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Cohomology mod p of symmetric products of spheres

By Minoru NAKAOKA

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Throughout this paper, we denote by $\mathfrak{S}_m$ the symmetric group of degree m, K a finite simplicial complex and p a fixed prime integer. The group $\mathfrak{S}_m$ operates in a natural way on the m-fold cartesian product $\mathfrak{X}_m(K)=K\times K\times\cdots\times K$. The orbit space $\mathfrak{S}_m(K)$ over $\mathfrak{X}_m(K)$ relative to $\mathfrak{S}_m$ is called the m-fold symmetric product. We study in the present paper the cohomology mod p of the symmetric product $\mathfrak{S}_m(S^n)$ of an n-sphere S^n. However the method we use will be applicable for calculation of cohomology of the symmetric product of more general complexes.

Let St^I denote the iterated Steenrod reduced powers, and $v_{0,m}$ a generator of $H^n(\mathfrak{S}_m(S^n); Z_p)\approx Z_p$. Then our main theorem is stated as follows[0]: If $q<n$ and $p^h\leqq m<p^{h+1}$, the vector space $H^{n+q}(\mathfrak{S}_m(S^n);Z_p)$ has a base formed by elements $\mathrm{St}^I v_{0,m}$, where I runs over the set of all admissible and special elements with degree q and length $\leqq h$. (See §3 for the precise definitions.)

The method we use is as follows.

Let $\mathfrak{S}_\infty(K)$ denote the infinite symmetric product of K. It follows from a result in my paper [7] that the injection homomorphism ι_m^*: $H^q(\mathfrak{S}_\infty(K)\,;\,Z_p)\longrightarrow H^q(\mathfrak{S}_m(K)\,;\,Z_p)$ is an epimorphism. As was proved by Dold-Thom [4], $\mathfrak{S}_\infty(K)$ is a product of the Eilenberg-MacLane complexes. Therefore we can describe a set of generators for $H^q(\mathfrak{S}_m(K)\,;\,Z_p)$ in virtue of the Cartan's computation [2]. In order to examine if these generators are linearly independent, we choose a particular p-Sylow subgroup $\mathfrak{G}_m$ of $\mathfrak{S}_m$, and consider the orbit space $\mathfrak{G}_m(K)$ over $\mathfrak{X}_m(K)$ relative to $\mathfrak{G}_m$. The natural projection defines a homomorphism $\rho^*: H^q(\mathfrak{S}_m(K)\,;\,Z_p)\to H^q(\mathfrak{G}_m(K)\,;\,Z_p)$. We prove it by using of the transfer homomorphism that ρ^* is a monomorphism. Let $m=a_0p^h+a_1p^{h-1}+\cdots+a_h$ $(0\leqq a_i<p)$ be the p-adic expansion of m, and denote by $\mathfrak{Z}_p(K)$ the p-fold cyclic product of K (i.e. the orbit space over $\mathfrak{X}_p(K)$ relative to the subgroup $\mathfrak{Z}_p\subset\mathfrak{S}_p$ of cyclic permutations). Then we have that $\mathfrak{G}_m(K)$ is homeomorphic with the space $\mathfrak{X}_{a_0}(\mathfrak{Z}_p^h(K))\times\mathfrak{X}_{a_1}(\mathfrak{Z}_p^{h-1}(K))\times\cdots\times\mathfrak{X}_{a_h}(K)$, where $\mathfrak{Z}_p^r(K)$ denotes the iterated cyclic product $\mathfrak{Z}_p\mathfrak{Z}_p\cdots\mathfrak{Z}_p(K)$ (r-times) of K. As for the cohomology structure of $\mathfrak{Z}_p(K)$, I have studied in the paper [6]. By making use of some results there, we analyse the cohomology structure mod p of $\mathfrak{Z}_p^r(K)$, and we determine the dependence of the generators.

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0) (Added April 14, 1958) I have recently succeeded in determination of the cohomology ring $H^*(\mathfrak{G}_m(S^n)\,;\,Z_p)$.

1. The orbit space $\mathfrak{S}_{p^r}(K)$

In this and next sections, we study the orbit space over the m-fold cartesian product of K relative to a p-Sylow subgroup of $\mathfrak{S}_m$. The special case $m=p^r$ is dealt in this section, and the general case in next section.

Let q be an integer $\geqq 0$. Denote by Ω_q a set consisting of all sequences $(i_1, i_2, \cdots, i_q)$, where each i_j is an integer mod p. Ω_q has p^q elements. We shall associate to an element $(i_1, i_2, \cdots, i_q) \in \Omega_q$ an integer $A_{i_1, i_2, \dots, i_q}$ defined as follows:

$$A_{i_1, i_2, \dots, i_q} = i_1 p^{q-1} + i_2 p^{q-2} + \cdots + i_q + 1 \qquad (0 \leqq i_j < p).$$

This gives clearly a one-to-one correspondence of Ω_q onto the set $\{1, 2, \cdots, p^q\}$.

We shall regard $\mathfrak{S}_m$ as the group of all transformations of m letters $1, 2, \cdots, m$. For each q $(0 \leqq q < r)$ and each $(k_1, k_2, \cdots, k_q) \in \Omega_q$, we define an element $T^r_{k_1, k_2, \dots, k_q} \in \mathfrak{S}_{p^r}$ by

$$\begin{aligned}(1.1)\qquad & T^r_{k_1, \dots, k_q}(A_{i_1, \dots, i_r}) \\ &= A_{i_1, \dots, i_q, i_{q+1}+1, i_{q+2}, \dots, i_r} \quad \text{if } (i_1, \cdots, i_q) = (k_1, \cdots, k_q), \\ &= A_{i_1, \dots, i_r} \quad \text{otherwise.}\end{aligned}$$

Obviously we have

$$(1.2)\qquad (T^r_{k_1, \dots, k_q})^p = 1.$$

We shall prove

LEMMA 1. *$T^r_{j_1, \dots, j_m} T^r_{k_1, \dots, k_q}$*

$$\begin{aligned}&= T^r_{k_1, \dots, k_q} T^r_{j_1, \dots, j_m} \quad \textit{if } m \leqq q \textit{ and } (j_1, \cdots, j_m) \neq (k_1, \cdots, k_m), \\ &= T^r_{k_1, \dots, k_m, k_{m+1}+1, k_{m+2}, \dots, k_q} T^r_{j_1, \dots, j_m} \\ &\qquad \textit{if } m < q \textit{ and } (j_1, \cdots, j_m) = (k_1, \cdots, k_m).\end{aligned}$$

Proof. The following can be easily proved from the definition (1.1).

Case I: $m \leqq q$ and $(j_1, \cdots, j_m) \neq (k_1, \cdots, k_m)$

$$\begin{aligned}& T^r_{j_1, \dots, j_m} T^r_{k_1, \dots, k_q}(A_{i_1, \dots, i_r}) \\ &= T^r_{k_1, \dots, k_q} T^r_{j_1, \dots, j_m}(A_{i_1, \dots, i_r}) \\ &= \begin{cases} A_{i_1, \dots, i_{m+1}+1, \dots, i_r} & \text{if } (i_1, \cdots, i_m) = (j_1, \cdots, j_m), \\ A_{i_1, \dots, i_{q+1}+1, \dots, i_r} & \text{if } (i_1, \cdots, i_m) \neq (j_1, \cdots, j_m) \text{ and } (i_1, \cdots, i_q) = (k_1, \cdots, k_q), \\ A_{i_1, \dots, i_r} & \text{if } (i_1, \cdots, i_m) \neq (j_1, \cdots, j_m) \text{ ahd } (i_1, \cdots, i_q) \neq (k_1, \cdots, k_q). \end{cases}\end{aligned}$$

Case II: $m < q$ and $(j_1, \cdots, j_m) = (k_1, \cdots, k_m)$

$$\begin{aligned}& T^r_{j_1, \dots, j_m} T^r_{k_1, \dots, k_q}(A_{i_1, \dots, i_r}) \\ &= T^r_{k_1, \dots, k_{m+1}+1, \dots, k_q} T_{j_1, \dots, j_m}(A_{i_1, \dots, i_r}) \\ &= \begin{cases} A_{i_1, \dots, i_{m+1}+1, \dots, i_{q+1}+1, \dots, i_r} & \text{if } (i_1, \cdots, i_q) = (k_1, \cdots, k_q), \\ A_{i_1, \dots, i_{m+1}+1, \dots, i_r} & \text{if } (i_1, \cdots, i_q) \neq (k_1, \cdots, k_q) \text{ and } (i_1, \cdots, i_m) = (k_1, \cdots, k_m), \\ A_{i_1, \dots, i_r} & \text{if } (i_1, \cdots, i_m) \neq (k_1, \cdots, k_m) \end{cases}\end{aligned}$$

Q. E. D.

Let $\pi^r_{k_1,\dots,k_q} \subset \mathfrak{S}_{p^r}$ denote a cyclic subgroup generated by $T^r_{k_1,\dots,k_q}$. The order of $\pi^r_{k_1,\dots,k_q}$ is p. Since

$$T^r_{j_1,\dots,j_q} T^r_{k_1,\dots,k_q} = T^r_{k_1,\dots,k_q} T^r_{j_1,\dots,j_q} \quad \text{if } (j_1,\cdots,j_q) \neq (k_1,\cdots,k_q), \tag{1.3}$$

we may define $\rho^r_{q+1} \subset \mathfrak{S}_{p^r}$ $(0 \leqq q < r)$ by

$$\rho^r_{q+1} = \prod_{(k_1,\cdots,k_q) \in \Omega_q} \pi^r_{k_1,\dots,k_q},$$

the product of $\pi^r_{k_1,\dots,k_q}$'s as subgroups of $\mathfrak{S}_{p^r}$. ρ^r_{q+1} is the direct product of $\pi^r_{k_1,\dots,k_q}$'s, and its order is the p^q-th power of p.

