Journal of the Institute of Polytechnics
Osaka City University,
Vol. 8. No. 2, Series A

A contribution to the theory of metrization

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(Received March 18, 1957)

Since Alexandroff and Urysohn's work various theorems concerning metrizability of a toplogical space were gotten by many mathematicians. Their methods of proofs, however, are generally various and rather complicated. The purpose of this paper is to give a systematic account of the theory of metrization. A main theorem on the metrizability of a T_1-space will be proved first, and then it will be shown that this theorem contains a large number of metrization theorems as direct consequences.

To prove our main theorem we use the following theorem due to E. Michael[1] as well as the well-known theorem of P. Alexandroff and P. Urysohn.

Michael's theorem. A regular topological space R is paracompact if and only if every open covering of R has an open refinement $\mathfrak{V}=\bigcup_{i=1}^{\infty}\mathfrak{V}_i$, where each $\mathfrak{V}_i$ is a locally finite collection of open subsets of R.

THEOREM 1. *In order that a T_1-topological space R is metrizable it is necessary and sufficient that one can assign a nbd (=neighborhood) basis $\{U_n(x)\mid n=1, 2, \cdots\}$ for every point x of R such that for every n and each point x of R there exist nbds $S_n^1(x)$, $S_n^2(x)$ of x satisfying*

$$\text{i)}\quad y\notin U_n(x) \text{ implies } S_n^2(y)\cap S_n^1(x)=\phi,$$
$$\text{ii)}\quad y\in S_n^1(x) \text{ implies } S_n^2(y)\subseteq U_n(x).\text{[2]}$$

Proof. Since the necessity is clear, we prove only the sufficiency. To begin with, R is regular because the condition i) of this proposition implies $\overline{S_n^1(x)}\subseteq U_n(x)$. Next, to show that R is paracompact, we take an arbitrary open covering $\mathfrak{V}=\{V_\alpha\mid\alpha<\tau\}$ of R[3]. If we let

$$V_{n,\alpha}=\cup\left\{(S_n^1(x))^\circ\mid U_n(x)\subseteq V_\alpha\right\},$$
$$V_{m,n,\alpha}=\cup\left\{U_m(x)\quad\mid U_m(x)\subseteq V_{n,\alpha}\right\},$$
$$V'_{m,n,\alpha}=\cup\left\{S_m^1(x)\quad\mid U_m(x)\subseteq V_{n,\alpha}\right\},$$
$$M_{m,n,\alpha}=(V'_{m,n,\alpha}-\bigcup_{\beta<\alpha}V_{n,\beta})^\circ\qquad(m,n=1,2,\cdots,\alpha<\tau),$$

then $\mathfrak{M}_{m,n}=\{M_{m,n,\alpha}\mid\alpha<\tau\}$ for each m, n is a locally finite open collection.

To show the local finiteness of $\mathfrak{M}_{m,n}$ we choose, for an arbitrary point p of

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The content of this paper is the datail of our brief note in Proc. of Japan Acad , vol. 33,

1) See [4]. 2) ii) can't be omitted. Let $R=\{(x,y)\mid 0\leqq x\leqq 1, y\geqq 0\}$, $U_n(p)=\{q\mid d(p,q)<1/n\}$ for $p=(x,y)$ with $y>0$, $U_n(p)=\{(x',y')\mid y'<n-\sqrt{n^2-(x'-x)^2}, |x'-x|<1/n\}\cup\{p\}$ for $p=(x,0)$. Then R with the nbd basis $\{U_n(p)\}$ is a non- metrizable T_1-space satisfying i).

3) In this proof we denote by α, β, γ, τ ordinal numbers and by A° the interior of A.

R, α $(\leqq\tau)$ such that $p\in V_{m,n,\alpha}$, $p\notin V_{m,n,\beta}$ $(\beta<\alpha)$. Then it follows from the condition i) that $S_m^2(p)\cap V'_{m,n,\beta}=\phi$ $(\beta<\alpha)$, which implies $S_m^2(p)\cap M_{m,n,\beta}=\phi$ $(\beta<\alpha)$. If $\alpha<\tau$, then since $p\in V_{m,n,\alpha}\subseteq V_{n,\alpha}$ and $V_{n,\alpha}$ is open, we obtain a nbd $V_{n,\alpha}$ of p satisfying $V_{n,\alpha}\cap M_{m,n,\gamma}=\phi$ $(\gamma>\alpha)$. Therefore the nbd $S_m^2(p)\cap V_{n,\alpha}$ of p intersects at most one of elements of $\mathfrak{M}_{m,n}$, proving the local finiteness of $\mathfrak{M}_{m,n}$[4].

