Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 8, No. 1, Series A

Non-existence of mappings of S^{31} into S^{16} with Hopf invariant 1

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(Received Oct. 10, 1956)

1. Statement of results

It was proved in [1] that if there exists an element of $\pi_{2n-1}(S^n)$ with Hopf invariant 1, then n has to be a power of 2. The existence of an element of $\pi_{2n-1}(S^n)$ with Hopf invariant 1 is established for the case $n=2$, 4 or 8 by the Hopf fibre mapping. [3].

The purpose of this paper is to give a proof of the following theorem.

THEOREM (1·1) *There exist no elements of* $\pi_{31}(S^{16})$ *with Hopf invariant* 1.

This theorem is equivalent to one of the following statements:

(1·2) *the Whitehead product* $[\iota_{15}, \iota_{15}]$ *of the class* $\iota_{15}\in\pi_{15}(S^{15})$ *of the identity of* S^{15} *is not zero*;

(1·3) *there are no mappings* $S^{15}\times S^{15}\to S^{15}$ *of type* (ι_{15}, ι_{15});

(1·4) *for a complex* $S^n\cup e^{n+16}$ *the squaring operation* Sq^i *is trivial for all* $i>0$.

For the proof, see (3·49) and (3·72) of [9] and [5].

2. Auxiliary results

To prove the theorem we apply the following results without proofs.

(2·1) $\pi_i(S^n)=0$ *for* $i<n$. $\pi_n(S^n)=Z$ *which is generated by the class* ι_n *of the identity of* S^n.

(2·2) $\pi_3(S^2)=Z$ *wihch is generated by the class* η_2 *of the Hopf fibre mapping.* $H(\eta_2)=\iota_3$. $\pi_{n+1}(S^n)=Z_2$ *for* $n>2$, *the generator of which is* $\eta_n=E^{n-2}\eta_2$. (cf. [2]).

(2·3) $\pi_{n+2}(S^n)=Z_2$ *for* $n\geqq 2$. *Generator*: $\eta_n\circ\eta_{n+1}$. (cf. [10]).

(2·4) $\pi_{n+3}(S^n)=Z_{24}$ *for* $n\geqq 5$. *We denote by* ν_n *a generator of* $\pi_{n+3}(S^n)$ *such that* $E\nu_n=\nu_{n+1}$. $12\nu_n=\eta_n\circ\eta_{n+1}\circ\eta_{n+2}$. (cf. [7]).

(2·5) $\pi_{n+4}(S^n)=0$ *for* $n\geqq 6$ *and* $\pi_{n+5}(S^n)=0$ *for* $n\geqq 7$. (cf. [7]).

(2·6) $\pi_{n+6}(S^n)=Z_2$ *for* $n\geqq 5$. *Generator*: $\nu_n\circ\nu_{n+3}$. (cf. [4]).

(2·7) $\pi_{12}(S^5)=Z_{60}$, $\pi_{14}(S^7)=Z_{120}$ *and* $\pi_{n+7}(S^n)=Z_{240}$ *for* $n\geqq 9$. (cf. [4]).

(2·8) $\sigma_n\circ\eta_{n+7}\neq 0$ *in* $\pi_{n+8}(S^n)$ *for a generator* σ_n *of* 2-*component of* $\pi_{n+7}(S^n)$, $n\geqq 9$. (cf. [1]).

(2·9) The following sequence of homotopy groups *is exact for* $i<3n-1$ [11] and *exact* mod. 2 *for* $i<4n-3$ (cf. [12]);

$$\cdots\xrightarrow{\Delta}\pi_i(S^n)\xrightarrow{E}\pi_{i+1}(S^{n+1})\xrightarrow{H}\pi_{i+1}(S^{2n+1})\xrightarrow{\Delta}\pi_{i-1}(S^n)\xrightarrow{E}\cdots,$$

where E is the Freudenthal suspension homomorphism, H is the generalized Hopf homomorphism [9] [12] and Δ is defined by

$$\Delta(E^2\alpha)=[\iota_n, \iota_n]\circ\alpha .$$

We shall use the following properties of the homomorphism H from [9]

(2·10). $$H(\alpha\circ E\beta)=H\alpha\circ E\beta ,$$
$$H((E\alpha)\circ\beta)=(\alpha*\alpha)\circ H\beta ,$$
$$H([\iota_n, \iota_n])=2\iota_{2n-1} \text{ if } n \text{ is even.}$$

Let σ_8 be the class of the Hopf fibre mapping of S^{15} onto S^8, then it is well known that

(2·11) the correspondence $(\alpha, \beta)\longrightarrow E\alpha+\sigma_8\circ\beta$ induces an isomorphism of $\pi_i(S^7)+\pi_{i+1}(S^{15})$ onto $\pi_{i+1}(S^8)$.