Next, for $q=1, 2, \cdots, r$, define

$$\sigma^r_q = \rho^r_1 \rho^r_2 \cdots \rho^r_q,$$

the product of ρ^r_m's as subgroups of $\mathfrak{S}_{p^r}$. Since Lemma 1 yields that $\rho^r_m \rho^r_n = \rho^r_n \rho^r_m$ $(1 \leqq m, n \leqq q)$, it follows that σ^r_q is a subgroup of $\mathfrak{S}_{p^r}$. Furthermore Lemma 1 shows that ρ^r_q is an invariant subgroup of σ^r_q. We have

$$\sigma^r_q / \rho^r_q \approx \sigma^r_{q-1}. \tag{1.4}$$

Actually, σ^r_q is a split extension of ρ^r_q by σ^r_{q-1}, where σ^r_{q-1} operates non-trivially on ρ^r_q. From (1.4), we obtain by induction on q that the order of σ^r_q is the $(p^{q-1}+p^{q-2}+\cdots+1)$-th power of p.

We write $\mathfrak{G}_{p^r} = \sigma^r_r$. The order of $\mathfrak{G}_{p^r}$ is the $(p^{r-1}+p^{r-2}+\cdots+1)$-th power of p. Since this is the highest order of p in $p^r!$, the group $\mathfrak{G}_{p^r}$ is a p-Sylow subgroup of $\mathfrak{S}_{p^r}$.[1)]

We note here the following

LEMMA 2. *If* $0 \leqq q < r-1$ *and* $T^{r-1}_{k_1,\dots,k_q}(A_{i_1,\dots,i_{r-1}}) = A_{j_1,\dots,j_{r-1}}$, *then* $T^r_{k_1,\dots,k_q}(A_{i_1,\dots,i_r}) = A_{j_1,\dots,j_{r-1},i_r}$.

This is clear from the definition (1.1).

Let $\mathfrak{X}_{p^r}(K)$ be the p^r-fold cartesian product of K. A point x of $\mathfrak{X}_{p^r}(K)$ is given as a function x defined for each $A_{i_1,\dots,i_r}$ and takes values in K. The symmetric group $\mathfrak{S}_{p^r}$ operates on $\mathfrak{X}_{p^r}(K)$ in a natural manner:

$$(\alpha x)(A_{i_1,\dots,i_r}) = x(\alpha(A_{i_1,\dots,i_r})), \quad \alpha \in \mathfrak{S}_{p^r}.$$

Define a map $f\colon \mathfrak{X}_{p^r}(K) \longrightarrow \mathfrak{X}_{p^{r-1}}(\mathfrak{X}_p(K))$ by

$$(fx)(A_{i_1,\dots,i_{r-1}}) = x(A_{i_1,\dots,i_{r-1},0}) \times x(A_{i_1,\dots,i_{r-1},1}) \times \cdots \times x(A_{i_1,\dots,i_{r-1},p-1}) \in \mathfrak{X}_p(K).$$

It is obvious that f is an onto-homeomorphism.

LEMMA 3. *If* $0 \leqq q < r-1$, *then*

$$f T^r_{k_1,\dots,k_q} = T^{r-1}_{k_1,\dots,k_q} f.$$

1) Such a subgroup for $p=2$ is studied in [1] by J. Adem.

Proof. Let $x \in \mathfrak{X}_{p^r}(K)$ and put $A_{j_1, \cdots, j_{r-1}} = T^{r-1}_{k_1, \cdots, k_q}(A_{i_1, \cdots, i_{r-1}})$. Then we have

$$
\begin{aligned}
&(fT^r_{k_1, \cdots, k_q}x)\ (A_{i_1, \cdots, i_{r-1}}) \\
&= (T^r_{k_1, \cdots, k_q}x)(A_{i_1, \cdots, i_{r-1}, 0}) \times \cdots \times (T^r_{k_1, \cdots\ k_q}x)(A_{i_1, \cdots, i_{r-1}, p-1}) \\
&= x(A_{j_1, \cdots, j_{r-1}, 0}) \times \cdots \times x(A_{j_1, \cdots, j_{r-1}, p-1}) \qquad \text{(cf. Lemma 2)} \\
&= (fx)(A_{j_1, \cdots, j_{r-1}}) \\
&= (T^{r-1}_{k_1, \cdots\ k_q} fx)(A_{i_1, \cdots, i_{r-1}})\,.
\end{aligned}
$$

Q. E. D.

Denote by $\mathfrak{Z}_p(K)$ the p-fold cyclic product of K. Let $\mathfrak{Z}_p \subset \mathfrak{S}_p$ be the subgroup of cyclic permutations. Then, by definition, $\mathfrak{Z}_p(K)$ is the orbit space $O(\mathfrak{X}_p(K), \mathfrak{Z}_p)$ over $\mathfrak{X}_p(K)$ relative to $\mathfrak{Z}_p$[2]. Write $\bar{I}: \mathfrak{X}_p(K) \longrightarrow \mathfrak{Z}_p(K)$ for the identification map.

Let $g: \mathfrak{X}_{p^{r-1}}(\mathfrak{X}_p(K)) \longrightarrow \mathfrak{X}_{p^{r-1}}(\mathfrak{Z}_p(K))$ be a continuous map defined by

$$g = \bar{I} \times \bar{I} \times \cdots \times \bar{I} \quad (p^{r-1}\text{-fold})\,,$$

namely

$$(gy)\ (A_{i_1, \cdots, i_{r-1}}) = \bar{I}(y(A_{i_1, \cdots, i_{r-1}}))\,, \quad y \in \mathfrak{X}_{p^{r-1}}(\mathfrak{X}_p(K))\,.$$

It follows immediately that

$$\beta g = g\beta \qquad (\beta \in \mathfrak{S}_{p^{r-1}})\,. \tag{1.5}$$

LEMMA 4.
$$
\begin{aligned}
gfT^r_{k_1, \cdots, k_q} &= gf && \textit{for} \quad q = r-1\,, \\
&= T_{k_1, \cdots, k_q} gf && \textit{for} \quad q < r-1\,.
\end{aligned}
$$

Proof. The formula for $0 \leqq q < r-1$ is obvious from Lemma 3 and (1.5). We shall prove $gfT^r_{k_1, \cdots, k_{r-1}} = gf$.

For $x \in \mathfrak{X}_{p^r}(K)$, we have

$$
\begin{aligned}
&(fT^r_{k_1, \cdots, k_{r-1}}x)\ (A_{i_1, \cdots, i_{r-1}}) \\
&= (T^r_{k_1, \cdots, k_{r-1}}x)(A_{i_1, \cdots, i_{r-1}, 0}) \times \cdots \times (T^r_{k_1, \cdots, k_{r-1}}x)(A_{i_1, \cdots, i_{r-1}, p-1}) \\
&= \begin{cases}
x(A_{i_1, \cdots, i_{r-1}, 1}) \times \cdots \times x(A_{i_1, \cdots, i_{r-1}, p-1}) \times x(A_{i_1, \cdots, i_{r-1}, p}) \\
\qquad\qquad \text{if} \quad (i_1, \cdots, i_{r-1}) = (k_1, \cdots, k_{r-1})\,, \\
x(A_{i_1, \cdots, i_{r-1}, 0}) \times \cdots \times x(A_{i_1, \cdots, i_{r-1}, p-2}) \times x(A_{i_1\ \cdots, i_{r-1}, p-1}) \\
\qquad\qquad \text{if} \quad (i_1, \cdots, i_{r-1}) \neq (k_1, \cdots, k_{r-1})\,.
\end{cases}
\end{aligned}
$$

Therefore it follows that

$$
\begin{aligned}
&(gfT^r_{k_1, \cdots, k_{r-1}}x)\ (A_{i_1, \cdots, i_{r-1}}) \\
&= \bar{I}((fT^r_{k_1, \cdots, k_{r-1}}x)\ (A_{i_1, \cdots, i_{r-1}})) \\
&= \bar{I}(x(A_{i_1, \cdots, i_{r-1}, 0}) \times \cdots \times x(A_{i_1, \cdots, i_{r-1}, p-1})) \\
&= \bar{I}((fx)\ (A_{i_1, \cdots, i_{r-1}})) \\
&= (gfx)\ (A_{i_1, \cdots, i_{r-1}})\,.
\end{aligned}
$$

Q. E. D.

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2) Let Y be a space on which a group Γ operates. Then the orbit space $O(Y, \Gamma)$ over Y relative to Γ is defined as a space obtained from Y by identifying each point $y \in Y$ with its image $\gamma(y)$ $(\gamma \in \Gamma)$.

