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Reduced join and Whitehead product

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Introduction

Barratt and Hilton [1]* proved the formula

$$E^{n+1}\alpha \circ E^{p+1}\beta = (-1)^{(p+m)(q+n)} E^{m+1}\beta \circ E^{q+1}\alpha$$

for $\alpha \in \pi_{p+1}(S^{m+1})$ and $\beta \in \pi_{q+1}(S^{n+1})$, by making use of the reduced join operation " $\divideontimes$ ". Then the element

$$E^n\alpha \circ E^p\beta - (-1)^{(p+m)(q+n)} E^m\beta \circ E^q\alpha$$

is in the kernel of the Freudenthal suspension homomorphism $E: \pi_{p+q+1}(S^{m+n+1}) \longrightarrow \pi_{p+q+2}(S^{m+n+2})$ which is closely related with the Whitehead product.

We prove here the following formula

$$E^n\alpha \circ E^p\beta - (-1)^{(p+m)(q+n)} E^m\beta \circ E^q\alpha = \pm[\iota_{m+n+1}, \iota_{m+n+1}] \circ E^{2n}H\alpha \circ E^pH\beta$$

under some conditions. This formula will be applied, in the next paper, to prove the non-existence of mappings: $S^{31} \longrightarrow S^{16}$ of the Hopf invariant 1.

1. Reduced join and preliminaries

In the following, for each space X we fix a base point $x_0 \in X$. When X is a cell complex, we take a vertex v_0 of X as a basepoint, and when X is the unit sphere

$$S^n = \{(t_1, \cdots, t_{n+1}) \mid t_1^2 + \cdots + t_{n+1}^2 = 1\}$$

of dimension n we take a point $e_0 = (-1, 0, \cdots, 0)$ as the base point.

Consider two spaces X and Y with base points $x_0 \in X$ and $y_0 \in Y$. Let $X \vee Y$ denote the subspace

$$X \times y_0 \cup x_0 \times Y$$

of $X \times Y$. A space Z, with a basepoint z_0, is called *a reduced join* of X and Y if there exists a mapping

$$\phi : (X \times Y, X \vee Y) \longrightarrow (Z, z_0)$$

which maps $X \times Y - X \vee Y = (X - x_0) \times (Y - y_0)$ homeomorphically onto $Z - z_0$, and we denote that

$$Z = X \divideontimes Y \qquad \text{and} \qquad \phi(x, y) = x \divideontimes y.$$

As is easily seen, the spaces $(X \divideontimes Y) \divideontimes Z$ and $X \divideontimes (Y \divideontimes Z)$ are naturally homeomorphic, and we denote these spaces by the same symbol $X \divideontimes Y \divideontimes Z$.

For two mappings

$$f : (X, x_0) \longrightarrow (X', x_0') \text{ and } g : (Y, y_0) \longrightarrow (Y', y_0'),$$

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* Numbers in bracket refer to the references at the end of the paper.

we define their *reduced join*

$$f\divideontimes g : X\divideontimes Y \longrightarrow X'\divideontimes Y'$$

by setting

$$(f\divideontimes g)(x\divideontimes y)=(f\divideontimes g)(\phi(x,y))=\phi'(f(x),\ g(y))=f(x)\divideontimes g(y)$$

for $x\in X$ and $y\in Y$, where ϕ and ϕ' are shrinking maps defining the reduced joins $X\divideontimes Y$ and $X'\divideontimes Y'$. The following formulas are easily verified:

(1·1), i) $\quad (f\divideontimes g)\divideontimes h=f\divideontimes(g\divideontimes h)$,

ii) $\quad (f'\circ f)\divideontimes(g'\circ g)=(f'\divideontimes g')\circ(f\divideontimes g)$,

iii) $\quad \sigma'\circ(f\divideontimes g)=(g\divideontimes f)\circ\sigma$,

where $\sigma: X\divideontimes Y\longrightarrow Y\divideontimes X$ and $\sigma': X'\divideontimes Y'\longrightarrow Y'\divideontimes X'$ are homeomorphisms given by $\sigma(x\divideontimes y)=y\divideontimes x$ and $\sigma'(x'\divideontimes y')=y'\divideontimes x'$.

Denote by V^{n+1} the cube bounded by S^n, i.e.,

$$V^{n+1}=\{(t_1,\cdots,t_{n+1})\mid t_1^2+\cdots+t_{n+1}^2\leqq 1\}.$$

Define a mapping

(1·2) $\quad d'_n: (S^n\times V^1,\ S^n\times e_0\cup e_0\times V^1)\longrightarrow(V^{n+1},\ e_0)$

which maps $(S^n-e_0)\times(V^1-e_0)$ homeomorphically onto $V^{n+1}-e_0$ by the formula

$$d'_n((t_1,\cdots,t_{n+1}),t)=((t_1+1)(t+1)/2-1,\ t_2(t+1)/2,\cdots,t_{n+1}(t+1)/2),$$

$(t_1,\cdots,t_{n+1})\in S^n$, $t\in V^1$. The mapping d'_n shows that $V^{n+1}=S^n\divideontimes V^1$.

Denote by E_+^{n+1} and E_-^{n+1} the upper and lower hemi-spheres of S^{n+1}, i.e., $E_+^{n+1}=\{(t_1,\cdots,t_{n+2})\in S^{n+1}\mid t_{n+2}\geqq 0\}$ and $E_-^{n+1}=\{(t_1,\cdots,t_{n+2})\in S^{n+1}\mid t_{n+2}\leqq 0\}$. Define a mapping

(1·2)' $\quad d_n: (S^n\times V^1,\ S^n\times S^0\cup e_0\times V^1)\longrightarrow(S^{n+1},\ e_0)$

by setting

$$d_n(x,t)=\begin{cases}p_+(d'_n(x,\ 1-2t)) & \text{for } 0\leqq t\leqq 1,\\ p_-(d'_n(x,\ 2t+1)) & \text{for } -1\leqq t\leqq 0,\end{cases}$$

where $p_+: V^{n+1}\longrightarrow E_+^{n+1}$ and $p_-: V^{n+1}\longrightarrow E_-^{n+1}$ are the projections (homeomorphisms) along the $(n+2)$-axis. The mapping d_n maps $(S^n-e_0)\times(V^1-S^0)$ homeomorphically onto $S^{n+1}-e_0$.

Define a mapping

(1·3) $\quad \phi_{m,n}: (S^m\times S^n,\ S^m\vee S^n)\longrightarrow(S^{m+n},\ e_0)$

inductively by the formulas

$$\phi_{m,0}(x,\ 1)=x,\quad \phi_{m,0}(x,\ -1)=e_0,$$

$$\phi_{m,n}(x,\ d_{n-1}(y,\ t))=d_{m+n-1}(\phi_{m,n-1}(x,\ y),\ t),$$

$x\in S^m$, $y\in S^{n-1}$, $n\geqq 1$, $t\in V^1$. As is easily seen, $\phi_{m,n}$ maps $S^m\times S^n-S^m\vee S^n$ homeomorphically onto $S^{m+n}-e_0$. Then

$$S^{m+n}=S^m\divideontimes S^n$$

with respect to the mapping $\phi_{m,n}$. From the definition of $\phi_{m,n}$, the equality

$$\phi_{l+m,n}(\phi_{l,m}(u,\ x),\ y))=\phi_{l,m+n}(u,\ \phi_{m,n}(x,\ y))$$

is verified directly. Then we have the identification

$$(S^l\divideontimes S^m)\divideontimes S^n=S^l\divideontimes(S^m\divideontimes S^n)\quad(=S^{l+m+n}).$$

Define a homeomorphism

(1·4) $\sigma_{m,n}: S^{m+n} \longrightarrow S^{m+n}$

by setting $\sigma_{m,n}(\phi_{m,n}(x, y)) = \phi_{n,m}(y, x)$, $x \in S^m$, $y \in S^n$.

