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Harmonic functions with two singular points II

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Let $\mathfrak{M}$ be a compact real analytic Riemannian manifold of dimesion n. Then, in a previous paper, we have proved the existence of a function φ which is harmonic in $\mathfrak{M}-\xi_1-\xi_2$ and in a neighbourhood of ξ_i, and which satisfies

$$(1)\qquad |\varphi(x)| = \begin{cases} O\left(\dfrac{1}{r^{n-2}(x, \xi_i)}\right) & (n>2), \\ O\left(\log r(x, \xi_i)\right) & (n=2), \end{cases} \qquad i=1,2,$$

where $r(x, \xi_i)$ is the geodesic distance between x and ξ_i.

Let $G(\xi_i, \delta)$ be a geodesic sphere with center ξ_i and radius δ, where δ is so small that $G(\xi_1, \delta) \cap G(\xi_2, \delta) = \phi$. Denote by $BG(\xi_i, \delta)$ the boundary of $G(\xi_i, \delta)$. Then, since

$$\int_{BG(\xi_i, \delta)} * d\varphi - \int_{BG(\xi_i, \delta')} * d\varphi = \int_{G(\xi_i, \delta)-G(\xi_i, \delta')} d * d\varphi = 0 \qquad (\delta' < \delta),$$

the integral of $*d\varphi$ on $BG(\xi_i, \delta)$:

$$(2)\qquad \int_{BG(\xi_i, \delta)} * d\varphi = E_i, \qquad i=1,2$$

is independent of the choice of δ. Furthermore, since

$$\int_{BG(\xi_1,\delta)} * d\varphi + \int_{BG(\xi_2,\delta)} * d\varphi = \int_{B(\mathfrak{M}-G(\xi_1, \delta)-G(\xi_2, \delta))} * d\varphi = 0,$$

we have

$$(3)\qquad E_1 + E_2 = 0.$$

THEOREM 1. *A harmonic function satisfying* (1) *at its two singular points is determined uniquely by* E_1 *of* (2) *up to a constant.*

Proof. Suppose there exist two harmonic functions φ_1 and φ_2 satisfying (1) and (2), then $\varphi = \varphi_1 - \varphi_2$ also satisfies (1) and

$$(2)'\qquad \int_{BG(\xi_1, \delta)} * d\varphi = 0.$$

Hence, in order to prove Theorem 1, it is sufficient to prove that a harmonic function satisfying (1) and (2)′ is a constant.

Now, in a small domain G containing ξ_1, we consider an elementary solution Ξ of $\Delta\Xi=0$, i.e., Ξ is the function of the following form, defined for x, ξ in G :

$$(4)\qquad \begin{cases} \Xi(x, \xi) = \dfrac{1}{(n-2)\omega_n} r^{2-n}(x, \xi)\, u(x, \xi) + \log r(x, \xi)\, v(x, \xi) + w(x, \xi) & \text{for } n>2, \\ \qquad = -\dfrac{1}{2\pi} \log r(x, \xi)\, u(x, \xi) + w(x, \xi) & \text{for } n=2, \end{cases}$$

where ω_n is the surface area of the n-dimensional unit sphere, u, v and w are holomorphic with respect to x, ξ, and $u(\xi, \xi)=1$. Suppose $\xi \neq \xi_1$, and let δ and δ' be so small that $G(\xi_1, \delta) \cup G(\xi, \delta') \subset G$ and $G(\xi_1, \delta) \cap G(\xi, \delta')=\phi$. Putting

$$G' = G-G(\xi_1, \delta)-G(\xi, \delta'),$$

we have by the Green's formula in [3]

$$\int_{BG'} *(\varphi d\Xi - \Xi\, d\varphi) = (\varphi, \Delta\, \Xi)_{G'} - (\Delta\, \varphi, \Xi)_{G'} = 0,$$

which implies

$$\int_{BG} *(\varphi d\Xi - \Xi\, d\varphi) = \int_{BG(\xi_1, \delta)} *(\varphi d\Xi - \Xi d\varphi) + \int_{BG(\xi, \delta')} *(\varphi d\Xi - \Xi\, d\varphi). \tag{5}$$