For the non-zero element of $\pi_{14}(S^7)$, which is given in §8 of [9] and here denoted by τ, we have from [6]

(2·12) $[\iota_8, \iota_8]=2\sigma_8-E\tau$.

We denote that $\sigma_{n+8}=E^n\sigma_8$, then $E^2\tau=2\sigma_9$ since $E[\iota_8, \iota_8]=0$. Since $H(\tau)=\eta_{13}\neq 0$ [9], the homomorphism H: $\pi_{14}(S^7)=Z_{120}\longrightarrow\pi_{14}(S^{13})=Z_2$ is onto. From the exactness (2·9) of the sequence

$$\pi_{13}(S^6)=Z_{60}\xrightarrow{E}\pi_{14}(S^7)=Z_{120}\xrightarrow{H}\pi_{14}(S^{13})=Z_2 ,$$

we have that E is an isomorphism into and there exists an element τ' of $\pi_{13}(S^6)$ such that $E\tau'=2\tau$. Also from the exactness (2·9) of the sequence

$$\pi_{12}(S^5)=Z_{30}\xrightarrow{E}\pi_{13}(S^6)=Z_{60}\xrightarrow{H}\pi_{13}(S^{11})=Z_2 ,$$

we have that $H(\tau')=\eta_{11}\circ\eta_{12}$.

Consequently,

(2·13) *there are elements* $\tau'\in\pi_{13}(S^6)$ *and* $\tau\in\pi_{14}(S^7)$ *such that*

$$H\tau=\eta_{13} , \qquad E^{n+2}\tau=2\sigma_{n+9} ,$$
$$H\tau'=\eta_{11}\circ\eta_{12} , \qquad E^{n+3}\tau=4\sigma_{n+9} ,$$

and the order of $\sigma_{n+9}\geqq 0$, *is* $16k$ *for an odd integer* k.

Consider the element $\eta_7\circ\sigma_8+\tau\circ\eta_{14}$. By (2·10), $H(\eta_7\circ\sigma_8+\tau\circ\eta_{14})=\eta_{13}\circ\eta_{14}+\eta_{13}\circ\eta_{14}=0$. From the exactness (2·9) of the sequence

$$\pi_{14}(S^6)\xrightarrow{E}\pi_{15}(S^7)\xrightarrow{H}\pi_{15}(S^{13}),$$

we have that $\eta_7\circ\sigma_8+\tau\circ\eta_{14}\in E\pi_{14}(S^6)$. Since $E^2(\tau\circ\eta_{14})=2\sigma_9\circ\eta_{16}=\sigma_9\circ(2\eta_{16})=0$, $E^2(\eta_7\circ\sigma_8+\tau\circ\eta_{14})=\eta_9\circ\sigma_{10}$. Therefore

(2·14) $$\eta_n\circ\sigma_{n+1}\in E^{n-6}\pi_{14}(S^6) \quad \text{for } n\geqq 9.$$

3. Proof of the Theorem

Applying theorem (4·6) of the previous paper [8] to the pairs (σ_8, η_2), (τ, τ'), (σ_8, τ'), (σ_8, τ), and (σ_8, σ_8), we have that

(3・1). i) $[\iota_9, \iota_9] = \sigma_9 \circ \eta_{16} + \eta_9 \circ \sigma_{10}$,

ii) $[\iota_{12}, \iota_{12}] \circ \eta_{23} \circ \eta_{24} \circ \eta_{25} = 16\, \sigma_{12} \circ \sigma_{19} = 4E^6(\tau' \circ \sigma_{13}) \in E^6\pi_{20}(S^6)$,

iii) $[\iota_{13}, \iota_{13}] \circ \eta_{25} \circ \eta_{26} = 8\sigma_{13} \circ \sigma_{20} = 2E^7(\tau' \circ \sigma_{13}) \in E^7\pi_{20}(S^6)$,

iv) $[\iota_{14}, \iota_{14}] \circ \eta_{27} = 4\sigma_{14} \circ \sigma_{21} = E^8(\tau' \circ \sigma_{13}) \in E^8\pi_{20}(S^6)$,

v) $[\iota_{15}, \iota_{15}] = 2\sigma_{15} \circ \sigma_{22}$.

Now consider the element $\tau \circ \sigma_{14}$. Since $H(\tau \circ \sigma_{14}) = \eta_{13} \circ \sigma_{14} = \sigma_{13} \circ \eta_{20}$ is an element of order 2 by (2・8), we have from the exactness (2・9) of the sequence

$$\pi_{20}(S^6) \xrightarrow{E} \pi_{21}(S^7) \xrightarrow{H} \pi_{21}(S^{13})$$

that

(3・2) $\tau \circ \sigma_{14}$ *has order* 2 mod. $E\pi_{20}(S^6)$.