LEMMA (1·4)′. *The degree of $\sigma_{m,n}$ is $(-1)^{mn}$.*

Proof. Let E^r denote a cube such that $E^r = \{(t_1, \cdots, t_r) \mid -1 \leqq t_i \leqq 1,\ i=1, \cdots, r\}$. Define a mapping $\varphi_r: E^r \longrightarrow S^r$ inductively by setting $\varphi_1(t) = d_0(1, t)$ and $\varphi_r(t_1, \cdots, t_{r-1}, t_r) = d_{r-1}(\varphi_{r-1}(t_1, \cdots t_{r-1}), t_r)$, then φ_r shrinks the boundary of E^r to a single point e_0. Let $\sigma: E^{m+n} \longrightarrow E^{m+n}$ be a homeomorphism given by the permutation $\sigma(t_1, \cdots, t_m, t_{m+1}, \cdots, t_{m+n}) = (t_{m+1}, \cdots, t_{m+n}, t_1, \cdots, t_m)$, then it is well known that the degree of σ is $(-1)^{mn}$. It is calculated directly that

$$\sigma_{m,n} \circ \varphi_{m+n} = \varphi_{m+n} \circ \sigma.$$

Then the degree of $\sigma_{m,n}$ is $(-1)^{mn}$. q. e. d.

If f_t and g_t are homotopies fixing the base points, then $f_t \circledast g_t$ is a homotopy. Therefore, if $f: (S^p, e_0) \longrightarrow (X, x_0)$ and $g: (S^q, e_0) \longrightarrow (Y, y_0)$ represent $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(Y)$ respectively, then $f \circledast g: (S^{m+n}, e_0) \longrightarrow (X \circledast Y, x_0 \circledast y_0)$ belongs an element $\alpha \circledast \beta \in \pi_{m+n}(X \circledast Y)$, called the *reduced join* of α and β, which depends only on α and β. From (1·1), we have that

(1·5), i) $(\alpha \circledast \beta) \circledast \gamma = \alpha \circledast (\beta \circledast \gamma)$,

ii) $(f'_* \alpha) \circledast (g'_* \beta) = (f' \circledast g')_* (\alpha \circledast \beta)$,

iii) $\sigma'_*(\alpha \circledast \beta) = (-1)^{pq} (\beta \circledast \alpha)$.

The reduced join $X \circledast S^1$ is called *a suspension* of X, and we denote that

$$X \circledast S^1 = EX.$$

Let $\phi: X \times S^1 \longrightarrow X \circledast S^1 = EX$ be the mapping which defines the reduced product $X \circledast S^1$. Define a mapping

(1·6) $d_X: (X \times V^1,\ X \times S^0 \cup x_0 \times V^1) \longrightarrow (EX, x_0)$

by the formula $d_X(x, t) = \phi(x, d_0(1, t))$, then d_X maps $(X - x_0) \times (V^1 - S^0)$ homeomorphically onto $EX - x_0$. Conversely a suspension EX of X is defined by a shrinking map d_X of (1·6). We denote

$$C_+(X) = d_X(X \times [0, 1]) \quad \text{and} \quad C_-X = d_X(X \times [-1, 0])$$

and identify each point x of X with a point $d_X(x, 0)$ of EX. Then C_+X and C_-X are contractible to the point $x_0 = x_0 \circledast e_0$ and $C_+X \cap C_-X = X$. With respect to the mapping d_n, we have $S^{n+1} = ES^n = S^n \circledast S^1$, $E^{n+1}_+ = C_+S^n$ and $E^{n+1}_- = C_-S^n$.

For a mapping $f: (X, x_0) \longrightarrow (Y, y_0)$, let

$$Ef: EX \longrightarrow EY$$

denote the mapping $f \circledast i_1$ and it is called *a suspension* of f. The mapping $Ef = f \circledast i_1$ is also defined by the formula

$$Ef(d_X(x, t)) = d_Y(f(x), t),$$

$x \in X$, $t \in V^1$. Obviously, $Ef(C_+X) \subset C_+Y$, $Ef(C_-X) \subset C_-Y$ and $Ef \mid X = f$, and conversely, a mapping satisfying these three conditions is homotopic to Ef.

We denote that

$$X \circledast S^n = E^n X.$$

Since $E^nX = X\circledast S^n = X\circledast S^{n-1}\circledast S^1 = E(X\circledast S^{n-1}) = E(E^{n-1}X)$, the space E^nX is an n-fold suspension of X. Also we denote by E^nf the n-fold suspension of f, then

$$E^n f = f \circledast i_n$$

for the indentity i_n of S^n. For the class $\alpha\in\pi_p(X)$ of a mapping $f: (S^p, e_0)\longrightarrow (X, x_0)$, the n-fold suspension

$$E^n\alpha \in \pi_{p+n}(E^nX)$$

is the class of E^nf. Then

$$E^n\alpha = \alpha\circledast\iota_n \quad (E\alpha = E^1\alpha = \alpha\circledast\iota_1)$$

for the class ι_n of i_n.

The following formula is verified in [1].

PROPOSITION (1·7)

$$\alpha\circledast\beta = (-1)^{p(q+n)}E^n\alpha\circ E^p\beta = (-1)^{m(q+n)}E^m\beta\circ E^q\alpha$$

for $\alpha\in\pi_p(S^m)$ *and* $\beta\in\pi_q(S^n)$.

Proof. First we remark that $(-\iota_{r+s})\circ E^s\gamma = -E^s\gamma$ for $s\geqq 1$ and for $\gamma\in\pi_k(S^r)$. Then $\iota_s\circledast\gamma = (-1)^{rs}\sigma_{k,s_*}(\gamma\circledast\iota_s) = (-1)^{s(k+r)}E^s\gamma$ by (1·5), iii) and (1·4)′.

By (1·5), ii),

$$\begin{aligned}\alpha\circledast\beta &= (\alpha\circ\iota_p)\circledast(\iota_n\circ\beta)\\ &= (\alpha\circledast\iota_n)\circ(\iota_p\circledast\beta)\\ &= (-1)^{p(q+n)}E^n\alpha\circ E^p\beta.\end{aligned}$$

Also

$$\begin{aligned}\alpha\circledast\beta &= (\iota_m\circ\alpha)\circledast(\beta\circ\iota_q)\\ &= (\iota_m\circledast\beta)\circ(\alpha\circledast\iota_q)\\ &= (-1)^{m(q+n)}E^m\beta\circ E^q\alpha.\end{aligned}$$

q. e. d.

Define a homeomorphism

(1·8) $\tau_{m,n}: (V^m\times V^n, V^m\times S^{n-1}\cup S^{m-1}\times V^n)\longrightarrow (V^{m+n}, S^{m+n-1})$

by the formula

$$\tau_{m,n}((t_1,\cdots,t_m),(s_1,\cdots,s_n)) = (\lambda t_1,\cdots,\lambda t_m,\lambda s_1,\cdots,\lambda s_n),$$

where $\lambda = \{\text{Max.}\,(t_1^2+\cdots+t_m^2,\ s_1^2+\cdots+s_n^2)/(t_1^2+\cdots+t_m^2+s_1^2+\cdots s_n^2)\}^{\frac{1}{2}}$

For a mapping $f: S^m\times S^n\longrightarrow X$, *a Hopf construction*

$$\bar f: S^{m+n+1}\longrightarrow EX$$

of f is a mapping which satisfies the following conditions.

$$\begin{aligned}&\bar f(\tau_{m+1,n+1}(V^{m+1}\times S^n))\subset C_+X,\\ (1\cdot 9)\qquad &\bar f(\tau_{m+1,n+1}(S^m\times V^{n+1}))\subset C_-X,\\ &\bar f\circ\tau_{m+1,n+1}\,|\,S^m\times S^n = f\end{aligned}$$

It is easy to see that

(1·9)′ *mappings which satisfy* (1·9) *are homotopic to each other.*

LEMMA (1·10) *Let* $\bar\phi_{m,n}: S^{m+n+1}\longrightarrow S^{m+n+1}$ *be a Hopf construction of the mapping* $\phi_{m,n}$ *of* (1·3). *Then the degree of* $\bar\phi_{m,n}$ *is* $(-1)^n$.