To assert that $\bigcup_{m,n=1}^{\infty}\mathfrak{M}_{m,n}=\mathfrak{M}$ covers R, we consider an arbitrary point p of R. Let $p\in V_\alpha$, $p\notin V_\beta$ $(\beta<\alpha)$, $\alpha<\tau$, then we can choose n such that $U_n(p)\subseteq V_\alpha$. Since $p\in(S_n^1(p))^\circ\subseteq V_{n,\alpha}$ for this n, we can choose m satisfying $U_m(p)\subseteq V_{n,\alpha}$. In consequence we have $S_m^1(p)\subseteq V'_{m,n,\alpha}$. On the other hand, it follows from $p\notin V_\beta$ $(\beta<\alpha)$ and from i) that $S_n^2(p)\cap V_{n,\beta}=\phi$ $(\beta<\alpha)$. This implies

$$S_m^1(p)\cap S_n^2(p)\subseteq V'_{m,n,\alpha}-\bigcup_{\beta<\alpha}V_{n,\beta}$$

and consequently $p\in M_{m,n,\alpha}$, i.e. $\mathfrak{M}$ covers R.

Since $\mathfrak{M}<\mathfrak{V}$ is obvious, we can conclude, from Michael's theorem, the paracompactness of R. Thus it follows from A. H. Stone's theorem[5] that R is fully normal.

To complete the proof, let us show that $\{S(p,\mathfrak{S}_m)\,|\,m=1,2,\cdots\}$[6] for $\mathfrak{S}_m=\{(S_m^2(y))^\circ\,|\,y\in R\}$ is a nbd basis of each point p of R. Let $U(p)$ be an arbitrary nbd of p, and choose n satisfying $U_n(x)\subseteq U(x)$, then we can find $m\geqq n$ such that $U_m(x)\subseteq S_n^1(x)$. If $(S_m^2(y))^\circ\ni x$, then considering $S_m^2(y)\cap S_m^1(x)\neq\phi$, we have $y\in U_m(x)\subseteq S_n^1(x)$ from i). Hence it follows from ii) that $S_m^2(y)\subseteq S_n^2(y)\subseteq U_n(x)$ because we can assume, without loss of generality, that $m\geqq n$ implies $S_m^2(y)\subseteq S_n^2(y)$ for every $y\in R$. Therefore we have $S(x,\mathfrak{S}_m)\subseteq U_n(x)\subseteq U(x)$, i.e. $\{S(p,\mathfrak{S}_m)\,|\,m=1,2,\cdots\}$ is a nbd basis of p. Thus we conclude the metrizability of R by the metrization theorem of P. Alexandroff and P. Urysohn.

Now we turn our attention to the application of this theorem. Some of the following theorems are well-known, and some of them are probably unknown.

THEOREM 2. (*Yu. Smirnov and the author*)[7] *A regular space R is metrizable if and only if there exists an open basis $\mathfrak{V}=\bigcup_{n=1}^{\infty}\mathfrak{V}_n$ of R, where each $\mathfrak{V}_n$ is a locally finite collection of open sets.*

Proof.[8] We define nbds $U_{n,m}(x)$ and $S_{n,m}^1(x)$ of each point x of R for every pair n,m of positive integers as follows: Let $V_n(x)=\cap\{V\,|\,x\in V\in\mathfrak{V}_n\}$. Then if $x\in U\subseteq\overline{U}\subseteq V_n(x)$ for some $U\in\mathfrak{V}_m$, we let $U_{n,m}(x)=V_n(x)$, $S_{n,m}^1(x)=U$; otherwise we let $U_{n,m}(x)=S_{n.m}^1(x)=R$. Moreover we put

$$S_{n,m}^2(x)=V_n(x)\cap\left[\cap\{(\overline{V})^c\,|\,x\notin\overline{V},\ V\in\mathfrak{V}_m\}\right].^{9)}$$

4) If $\alpha=\tau$, $S_m^2(p)$ intersects no set of $\mathfrak{M}_{m,n}$.