LEMMA 5. *If* $gf(x)=gf(x')$ *for* $x,\ x'\in\mathfrak{X}_{p^r}(K)$, *then* $x'=\alpha x$ *with* $\alpha\in\rho_r^r$.

Proof. Since $(gfx)(A_{i_1,\cdots,i_{r-1}})=\bar{\mathrm{I}}(x(A_{i_1,\cdots,i_{r-1},0})\times\cdots\times x(A_{i_1,\cdots,i_{r-1},p-1}))$ and $(gfx')(A_{i_1,\cdots,i_{r-1}})=\bar{\mathrm{I}}(x'(A_{i_1,\cdots,i_{r-1},0})\times\cdots\times x'(A_{i_1,\cdots,i_{r-1},p-1}))$, it follows that

$$x'(A_{i_1,\cdots,i_{r-1},i_r})=x(A_{i_1,\cdots,i_{r-1},i_r+n})\ (i_r=0,\ 1,\ \cdots,\ p-1)\,,$$

where $n=n(i_1,\ \cdots,\ i_{r-1})$ is an integer mod p depending on $(i_1,\ \cdots,\ i_{r-1})$.

Let α be an element of the abelian group ρ_r^r defined by

$$\alpha=\prod_{(k_1,\cdots,k_{r-1})\in\Omega_{r-1}}(T^r_{k_1,\cdots,k_{r-1}})^{n(k_1,\cdots,k_{r-1})}\,.$$

Then it follows that

$$\begin{aligned}&(\alpha x)\ (A_{i_1,\cdots,i_r})\\&=x((T^r_{i_1,\cdots,i_{r-1}})^{n(i_1,\cdots,i_r)}(A_{i_1,\cdots,i_r}))\,.\\&=x(A_{i_1,\cdots,i_{r-1},i_r+n})\quad(n=n(i_1,\ \cdots,\ i_{r-1}))\,.\end{aligned}$$

Therefore $x'(A_{i_1,\cdots,i_r})=(\alpha x)(A_{i_1,\cdots,i_r})$, and hence $x'=\alpha x$. Q. E. D.

Write $\mathfrak{S}_{p^r}(K)$ for the orbit space $\mathrm{O}(\mathfrak{X}_{p^r}(K),\ \mathfrak{S}_{p^r})$, and consider the identification maps

$$\begin{aligned}\varphi:&\quad\mathfrak{X}_{p^r}(K)\longrightarrow\mathfrak{S}_{p^r}(K)\,,\\ \psi:&\quad\mathfrak{X}_{p^{r-1}}(\mathfrak{Z}_p(K))\longrightarrow\mathfrak{S}_{p^{r-1}}(\mathfrak{Z}_p(K))\,.\end{aligned}$$

Then it follows from Lemma 4 that $gf\colon\mathfrak{X}_{p^r}(K)\to\mathfrak{X}_{p^{r-1}}(\mathfrak{Z}_p(K))$ defines a continuous map $h\colon\mathfrak{S}_{p^r}(K)\longrightarrow\mathfrak{S}_{p^{r-1}}(\mathfrak{Z}_p(K))$ such that

$$\psi gf=h\varphi\,. \tag{1.6}$$

PROPOSITION 1. *h is an onto-homeomorphism.*

Proof. Since gf and φ are onto, it follows from (1.6) easily that h is onto. We shall next prove that h is one-to-one. Since φ is onto, it is sufficient for this purpose to prove that if $h\varphi(x)=h\varphi(x')$ for $x,\ x'\in\mathfrak{X}_{p^r}(K)$ then $x'=\gamma x$ with $\gamma\in\mathfrak{S}_{p^r}$. Under this assumption, it follows from (1.6) that $\psi gf(x)=\psi gf(x')$. Therefore $gf(x')=\beta gf(x)$ with $\beta\in G_{p^{r-1}}$. Let $\beta=T^{r-1}_{I_1}T^{r-1}_{I_2}\cdots T^{r-1}_{I_w}$, where each $I_j\in\Omega_q$ $(q<r-1)$. Put $\bar\beta=T^r_{I_1}T^r_{I_2}\cdots T^r_{I_w}\in G_{p^r}$. Then it follows from Lemma 4 that $gf(x')=gf\bar\beta(x)$. Therefore Lemma 5 implies that $x'=\alpha\bar\beta x$ with $\alpha\in\rho_r^r$. Put $\gamma=\alpha\bar\beta$. Since $\gamma\in\mathfrak{S}_{p^r}$, we obtain $x'=\gamma x$ $(\gamma\in\mathfrak{S}_{p^r})$.

Since h is continuous and $\mathfrak{S}_{p^r}(K)$ is compact, it follows that h is an onto-homeomorphism. Q. E. D.

Define the iterated cyclic product $\mathfrak{Z}_p^r(K)\,(r=0,1,\cdots)$ by

$$\mathfrak{Z}_p^r(K)=\mathfrak{Z}_p(\mathfrak{Z}_p^{r-1}(K)),\quad\mathfrak{Z}_p^0(K)=K\,.$$

We have

THEOREM 1. *The space* $\mathfrak{S}_{p^r}(K)$ *is homeomorphic with the iterated cyclic product* $\mathfrak{Z}_p^r(K)$.

Proof. For $r=0$ the theorem is trivial. To establish the general case we proceed by induction. Assume that $\mathfrak{G}_{p^{r-1}}(\mathfrak{Z}_p(K))$ is homeomorphic with $\mathfrak{Z}_p^{r-1}(K)$ for every K. Then $\mathfrak{G}_{p^{r-1}}(\mathfrak{Z}_p(K))$ is homeomorphic with $\mathfrak{Z}_p^{r-1}(\mathfrak{Z}_p(K))=\mathfrak{Z}_p^r(K)$. Therefore it follows from Proposition 1 that $\mathfrak{G}_{p^r}(K)$ is homeomorphic with $\mathfrak{Z}_p^r(K)$. Q. E. D.

2. The orbit space $\mathfrak{G}_m(K)$

Let m be an integer, and let

$$m=\sum_{r=0}^{h} a_{h-r}p^r \qquad (0\leqq a_i<p)$$

be the p-adic expansion of m. Denote by $W(m)$ a set consisting of all pairs (r, j) of integers such that $0\leqq r\leqq h$, $1\leqq j\leqq a_{h-r}$. To each $(r, j)\in W(m)$, we shall associate a monotone map θ_r^j: $\{1, 2, \cdots, p^r\}\longrightarrow\{1, 2, \cdots, m\}$ defined by

$$\theta_r^j(s)=\sum_{q=r+1}^{h} a_{h-q}p^q+(j-1)p^r+s \qquad (1\leqq s\leqq p^r),$$

and define a monomorphism $\bar{\theta}_r^j$: $\mathfrak{S}_{p^r}\longrightarrow\mathfrak{S}_m$ by

$$\begin{aligned}(\bar{\theta}_r^j\alpha)(t)&=\theta_r^j\alpha(s) && \text{if } t=\theta_r^j(s) \text{ with } 1\leqq s\leqq p^r,\\ &=t && \text{otherwise,}\end{aligned}$$

where $\alpha\in\mathfrak{S}_{p^r}$ and $1\leqq t\leqq m$. Write ${}^j\mathfrak{G}_{p^r}$ for the image group $\bar{\theta}_r^j(\mathfrak{G}_{p^r})$, where $\mathfrak{G}_{p^r}$ is the p-Sylow subgroup of $\mathfrak{S}_{p^r}$ mentioned in §1. If $(r, j)\neq(q, k)$, then $\alpha\beta=\beta\alpha$ for $\alpha\in{}^j\mathfrak{G}_{p^r}$, $\beta\in{}^k\mathfrak{G}_{p^q}$. Therefore we may define a group $\mathfrak{G}_m\subset\mathfrak{S}_m$ by

$$\mathfrak{G}_m=\prod_{(r,j)\in W(m)} {}^j\mathfrak{G}_{p^r},$$

the product of ${}^j\mathfrak{G}_{p^r}$'s as subgroups of $\mathfrak{S}_m$. $\mathfrak{G}_m$ is the direct product of ${}^j\mathfrak{G}_{p^r}$'s, and hence its order is the $(\sum_{r=1}^{h} a_{h-r}(p^{r-1}+p^{r-2}+\cdots+1))$-th power of p. This is the highest power of p in $m!$, so that $\mathfrak{G}_m$ is a p-Sylow subgroup of $\mathfrak{S}_m$.