Proof. Set $F_+^{m+1} = \{(t_1,\cdots,t_{m+2})\in S^{m+1}\,|\,t_{m+2}\geqq 1/\sqrt{2}\}$ and $F_-^{m+1} = \{(t_1,\cdots,t_{m+2})\in S^{m+1}\,|\,t_{m+2}\leqq 1/\sqrt{2}\}$. $\bar\phi_{m,0}$ maps F_+^{m+1} and F_-^{m+1} into E_+^{m+1} and E_-^{m+1} respectively and the restriction $\bar\phi_{m,0}\,|\,F_+^{m+1}\cap F_-^{m+1}$ is given by $\bar\phi_{m,0}(t_1,\cdots,t_{m+1},1/\sqrt{2}) = (\sqrt{2}\,t_1,\cdots,\sqrt{2}\,t_{m+1},0)$. Then $\bar\phi_{m,0}$ is homotopic to the identity. Now we chose a Hopf construction $\bar{\phi}_{m,n}$ of $\phi_{m,n}$ such that

$$(1\cdot 10)' \quad \begin{aligned} &\bar{\phi}_{m,n}(\tau_{m+1,n+1}(d'_m(x,t),y)) = d_{m+n}(\phi_{m,n}(x,y),\ (1-t)/2),\\ &\bar{\phi}_{m,n}(\tau_{m+1,n+1}(x,d'_n(y,t))) = d_{m+n}(\phi_{m,n}(x,y),\ (t-1)/2). \end{aligned}$$

Let $\sigma: S^{m+n+1} \longrightarrow S^{m+n+1}$ be a homeomorphism given by $\sigma(d_{m+n}(d_{m+n-1}(z,t_1),t_2)) = d_{m+n}(d_{m+n-1}(z,t_2),t_1)$, then $\sigma = i_{m+n-1} \divideontimes \sigma_{1,1}$ and its degree is -1. Since $E_+^{m+n+1} = \tau_{m+1,n+1}(V^{m+1}\times E_+^n \cup S^m \times d'_n(E_+^n \times V^1))$ and $\phi_{m,n}(S^m\times E_+^n)\subset E_+^{m+n}$, we have that $(\sigma\circ\bar{\phi}_{m,n})(E_+^{m+n+1}) \subset \sigma(d_{m+n}(E_+^{m+n}\times V^1)) = E_+^{m+n+1}$. Similarly $(\sigma\circ\bar{\phi}_{m,n})(E_-^{m+n+1})\subset E_-^{m+n+1}$. Since $\tau_{m+1,n} = \tau_{m+1,n+1} \mid V^{m+1}\times V^n$, $\phi_{m,n-1} = \phi_{m,n} \mid S^m \times S^{n-1}$ and since $d'_{n-1} = d'_n \mid S^{n-1}\times V^1$, we have that $\bar{\phi}_{m,n-1} = (\sigma\circ\bar{\phi}_{m,n}) \mid S^{m+n}$. Therefore $\sigma\circ\bar{\phi}_{m,n}$ is homotopic to the suspension $E\bar{\phi}_{m,n-1}$. If the degree of $\bar{\phi}_{m,n-1}$ is $(-1)^{n-1}$, the degree of $\bar{\phi}_{m,n}$ is $(-1)^n$. Then $(1\cdot 10)$ is proved by the induction. q. e. d.

PROPOSITION $(1\cdot 11)$, i). *Let $\bar{\gamma}$ be an element of $\pi_{p+q+1}(EX)$ which is represented by a Hopf construction $\bar{h}: S^{p+q+1}\longrightarrow EX$ of a mapping $h: (S^p\times S^q, S^p\vee S^q) \longrightarrow (X, x_0)$. Let γ' be an element of $\pi_{p+q}(X)$ which is represented by a mapping $h': S^{p+q}\longrightarrow X$ such that $h'\circ\phi_{p,q} = h$. Then $\bar{\gamma} = (-1)^q E\gamma'$.*

ii) *For the cace that $X = K\divideontimes L$ and $h(x,y) = f(x)\divideontimes g(y) = \phi(f(x), g(y))$ for representatives f and g of $\alpha\in\pi_p(X)$ and $\beta\in\pi_q(Y)$ respectively, we have that $\bar{\gamma} = (-1)^q E(\alpha\divideontimes\beta)$.*

Proof. Consider a mapping $H = Eh'\circ\bar{\phi}_{p,q}$, then H is a Hopf construction of h. By $(1\cdot 9)'$ and $(1\cdot 10)$, we have that $\bar{\gamma} = (-1)^q E\gamma'$. In ii), $\gamma' = \alpha\divideontimes\beta$. q. e. d.

Define a mapping

$$(1\cdot 12) \qquad \psi_n: (V^n, S^{n-1}) \longrightarrow (S^n, e_0)$$

by the formula

$$\psi_n(d'_{n-1}(x,t)) = d_{n-1}(x,t), \quad x\in S^{n-1},\ t\in V^1,$$

then ψ_n maps $V^n - S^{n-1}$ homeomorphically onto $S^n - e_0$.

To consider homotopy groups $\pi_n(X, A)$ and $\pi_n(X)$, we take the orientations of the anti-images (V^n, S^{n-1}) and S^n such that the mapping ψ_n preserves the orientations. Then we remark that the following diagram is commutative:

$$(1\cdot 12)' \qquad \begin{array}{ccc} \pi_i(V^n, S^{n-1}) & \xrightarrow{\partial} & \pi_{i-1}(S^{n-1}) \\ \downarrow \psi_{n*} & \swarrow E & \\ \pi_i(S^n)\,. & & \end{array}$$

Consider mappings $f: (S^p, e_0)\longrightarrow (S^m, e_0)$ and $g: (S^p, e_0)\longrightarrow (S^n, e_0)$. Define extensions $F: V^{p+1}\longrightarrow V^{m+1}$ and $G: V^{q+1}\longrightarrow V^{n+1}$ of $f = F\mid S^p$ and $g = G\mid S^q$ respectively, by setting

$$E(d'_p(x,t)) = d'_m(f(x),t) \quad \text{and} \quad G(d'_q(x,t)) = d'_n(g(x),t).$$

We define a join

$$f * g:\ S^{p+q+1}\longrightarrow S^{m+n+1}$$

of f and g by the formula

$$(f * g)\,(\tau_{p+1,q+1}(x, y)) = \tau_{m+1,n+1}(F(x), G(y)),$$

then, for homotopies f_t and g_t, the join $f_t * g_t$ is also a homotopy. Let $\alpha \in \pi_p(S^m)$ and $\beta \in \pi_q(S^n)$ be the classes of f and g, then the class $\alpha * \beta \in \pi_{p+q+1}(S^{m+n+1})$ of $f * g$ is independent of representatives f and g. This operation "$*$" coincides with that of [9]. We have the formula (*cf.* [1])

(1·13) $$\alpha * \beta = (-1)^{q+n} E(\alpha \divideontimes \beta).$$

Proof. It is easily verified that

$$\bar{\phi}_{m,n} \circ (f * g) = E(f \divideontimes g) \circ \bar{\phi}_{p,q}$$

for the Hopf constructions $\bar{\phi}_{m,n}$ and $\bar{\phi}_{p,q}$ defined by (1.10)′. Then by (1·10),

$$\alpha * \beta = (-1)^n \iota_{m+n+1} \circ E(\alpha \divideontimes \beta) \circ (-1)^q \iota_{p+q+1}$$
$$= (-1)^{q+n} E(\alpha \divideontimes \beta). \qquad \text{q. e. d.}$$

Combining this to (1·7), we have that

(1·13)′ $$\alpha * \beta = (-1)^{(p+1)(q+n)} E^{n+1}\alpha \circ E^{p+1}\beta$$
$$= (-1)^{(m+1)(q+n)} E^{m+1}\beta \circ E^{q+1}\alpha$$

for $\alpha \in \pi_p(S^m)$ and $\beta \in \pi_q(S^n)$.

For two mappings $f' : (S^{m+1}, e_0) \longrightarrow (X, x_0)$ and $g' : (S^{n+1}, e_0) \longrightarrow (X, x_0)$, we define their *Whitehead product*

$$[f', g'] : S^{m+n+1} \longrightarrow X$$

by setting

$$[f', g']\,(\tau_{m+1,n+1}(x, y)) = \begin{cases} f'(\psi_{m+1}(x)), & (x, y) \in V^{m+1} \times S^n, \\ g'(\psi_{n+1}(y)), & (x, y) \in S^m \times V^{n+1}. \end{cases}$$

Let $\alpha' \in \pi_{m+1}(X)$ and $\beta' \in \pi_{n+1}(X)$ be the classes of f' and g' respectively, then the class $[\alpha', \beta'] \in \pi_{m+n+1}(X)$ of $[f', g']$ is independent of representatives f' and g'. From the definition of ψ_{r+1}, $*$ and E, we have the formula

$$[f' \circ Ef, g' \circ Eg] = [f', g'] \circ (f * g).$$

Then by (1·13)′ (*cf.* (3·59) of [9])

(1·14) $$[\alpha' \circ E\alpha, \beta' \circ E\beta] = [\alpha', \beta'] \circ (\alpha * \beta)$$
$$= (-1)^{(p+1)(q+n)} [\alpha', \beta'] \circ E^{n+1}\alpha \circ E^{p+1}\beta$$
$$= (-1)^{(m+1)(q+n)} [\alpha', \beta'] \circ E^{m+1}\beta \circ E^{q+1}\alpha,$$

$\alpha' \in \pi_{m+1}(X)$, $\beta' \in \pi_{n+1}(X)$, $\alpha \in \pi_p(S^m)$, $\beta \in \pi_q(S^n)$.