5) Full normality and paracompactness are equivalent in T_2-spaces. See [11].

6) $S(p,\mathfrak{U})=\cup\{U\,|\,p\in U\in\mathfrak{U}\}$, $S(A,\mathfrak{U})=\cup\{U\,|\,U\cap A\neq\phi,\ U\in\mathfrak{U}\}$ for a covering $\mathfrak{U}$, a point p and a subset A.

7) See [10] and [7].

8) From now forth we prove only the sufficiencies since the necessities are generally obvious.

9) We denote by V^c or $C(V)$ the complement set of V.

Since $\mathfrak{B}_n$ and $\mathfrak{B}_m$ are locally finite, we have nbds $U_{n,m}(x)$, $S^1_{n,m}(x)$, $S^2_{n,m}(x)$ of x satisfying the conditions i) ii) of Theorem 1. For every nbd $U(x)$ of an arbitrary point x of R we can choose n, m and $V \in \mathfrak{B}_m$ such that

$$U(x) \supseteq V_n(x) \supseteq \bar{V} \supseteq V \ni x$$

because R is regular and $\mathfrak{B}$ is an open basis of R. Hence $U_{n,m}(x) = V_n(x) \subseteq U(x)$, i.e. $\{U_{n,m}(x) \mid n,m = 1,2 \cdots\}$ is a nbd basis of x. Thus we conclude, by Theorem 1, the metrizability of R.

THEOREM 3. (*K. Morita*)[10] *A T_1-space R is metrizable if and only if there exists a countable collection $\{\mathfrak{F}_n \mid n = 1,2 \cdots\}$ of locally finite closed coverings of R such that $S(x, \mathfrak{F}_n) \subseteq U(x)$ for any nbd $U(x)$ of any point x of R and for some n.*

Proof. Since $\mathfrak{F}_n$ is locally finite, $N_n(x) = \cap \{F^c \mid x \in F^c, F \in \mathfrak{F}_n\}$ is an open nbd of any point x of R. If $S(x, \mathfrak{F}_n) \subseteq N_n(x)$, then we let

$$U_{n,m}(x) = N_n(x), \qquad S^1_{n,m}(x) = N_m(x).$$

If $S(x, \mathfrak{F}_m) \not\subseteq N_n(x)$, then we let

$$U_{n,m}(x) = S^1_{n,m}(x) = R.$$

Moreover we let generally $S^2_{n,m}(x) = N_m(x)$.

Then $U_{n,m}(x)$, $S^1_{n,m}(x)$, $S^2_{n,m}(x)$ are nbds of x satisfying the conditions i) ii) of Theorem 1. Since $\mathfrak{F}_m$ covers R, $N_m(y) \subseteq S(y, \mathfrak{F}_m)$ for every $y \in R$; hence $y \notin U_{n,m}(x)$ implies $S^2_{n,m}(y) = N_m(y) \subseteq (S(x, \mathfrak{F}_m))^c \subseteq (N_m(x))^c = (S^1_{n,m}(x))^c$.
$y \in S^1_{n,m}(x)$ implies

$$S^2_{n,m}(y) = N_m(y) \subseteq N_m(x) \subseteq N_n(x) = U_{n,m}(x).$$

Thus this theorem is a direct consequence of Theorem 1.

THEOREM 4. (*A. H. Frink*)[11] *A T_1-space R is metrizable if and only if one can assign a nbd basis $\{V_n(x) \mid n = 1, 2 \cdots\}$ for every point x of R such that for every n and $x \in R$ there exists $m = m(n, x)$ satisfying the condition: $V_m(x) \cap V_m(y) \neq \phi$ implies $V_m(y) \subseteq V_n(x)$.*

Proof. Letting $\mathfrak{B}_n = \{V_n(x) \mid x \in R\}$, $\mathfrak{W}_n = \{V_{m(n,x)}(x) \mid x \in R\}$, we define nbds $U_n(x)$, $S^1_n(x)$, $S^2_n(x)$ of any point x of R by

$$U_n(x) = S(x, \mathfrak{B}_n), S^1_n(x) = S(x, \mathfrak{W}_n), S^2_n(x) = V_{m(n,x)}(x).$$