We shall represent points of $\mathfrak{X}_m(K)$ as functions y defined on $\{1, 2, \cdots, m\}$ and take values in K. The operation of $\mathfrak{S}_m$ on $\mathfrak{X}_m(K)$ is written as follows:

$$(\beta y)(t)=y(\beta t) \qquad \beta\in\mathfrak{S}_m,\ y\in\mathfrak{X}_m(K),\ 1\leqq t\leqq m.$$

To each $(r, j)\in W(m)$, we shall associate two maps

$$\tilde{\xi}_r^j:\ \mathfrak{X}_{p^r}(K)\longrightarrow\mathfrak{X}_m(K), \qquad \tilde{\eta}_r^j:\ \mathfrak{X}_m(K)\longrightarrow\mathfrak{X}_{p^r}(K)$$

defined by

$$\begin{aligned}(\tilde{\xi}_r^j x)(t)&=x(s) \quad \text{if } t=\theta_r^j s, \text{ and } =* \text{ otherwise,}\\ (\tilde{\eta}_r^j y)(s)&=y(\theta_r^j s),\end{aligned}$$

where $1\leqq s\leqq p^r$, $1\leqq t\leqq m$, $x\in\mathfrak{X}_{p^r}(K)$, $y\in\mathfrak{X}_m(K)$ and $*$ is a base vertex of K. It is obvious that for any $\alpha\in\mathfrak{S}_{p^r}$

$$\tilde{\xi}_r^j \alpha = (\bar{\theta}_r^j \alpha) \tilde{\xi}_r^j ,$$

$$\tilde{\eta}_r^j (\bar{\theta}_q^k \alpha) = \alpha \tilde{\eta}_r^j \text{ if } (q, k) = (r, j), \quad \text{and } = \tilde{\eta}_r^j \text{ otherwise.}$$

Therefore the maps $\tilde{\xi}_r^j$ and $\tilde{\eta}_r^j$ yield respectively maps $\xi_r^j : \mathfrak{G}_{p^r}(K) \longrightarrow \mathfrak{G}_m(K)$ and $\eta_r^j : \mathfrak{G}_m(K) \longrightarrow \mathfrak{G}_{p^r}(K)$. It follows immediately that

$$(2.1) \qquad \begin{aligned} \eta_q^k \xi_r^j &= \text{identity map} && \text{if } (r, j) = (q, k) , \\ &= \text{constant map} && \text{if } (r, j) \neq (q, k) . \end{aligned}$$

Define a map $\eta : \mathfrak{G}_m(K) \longrightarrow \mathfrak{X}_{a_0}(\mathfrak{G}_{p^h}(K)) \times \mathfrak{X}_{a_1}(\mathfrak{G}_{p^{h-1}}(K)) \times \cdots \times \mathfrak{X}_{a_h}(K)$ by

$$\eta(z) = (\eta_h^1(z) \times \cdots \times \eta_h^{a_0}(z)) \times (\eta_{h-1}^1(z) \times \cdots \times \eta_{h-1}^{a_1}(z)) \times \cdots \times (\eta_0^1(z) \times \cdots \times \eta_0^{a_h}(z)) .$$

It is easily seen that η is an onto-homeomorphism. Therefore by Theorem 1 we have

THEOREM 1′. *The space $\mathfrak{G}_m(K)$ is homeomorphic with the space $\mathfrak{X}_{a_0}(\mathfrak{Z}_p^h(K)) \times \mathfrak{X}_{a_1}(\mathfrak{Z}_p^{h-1}(K)) \times \cdots \times \mathfrak{X}_{a_h}(K)$*[3].

For $q > 0$, let

$$\begin{aligned} \xi_r^{j*} &: H^q(\mathfrak{G}_m(K) ; Z_p) \longrightarrow H^q(\mathfrak{G}_{p^r}(K) ; Z_p) , \\ \eta_r^{j*} &: H^q(\mathfrak{G}_{p^r}(K) ; Z_p) \longrightarrow H^q(\mathfrak{G}_m(K) ; Z_p) \end{aligned}$$

be the homomorphisms induced by ξ_r^j and η_r^j respectively. We have then by (2.1)

$$\begin{aligned} \xi_r^{j*} \eta_q^{k*} &= \text{identity} && \text{if } (r, j) = (q, k) , \\ &= 0 && \text{if } (r, j) \neq (q, k) . \end{aligned}$$

Therefore, by the Künneth relation, we have

COROLLARY. *Assume that K is $(n-1)$-connected and $q < 2n$. Then a set of the homomorphisms ξ_r^{j*} (resp. η_r^{j*}), $(r, j) \in W(m)$, provides a projective (resp. injective) representation of $H^q(\mathfrak{G}_m(K) ; Z_p)$ as a direct sum.*

Let

$$\rho : \mathfrak{X}_{a_0}(\mathfrak{Z}_p^h(K)) \times \mathfrak{X}_{a_1}(\mathfrak{Z}_p^{h-1}(K)) \times \cdots \times \mathfrak{X}_{a_h}(K) \longrightarrow \mathfrak{S}_m(K)$$

be the natural projection of $\mathfrak{G}_m(K)$ onto $\mathfrak{S}_m(K)$. Then we have

THEOREM 2. *The homomorphism*

$$\rho^* : H^q(\mathfrak{S}_m(K) ; Z_p) \to H^q(\mathfrak{X}_{a_0}(\mathfrak{Z}_p^h(K)) \times \mathfrak{X}_{a_1}(\mathfrak{Z}_p^{h-1}(K)) \times \cdots \times \mathfrak{X}_{a_h}(K) ; Z_p)$$

induced by ρ is a monomorphism for any q.

More generally we have

THEOREM 2′. *Let Γ_1, Γ_2 ($\Gamma_1 \subset \Gamma_2$) be two subgroups of $\mathfrak{S}_m$ such that the index*

3) Let $\mathfrak{G}'_m$ be any p-Sylow subgroup of $\mathfrak{S}_m$. By the well-known fact, $\mathfrak{G}_m$ and $\mathfrak{G}'_m$ are conjugate. Therefore the space $\mathfrak{G}_m(K)$ and $\mathfrak{G}'_m(K)$ are homeomorphic. In general the following holds: Let Y be a space on which a group Γ operates, and Γ', Γ'' be conjugate subgroups of Γ. Then the orbit space $O(Y, \Gamma')$ and $O(Y, \Gamma'')$ are homeomorphic. In fact, if $\Gamma'' = \alpha \Gamma' \alpha^{-1}$ with $\alpha \in \Gamma$, the map $\bar{\alpha} : O(Y, \Gamma') \longrightarrow O(Y, \Gamma'')$ induced by the transformation $\alpha : Y \longrightarrow Y$ gives a homeomorphism.

$(\Gamma_2:\Gamma_1)$ of Γ_1 in Γ_2 is prime to p. Then the homomorphism $\rho^*: H^q(\mathrm{O}(\mathfrak{X}_m(K), \Gamma_2); Z_p)\longrightarrow H^q(\mathrm{O}(\mathfrak{X}_m(K), \Gamma_1); Z_p)$ induced by the natural projection ρ is a monomorphism for any q.

Since $\mathfrak{G}_m$ is a p-Sylow subgroup of $\mathfrak{S}_m$, the index $(\mathfrak{S}_m:\mathfrak{G}_m)$ is prime to p. Therefore Theorem 2′ implies Theorem 2.

