+m$.