If $S^2_n(y) \cap S^1_n(x) \neq \phi$, then there exists z such that $x \in V_{m(n,z)}(z) \not\subseteq (V_{m(n,y)}(y))^c$. In the case of $m(n,z) \leqq m(n,y)$ we have $V_{m(n,y)}(y) \subseteq V_n(z) \ni x$ since we can assume, without loss of generality, that $m \geqq n$ implies $V_m(x) \subseteq V_n(x)$. Therefore

$$S^2_n(y) = V_{m(n,y)}(y) \subseteq S(x, \mathfrak{B}_n) = U_n(x).$$

In the case of $m(n, z) \geqq m(n, y)$ we have $x \in V_{m(n,z)}(z) \subseteq V_n(y)$ and consequently

$$S^2_n(y) \subseteq V_n(y) \subseteq U_n(x).$$

For a given nbd $U(x)$ of x we take n with $V_n(x) \subseteq U(x)$. Then we have

10) See [5].
11) See [3].

$$S(x, \mathfrak{V}_{m(n,x)}) = U_{m(n,x)}(x) \subseteq U(x).$$

Hence this theorem is also a direct consequence of Theorem 1.

THEOREM 5. (*the author*)[12] *A T_1-space R is metrizable if and only if there exists a family $\{f_\alpha | \alpha \in A\}$ of real valued functions of R such that*

i) $\bigcup_{\beta \in B} f_\beta$ *and* $\bigcap_{\beta \in B} f_\beta$ *are continuous for every* $B \subseteq A$,

ii) *for any nbd $U(x)$ of any point x of R there exists $\alpha \in A$ and a real number ε: $f_\alpha(x) < \varepsilon$, $f_\alpha((U(x))^c) \geqq \varepsilon$.*[13]

Proof. We let $A(x) = \left\{\alpha | f_\alpha(x) < \frac{r+r'}{2}\right\}$ for each point x of R. For every pair $r < r'$ of rational numbers we let

$$U_{r,r'}(x) = \{y | \bigcup_{\alpha \in A(x)} f_\alpha(y) < r'\} \cap \{y | \bigcap_{\alpha \in C(A(x))} f_\alpha(y) > r\},$$

$$S^1_{r,r'}(x) = \{y | \bigcup_{\alpha \in A(x)} f_\alpha(y) < \frac{r+r'}{2} + \frac{r'-r}{4}\} \cap \left\{y | \bigcap_{\alpha \in C(A(x))} f_\alpha(y) > \frac{r+r'}{2} - \frac{r'-r}{4}\right\}.$$

Let $p \notin U_{r,r'}(x)$ $(x \in S \subseteq R)$ and let $\bigcup_{\alpha \in A(x)} f_\alpha(p) \geqq r'$ $(x \in S_1)$, $\bigcap_{\alpha \in C(A(x))} f_\alpha(p) \leqq r$ $(x \in S_2)$. Then we can choose $\alpha(x) \in A(x)$ for every $x \in S_1$ such that $f_{\alpha(x)}(p) > \frac{r+r'}{2} + \frac{r'-r}{3}$ $(< r')$. Hence we have, by the continuity of $\bigcap_{x \in S_1} f_{\alpha(x)}(p)$, a nbd $U(p)$ of p satisfying

$$\bigcap_{x \in S_1} f_{\alpha(x)}(U(p)) > \frac{r+r'}{2} + \frac{r'-r}{4}.$$

In the same way we can choose $\alpha(x) \in C(A(x))$ for every $x \in S_2$ and a nbd $V(p)$ of p satisfying

$$\bigcup_{x \in S_2} f_{\alpha(x)}(V(p)) < \frac{r+r'}{2} - \frac{r'-r}{4}.$$

Thus we get a nbd $U(p) \cap V(p)$ of p with

$$U(p) \cap V(p) \cap S^1_{r,r'}(x) = \phi \quad (x \in S).$$

Similarly we have, by the continuities of $\cup f_\alpha$ and $\cap f_\alpha$, a nbd $W(p)$ for any point p such that $p \in S^1_{r,r'}(x)$ implies $W(p) \subseteq U_{r,r'}(p)$.