Proof of Theorem 2′. As in the proof of Proposition 1 in [7], it is given by means of the *special cohomology groups* and the *transfer homomorphism.*

Let $C^q(\mathfrak{X}_m(K); Z_p)^{\Gamma_j}$ be the subgroup of the (alternative) cochain group $C^q(\mathfrak{X}_m(K); Z_p)$ which consist of all cochains u such that $\gamma u=u$ for all $\gamma\in\Gamma_j$ $(j=1, 2)$. Then $\{C^q(\mathfrak{X}_m(K); Z_p)^{\Gamma}, \delta\}$ is a cochain complex, where δ denotes the coboundary operator of the simplicial complex $\mathfrak{X}_m(K)$. The cohomology group of this complex is denoted by ${}^{\Gamma_j^{-1}}H^q(\mathfrak{X}_m(K); Z_p)$ (the special cohomology group). Let $i: C^q(\mathfrak{X}_m(K); Z_p)^{\Gamma_2}\longrightarrow C^q(\mathfrak{X}_m(K); Z_p)^{\Gamma_1}$ be the inclusion, and $t: C^q(\mathfrak{X}_m(K); Z_p)^{\Gamma_1}\longrightarrow C^q(\mathfrak{X}_m(K); Z_p)^{\Gamma_2}$ the transfer homomorphism (cf. p. 254 of [3]). We have then

$$ti(c) = (\Gamma_2:\Gamma_1)c, \quad c\in C^q(\mathfrak{X}_m(K); Z_p)^{\Gamma_2}.$$

The cochain maps i and t induce the homomorphisms $i^*: {}^{\Gamma_2^{-1}}H^q(\mathfrak{X}_m(K); Z_p)\longrightarrow {}^{\Gamma_1^{-1}}H^q(\mathfrak{X}_m(K); Z_p)$ and $t^*: {}^{\Gamma_1^{-1}}H^q(\mathfrak{X}_m(K); Z_p)\longrightarrow {}^{\Gamma_2^{-1}}H^q(\mathfrak{X}_m(K); Z_p)$ respectively, and we have

$$t^*i^* = (\Gamma_2:\Gamma_1).$$

Since $(\Gamma_2:\Gamma_1)$ is prime to p, it follows that t^*i^* is an automorphism, and hence i^* is a monomorphism. Denote by $\varphi_j: \mathfrak{X}_m(K)\longrightarrow \mathrm{O}(\mathfrak{X}_m(K), \Gamma_j)$ the natural projection $(j=1, 2)$. Obviously φ_j induces an isomorphism $\varphi_j^*: H^q(\mathrm{O}(\mathfrak{X}_m(K), \Gamma_j); Z_p)\longrightarrow {}^{\Gamma_j^{-1}}H^q(\mathfrak{X}_m(K); Z_p)$, and the commutativity $i^*\varphi_2^*=\varphi_1^*\rho^*$ holds. Consequently ρ^* is a monomorphism. Q. E. D.

3. Prerequisites: notations, cohomology of cyclic product

Let Z_+ denote the set of all non-negative integers. We denote by Z_+^∞ the set consisting of all sequences

$$I = (i_1, i_2, \cdots, i_k \cdots), \qquad (i_k\in Z_+)$$

such that $i_k=0$ for sufficiently large k. In Z_+^∞, we shall consider the following relation of order $<$ (lexicographic order from the left): For any two elements $I=(i_1, i_2, \cdots, i_k, \cdots)$ and $J=(j_1, j_2, \cdots, j_k, \cdots)$ of Z_+^∞, we write $I<J$ if and only if

$$i_1 = j_1, \cdots, i_k = j_k,\ i_{k+1}<j_{k+1}$$

for some k.

For any element $I=(i_1, i_2, \cdots, i_k, \cdots)\in Z_+^\infty$, the *length* $l(I)$, the *height* $h(I)$ and the *degree* $d(I)$ are defined as follows;

$l(I)$ = the least number of l such that $i_k=0$ for all $k>l$,
$h(I)$ = number of the set $\{k \mid i_k \neq 0\}$,
$d(I)=\sum_{k=1}^{\infty} i_k$.

An element $(i_1, i_2, \cdots, i_k, \cdots) \in Z_+^\infty$ is called to be *proper* if $i_k \equiv 0$ or 1 mod $2(p-1)$ for any k. A proper element $(i_1, i_2, \cdots, i_k, \cdots)$ is called to be *admissible* if $i_k \geqq p i_{k+1}$ is satisfied for any $k \geqq 1$. For an admissible element I, we have $h(I)=l(I)$. We say that an element $(i_1, i_2, \cdots, i_k, \cdots)$ is *special* if $i_k \neq 1$ for any k.

The element $(0, 0, \cdots, 0, \cdots) \in Z_+^\infty$ is denoted by O. This is a unique element such that the length is 0. O is admissible and special.

For each $r \in Z_+$, we define a subset $Z_+^r \subset Z_+^\infty$ by

$$Z_+^r = \{I \in Z_+^\infty \mid l(I) \leqq r\}.$$

For any complex K, the Steenrod operations are homomorphisms

$$\mathrm{Sq}^s : H^q(K; Z_2) \longrightarrow H^{q+s}(K; Z_2) \qquad \text{for} \quad p=2,$$
$$\mathcal{P}^s : H^q(K; Z_p) \longrightarrow H^{q+2s(p-1)}(K; Z_p) \quad \text{for} \quad p>2.$$

We shall denote by

$$\Delta : H^q(K; Z_p) \longrightarrow H^{q+1}(K; Z_p)$$

the coboundary operation associated with the coefficient sequence $0 \longrightarrow Z_p \longrightarrow Z_{p^2} \longrightarrow Z_p \longrightarrow 0$ (the Bockstein homomorphism).

Let $i=2s(p-1)+\varepsilon$, where $s \in Z_+$ and $\varepsilon=0$ or 1. Then, following H. Cartan [2], we put

$$St^i = Sq^i,$$
$$= \mathcal{P}^s \text{ if } \varepsilon=0, \qquad = \Delta\mathcal{P}^s \text{ if } \varepsilon=1$$

according as $p=2$ or $p>2$, and we associate to each proper element $I=(i_1, i_2, \cdots, i_k, \cdots)$ a homomorphism

$$St^I : H^q(K; Z_p) \rightarrow H^{q+d(I)}(K; Z_p)$$

defined by

$$St^I = St^{i_1} St^{i_2} \cdots St^{i_k} \cdots.$$

With J. Adem [1], we make the following convention on the binomial coefficient: For any integers i and j, we put

$$\binom{i}{j} = \frac{i(i-1)\cdots(i-j+1)}{j!} \quad \text{if} \quad j>0,$$
$$=1 \text{ if } j=0, \text{ and } =0 \text{ if } j<0.$$

It should be noted that $\binom{-1}{j}=(-1)^j$ if $j \in Z_+$. The definition implies directly

LEMMA 6. *If* $\binom{i}{j} \neq 0$ *and* $i \geqq 0$, *then* $i \geqq j$.

The cohomology of the p-fold cyclic product $\mathfrak{Z}_p(K)$ of K is studied by the author in his paper [6]. In the study, the homomorphisms

$$\phi_0^*: H^q(\mathfrak{X}_p(K);Z_p) \longrightarrow H^q(\mathfrak{Z}_p(K), \mathfrak{d}_p(K); Z_p),$$
$$E_m: H^q(K; Z_p) \longrightarrow H^{q+m}(\mathfrak{Z}_p(K), \mathfrak{d}_p(K); Z_p)$$

are fundamental. By using of these homomorphisms, we shall now define a homomorphism

(3.1) $$\Phi_m: H^q(K; Z_p) \longrightarrow H^{q+m}(\mathfrak{Z}_p(K); Z_p)$$

for each $m \in Z_+$ as follows:

$$\Phi_0(c) = j^*\phi_0^*((-c)\times 1 \times \cdots \times 1),$$
$$\Phi_m(c) = j^*E_m(c) \qquad (m>0),$$

where $c \in H^q(K; Z_p)$, 1 denotes the unit class of $H^*(K; Z_p)$ and $j^*: H^{q+m}(\mathfrak{Z}_p(K), \mathfrak{d}_p(K); Z_p) \to H^{q+m}(\mathfrak{Z}_p(K); Z_p)$ is the injection homomorphism. $\Phi_1=0$ is a direct consequence of the definition of E_1.

Theorem (11.4) in [6] yields

PROPOSITION 2. *Let B be a basis of the vector space $H^*(K; Z_p)$. Then a set*

$$\Phi(B) = \{\Phi_m(b) \mid b \in B,\ 0 \leqq m \leqq (p-1)\dim b,\ m \neq 1\}$$

of elements of the vector space $H^*(\mathfrak{Z}_p(K); Z_p)$ is independent. If K is $(n-1)$-connected and $q<2n$, then a base for the vector space $H^q(\mathfrak{Z}_p(K); Z_p)$ can be formed by a set $\{c \in \Phi(B) \mid \dim c = q\}$.