From (2)′ we have

$$\lim_{\delta\to 0} \int_{BG(\xi_1, \delta)} *\Xi\, d\varphi = \Xi(\xi_1, \xi) \lim_{\delta\to 0}\int_{BG(\xi_1, \delta)} *d\varphi = 0,$$

and since $\lim_{\delta'\to 0}\int_{BG(\xi, \delta')} *d\Xi = -1$, we have

$$\lim_{\delta'\to 0}\int_{BG(\xi, \delta')} *\varphi d\Xi = \varphi(\xi) \lim_{\delta'\to 0}\int_{BG(\xi, \delta')} *d\Xi = -\varphi(\xi).$$

Furthermore, for $n>2$, we have

$$\varphi = O(1/\delta^{n-2}), \qquad \int_{BG(\xi_1, \delta)} *d\,\Xi = O(\delta^{n-1}) \text{ on } BG(\xi_1, \delta),$$

$$\text{and } \Xi(x, \xi) = O(1/\delta'^{n-2}), \qquad \int_{BG(\xi, \delta')} *d\varphi = O(\delta'^{n-1}) \text{ on } BG(\xi, \delta').$$

Therefore we have

$$\int_{BG(\xi_1, \delta)} \varphi * d\,\Xi = O\left(\frac{1}{\delta^{n-2}}\right)\cdot O(\delta^{n-1}) \to 0, \qquad (\delta \to 0),$$

$$\int_{BG(\xi, \delta')} \Xi * d\varphi = O\left(\frac{1}{\delta'^{n-2}}\right)\cdot O(\delta'^{n-1}) \to 0, \qquad (\delta' \to 0).$$

Combining these results with (5) we have immediately

$$\varphi(\xi) = -\int_{BG} *(\varphi d\Xi(\ \ ,\xi) - \Xi(\ \ ,\xi)\, d\varphi). \tag{6}$$

This representation of φ is also valid in the case of $n=2$.

Hence $\varphi(\xi)$ can be defined at ξ_1. Denote by Δ_ξ the Laplacian operator with respect to ξ, then we obtain from (6)

$$\Delta_\xi\, \varphi(\xi) = -\int_{BG} *(\varphi \Delta_\xi\, d\,\Xi(\ \ ,\xi) - \Delta_\xi\, \Xi(\ \ ,\xi)\, d\varphi). \tag{7}$$

Since the left side of (7) is equal to 0 exept for $\xi=\xi_1$ and the right side of (7) is continuous with respect to ξ in a neighbourhood of ξ_1, we get

$$\Delta_\xi\, \varphi(\xi) = 0 \text{ at } \xi = \xi_1.$$

Thus φ is harmonic at ξ_1. Similary φ is harmonic at ξ_2. Hence φ is harmonic in the whole $\mathfrak{M}$, and hence, by the Corollary of the Maximum principle in the paper [3], φ must be a constant. q.e.d.

THEOREM 2. *Let ψ be an arbitrary harmonic function having two singular points ξ_1 and ξ_2 and satisfying* (1). *Then ψ can be written as $a\varphi+b$, where a and b are constants.*

Proof. Putting

$$\int_{BG(\xi_1,\delta)} *d\psi = E'_1 \quad \text{and} \quad a = E_1'/E_1,$$

then by Theorem 1 we conclude immediately $\psi = a\varphi + b$. q.e.d.

Since ψ has the same singularities as φ, we obtain

THEOREM 3. *Let ψ be a harmonic function with two singular points ξ_1 and ξ_2 and satisfying* (1) *at $\xi_i (i=1,2)$. Then, in a neighbourhood of ξ_i, ψ can be written as*

$$\psi = \begin{cases} r^{2-n}(x,\xi_i)\,u + \log r(x,\xi_i)\cdot v + w, & (n > 2), \\ \log r(x,\xi_i)\cdot u + v, & (n = 2), \end{cases}$$

where u, v and w are holomorphic functions of x.

References

[1] K. Kodaira, Harmonic fields in Riemannian manifolds, Annals of Mathematics, vol. 50 (1949), 587-665.

[2] H. Yamasuge, Harmonic functions with two singular points. Jour. Inst. Poly. Osaka City Univ. vol. 8. No. 1 (1957), 39-42.

[3] H. Yamasuge, Maximum principle for harmonic functions in Riemannian manifolds, Jour. Inst. Poly. Osaka City Univ. vol. 8. No. 1 (1957), 35-38.