A mapping

$$h : (S^{m+1} \times S^{n+1}, S^{m+1} \vee S^{n+1}) \longrightarrow (X, A)$$

is called to have a *type* (α, β) if $h \mid S^{m+1} \times e_0$ and $h \mid e_0 \times S^{n+1}$ represent α and β respectively. Let a mapping

$$H : (V^{m+n+2}, S^{m+n+1}) \longrightarrow (X, A)$$

be defined by the formula $H(\tau_{m+1,n+1}(x, y)) = h(\psi_{m+1}(x), \psi_{n+1}(y))$. Then we have easily

(1·15). *$\partial\gamma = [\alpha, \beta]$ for the class $\gamma \in \pi_{m+n+2}(X, A)$ of H. In the case $X = A$, $[\alpha, \beta] = 0$ if and only if these exists a mapping $h : S^{m+1} \times S^{n+1} \longrightarrow X$ of type (α, β).*

Next we prove that

(1·16) *a mapping* $f_{m,n}: (V^{m+n}, S^{m+n-1}) \longrightarrow (S^{m+n}, e_0)$ *which is given by the formula* $f_{m,n}(\tau_{m,n}(x, y)) = \phi_{m,n}(\psi_m(x), \psi_n(y))$ *is homotopic to* ψ_{m+n}.

Proof. It is sufficient to prove that the composition $f_{m,n}\circ\psi_{m+n}^{-1} = f'_{m,n}: S^{m+n} \longrightarrow S^{m+n}$ is homotopic to the identity. Let $\rho_r: S^r \longrightarrow S^r$ be a permutation given by $\rho_r(t_1, \cdots, t_{r-1}, t_r, t_{r+1}) = (t_1, \cdots, t_{r-1}, t_{r+1}, t_r)$, then $\rho_r\circ\psi_r \mid V^{r-1} = \psi_{r-1}$. Since the degree of ρ_r is -1, the composition $(i_m ⋇ \rho_n)\circ f'_{m,n}\circ\rho_{m+n}$ is homotopic to $f'_{m,n}$. On the other hand, $(i_m ⋇ \rho_n)\circ f'_{m,n}\circ\rho_{m+n}$ maps E_+^{m+n} and E_-^{m+n} into themselves respectively and coincides with $f'_{m,n-1}$ on S^{m+n-1}. Therefore $(i_m ⋇ \rho_n)\circ f'_{m,n}\circ\rho_{m+n} \simeq f'_{m,n} \simeq Ef'_{m,n-1}$. This is true for $n=1$ if we regard that $f'_{m,0}$ is the identity. By the induction, we have that $f'_{m,n}$ is homotopic to the identity. q. e. d.

Finally we prove the following lemma.

LEMMA (1·17) *Let* $\alpha \in \pi_m(X)$ *be represented by a mapping* $f: (S^m, e_0) \longrightarrow (X, x_0)$, *and define mappings* $F_1: S^{m+n+1} \longrightarrow X ⋇ S^{n+1}$ *and* $F_2: S^{m+n+1} \longrightarrow S^{n+1} ⋇ X$ *by setting*

$$F_1(\tau_{m+1,n+1}(x, y)) = \begin{cases} f(x) ⋇ \psi_{n+1}(y), & (x, y) \in S^m \times V^{n+1}, \\ x_0 ⋇ e_0, & (x, y) \in V^{m+1} \times S^n, \end{cases}$$

$$F_2(\tau_{n+1,m+1}(x, y)) = \begin{cases} e_0 ⋇ x_0, & (x, y) \in S^n \times V^{m+1}, \\ \psi_{n+1}(x) ⋇ f(y) & (x, y) \in V^{n+1} \times S^m, \end{cases}$$

then F_1 *and* F_2 *represent* $(-1)^n(\alpha ⋇ \iota_{n+1})$ *and* $-(\iota_{n+1} ⋇ \alpha)$ *respectively.*

Proof. Define mappings k_1 and k_2 of S^{m+n+1} on itself by the formula

$$k_1(\tau_{m+1,n+1}(x, y)) = \begin{cases} \phi_{m,n+1}(x, \psi_{n+1}(y)), & (x, y) \in S^m \times V^{n+1}, \\ e_0, & (x, y) \in V^{m+1} \times S^n, \end{cases}$$

$$k_2(\tau_{n+1,m+1}(y, x)) = \begin{cases} e_0, & (y, x) \in S^n \times V^{m+1}, \\ \phi_{n+1,m}(\psi_{n+1}(y), x), & (y, x) \in V^{n+1} \times S^m. \end{cases}$$

Then $F_1 = (f ⋇ i_{n+1})\circ k_1$ and $F_2 = (i_{n+1} ⋇ f)\circ k_2$. Therefore it is sufficient to prove that

(1·17)′ *the degrees of* k_1 *and* k_2 *are* $(-1)^n$ *and* -1 *respectively.*

Let $\{x, y, t\}$ denote a point of S^{m+n+1} such that

$$\{x, y, t\} = \begin{cases} \tau_{m+1,n+1}(x, d'_n(y, 2t+1)) & \text{for} \quad -1 \leqq t \leqq 0, \\ \tau_{m+1,n+1}(d'_m(x, -2t+1), y) & \text{for} \quad 0 \leqq t \leqq 1, \end{cases}$$

$x \in S^m, y \in S^n, t \in V^1$. Then $k_1(\{x, y, t\}) = \phi_{m,n+1}(x, d_n(y, 2t+1))$ for $-1 \leqq t \leqq 0$ and $k_1(\{x, y, t\}) = e_0$ for $0 \leqq t \leqq 1$. It is easy to see that k_1 is homotopic to a mapping k' which is given by $k'(\{x, y, t\}) = \phi_{m,n+1}(x, d_n(y, t)) = d_{m+n}(\phi_{m,n}(x, y), t)$. k' is a Hopf construction of the mapping $\phi_{m,n}$. Then the degree of k' is $(-1)^n$ by (1·10) and the degree of k_1 is $(-1)^n$. Also we denote by $\{y, x, t\}$ a point of S^{m+n+1} such that

$$\{y, x, t\} = \begin{cases} \tau_{n+1,m+1}(y, d'_m(x, 2t+1)) & \text{for} \quad -1 \leqq t \leqq 0, \\ \tau_{n+1,m+1}(d'_n(y, -2t+1), x) & \text{for} \quad 0 \leqq t \leqq 1. \end{cases}$$

Then $k_2(\{y, x, t\}) = e_0$ for $-1 \leqq t \leqq 0$ and $k_2(\{y, x, t\}) = \phi_{n+1,m}(d_n(y, -2t+1), x)$, for $0\leqq t\leqq 1$, and k_2 is homotopic to a mapping k'' which is given by $k''(\{y, x, t\})$ $=\phi_{n+1,m}(d_n(y, -t), x)=\sigma_{m,n+1}(\phi_{m,n+1}(x, d_n(y, -t)))=\sigma_{m,n+1}(d_{m+n}(\phi_{m,n}(x, y), -t))$ $=\sigma_{m,n+1}(d_{m+n}(\sigma_{n,m}(\phi_{n,m}(y, x)), -t))=(\sigma_{m,n+1}\circ\rho)(d_{m+n}((\sigma_{n,m}\circ\phi_{n,m})(y, x), t))$, where ρ is a reflection giben by $\rho(d_{m+n}(z, t))=d_{m+n}(z, -t)$. Then $\rho\circ\sigma_{n+1,m}\circ k''=E\sigma_{n,m}\circ\overline{\phi}_{n,m}$ for a Hopf construc-tion $\overline{\phi}_{n,m}$ of $\phi_{n,m}$ such that $\overline{\phi}_{n,m}(\{y, x, t\})=d_{m+n}(\phi_{n,m}(y, x), t)$. Then the degree of k'' is $(-1)^{m+(n+1)m+nm+1}=-1$ by (1·10), and the degree of k_2 is -1. q. e. d.