Theorems (11.6) and (11.7) in [6] give

PROPOSITION 3. *Let $m, s \in Z_+$ and $m=2t+\eta$ with $t \in Z_+$, $\eta=0$ or 1. Then it holds that*

$$\mathrm{Sq}^s \Phi_m = \sum_{j=0}^{s} \binom{m-1}{j} \Phi_{m+j}\mathrm{Sq}^{s-j} \qquad \text{for} \quad p=2,$$
$$\mathcal{P}^s \Phi_m = \sum_{j=0}^{s} \binom{t+\eta-1}{j} \Phi_{m+2j(p-1)}\mathcal{P}^{s-j} \qquad \text{for} \quad p>2,$$
$$\Delta\Phi_m = (1+(-1)^m)/2\, \Phi_{m+1} + (-1)^m \Phi_m \Delta.$$

4. Cohomology of iterated cyclic products of spheres

Let S^n denote an n-sphere $(n \geqq 1)$, and e^n be a fixed generator of $H^n(S^n; Z_p)$. Let $r \in Z_+$, and $M=(m_1, m_2, \cdots, m_k, \cdots)$ be an element of Z_+^∞. Then we shall associate to M an element $[M]_r = [m_1, \cdots, m_r] \in H^{n+d(M)}(\mathfrak{Z}_p^r(S^n); Z_p)$ defined as the image of e^n by the composite homomorphism

$$H^n(S^n; Z_p) \xrightarrow{\Phi_{m_r}} H^{n+m_r}(\mathfrak{Z}^1(S^n); Z_p) \xrightarrow{\Phi_{m_{r-1}}} H^{n+m_{r-1}+m_r}(\mathfrak{Z}^2(S^n); Z_p) \longrightarrow \cdots$$
$$\xrightarrow{\Phi_{m_1}} H^{n+m_1+\cdots+m_r}(\mathfrak{Z}^r(S^n); Z_p).$$

It is clear that $\dim[M]_r \geqq n$ for any $M \in Z_+^r$, and that

$$(4.1) \qquad \Phi_{m_1}[m_2, \cdots, m_r] = [m_1, m_2, \cdots, m_r], \quad \Phi_{m_1}(e^n) = [m_1].$$

It follows from the fact $\Phi_1 = 0$ that $[M]_r = 0$ unless $M \in Z_+^r$ is special.

From Proposition 2 we have immediately

PROPOSITION 4. *Put* $\mathfrak{B}_r = \{[M]_r \mid M \in Z_+^r,\ M\colon \textit{special}\}$. *If* $q < n$, *a basis for the vector space* $H^{n+q}(\mathfrak{Z}_p^r(S^n)\,;\,Z_p)$ *can be formed by a set* $\{c \in \mathfrak{B}_r \mid \dim c = n+q\}$. *Especially* $H^n(\mathfrak{Z}_p^r(S^n)\,;\,Z_p)$ *is generated by the element* $[O]_r$.

Throughout this section we assume that every cohomology class has dimension less than $2n$.

The following proposition can be proved from Proposition 3 and (4.1) by induction on r. The proof is straightforward.

PROPOSITION 5. *We have in* $H^*(\mathfrak{Z}_p^r(S^n)\,;\,Z_p)$ *the formulas*:

$$(4.2) \qquad \mathrm{Sq}^s[m_1, m_2, \cdots, m_r]$$
$$= \sum_S \binom{m_1-1}{s_1}\binom{m_2-1}{s_2}\cdots\binom{m_r-1}{s_r}[m_1+s_1, m_2+s_2, \cdots, m_r+s_r] \quad \text{for } p=2,$$
$$\mathcal{P}^s[m_1, m_2, \cdots, m_r]$$
$$= \sum_S \binom{t_1+\eta_1-1}{s_1}\binom{t_2+\eta_2-1}{s_2}\cdots\binom{t_r+\eta_r-1}{s_r}[m_1+2s_1(p-1),$$
$$m_2+2s_2(p-1), \cdots, m_r+2s_r(p-1)] \quad \text{for } p>2,$$

where $S = (s_1, s_2, \cdots, s_r, 0, 0, \cdots) \in Z_+^r$, $d(S) = s$, *and we put* $m_k = 2t_k + \eta_k$ *with* $t_k \in Z_+$, $\eta_k = 0$ *or* 1.

$$(4.3) \qquad \Delta[m_1, m_2, \cdots, m_r]$$
$$= \sum_{k=1}^{r} (-1)^{m_1+\cdots+m_{k-1}}(1+(-1)^{m_k})/2\,[m_1, \cdots, m_k+1, \cdots, m_r].$$

Let $M \in Z_+^r$, and let $I \in Z_+^\infty$ be proper. Then it follows from Propositions 4 and 5 that $St^I[M]_r$ has a unique representation:

$$St^I[M]_r = \sum_N a_N[N]_r \qquad (a_N \in Z_p),$$

where N is extended over all special elements of Z_+^r with $d(N) = d(M) + d(I)$. If $a_N \neq 0$ in this expression, we write

$$[N]_r \subset St^I[M]_r.$$

LEMMA 7. *Let* $M, N \in Z_+^r$ *and* $i \equiv 0$ *or* 1 *mod* $2(p-1)$. *Then if*

$$(p-1)d(M) \leqq i, \qquad i > 0, \qquad [N]_r \subset St^i[M]_r,$$

we have

$$h(N) \geqq h(M) + 1.$$

Proof. Since the result for $p=2$ are proved similarly, as an illustration we write the proof for $p>2$. Let $i=2s(p-1)+\varepsilon$ ($s\in Z_+$, $\varepsilon=0$ or 1).

Case 1: $\varepsilon=0$. Let $M=(m_1, m_2, \cdots, m_k, \cdots)$, $N=(n_1, n_2, \cdots, n_k, \cdots)$. Then, by Proposition 5, we may assume that

$$n_k = m_k+2s_k(p-1) \qquad (k=1, 2, \cdots),$$
$$S=(s_1, s_2, \cdots, s_k, \cdots)\in Z_+^r, \quad d(S)=s.$$

Put $h=h(M)$, and let $m_k>0$ for $k=\alpha_1, \alpha_2, \cdots, \alpha_h$. The proposition is clear for $h=0$, and hence we may assume $h>0$.

Since $n_k\geqq m_k$ for any k, we have $h(N)\geqq h(M)$. Assume now $h(N)=h(M)$. Then we have $n_k=0$ for $k\neq\alpha_1, \alpha_2, \cdots, \alpha_h$, and hence $s_k=0$ for $k\neq\alpha_1, \alpha_2, \cdots, \alpha_h$, Therefore if we put $m_k=2t_k+\eta_k$ ($k=1, 2, \cdots$), it follows from (4.2) and the assumption that

$$\binom{t_{\alpha_1}+\eta_{\alpha_1}-1}{s_{\alpha_1}}\binom{t_{\alpha_2}+\eta_{\alpha_2}-1}{s_{\alpha_2}}\cdots\binom{t_{\alpha_h}+\eta_{\alpha_h}-1}{s_{\alpha_h}}\not\equiv 0 \bmod p.$$

Since $t_{\alpha_k}+\eta_{\alpha_k}-1\geqq 0$ for $k=1, 2, \cdots, h$, it follows from Lemma 6 that

$$t_{\alpha_k}+\eta_{\alpha_k}-1\geqq s_{\alpha_k} \qquad (k=1, 2, \cdots, h),$$

and hence

$$m_{\alpha_k}=2t_{\alpha_k}+\eta_{\alpha_k}\geqq 2s_{\alpha_k}-\eta_{\alpha_k}+2>2s_{\alpha_k}.$$

Therefore we have

$$d(M)=\sum_{k=1}^{h} m_{\alpha_k}>2\sum_{k=1}^{h} s_{\alpha_k}=2s.$$

and so $(p-1)d(M)>2s(p-1)=i$, which contradicts with our assumption. Thus $h(N)\geqq h(M)+1$.

Case 2: $\varepsilon=1$. Let $i=1$. Then $M=O$, and hence the lemma is clear by (4.3). Therefore we shall assume $i>1$.

The assumption $(p-1)d(M)\leqq i=2s(p-1)+1$ implies $(p-1)d(M)\leqq 2s(p-1)$. And, since $i>1$, we have $2s(p-1)>0$.

Since $[N]_r\subset \Delta St^{2s(p-1)}[M]_r$, there exists an element $L\in Z_+^r$ such that

$$[L]_r\subset St^{2s(p-1)}[M]_r, \tag{4.4}$$
$$[N]_r\subset \Delta[L]_r. \tag{4.5}$$

Since $(p-1)d(M)\leqq 2s(p-1)$ and $2s(p-1)>0$, it follows from (4.4) and the fact just proved above that

$$h(L)\geqq h(M)+1.$$

It follows from (4.3) and (4.5) that

$$h(N)\geqq h(L).$$

Therefore we have $h(N)\geqq h(M)+1$. Q. E. D.

PROPOSITION 6. *Let $I\in Z_+^\infty$ be admissible, and let*

$$[N]_r \subset St^I[O]_r \qquad (N\in Z_+^r)\,.$$

Then we have

$$h(N)\geqq h(I) = l(I)\,.$$

Proof. The proof is by induction on $l(I)$. If $l(I)=0$ the proposition is trivial. Therefore we assume the proposition for I with $l(I)=l-1$, and shall prove it for I with $l(I)=l>0$.

