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Harmonic functions with two singular points

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In this paper we assume that $\mathfrak{M}$ is a closed orientable analytic Riemannian manifold with a positive-definite metric $ds^2=g_{ik}dx^i dx^k$ where g_{ik} are holomorphic functions of $x^1, \cdots, x^n$.

In 1 we shall prove the existence of a harmonic function φ with two singular points such that

$$\varphi \text{ is harmonic in } \mathfrak{M}-\xi_1-\xi_2,$$

$$\text{if } n>2, \quad \lim_{x\to\xi_i}(n-2)\omega_n r^{n-2}(x, \xi_i)\varphi(x)=(-1)^{i-1},\ i=1, 2,$$

$$\text{and if } n=2, \quad \lim_{x\to\xi_i} 2\pi\varphi(x)/\log r(x, \xi_i) = (-1)^i,\ i=1, 2,$$

where $r(x, \xi_i)$ is the geodesic distance between x and ξ_i, and ω_n is the surface area of the n-dimensional unit sphere. Hence we may consider that $\varphi(x)$ is the potential at x of the pair of masses which has the mass 1 at ξ_1 and the mass -1 at ξ_2.

Now we consider the equipotential surface U_C given by $\varphi=C$. We shall say that a point is stational if all first partial derivatives of φ are zero at this point, and say that a stational point is non-degenerate if at this point the determinent $\left|\dfrac{\partial^2\varphi}{\partial x^i\partial x^k}\right|$ is not zero. We change C from $+\infty$ to $-\infty$, then U_C is homeomorphic with a sphere if $|C|$ is sufficiently large, and the topological structure of U_C changes only when U_C passes stational points. Hence if we assume that all stational points are non-degenerate, there are close relations between the number of all stational points and the topological structure of $\mathfrak{M}$. We shall state about them in 2.

1. Existence of a harmonic function with two singular points

Let G be a sufficiently small geodesic sphere in $\mathfrak{M}$ and ξ an arbitrary interior point of G. Then the Laplace's equation $\Delta\Xi=0$ has a solution

$$(1) \qquad \Xi(x, \xi)=\begin{cases} -\dfrac{1}{2\pi}\log r(x, \xi)\cdot u(x, \xi)+v(x, \xi), & (n=2), \\ \dfrac{1}{(n-2)\omega_n} r^{2-n}(x, \xi)u(x, \xi)+\log r(x, \xi)\cdot v(x, \xi), & (n>2), \end{cases}$$

defined for x, ξ in G, where u, v are holomorphic with respect to x, ξ, and $u(\xi, \xi)=1$. See [1].

Let ξ_1 and ξ_2 be two interior points given in G.

Putting

$$h = \Xi(x, \xi_1) - \Xi(x, \xi_2),$$

we have

$$\Delta h = 0 \text{ in } G - \xi_1 - \xi_2.$$

We shall say that a form α is regular harmonic in a domain D if $d\alpha = 0$ and $\delta\alpha = 0$ in D. Then we have

LEMMA. *There exists* 1*-form e possessing the following properties:*

e is regular harmonic in $\mathfrak{M} - \xi_1 - \xi_2$

and

$$e = dh + f \qquad \textit{in} \quad G - \xi_1 - \xi_2$$

where f is a regular harmonic 1*-from in G.*

Proof. By the general existence theorem shown in [1] it is sufficient to prove that $dh \times BG = \int_{BG} * dh = 0$ for the surface BG of a geodesic sphere G.

For an arbitrary function g

$$d * dg = - * \delta dg = - * \Delta g.$$

Hence if g is harmonic in a domain D and its first derivatives are continuous in $\bar{D} = D + BD$, then

$$\int_{BD} * dg = \int_D d * dg = \int_D - * \Delta g = 0.$$

Applying the above to dh, $d\Xi(x, \xi_1)$ and $d\Xi(x, \xi_2)$,

we have $$\int_{BG} * dh = \int_{BG'_\delta} * d\Xi(x, \xi_1) - \int_{BG''_\delta} * d\Xi(x, \xi_2)$$

where G'_δ, G''_δ are geodesic spheres of the center ξ_1, ξ_2 and of the radius δ respectively. Moreover we can verify that

$$\lim_{\delta \to 0} \int_{BG^{(i)}_\delta} * d\Xi(x, \xi_i) = -1, \qquad i = 1, 2.$$

Hence $\int_{BG} * dh = 0$, q. e. d. [See the proof of Theorem 2 in [4]].

Now we consider the periods of e along loops which do not pass through the points ξ_1 and ξ_2. Since e is closed, the periods of e depend only on homology classes of loops in $\mathfrak{M}$. Let $\gamma_i (i = 1, \cdots, R_1)$ be a base for 1-cycles of $\mathfrak{M}$. Then by the theorem of de Rham there exists harmonic 1-form e' such that

$$\int_{\gamma_i} e = \int_{\gamma_i} e', \ (i = 1, \cdots, R_1).$$

Let P_0 be a fixed point in $\mathfrak{M}$ and P an arbitrary point in $\mathfrak{M}$.