To show that $\{U_{r,r'}(x) | r, r'$ are rational numbers with $r < r'\}$ is a nbd basis of x, we take, for a given nbd $U(x)$ of x, $\alpha \in A$, ε with $f_\alpha(x) < \varepsilon$, $f_\alpha((U(x))^c) \geqq \varepsilon$. Moreover we choose rational numbers r, r' with $f_\alpha(x) < r < r' < \varepsilon$. Then it follows from $\alpha \in A(x)$ that

$$U_{r,r'}(x) \subseteq \{y | \bigcup_{\alpha \in A(x)} f_\alpha(y) < r'\} \subseteq \{y | f_\alpha(y) < r'\} \subseteq U(x),$$

asserting the metrizability of R by Theorem 1.

THEOREM 6. (*A. H. Stone, K. Morita and S. Hanai*)[14] *Let f be a closed continuous mapping of a metric space R_1 onto a topological space R_2. Then R_2 is metrizable if and only if the boundary $Bf^{-1}(y)$ of the inverge image $f^{-1}(y)$ is*

12) See [8].

13) We denote by $f_\alpha(A) \geqq \varepsilon$ the fact that $f_\alpha(x) \geqq \varepsilon$ $(x \in A)$.

14) See [12] and [6]. With respect to the proof of the necessity, see also, those papers.

compact for every point y of R_2.

Proof. Let $U_{n,x}=S_{1/n}(Bf^{-1}(x))\cup f^{-1}(x)$[15], then $U_{n,x}$ is evidently an open set. We define nbds $U_n(x)$ and $S^1_n(x)$ of each point x of R by

$$U_n(x)=f(U_{n,x}),\ S^1_{n-1}(x)=\left(f(U^c_{n,x})\right)^c \quad (\subseteq U_n(x))$$

For a given nbd $U(x)$ of x we can choose n satisfying $S_{1/n}(Bf^{-1}(x))\subseteq f^{-1}(U(x))$ since $f^{-1}(U(x))$ is an open set containing $f^{-1}(x)$ and $Bf^{-1}(x)$ is compact. Hence $f(U_{n,x})=U_n(x)\subseteq U(x)$, i.e. $\{U_n(x)\,|\,n=1,2\cdots\}$ is a nbd basis of x.

Next $p\notin U_n(x)$ $(x\in S\subseteq R_2)$ and $p\in S^1_n(y)$ $(y\in T\subseteq R_2)$ imply $f^{-1}(p)\cap U_{n,x}=\phi$ $(x\in S)$ and $f^{-1}(p)\subseteq U_{n+1,y}$ $(y\in T)$; hence there exists an open set $U(f^{-1}(p))$ containing $f^{-1}(p)$ such that

$$U(f^{-1}(p))\cap U_{n+1,x}=\phi \quad (x\in S),\ U(f^{-1}(p))\subseteq U_{n,y} \quad (y\in T).$$

Thus we have, by the closedness of f, a nbd $f(U(f^{-1}(p)))$ of p with

$$f(U(f^{-1}(p)))\cap S^1_n(x)=\phi \quad (x\in S),\ f(U(f^{-1}(p)))\subseteq U_n(y) \quad (y\in T);$$

hence R_2 is metrizable by Theorem 1.

The following is a generalization of the theorem in our previous paper[16] and is a slight modification of Theorem 1 too.

THEOREM 7. *A T_1-space R is metrizable if and only if there exists a nbd basis $\{S_n(x)\,|\,n=1,2\cdots\}$ of every point x of R satisfying*

i) *$y\in S_n(x)$ implies $x\in S_n(y)$,*

ii) *for every n and $x\in R$ one can assign nbds $S'_n(x)$ $S''_n(x)$ such that $y\notin S_n(x)$ implies $S''_n(y)\cap S'_n(x)=\phi$ and such that $y\notin S'_n(x)$ implies $S''_n(y)\not\ni x$.*

Proof. Put $U_n(x)=S_n(x)$, $S^1_n(x)=S'_n(x)\cap S''_n(x)$, $S^2_n(x)=S''_n(x)$, then $y\notin U_n(x)$ implies $x\notin S_n(y)$ and consequently $S''_n(x)\cap S'_n(y)=\phi$. Therefore $z\in S^1_n(x)\subseteq S''_n(x)$ implies $y\notin S''_n(z)$, i.e. $S^2_n(z)\subseteq S_n(x)$. Since $y\notin U_n(x)$ implies $S^2_n(y)\cap S^1_n(x)\subseteq S''_n(y)\cap S'_n(x)=\phi$, we can conclude the validity of this theorem from Theorem 1.