Let $I=(i_1, i_2, \cdots, i_k, \cdots)$ and put $I'=(i_2, i_3, \cdots, i_k, \cdots)$. Then we have $[N]_r \subset St^{i_1}St^{I'}[O]_r$. Therefore there is an element $M\in Z_+^r$ such that

$$[M]_r \subset St^{I'}[O]_r\,, \tag{4.6}$$

$$[N]_r \subset St^{i_1}[M]_r\,. \tag{4.7}$$

Since $I'\in Z_+^\infty$ is admissible and $l(I')=l-1$, it follows from (4.6) and the hypothesis of induction that

$$h(M)\geqq h(I') = l-1\,. \tag{4.8}$$

Since I is admissible, we have by the definition

$$i_k\geqq p i_{k+1}\,, \quad k = 1, 2, \cdots.$$

Adding these inequalities, we have

$$i_1\geqq (p-1)(i_2+i_3+\cdots) = (p-1)d(I')\,.$$

Since $l(I)>0$, we have $i_1>0$. Therefore, by Lemma 7, it follows from (4.7) that

$$h(N)\geqq h(M)+1\,. \tag{4.9}$$

Together (4.8) with (4.9), we obtain $h(N)\geqq l=h(I)$. Q. E. D.

A direct consequence of Propositions 5 and 8, we have

THEOREM 3. *Let $I\in Z_+^\infty$ be admissible and $h(I)>r$. Then it holds that*

$$\mathrm{St}^I[O]_r = 0 \quad in \quad H^*(\mathfrak{Z}_p^r(S^n)\,;\, Z_p)\,.$$

Denote by $\alpha_i\in\mathfrak{S}_r$ $(1\leqq i<r)$ the permutation which interchanges i and $i+1$, and leaves fixed all the other letters. It is well known that $\mathfrak{S}_r$ is generated by $\alpha_1, \alpha_2, \cdots, \alpha_{r-1}$ with the defining relations:

$$\alpha_1^2=\alpha_2^2=\cdots=\alpha_{r-1}^2=1\,,\quad (\alpha_i\alpha_j)^2=1 \qquad \text{if} \quad i+1<j\,,$$
$$(\alpha_i\alpha_{i+1})^3=1$$

(See Dickson: Linear groups p. 287). Therefore it follows that if we define

$$\alpha_i[m_1, \cdots, m_i, m_{i+1}, \cdots, m_r] = (-1)^{m_i m_{i+1}}[m_1, \cdots, m_{i+1}, m_i, \cdots, m_r]$$
$$(i=1, 2, \cdots, r-1)\,,$$

then $\mathfrak{S}_r$ becomes an operator group on a vector space $H_0^*(\mathfrak{Z}_p^r(S^n)\,;\, Z_p)$ generated by the set $\mathfrak{B}_r$ (see Proposition 4). Let $c \in H_0^*(\mathfrak{Z}_p^r(S^n); Z_p)$. If $\alpha c = c$ for any $\alpha \in \mathfrak{S}_r$, we call that c is *symmetric*.

PROPOSITION 7. *If* $c \in H_0^*(\mathfrak{Z}_p^r(S^n); Z_p)$ *is symmetric, then so is* $\mathrm{St}^I c$ *for any proper* $I \in Z_+^\infty$. *Especially* $\mathrm{St}^I[O]_r$ *is symmetric.*

Proof. By straightforward calculation, it follows from Proposition 5 that $\alpha_i \in \mathfrak{S}_r$ commutes with Sq^s, $\mathcal{P}^s$ and Δ (i.e. $\alpha_i \mathrm{Sq}^s[M]_r = \mathrm{Sq}^s \alpha_i[M]_r$ etc). Therefore we have $\alpha\,\mathrm{Sq}^s = \mathrm{Sq}^s\alpha$, $\alpha\mathcal{P}^s = \mathcal{P}^s\alpha$ and $\alpha\Delta = \Delta\alpha$ for any $\alpha \in \mathfrak{S}_r$. This proves the proposition.

Q. E. D.

LEMMA 8. *Let* $M = (m_1, m_2, \cdots, m_k, \cdots)$, $N = (n_1, n_2, \cdots, n_k, \cdots) \in Z_+^r$, *and* $i \equiv 0$ *or* 1 *mod* $2(p-1)$. *Assume now* $[N]_r \subset \mathrm{St}^i[M]_r$. *Then, for* q *such that* $m_q > 0$, *we have* $n_q < pm_q$.

Proof. Since the proof for $p=2$ is similar, we write only the proof for $p>2$. Put $i = 2s(p-1)+\varepsilon$ ($s \in Z_+$, $\varepsilon = 0$ or 1).

Case 1: $\varepsilon = 0$. We may assume that $n_k = m_k + 2s_k(p-1)$, $S = (s_1, s_2, \cdots, s_k, \cdots) \in Z_+^r$, $d(S) = s$. Put $m_k = 2t_k + \eta_k$ ($t_k \in Z_+$, $\eta_k = 0$ or 1). Then it follows from Proposition 5 and the assumption that

$$\binom{t_1+\eta_1-1}{s_1}\binom{t_2+\eta_2-1}{s_2}\cdots\binom{t_r+\eta_r-1}{s_r} \not\equiv 0 \quad mod\ p.$$

Especially $\binom{t_q+\eta_q-1}{s_q} \neq 0$. Since $m_q > 0$, we have $t_q + \eta_q - 1 \geqq 0$. Therefore it follows from Lemma 6 that $t_q + \eta_q - 1 \geqq s_q$. From this, we have $m_q - 2s_q = 2t_q + \eta_q - 2s_q \geqq 2 - \eta_q > 0$. Hence $pm_q - n_q = (p-1)m_q + (m_q - n_q) = (p-1)m_q - 2s_q(p-1) = (p-1)(m_q - 2s_q) > 0$. Namely we have $pm_q > n_q$.

Case 2: $\varepsilon = 1$. The lemma follows easily from the result for $\varepsilon = 0$ and (4.3).

Q. E. D.

PROPOSITION 8. *Let* $I \in Z_+^r$ *be admissible, and* $N \in Z_+^r$. *Then if* $[N]_r \subset \mathrm{St}^I[O]_r$, *we have* $N \leqq I$. *Furthermore* $[I]_r \subset \mathrm{St}^I[O]_r$.

Proof.[4] We write only the proof for $p>2$. The proof for $p=2$ is similar.

Since the statement is trivial if $l(I)=0$, we proceed by induction on $l(I)$.

Assuming the statement for I with $l(I)=l-1$, we shall prove it for I with $l(I)=l>1$.

Let $I = (i_1, i_2, \cdots, i_k, \cdots)$, and put $I' = (i_2, i_3, \cdots, i_k, \cdots)$. Then if $[N]_r \subset \mathrm{St}^I[O]_r$, there exists an element $M \in Z_+^r$ such that

(4.10) $$[M] \subset \mathrm{St}^{I'}[O]_r,$$

(4.11) $$[N]_r \subset \mathrm{St}^{i_1}[M]_r.$$

4) I am indebted to my colleagues Mizuno and Toda for the improvement of this proof.

Since I' is admissible and $l(I')=l-1$, it follows from (4.10) and the hypothesis of induction that $M \leqq I'$. Let $M=(m_1, m_2, \cdots, m_k, \cdots)$ and $N=(n_1, n_2, \cdots, n_k, \cdots)$.

Case 1: $m_1>0$. It follows from Lemma 8 and (4.11) that $n_1<pm_1$. Since $M \leqq I'$, we have $m_1 \leqq i_2$. Therefore we obtain

$$n_1 < pm_1 \leqq pi_2 \leqq i_1 .$$

Thus we have $N<I$.

Case 2: $m_1=0$. It follows from Proposition 7 and (4.10) that

$$[m_2, m_3, \cdots, m_r, 0] = [m_2, m_3, \cdots, m_r, m_1] \subset \mathrm{St}^{I'}[O]_r .$$

Therefore, by the hypothesis of induction, we have

$$(m_2, m_3, \cdots, m_r, 0, 0, \cdots) \leqq I' = (i_2, i_3, \cdots) .$$

It follows from (4.11) and Proposition 5 that

$$N = (n_1, n_2, n_3, \cdots) \leqq (m_1+i_1, m_2, m_3, \cdots) = (i_1, m_2, m_3, \cdots) .$$

Therefore we obtain

$$N \leqq (i_1, m_2, m_3, \cdots) \leqq (i_1, i_2, i_3, \cdots) = I .$$

This completes the proof of the first part.

Assume that $[I]_r \subset \mathrm{St}^{i_1}[M]_r$ with $[M]_r \subset \mathrm{St}^{I'}[O]_r$. Then it follows from the above argument that $M=(0, i_2, i_3, \cdots)$. Thus, by the hypothesis of induction and Propositions 5 and 8, we have the second part. Q. E. D.

As a direct consequence of Propositions 4 and 9, we obtain

THEOREM 4. *A set of elements* $\mathrm{St}^I[O]_r \in H^{n+q}(\mathfrak{Z}_p^r(S^n)\,;\, Z_p)$ *is linearly independent, where* $I \in Z_+^\infty$ *is extended over all admissible and special elements such that* $d(I)=q<n$ *and* $l(I) \leqq r$.