Put

$$\varphi(P) = \int_{P_0}^{P} (e - e'),$$

then the periods of $\varphi(P)$ on every loops are always zero. Hence $\varphi(P)$ is an one valued function defined in $\mathfrak{M}$. From the construction of φ we have

THEOREM 1. *For arbitrary two points ξ_1 and ξ_2 given in a small subdomain*

G, there exists a harmonic function φ *such that*

$$\Delta\varphi = 0 \quad in \quad \mathfrak{M} - \xi_1 - \xi_2 ,$$

and in a neighbourhood of ξ_i $(i=1, 2)$

$$(2) \qquad \varphi(x) = \begin{cases} \dfrac{1}{(n-2)\omega_n} r^{2-n}(x, \xi_i)u_i + \log r(x, \xi_i) \cdot v_i + w_i , & (n > 2) , \\ -\dfrac{1}{2\pi} \log r(x, \xi_i) \cdot u_i + v_i , & (n = 2) , \end{cases}$$

where u_i, v_i *and* w_i *are holomorphic functions of* x, *and* $u_i(\xi_i) = 1$.

2. Relations between the number of stational points and the topological structure of $\mathfrak{M}$.

From now on let us assume that all stational points are non-degenerate. Then in a suitable coordinate system, the Taylor's expansion of φ at every stational point becomes

$$(3) \qquad \varphi(x) = C + (-x_1^2 - \cdots - x_\nu^2 + x_{\nu+1}^2 + \cdots + x_n^2) + \chi(x).$$

Hence the stational point $x=0$ is isolated, and since $\mathfrak{M}$ is closed, the number of stational points is finite.

Suppose $\nu=0$ and take a sufficiently small positive number δ. Then the closed subdomain G_δ given by the inequality $\varphi(x) \leqq C+\delta$ is homeomorphic with a sphere. Using maximum principle for φ, we see that φ is the constant $C+\delta$ in G_δ. By [1], φ is holomorphic in $\mathfrak{M}-\xi_1-\xi_2$. Therefore φ would be identically the constant $C+\delta$, contrary to (2) of Theorem 1. Hence $\nu \geqq 1$.

Similarly we have $\nu \leqq n-1$.

Now let us consider the equipotential surface $U=C$ and denote it by U_C. If U_C has no stational point, U_C is an orientable $(n-1)$-dimensional manifold.

Change C from $+\infty$ to $-\infty$. Then U_C moves in $\mathfrak{M}$ but the topological structure of U_C changes only when U_C passes the stational points of φ.

The case of $n=2$. In this case (3) becomes

$$(4) \qquad \varphi(x) = C - x_1^2 + x_2^2 + \chi(x).$$

Let δ be a sufficiently small positive number. Then in a neighbourhood V of the stational point $x=0$, $U_{C\pm\delta}$ are hyperbolas and U_C is two straight lines. If U_C passes no stational point, U_C cosists of some loops, and the number of the loops increases by 1 or decreases by 1 whenever U_C passes a stational point. Suppose $\{P_i, Q_j; i=1, \cdots, g, j=1, \cdots g'\}$ is the complete set of all stational points such that the number of the loops increases by 1 when U_C passes P_i and decreases by 1 when U_C passes Q_j. And if C is sufficiently large, by (2) $U_{\pm C}$ consists of one loops. Hence we have $g = g'$. Moreover we can assume without loss of generality that $\varphi(P_i) > \varphi(Q_j)$ for $i, j=1, \cdots, g$. Take C so that $\varphi(P_i) > C > \varphi(Q_j)$ for $i, j=1, \cdots, g$.

Put

(5) $$\mathfrak{M}(a, b) = \{P \mid a \leqq \varphi(P) \leqq b\}$$

then $$\mathfrak{M} = \mathfrak{M}(C, \infty) + \mathfrak{M}(-\infty, C)$$

where $\mathfrak{M}$ (C, ∞) and $\mathfrak{M}(-\infty, C)$ are homeomorphic to a sphere with g holes. Hence we have

THEOREM 2. *If all stational points of φ are non-degenerate, then the number of these points is equal to twice the genus of $\mathfrak{M}$.*

The case of $n=3$. In this case ν of (3) is 2 or 1. Let $P_1, \cdots, P_g$ be all stational points at which $\nu=2$ and $Q_1, \cdots, Q_{g'}$ all stational points at which $\nu=1$. Then in a neighbourhood of P_i, U_C changes from a hyperboloid of two sheets to a hyperboloid of one sheet, and in a neighbourhood of Q_j, U_C changes from a hyperboloid of one sheet to a hyperboloid of two sheets. Thus we see easily that the genus of a connected component of U_C increases by 1 or decreases by 1 according as the component passes point P_i or Q_j. Moreover if C is sufficiently large, then by (2), $U_{\pm C}$ is homeomorphic with a sphere, and hence $g = g'$.

We may assume without loss of generality that $\varphi(P_i) > \varphi(Q_j)$, $i, j=1, \cdots, g$. Then similarly to (5) we have

(6) $$\mathfrak{M} = \mathfrak{M}(C, \infty) + \mathfrak{M}(-\infty, C)$$

where $\mathfrak{M}(C, \infty)$ and $\mathfrak{M}(-\infty, C)$ are homeomorphic with a closed subdomain bounded by a surface of genus g in E^3. Thus we have

THEOREM 3. *If all stational points of φ are non-degenerate, then the number g of these points is even. Take two closed domains bounded by a surface of genus $g/2$ in E^3. Then $\mathfrak{M}$ is obtained from these two domains by identifying their boundaries by a homeomorphism.*

Reference

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