THEOREM 8.[17] *A T_1-space R is metrizable if and only if there exists a non-negative function $\varphi(x,y)$ of $R\times R$ satisfying*

i) $\varphi(x,y)=\varphi(y,x)$

ii) *$d(x,A)=\inf\{\varphi(x,y)\,|\,y\in A\}$ is, for every subset A of R, a continuous function of x,*

iii) *$\{S_n(x)\,|\,n=1,2\cdots\}$ for $S_n(x)=\{y\,|\,\varphi(x,y)<1/n\}$ is a nbd basis of any point x of R.*

Proof. We let $S'_n(x)=S_{n+1}(x)$. Then $p\notin U_n(x)$ $(x\in A)$ and $p\notin S'_n(y)$ $(y\in B)$ imply $d(p,A)\geqq 1/n$ and $d(p,B)\geqq 1/(n+1)$ and hence there exists a nbd $S''_n(p)$ of p satisfying

$$d(S''_n(p),A)>1/(n+1),\ d(S''_n(p),B)>0.$$

In consequence

15) $S_{1/n}(A)=\{y\,|\,\inf\{\rho(y,x)\,|\,x\in A\}<1/n\}$.

16) Lemma 2 of [8].

17) This theorem is an extension of Corollary 6 of [8].

$$S''_n(p) \cap S'_n(x) = \phi \ (x \in A), S''_n(p) \not\ni y \qquad (y \in B).$$

Therefore R is metrizable by Theorem 7.

COROLLARY. *A T_1-space R is metrizable if and only if there exists a non-negative function $\varphi(x, y)$ of $R \times R$ satisfying*

i) $\varphi(x, y) = \varphi(y, x)$

ii) $d(x, F) = inf \{\varphi(x, y) \mid y \in F\}$ *is, for every closed set F of R, a continuous function of x,*

iii) $\{S_n(x) \mid n = 1, 2 \cdots\}$ *for* $S_n(x) = \{y \mid \varphi(x, y) < 1/n\}$ *is a nbd basis of any point x of R.*

Proof. Since $d(x, A) = d(x, \bar{A})$ is obvious for any subset A of R, this proposition is a direct consequence of Theorem 8.

The following two theorems are directly deduced from Theorem 1 and from Theorem 7 respectively and can be considered extensions of Chittenden's theorem.

THEOREM 9. *A T_1-space R is metrizable if and only if there exists a non-negative function $\varphi(x, y)$ of $R \times R$ satisfying*

i) $\varphi(x, y) = \varphi(y, x)$

ii) *for every $\varepsilon > 0$ and every $x \in R$ one can assign $\delta(\varepsilon, x) > 0$ such that $\varphi(x, z) < \delta(\varepsilon, x)$ and $\varphi(y, z) < \delta(\varepsilon, y)$ imply $\varphi(x, y) < \varepsilon$ and such that $\varphi(x, z) < \delta(\varepsilon, x)$ and $\varphi(z, y) < \delta(\varepsilon, z)$ imply $\varphi(x, y) < \varepsilon$,*

iii) $\{S_n(x) \mid n = 1, 2 \cdots\}$ *for* $S_n(x) = \{y \mid \varphi(x, y) < 1/n\}$ *is a nbd basis of any point x of R.*

Proof. It is obvious.

THEOREM 10. *A T_1-space R is metrizable if and only if there exists a non-negative valued function $\varphi(x, y)$ of $R \times R$ satisfying*

i) $\varphi(x, y) = \varphi(y, x)$

ii) *for every $\varepsilon > 0$ and $x \in R$ one can assign $\delta_1(\varepsilon) > 0$ and $\delta_2(\varepsilon, x) > 0$ such that $\varphi(x, z) < \delta_1(\varepsilon)$ and $\varphi(y, z) < \delta_2(\varepsilon, y)$ imply $\varphi(x, y) < \varepsilon$,*

iii) $\{S_n(x) \mid n = 1, 2 \cdots\}$ *for* $S_n(x) = \{y \mid \varphi(x, y) < 1/n\}$ *is a nbd basis of any point x of R.*

Proof. It is obvious.

The following is an extension of Alexandroff and Urysohn's theorem.