5. Proof of main theorem

A point of the m-fold symmetric product $\mathfrak{S}_m(K)$ is represented by an unordered set $\{t_1, t_2, \cdots, t_m\}$ with $t_j \in K$ for $j=1, 2, \cdots, m$. Let $* \in K$ be a fixed vertex. For any integers m, n with $m \leqq n$, define a map $\iota_{m,n}\colon \mathfrak{S}_m(K) \longrightarrow \mathfrak{S}_n(K)$ by

$$\iota_{m,n}\{t_1, t_2, \cdots, t_m\} = \{t_1, t_2, \cdots, t_m, *, *, \cdots\} .$$

Obviously $\iota_{m,n}$ maps $\mathfrak{S}_m(K)$ into $\mathfrak{S}_n(K)$ homeomorphically. The inductive limit of the sequence

$$K = \mathfrak{S}_1(K) \xrightarrow{\iota_{1,2}} \mathfrak{S}_2(K) \longrightarrow \cdots \longrightarrow \mathfrak{S}_m(K) \xrightarrow{\iota_{m,m+1}} \mathfrak{S}_{m+1}(K) \longrightarrow \cdots$$

is called the *infinite symmetric product* of K, and is denoted by $\mathfrak{S}_\infty(K)$.

The following theorem was established in [7] by the author.

THEOREM 5. *Let $m \leq n$, then the injection homomorphism*

$$\iota^*_{m,n} : H^q(\mathfrak{S}_n(K)\,;\, Z_p) \longrightarrow H^q(\mathfrak{S}_m(K)\,;\, Z_p)$$

is an epimorphism for any q.

From this, we have

THEOREM 6. *Let $\iota_m : \mathfrak{S}_m(K) \longrightarrow \mathfrak{S}_\infty(K)$ be the inclusion map, then the injection homomorphism*

$$\iota^*_m : H^q(\mathfrak{S}_\infty(K)\,;\, Z_p) \longrightarrow H^q(\mathfrak{S}_m(K)\,;\, Z_p)$$

is an epimorphism for any q.

Proof. It follows from Theorem 5 that the homomorphism of homology

$$\iota_{m,n_*} : H_q(\mathfrak{S}_m(K)\,;\, Z_p) \longrightarrow H_q(\mathfrak{S}_n(K)\,;\, Z_p)$$

is a monomorphism for any $n \geq m$. Let $a \in H_q(\mathfrak{S}_m(K)\,;\, Z_p)$ be an element such that $\iota^*_m(a) = 0$, and c a cocycle mod p in $\mathfrak{S}_m(K)$ representing a. Then c is a bounding cycle in $\mathfrak{S}_n(K)$ for sufficiently large n. Therefore we have $\iota_{m,n_*}(a) = 0$, and hence $a = 0$ by the fact above-mentioned. Thus it follows that the homomorphism of homology

$$\iota_{m_*} : H_q(\mathfrak{S}_m(K)\,;\, Z_p) \longrightarrow H_q(\mathfrak{S}_\infty(K)\,;\, Z_p)$$

is a monomorphism. From this we have immediately Theorem 6. Q. E. D.

The following theorem was established by A.Dold–R.Thom [4] and others.

THEOREM 7. *$\mathfrak{S}_\infty(S^n)$ is an Eilenberg-MacLane complex $K(Z, n)$ (i.e. the homotopy group $\pi_i(\mathfrak{S}_\infty(S^n)) \approx Z$ for $i = n$, and $= 0$ for $i \neq n$), where Z denotes the additive group of integers.*

The mod p cohomology group $H^*(Z, n\,;\, Z_p)$ of $K(Z, n)$ was calculated by H. Cartan [2] (See also J-P. Serre [9] for $p = 2$):

THEOREM 8. *Denote by u_0 a fixed generator of $H^n(Z, n\,;\, Z_p) \approx Z_p$. Then if $q < n$ the vector space $H^{n+q}(Z, n\,;\, Z_p)$ has a base formed by elements $\mathrm{St}^I u_0$, where $I \in Z^\infty_+$ is extended over all admissible and special elements with $d(I) = q$.*

We have

PROPOSITION 10. *Put $v_{0,m} = \iota^*_m(u_0)$, then $H^n(\mathfrak{S}_m(S^n)\,;\, Z_p)$ is a cyclic group of order p whose generator is $v_{0,m}$. If $m = p^r$ then $\rho^* v_{0,m} = a[O]_r$ with $a \not\equiv 0 \bmod p$, where ρ^* is the homomorphism in Theorem 2.*

Proof. It is known [5, 7] that $H^n(\mathfrak{S}_m(S^n)\,;\, Z_p)$ has a subgroup isomorphic with $H^n(S^n\,;\, Z_p) \approx Z_p$. Therefore the first part of Proposition 10 follows from Theorems 6 and 7. The second part follows from Theorem 2 and Proposition 4.

Q. E. D.

We shall now prove

MAIN THEOREM. *Let $p^h \leqq m < p^{h+1}$ and $q < n$. Then the vector space $H^{n+q}(\mathfrak{S}_m(S^n); Z_p)$ has a base formed by elements $\mathrm{St}^I v_{0,m}$, where $I \in Z_+^\infty$ is extended over all admissible and special elements with $d(I)=q$ and $l(I) \leqq h$.*

Proof. It follows from Theorems 6, 7 and 8 using the naturality of St^I that the vector space $H^{n+q}(\mathfrak{S}_m(S^n); Z_p)$ is generated by elements $\mathrm{St}^I v_{0,m}$, where $I \in Z_+^\infty$ is extended over all admissible and special elements with $d(I)=q$. Therefore, for the proof of the theorem, it is sufficient to prove the following (A) and (B).

(A) If I is an admissible element with $l(I) > h$, then $\mathrm{St}^I v_{0,m}=0$.

(B) If $\sum_i a_i \mathrm{St}^{I_i} v_{0,m}=0$ $(a_i \in Z_p)$ for admissible and special elements I_i with $d(I_i)=q$ and $l(I_i) \leqq h$, then we have $a_i=0$.

For a proof of (A), let $m=\sum_{r=0}^{h} a_{h-r} p^r$ $(a_0 \neq 0)$ be the p-adic expansion of m, and consider the diagram

$$\begin{array}{ccc} H^{n+q}(\mathfrak{Z}_p^r(S^n); Z_p) & \xleftarrow{\xi_r^{j*}} & H^{n+q}(\mathfrak{G}_m(S^n); Z_p) \\ \uparrow \rho^* & & \uparrow \rho^* \\ H^{n+q}(\mathfrak{S}_{p^r}(S^n); Z_p) & \xleftarrow{\iota_{p^r,m}^*} & H^{n+q}(\mathfrak{S}_m(S^n); Z_p), \end{array}$$

where ρ^* and ξ_r^{j*} are the homomorphisms mentioned in §2. It follows from definitions that the commutativity holds in this diagram. Therefore we have

$$\xi_r^{j*} \rho^* \mathrm{St}^I v_{0,m} = \rho^* \iota_{p^r,m}^* \mathrm{St}^I v_{0,m} = \rho^* \mathrm{St}^I v_{0,p^r}.$$

Since $r \leqq h < l(I)$, Proposition 10 and Theorem 3 imply that $\rho^* \mathrm{St}^I v_{0,p^r} = a \mathrm{St}^I |O|_r = 0$. Namely we have

$$\xi_r^{j*} \rho^* \mathrm{St}^I v_{0,m} = 0 \quad \text{for every } (r, j) \in W(m).$$

Thus it follows from Corollary of Theorem 1′ that $\rho^* \mathrm{St}^I v_{0,m}=0$. By Theorem 2, we have $\mathrm{St}^I v_{0,m}=0$. This completes the proof of (A).

From the assumption of (B), we have $\sum_i a_i \mathrm{St}^{I_i} v_{0,p^h} = \iota_{p^h,m}^* (\sum_i a_i \mathrm{St}^{I_i} v_{0,m}) = 0$. Therefore we obtain by Proposition 10 that $\sum_i a_i \mathrm{St}^{I_i}[O]_h=0$. Then Theorem 4 implies that $a_i=0$ for each i, and we have (B). Q. E. D.

Together with Proposition 7, we have

COROLLARY 1. *If $q < n$, the image of $H^{n+q}(\mathfrak{S}_{p^h}(S^n); Z_p)$ by the monomorphism ρ^* is contained in the subspace of $H^{n+q}(\mathfrak{Z}_p^h(S^n); Z_p)$ formed by all the symmetric elements.*

We have also

COROLLARY 2. *If $p^h \leqq m < p^{h+1}$ and $q < n$, then the homomorphism $\iota_{p^h,m}^*: H^{n+q}(\mathfrak{S}_m(S^n); Z_p) \longrightarrow H^{n+q}(\mathfrak{S}_{p^h}(S^n); Z_p)$ is an isomorphism.*

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