2. Hopf invariant

In the following we suppose that each complex is finite and has only one vertex.

According to [3], we define the reduced product complex K_∞ of K which is canonically imbedded in the loop-space $\Omega(EK)$ of EK. A point of K_∞ is represented by the product $x_1\cdots x_k$ for some $x_1, \cdots, x_k\in K$, and the injection $K\subset\Omega(EK)$ associates with a point x of K a loop $l_x: V^1\longrightarrow EK$ given by $l_x(t)=d_K(x, t)$. The imbedding $\tilde{i}: K_\infty\longrightarrow\Omega(EK)$ induces isomorphisms of the homotopy groups [3] [7]

(2·1) $$\tilde{i}_*: \pi_i(K_\infty)\approx\pi_i(\Omega(EK)).$$

For a mapping $f: (S^{i+1}, e_0)\longrightarrow(EK, u_0)$, we define a mapping $\Omega f: (S^i, e_0)\longrightarrow(\Omega(EK), u_0)$ by the formula

$$\Omega f(x)(t)=f(d_i(x, t)),$$

$x\in K$, $t\in V^1$. The correspondence $f\longrightarrow\Omega f$ induces an isomorphism

(2·2) $$\Omega: \pi_{i+1}(EK)\approx\pi_i(\Omega(EK)).$$

Then we have that

(2·3) $$E=(\Omega^{-1}\circ\tilde{i}_*)\circ i_*: \pi_i(K)\longrightarrow\pi_i(K_\infty)\approx\pi_{i+1}(EK),$$

that is to say, the suspension homomorphism E is equivalent to the injection homomorhism $i_*: \pi_i(K)\longrightarrow\pi_i(K_\infty)$. From the exact sequence for the pair (K_∞, K), we have *an exact sequence*

(2·4) $$\cdots\longrightarrow\pi_i(K)\xrightarrow{E}\pi_{i+1}(EK)\xrightarrow{J}\pi_i(K_\infty, K)\xrightarrow{\partial}\pi_{i-1}(K)\longrightarrow\cdots,$$

where $J=j_*\circ\tilde{i}_*^{-1}\circ\Omega$ for the injection homomorphism $j_*: \pi_i(K_\infty)\longrightarrow\pi_i(K_\infty, K)$.

Define a mapping

$$h': (K_2, K)\longrightarrow(K\divideontimes K, u_0\divideontimes u_0)$$

by setting

$$h'(x\cdot y)=x\divideontimes y,$$

where $K_2=\{x\cdot y\in K_\infty\mid x, y\in K\}$. Let

(2·5) $$h: (K_\infty, K)\longrightarrow((K\divideontimes K)_\infty, u_0\divideontimes u_0)$$

be the *combinatorial extension* [3] of h'. Then h defines two generalizations of the Hopf invariant:

(2·6), i) $H'=(\Omega^{-1}\circ\tilde{i}_*)\circ h_*: \pi_i(K_\infty, K)\longrightarrow\pi_i((K\divideontimes K)_\infty)\approx\pi_{i+1}(E(K\divideontimes K))$;

ii) $H=H'\circ J=(\Omega^{-1}\circ\tilde{i}_*)\circ h_*\circ(\tilde{i}_*^{-1}\circ\Omega): \pi_{i+1}(EK)\approx\pi_i(K_\infty)\longrightarrow\pi_i((K\divideontimes K)_\infty)\approx\pi_{i+1}(E(K\divideontimes K))$.

The following proposition is proved without difficulties (*cf.* [2]).

PROPOSITION (2·7) *If K is $(r-1)$-connected $(r>1)$, then H' is an isomorphism for $i\leqq 3r-2$ and a homomorphism onto for $i=3r-1$.*

In the case $K=S^r$, we have that

PROPOSITION (2·8), i), *if r is odd, then H' is an isomorphism for all i:*

ii), *if r is even, then H' is an isomorphism of the 2-components for all i.*

For the proof, see [5] and [8].

For two mappings $f:(S^p, e_0)\longrightarrow(K, u_0)$ and $g:(S^q, e_0)\longrightarrow(K, u_0)$, define a mapping

$$\{f, g\}:(V^{p+q}, S^{p+q+1})\longrightarrow(K_\infty, K)$$

by the formula

$$\{f, g\}(\tau_{p,q}(x, y))=f(\psi_p(x))\cdot g(\psi_q(y)),\qquad (x, y)\in V^p\times V^q.$$

Then the homotopy class of $\{f, g\}$ is an element $\{\alpha, \beta\}\in\pi_{p+q}(K_\infty, K)$ such that

(2·9) $$\partial\{\alpha, \beta\}=[\alpha, \beta]$$

for the classes α and β of f and g respectively.

From the exactness of the sequence (2·4), we have that

(2·10) $$E[\alpha, \beta]=0.$$

From (2·3), (2·6) and from the definition of the mappings, we have easily that

(2·11) $$H'\{\alpha, \beta\}=E(\alpha\divideontimes\beta).$$

We introduce the following results of James from [4, Theorem (2·17)].

(2·12) *An element γ of $\pi_{p+q+1}(EK)$ is represented by a Hopf construction of a mapping of a type (α, β) if and only if*

$$J\gamma=\{\alpha, \beta\}.$$

By (2·12) and (2·11),

(2·12)′ $$H\gamma=E(\alpha\divideontimes\beta).$$

In the case $K=S^r$, we have that

(2·13) *if $i\leqq 3r-2$, then an element γ of $\pi_{i+1}(S^{r+1})$ is represented by a Hopf construction of a mapping $f: S^{i-r}\times S^r\longrightarrow S^r$ of a type (α, ι_r) where α is an element of $\pi_{i-r}(S^r)$ such that $E^{r+1}\alpha=H\gamma$. (See* [10]).

Proof. Since $E^{r+1}:\pi_{i-r}(S^r)\longrightarrow\pi_{i+1}(S^{2r+1})$ is an isomorphism for $i-r\leqq 2r-2$, there is an element γ of $\pi_{i-r}(S^r)$ such that $E^{r+1}\alpha=H\gamma=E(\alpha\divideontimes\iota_r)$. By (2·7), $H'\{\alpha, \iota_r\}=E(\alpha\divideontimes\iota_r)=H\gamma=H'J\gamma$ implies that $\{\alpha, \iota_r\}=J\gamma$. Therefore γ is represented by a Hopf construction of a mapping of the type (α, ι_r), by (2·12). q. e. d.

3. Reduced join and Hopf construction

Let K and L be finite cell complexes with only vertices $u_0=K^0$ and $v_0=L^0$. Consider suspensions EK and EL of K and L, and let

$$d_K:(K\times V^1, K\times S^0\cup u_0\times V^1)\longrightarrow(EK, u_0)$$

and $$d_L:(L\times V^1, L\times S^0\cup v_0\times V^1)\longrightarrow(EL, v_0)$$

be mappings defining the suspensions. Let

$$EK^* = EK \cup e^{p+2} \quad \text{and} \quad EL^* = EL \cup e^{q+2}$$

be cell complexes with *characteristic maps*

$$(3\cdot1) \qquad \begin{aligned} F &: (V^{p+2}, S^{p+1}) \longrightarrow (EK^*, EK), \\ G &: (V^{q+2}, S^{q+1}) \longrightarrow (EL^*, EL). \end{aligned}$$

Let

$$\phi : (EK^* \times EL^*, EK^* \vee EL^*) \longrightarrow (EK^* \divideontimes EL^*, u_0 \divideontimes v_0)$$

be a shrinking map defining the reduced join $EK^* \divideontimes EL^*$, then ϕ defines $EK \divideontimes EL^*$, $EK \divideontimes EL$, $K \divideontimes EL$, etc., and we denote that $\phi(x, y) = x \divideontimes y$ for points $x \in EK^*$ and $y \in EL^*$. Define subspaces M, M_+, M_- and M_0 of $EK^* \divideontimes EL^*$ as follows:

$$(3\cdot2) \qquad \begin{aligned} &M_+ = C_+K \divideontimes EL^* \cup EK^* \divideontimes C_+L, \qquad M_- = C_-K \divideontimes EL^* \cup EK^* \divideontimes C_-L, \\ &M = M_+ \cup M_- = EK \divideontimes EL^* \cup EK^* \divideontimes EL, \\ &M_0 = M_+ \cap M_- = K \divideontimes EL^* \cup C_+K \divideontimes C_-L \cup C_-K \divideontimes C_+L \cup EK^* \divideontimes L. \end{aligned}$$

Consider a homeomorphism

$$\sigma : EK \divideontimes L \longrightarrow K \divideontimes EL$$

given by the formula

$$(3\cdot3) \qquad \sigma(d_K(x, t) \divideontimes y) = x \divideontimes d_L(y, -t), \qquad x \in K,\ y \in L,\ t \in X^1,$$

then σ is identical on $K \divideontimes L = EK \divideontimes L \cap K \divideontimes EL$. Attaching the subcomplex $EK \divideontimes L$ of $EK^* \divideontimes L$ to the subcomplex $K \divideontimes EL$ of $K \divideontimes EL^*$ by the homeomorphism σ, we obtain a complex

$$(3\cdot3)' \qquad N = K \divideontimes EL^* \cup \bar{\sigma}(EK^* \divideontimes L)$$

where $\bar{\sigma}$ is a homeomorphism into N such that $\bar{\sigma} \mid EK \divideontimes L = \sigma$.