By (2・11), $\pi_{22}(S^8) = \sigma_8 \circ \pi_{22}(S^{15}) + E\pi_{21}(S^7)$. By (2・9) the kernel of $E: \pi_{22}(S^8) \longrightarrow \pi_{23}(S^9)$ is $\Delta(\pi_{24}(S^{17})) = \Delta(E^3\pi_{21}(S^{14}))$. Since $H(\tau \circ 2\alpha) = 2(\eta_{13} \circ \eta_{14}) \circ \alpha = 0$ for $\alpha \in \pi_{21}(S^{14})$, we have $\tau \circ 2\alpha \in E\pi_{20}(S^6)$ by (2・9). Then $2\Delta(\pi_{24}(S^{17})) \subset 4\sigma_8 \circ \pi_{22}(S^{15}) + E^2\pi_{20}(S^6)$. Since $\Delta(\sigma_{17}) = 2\sigma_8 \circ \sigma_{15} - E\tau \circ \sigma_{15}$ and since σ_{15} generates $\pi_{22}(S^{15})/2\pi_{22}(S^{15})$, we have from (2・9) that

(3・3) $E^2\tau \circ \sigma_{16} = 2\sigma_9 \circ \sigma_{16}$ *has order* 2 mod. $E^3\pi_{20}(S^6)$.

By (2・5), $\eta_n \circ \nu_{n+1} = 0$ for $n \geqq 6$. By (3・1), i), $\Delta(\nu_{19} \circ \nu_{22}) = [\iota_9, \iota_9] \circ \nu_{17} \circ \nu_{20} = \eta_9 \circ \sigma_{10} \circ \nu_{17} \circ \nu_{20} + \sigma_9 \circ \eta_{16} \circ \nu_{17} \circ \nu_{20} = \eta_9 \circ \sigma_{10} \circ \nu_{17} \circ \nu_{20}$. Then by (2・14) and (2・9), the kernel of $E: \pi_{23}(S^9) \longrightarrow \pi_{24}(S^{10})$ is in $E^3\pi_{14}(S^6) \circ \nu_{17} \circ \nu_{20} \subset E^3\pi_{20}(S^6)$.

By (2・5), $\Delta(\pi_{26}(S^{21})) = \Delta(\pi_{27}(S^{23})) = 0$. Then the suspension homomorphisms $E: \pi_{24}(S^{10}) \longrightarrow \pi_{25}(S^{11})$ and $E: \pi_{25}(S^{11}) \longrightarrow \pi_{26}(S^{12})$ are isomorphisms into. Therefore

(3・5) $2\sigma_{12} \circ \sigma_{19}$ *has order* 2 mod. $E^6\pi_{20}(S^6)$.

By (2・10), $H(\Delta(\nu_{25})) = H[\iota_{12}, \iota_{12}] \circ \nu_{23} = 2\nu_{23}$. If $\Delta\alpha$ is an image of E for $\alpha \in \pi_{28}(S^{25})$, then $2\alpha = 0$ since $H \circ E = 0$. Then $E\pi_{25}(S^{11}) \cap \Delta(\pi_{28}(S^{25}))$ is generated by $\Delta \eta_{25} \circ \eta_{26} \circ \eta_{27}) = 4E^6(\tau' \circ \sigma_{13})$ since (3・1), ii). Therefore

(3・6) $2\sigma_{13} \circ \sigma_{20}$ *has order* 2 mod. $E^7\pi_{20}(S^6)$.

By (2・9) and by (3・1), iii), the kernel of $E: \pi_{27}(S^{13}) \longrightarrow \pi_{28}(S^{14})$ is in $E^7\pi_{20}(S^6)$. Then

(3・7) $2\sigma_{14} \circ \sigma_{21}$ *has order* 2 mod. $E^8\pi_{20}(S^6)$.

Similarly, from (2・9) and (3・1), iv), we have that

(3・8) $2\sigma_{15} \circ \sigma_{22}$ *has order* 2 mod. $E^9\pi_{20}(S^6)$.

Since $4\sigma_{15} \circ \sigma_{22} = E(4\sigma_{14} \circ \sigma_{21}) = E([\iota_{14}, \iota_{14}] \circ \eta_{27}) = 0$, we have from (3・1), v) that

(3・9) $[\iota_{15}, \iota_{15}] = 2\sigma_{15} \circ \sigma_{22}$ *has order* 2.

Consequently, $[\iota_{15}, \iota_{15}] \neq 0$ and (2・2) is proved. By (3・49) of [9] we have Theorem (1・1).

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