THEOREM 11. *A T_1-space R is metrizable if and only if there exists a countable collection $\{\mathfrak{U}_n \mid n = 1, 2 \cdots\}$ of open coverings such that*

i) *for every $\mathfrak{U}_n$ we can choose an open covering $\mathfrak{V}_n$ satisfying $\overline{S(A, \mathfrak{V}_n)} \subseteq S(A, \mathfrak{U}_n)$ for every closed subset A of R,*

ii) $\{S(x, \mathfrak{U}_n) \mid n = 1, 2 \cdots\}$ *is a nbd basis of any point x of R.*

Proof. Letting

$$S_n(x) = S(x, \mathfrak{U}_n), \ S'_n(x) = S(x, \mathfrak{V}_n),$$

for $p \notin S_n(x)$ $(x \in A)$ we have, by $\overline{S(A, \mathfrak{V}_n)} \subseteq S(A, \mathfrak{U}_n)$, a nbd $W_n(p)$ of p satisfying $W_n(p) \cap S'_n(x) = \phi$ $(x \in A)$. Since $S''_n(p) = W_n(p) \cap S(p, \mathfrak{V}_n)$ evidently satisfies the condition of Theorem 7, we conclude the metrizability of R.

THEOREM 12. (*R. H. Bing*)[18] *A regular space R is metrizable if and only if there exists a countable collection $\{\mathfrak{U}_n | n=1, 2\cdots\}$ of open collections such that*

i) *the sum of the closures of any subcollection of $\mathfrak{U}_n$ is closed,*

ii) $\{S(x, \mathfrak{U}_n) | n = 1, 2 \cdots, S(x, \mathfrak{U}_n) \neq \phi\}$ *is a nbd basis of each point x of R.*

Proof. For every triad l, m, n of positive integers and every point x of R we define nbds $S_{l,m,n}(x)$ and $S'_{l,m,n}(x)$ by

$$S_{l,m,n}(x) = S(x, \mathfrak{U}_n),\ S'_{l,m,n}(x) = S(x, \mathfrak{U}_m)$$

if

$$x \in \overline{S(x, \mathfrak{U}_l)} \subseteq S(x, \mathfrak{U}_m) \subseteq \overline{S(x, \mathfrak{U}_m)} \subseteq S(x, \mathfrak{U}_n),$$

and $S_{l,m,n}(x) = S'_{l,m,n}(x) = R$ otherwise.

It follows from the regularity of R that $\{S_{l,m,n}(x) | l, m, n = 1, 2 \cdots\}$ is a nbd basis of x.

If $p \notin S_{l,m,n}(x)$ $(x \in A)$, then

$$p \notin S(A, \mathfrak{U}_n) \supseteq \bigcup_{x\in A} \overline{S(x, \mathfrak{U}_m)} = \overline{S(A, \mathfrak{U}_m)},$$

which implies $V(p) \cap S'_{l,m,n}(x) = \phi$ $(x \in A)$ for some nbd $V(p)$ of p. If $p \notin S'_{l,m,n}(y)$ $(y \in B)$, then

$$p \notin S(B, \mathfrak{U}_m) \supseteq \bigcup_{y\in B} \overline{S(y, \mathfrak{U}_l)} = \overline{S(B, \mathfrak{U}_l)},$$

which implies $W(p) \not\ni y$ $(y \in B)$ for some nbd $W(p)$ of p. Letting $S''_{l,m,n}(p) = V(p) \cap W(p)$, we have three nbds satisfying the condition i), ii) of Theorem 7. Thus this theorem is also a direct consequence of Theorem 7.

With respect to a topologically complete space[19], we have a simpler condition for metrizability than those of Theorem 1 and Theorem 7.

THEOREM 13. *A topologically complete space R is completely metrizable if and only if one can assign a nbd basis $\{U_n(x) | n=1, 2\cdots\}$ for every point x of R such that for every n and each point x of R there exist nbds $S_n^1(x)$, $S_n^2(x)$ of x satisfying that $y \notin U_n(x)$ implies $S_n^2(y) \cap S_n^1(x) = \phi$.*

Proof. If we review the proof of Theorem 1, then we know that the condition ii) was not used to prove the fullnormality of R. Hence R is fully normal in the present instance. Since R is fully normal and topologically complete, there exists, by N. A. Shanin's theorem[20], a countable collection $\{\mathfrak{V}_n | n=1, 2\cdots\}$ of open coverings having the following property: If a maximum filter $\mathfrak{F} = \{\mathfrak{F}_\alpha | \alpha \in A\}$ of closed sets is divergent, then we can choose $\mathfrak{V}_n$ satisfying $\overline{S(x, \mathfrak{V}_n)} \subseteq F_\alpha^c$ for every $x \in R$ and some $F_\alpha \in \mathfrak{F}$.