Let

$$(3\cdot4) \qquad \psi_K : (EK^*, EK) \longrightarrow (S^{p+2}, e_0), \text{ and } \psi_L : (EL^*, EL) \longrightarrow (S^{q+2}, e_0)$$

be mappings such that $\psi_K \circ F = \psi_{p+2}$ and $\psi_L \circ G = \psi_{q+2}$, then ψ_K and ψ_L skrink EK and EL to a single point e_0. Define mappings

$$(3\cdot5) \qquad P_1 : N \longrightarrow K \divideontimes S^{q+2} \quad \text{and} \quad P_2 : N \longrightarrow S^{p+2} \divideontimes L$$

as follows;

$$P_1 \mid K \divideontimes EL^* = i_K \divideontimes \psi_L, \qquad P_1(\bar{\sigma}(EK^* \divideontimes L)) = e_0 \divideontimes v_0,$$

$$P_2 \circ \bar{\sigma} \mid EK^* \divideontimes L = \psi_K \divideontimes i_L, \qquad P_2(K \divideontimes EK^*) = u_0 \divideontimes e_0,$$

where i_K and i_L are the identities of K and L.

First we prove the following lemma.

LEMMA (3·6). *There exists a mapping*

$$\chi : (M, M_+, M_-) \longrightarrow (EN, C_+N, C_-N)$$

such that $\chi \mid K \divideontimes EL^* = $ *identity and* $\chi \mid EK^* \divideontimes L = \bar{\sigma}$. *Such mappings* χ *are homotopic to each other and homotopy equivalences.*

Proof. First consider the case $K = L = S^0$, then $EK \divideontimes EL = S^1 \divideontimes S^1 = ES^1 = S^2$ which is divided into four parts $C_+(E^1_+)$, $C_+(E^1_-)$, $C_-(E^1_+)$ and $C_-(E^1_-)$ by two circles $S^1 = S^1 \divideontimes S^0$ and $S^1_0 = S^0 \divideontimes S^1$. It is easy to see that S^1_0 is a deformation retract of $C_+(E^1_-) \cup C_-(E^1_+)$ and we may chose the retraction such that S^1 is mapped onto S_0^1 by the homeomorphism σ. Since $EK \divideontimes EL$ is naturally homeomorphic to $K \divideontimes L \divideontimes S^2$ such that $C_+K \divideontimes C_-L \cup C_-K \divideontimes C_+L$ corresponds to $(K \divideontimes L) \divideontimes (C_-(E_+^1) \cup C_+(E_-^1))$, the above deformation gives a deformation (retraction) of $C_+K \divideontimes C_-L \cup C_-K \divideontimes C_+L$ onto $K \divideontimes EL$ such that $EK \divideontimes L$ is mapped by the homeomorphism σ. This deformation shows that there exists a mapping of M_0 onto N carrying $K \divideontimes EL^* \cup EK^* \divideontimes L$ as in (3·6) and such mappings are homotopic to each other. Next since C_+N and C_-N are contractible to a single point, the above mapping of M_0 onto N is extended over the whole of M such that M_+ an M_- are mapped into C_+N and C_-N respectively, and such extensions are homotopic to each other. It is easy to see that this mapping induces isomorphisms of the homology groups. Since M and EN are simply connected, the mapping is an homotopy equivalence by Theorem 3 of [11]. q. e. d.

Now suppose that

(3·7) $$[\alpha', \alpha''] = 0 \quad \text{and} \quad [\beta', \beta''] = 0$$

for $\alpha' \in \pi_{p'}(K)$, $\alpha'' \in \pi_{p''}(K)$, $\beta' \in \pi_{q'}(L)$ *and* $\beta'' \in \pi_{q''}(L)$.

By (1·15), there exist mappings

$$f : (S^{p'} \times S^{p''}, e_0 \times e_0) \longrightarrow (K, u_0)$$

and $$g : (S^{q'} \times S^{q''}, e_0 \times e_0) \longrightarrow (L, v_0)$$

of *the types* (α', α'') *and* (β', β'') respectively. Set $p = p' + p''$ and $q = q' + q''$, and let

(3·7)′ $$\bar{f} : S^{p+1} \longrightarrow EK \quad \textit{and} \quad \bar{g} : S^{q+1} \longrightarrow EL$$

be *Hopf constructions* of f and g respectively. We construct complexes EK^* and EL^* such that

$$F \mid S^{p+1} = \bar{f} \quad \text{and} \quad G \mid S^{q+1} = \bar{g}$$

in (3·1).

THEOREM (3·8). *Let $\alpha \in \pi_{p+1}(EK)$ and $\beta \in \pi_{q+1}(EL)$ be the classes of $\bar{f}$ and $\bar{g}$ respectively, then there exists a Hopf construction*

$$H : S^{p+q+3} \longrightarrow EN$$

of a mapping

$$h : (S^{p'+q'+1} \times S^{p''+q''+1}, S^{p'+q'+1} \vee S^{p''+q''+1}) \longrightarrow (N, E(K \divideontimes L))$$

of a type $((-1)^{q'+1}E(\alpha' \divideontimes \beta'), (-1)^{q''}E(\alpha'' \divideontimes \beta''))$ *such that the compositions*

$$EP_1 \circ H : S^{p+q+3} \longrightarrow E(K \divideontimes S^{q+2}) = K \divideontimes S^{q+3} = E^{q+2}(EK)$$

and $$EP_2 \circ H : S^{p+q+1} \longrightarrow E(S^{p+2} \divideontimes L) = S^{p+2} \divideontimes EL$$

represent $(-1)^{p''q'+p''+q'}E^{q+2}\alpha$ *and* $(-1)^{p''q'+p''+q'} \iota_{p+2} \divideontimes \beta$ *respectively.*

Proof. Consider a mapping

$$H_0 : S^{p+q+3} \longrightarrow EN$$

which is defined by the formula

$$H_0(\tau_{p+2, q+2}(x, y)) = \chi(F(x) \divideontimes G(y))$$

for $(x, y) \in S^{p+1} \times V^{q+2} \cup V^{p+2} \times S^{q+1}$. Compare the composition $EP_1 \circ \chi : M \longrightarrow EN \longrightarrow K \divideontimes S^{q+3}$ and a mapping

$$Q_1 : M \longrightarrow K \divideontimes S^{q+3} = E(K \divideontimes S^{q+2})$$

which is given by setting

$$Q_1(EK^* \divideontimes EL) = u_0 \divideontimes e_0$$

and $$Q_1 \mid EK \divideontimes EL^* = (i_K \circ \sigma_{1,q+2}) \circ (i_{EK} \divideontimes \psi_L).$$

The mappings $EP_1 \circ \chi$ and Q_1 map M_+ and M_- into $C_+(K \divideontimes S^{q+2})$ and $C_-(K \divideontimes S^{q+2})$ respectively and they coincide on M_0. Therefore the mappings $EP_1 \circ \chi$ and Q_1 are homotopic to each other. Then the composition $(i_K \divideontimes \sigma_{1,q+2})^{-1} \circ EP_1 \circ H_0$ is homotopic to a mapping $R_1 : S^{p+q+3} \longrightarrow EK \divideontimes S^{q+2}$ which is given by

$$R_1(\tau_{p+2,q+2}(x, y)) = \begin{cases} \bar{f}(x) \divideontimes \psi_{q+2}(y) & \text{for } (x, y) \in S^{p+1} \times V^{q+2}, \\ u_0 \divideontimes e_0 & \text{for } (x, y) \in V^{p+2} \times S^{q+1}. \end{cases}$$