On the other hand let $\mathfrak{S}_n = \{(S_n^2(x))^\circ | x \in R\}$, then for any x, y with $x \neq y$ we

18) See [1].

19) A T_2-space R is called a topologically complete space if R is homeomorphic with a G_δ -set of some compact T_2-space.

20) A T_2-space R is topologically complete if and only if there exists a countable collection $\{\mathfrak{U}_n | n=1,2,\cdots\}$ of open coverings such that if a maximum filter $\mathfrak{F}$ of closed sets is divergent, then we can choose $\mathfrak{U}_n$ with $\mathfrak{U}_n < \mathfrak{W} = \{F^c | F \in \mathfrak{F}\}$. See [9].

have $U_m(x) \frown U_n(y) = \phi$ for some m, n. Let us assume $m \geqq n$, then $x \in (S_m^2(z))^\circ$ implies $z \in U_m(x)$ and consequently $z \notin U_n(y)$ because we can assume without loss of generality, that $m \geqq n$ implies $S_m^2(x) \subseteq S_n^2(x)$ $(x \in R)$. Hence $S_m^2(z) \frown S_n^1(y) = \phi$, which means $y \notin \overline{S(x, \mathfrak{S}_m)}$. In the same way we can show $x \notin \overline{S(y, \mathfrak{S}_m)}$.

Letting $\mathfrak{W}_n = \mathfrak{B}_n \wedge \mathfrak{S}_n$ we have a countable collection $\{\mathfrak{W}_n | n = 1, 2 \cdots\}$ of open coverings having the property of $\{\mathfrak{B}_n\}$ and that of $\{\mathfrak{S}_n\}$. Now we shall show that $\{S(x, \mathfrak{W}_{n_1} \wedge \mathfrak{W}_{n_2} \wedge \cdots \wedge \mathfrak{W}_{n_k}) | n_i = 1, 2 \cdots (i = 1, \cdots k), k = 1, 2 \cdots\}$ is a nbd basis of any point x of R. If we assume the contrary:

$$S(x, \mathfrak{W}_{n_1}) \cap S(x, \mathfrak{W}_{n_2}) \cap \cdots \cap S(x, \mathfrak{W}_{n_k}) = S(x, \mathfrak{W}_{n_1} \wedge \cdots \wedge \mathfrak{W}_{n_k}) \nsubseteq U(x)$$

for some open nbd $U(x)$ of x and every n_i, k, then $\{(U(x))^c, \overline{S(x, \mathfrak{W}_n)} | n=1, 2 \cdots\} = \mathfrak{G}$ is a family of closed sets having the finite intersection property. Since all the sets of $\mathfrak{G}$ have obviously no common point, if we construct a maximum filter $\mathfrak{F}$ of closed sets such that $\mathfrak{G} \subseteq \mathfrak{F}$, $\mathfrak{F}$ is divergent. Hence we can choose $\mathfrak{W}_n$ satisfying $\overline{S(x, \mathfrak{W}_n)} \subseteq F^c$ for some $F \in \mathfrak{F}$. This contradicts $\overline{S(x, \mathfrak{W}_n)} \in \mathfrak{G} \subseteq \mathfrak{F}$ since $\mathfrak{F}$ is a filter. Thus we have the countable collection $\{\mathfrak{W}_{n_1} \wedge \cdots \wedge \mathfrak{W}_{n_k} | n_i = 1, 2 \cdots (i = 1, \cdots, k), k = 1, 2 \cdots\}$ of open coverings such that $\{S(x, \mathfrak{W}_{n_1} \wedge \cdots \wedge \mathfrak{W}_{n_k})\}$ is a nbd basis of x, which combining with the full normality of R induces, by Alexandroff and Urysohn's theorem, the metrizability of R and consequently the complete metrizability of R by E. Čech's Theorem[21].

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21) A topological space R is completely metrizable if and only if R is metrizable and topologically complete. See [2].