By (1·17), R_1 represents $(-1)^{q+1}\alpha \divideontimes \iota_{q+2} = (-1)^{q+1}E^{q+2}\alpha$, and $EP_1 \circ H_0$ represents $(i_K \divideontimes \sigma_{1,q+2})_*((-1)^{q+1}E^{q+2}\alpha)$. Since $\sigma_{1,q+2}$ is homotopic to a reduced join $i_1 \divideontimes \lambda$ for a mapping $\lambda : S^{q+2} \longrightarrow S^{q+2}$ of the degree $(-1)^{q+2}$, we have from (1·5) that

$$(i_K \divideontimes \sigma_{1,q+2})_*((-1)^{q+1}E^{q+2}\alpha) = (i_{EK} \divideontimes \lambda)_*((-1)^{q+1}\alpha \divideontimes \iota_{q+2})$$
$$= (-1)^{q+1}\alpha \divideontimes ((-1)^{q+2}\iota_{q+2}) = -E^{q+2}\alpha.$$

Next we compare the composition $EP_2 \circ \chi$ and a mapping

$$Q_2 : M \longrightarrow S^{p+2} \divideontimes EL$$

which is given by setting

$$Q_2 \mid EK^* \divideontimes EL = \psi_K \divideontimes i_{EL}$$

and $$Q_2(EK \divideontimes EL^*) = e_0 \divideontimes v_0 .$$

The mappings $EP_2 \circ \chi$ and Q_2 map M_+ and M_- into $C_+(S^{p+2} \divideontimes L)$ and $C_-(S^{p+2} \divideontimes L)$ respectively and they coincide on M_0. Therefore the mappings $EP_2 \circ \chi$ and Q_2 are homotopic to each other. The composition $EP_2 \circ H_0$ is homotopic to a mapping R_2 which is given by

$$R_2(\tau_{p+2,q+2}(x, y)) = \begin{cases} e_0 \divideontimes v_0 & \text{for } (x, y) \in S^{p+1} \times V^{q+2}, \\ \psi_{p+2}(x) \divideontimes \bar{g}(y) & \text{for } (x, y) \in V^{p+2} \times S^{q+1}. \end{cases}$$

By (1·17), R_2 represents $-(\iota_{p+2} \divideontimes \beta)$. Therefore $EP_2 \circ H_0$ represents $-(\iota_{p+2} \divideontimes \beta)$.

Now define a homeomorphism

$$\zeta : (V^{p+q+4}, S^{p+q+3}) \longrightarrow (V^{p+q+4}, S^{p+q+3})$$

by the formula

$$\zeta(\tau_{p'+q'+2,\, p''+q''+2}(\tau_{p'+1,q'+1}(x', y'), \tau_{p''+1,q''+1}(x'', y'')))$$
$$= \tau_{p+2,q+2}(\tau_{p'+1,p''+1}(x', x''), \tau_{q'+1,q''+1}(y', y''))$$

then the degree of ζ is $(-1)^{(p''+1)(q'+1)}$. We set

$$H = H_0 \circ \zeta,$$

then $EP_1 \circ H$ and $EP_2 \circ H$ represent $(-1)^{p''q'+p''+q'}E^{q+2}\alpha$ and $(-1)^{p''q'+p''+q'}\iota_{p+2} \divideontimes \beta$ respectively. It is verified directly that H maps $\tau_{p'+q'+2,\, p''+q''+2}(V^{p'+q'+2} \times S^{p''+q''+1})$ and $\tau_{p'+q'+2,\, p''+q''+2}(S^{p'+q'+1} \times V^{p''+q''+2})$ into C_+N and C_-N respectively Then H is a Hopf construction of a mapping

$$h : S^{p'+q'+1} \times S^{p''+q''+1} \longrightarrow N$$

which is given by $h(x, y) = H(\tau_{p'+q'+1,\, p''+q''+1}(x, y))$. Let $h_1 : S^{p'+q'+1} \longrightarrow E(K \divideontimes L)$ and $h_2 : S^{p''+q''+1} \longrightarrow E(K \divideontimes L)$ be mappings given by

$$h_1(x) = h(x, \tau_{p''+1,q''+1}(e_0, e_0))$$

and

$$h_2(y) = h(\tau_{p'+1,q'+1}(e_0, e_0), y).$$

The mapping $h_1 \circ \tau_{p'+1,q'+1}$ maps $V^{p'+1} \times S^{q'}$ and $S^{p'} \times V^{q'+1}$ into $C_-(K \divideontimes L)$ and $C_+(K \divideontimes L)$ respectively and its restriction on $S^{p'} \times S^{q'}$ is given by $h_1(x', y') = f(x', e_0) \divideontimes g(y', e_0)$. Let $\rho: E(K \divideontimes L) \longrightarrow E(K \divideontimes L)$ be a reflection given by $\rho(d_{K\divideontimes L}(z, t)) = d_{K\divideontimes L}(z, -t)$. Then by (1·11), ii), $\rho \circ h_1$ represents $(-1)^{q'} E(\alpha' \divideontimes \beta')$, and h_1 represents $(-1)^{q'+1} E(\alpha' \divideontimes \beta')$. The mapping $h_2 \circ \tau_{q''+1,q''+1}$ maps $V^{p''+1} \times S^{q''}$ and $S^{q''} \times V^{q''+1}$ into $C_+(K \divideontimes L)$ and $C_-(K \divideontimes L)$ respectively and its restriction on $S^{p''} \times S^{q''}$ is given by $h_2(x'', y'') = f(e_0, x'') \divideontimes g(e_0, y'')$. By (1·11), ii), h_2 represents $(-1)^{q''} E(\alpha'' \divideontimes \beta'')$. Therefore h is a mapping of the type $((-1)^{q'+1} E(\alpha' \divideontimes \beta'), (-1)^{q''} E(\alpha'' \divideontimes \beta''))$.

q. e. d.

By (1·15),

Corollary (3·9) *if* $[\alpha', \alpha''] = 0$ *and* $[\beta', \beta''] = 0$, *then* $i_*[E(\alpha' \divideontimes \beta'), E(\alpha'' \divideontimes \beta'')] = 0$ *for the injection homomorphism* $i_*: \pi_{p+q+1}(E(K \divideontimes L)) \longrightarrow \pi_{p+q+1}(N)$.

By (2·12)′,

Corollary (3·10) *for the class* $\gamma \in \pi_{p+q+3}(EN)$ *of the mapping* H *of* (3·8), *we have that* $H\gamma = (-1)^{q+1} E(E(\alpha' \divideontimes \beta') \divideontimes E(\alpha'' \divideontimes \beta''))$.

4. Whitehead product

Here we consider the case that

$$K = S^m \quad \text{and} \quad L = S^n.$$

Then

$$EK^* = S^{m+1} \cup e^{p+2}, \qquad EL^* = S^{n+1} \cup e^{q+2},$$
$$N = K \divideontimes EL^* \cup \bar{\sigma}(EK^* \divideontimes L) = S^{m+n+1} \cup e^{p+n+2} \cup e^{m+q+2},$$
$$P_1: N \longrightarrow K \divideontimes S^{q+1} = S^{m+q+2}, \quad P_2: N \longrightarrow S^{p+2} \divideontimes L = S^{p+n+2}.$$

The homeomorphism $\sigma: S^{m+n+1} \longrightarrow S^{m+n+1}$ of (3·3) is given by $\sigma(\phi_{m+1,n}(d_m(x, t), y)) = \phi_{m,n+1}(x, d_n(y, -t))$. Then we have that

(4·1) *the degree of* σ *is* $(-1)^{n+1}$.

Proof. Let $\rho: S^{m+n+1} \longrightarrow S^{m+n+1}$ be a reflection given by $\rho(d_{m+n}(z, t)) = d_{m+n}(z, -t)$. It is calculated directly that $\sigma = \rho \circ E\sigma_{n,m} \circ \sigma_{m+1,n}$. Then the degree of σ is $(-1)^{1+mn+(m+1)n} = (-1)^{n+1}$. q. e. d.

Define characteristic maps

$$\mu_1: (V^{m+q+2}, S^{m+q+1}) \longrightarrow (N, S^{m+n+1})$$

and

$$\mu_2: (V^{p+n+2}, S^{p+n+1}) \longrightarrow (N, S^{m+n+1})$$

of e^{m+q+2} and e^{p+n+2} respectively by the formulas

$$\mu_1(\tau_{m,p+2}(x, y)) = \psi_m(x) \divideontimes G(y),$$

and

$$\mu_2(\tau_{p+2,n}(x', y')) = \bar{\sigma}(F(x') \divideontimes \psi_n(y')),$$

(4·2) *then* $\mu_1 | S^{m+q+1}$ *and* $\mu_2 | S^{p+n+1}$ *represent* $-\iota_m \divideontimes \beta$ *and* $\alpha \divideontimes \iota_n$ *respectively.*

Proof. Since

$$\mu_1(\tau_{m,q+2}(x, y)) = \begin{cases} e_0, & \text{for } (x, y) \in S^{m-1}\times V^{q+2}, \\ \psi_m(x) \divideontimes \bar{g}(y) & \text{for } (x, y) \in V^m \times S^{q+1}, \end{cases}$$

$\mu_1 | S^{m+q+2}$ represents $-\iota_m \divideontimes \beta$ by (1·17). Similarly, from (1·17), we have that $\bar{\sigma}^{-1}\circ\mu_2 | S^{p+n+1}$ represents $(-1)^{n-1}\alpha \divideontimes \iota_n$. By (4·1), $\mu_2 | S^{p+n+1}$ represents $\alpha\divideontimes\iota_n$. q. e. d.

Next we have that

(4·3) *the compositions $P_1\circ\mu_1$ and $P_2\circ\mu_2$ are homotopic to* ψ_{m+q+2} and ψ_{p+n+2} *respectively.*

Proof. We have that $(P_1\circ\mu_1)(\tau_{m,q+2}(x, y)) = \phi_{m,q+2}(\psi_m(x), \psi_{q+2}(y))$ and $(P_2\circ\mu_2)(\tau_{p+2,n}(x', y')) = \phi_{p+2,n}(\psi_{p+2}(x'), \psi_n(y'))$. Then (4·3) follows from (1·16) directly. q. e. d.

THEOREM (4·4) *Let $\alpha \in \pi_{p+1}(S^{m+1})$ and $\beta \in \pi_{q+1}(S^{n+1})$ be represented by Hopf constructions of mappings of the types (α', α'') and (β', β'') respectively. Then there exists an element ν of $\pi_{p+q+2}(N, S^{m+n+1})$ such that*

$$E(P_{1*}(\nu)) = (-1)^{p''q'+q+1}E^{q+2}\alpha,$$
$$E(P_{2*}(\nu)) = (-1)^{p''q'+q+1}(\iota_{p+2}\divideontimes\beta)$$

and

$$\partial\nu = (-1)^{q+1}[E(\alpha'\divideontimes\beta'), E(\alpha''\divideontimes\beta'')].$$

Proof. Let $\psi : (V^{p+q+2}, S^{p+q+1}) \longrightarrow (S^{p'+q'+1}\times S^{p''+q''+1}, S^{p'+q'+1}\vee S^{p''+q''+1})$ be a mapping given by setting $\psi(\tau_{p'+q'+1,p''+q''+1}(x, y)) = (\psi_{p'+q'+1}(x), \psi_{p''+q''+1}(y))$ and let ν be the class of the composition $h\circ\psi$, where h is the mapping of the theorem (3·8). Since h has the type $((-1)^{q'+1}E(\alpha'\divideontimes\beta'), (-1)^{q''}E(\alpha''\divideontimes\beta''))$, we have from (1·15) that $\partial\nu=(-1)^{q+1}[E(\alpha'\divideontimes\beta'), E(\alpha''\divideontimes\beta'')]$.

Consider a mapping $h' : S^{p+q+2} \longrightarrow S^{m+q+2}$ such that $P_1\circ h = h'\circ\phi_{p'+q'+1,p''+q''+1}$. By (1·16), $\phi_{p'+q'+1,p''+q''+1}\circ\psi$ is homotopic to ψ_{p+q+2}, then h' represents $P_{1*}(\nu)$. Since $EP_1\circ H$ is a Hopf construction of $P_1\circ h$, we have from (1·11), i), that $EP_1\circ H$ represents $(-1)^{p''+q''+1}E(P_{1*}(\nu))$. Then by (3·8) $E(p_{1*}(\nu))=(-1)^{p''q'+p''+q'+q''+q''+1}E^{p+2}\alpha=(-1)^{p''q'+q+1}E^{q+2}\alpha$. Similarly we have that $E(P_2^*(\nu))=(-1)^{p''q'+p''+q'+p''+q''+1}\iota_{p+2}\divideontimes\beta=(-1)^{p''q'+q+1}\iota_{p+2}\divideontimes\beta$.

PROPOSITION (4·5). *Let $\alpha \in \pi_{p+1}(S^{m+1})$ and $\beta \in \pi_{q+1}(S^{n+1})$ be the classes of Hopf constructions of mappings of the types (α', α'') and (β', β'') respectively Suppose that $p \leqq 2m+n-1$ and $q \leqq m+2n-1$, then we have the formula*

$$E^n\alpha\circ E^p\beta-(-1)^{(p+m)(q+n)}E^m\beta\circ E^q\alpha$$
$$= (-1)^{p''q'+p(q+n)}[E(\alpha'\divideontimes\beta'), E(\alpha''\divideontimes\beta'')]$$
$$= (-1)^{(p+m)n+p''+q''}[\iota_{m+n+1}, \iota_{m+n+1}]\circ E^{2n}H\alpha\circ E^pH\beta.$$

Proof. First we may suppose that $p \geqq m$ and $q \geqq n$ without the loss of generalities. Since $p \leqq 2m+n-1$, we have $p+q+2 \leqq 2m+n+q+1 < 2(m+q+2)-2$ and hence the suspension homomorphism $E: \pi_{p+q+2}(S^{m+q+2}) \longrightarrow \pi_{p+q+3}(S^{m+q+3})$ is an isomorphism. Then from (4·4) we have that

$$P_{1*}(\nu) = (-1)^{p''q'+q+1}E^{q+1}\alpha.$$

Similarly, from the condition $q \leqq m+2n-1$ and from (4·4), we have that

$$P_{2*}(\nu) = (-1)^{p''q'+q+1}E^{-1}(\iota_{p+2}\divideontimes\beta)$$
$$= (-1)^{p''q'+q+1}E^{-1}((-1)^{(p+2)(q+n)}E^{p+2}\beta)$$

$$= (-1)^{p''q'+q+1+p(q+n)}E^{p+1}\beta.$$

Let $P: N \longrightarrow S^{m+q+2}\vee S^{p+n+2}$ be a mapping defined by setting $P(x)=(P_1(x), P_2(x))$, then P shrinks S^{m+n+1} to a single point. Since S^{m+n+1} is $(m+n)$-connected and (N, S^{m+n+1}) is Min. $(m+q+1, p+n+1)$-connected, we have from Theorem II of [2] that the induced homomorphism

$$P_*: \pi_i(N, S^{m+n+1}) \longrightarrow \pi_i(S^{m+q+2}\vee S^{p+n+2})$$

is an isomorphism for $i \leqq \text{Min.}(m+q, p+n)+m+n+1$. In particular, when $i=p+q+2$, P_* is an isomorphism and the group $\pi_{p+q+2}(S^{m+q+2}\vee S^{p+n+2})$ is isomorphic to $\pi_{p+q+2}(S^{m+q+2})+\pi_{p+q+2}(S^{p+n+2})$. Then the correspondence $\gamma \longrightarrow P_{1*}(\gamma)+P_{2*}(\gamma)$ induces an isomorphism

$$\pi_{p+q+2}(N, S^{m+n+1}) \approx \pi_{p+q+2}(S^{m+q+2})+\pi_{p+q+2}(S^{p+n+2}).$$

In the diagram

$$\begin{array}{ccc} \pi_{p+q+2}(V^{m+q+2}, S^{m+q+1}) & \xrightarrow{\mu_{1*}} & \pi_{p+q+2}(N, S^{m+n+1}) \\ \downarrow \partial & & \downarrow P_{1*} \\ \pi_{p+q+1}(S^{m+q+1}) & \xrightarrow{E} & \pi_{p+q+2}(S^{m+q+2}) \end{array}$$

the commutativity holds, from (4·3) and from the commutativity of (1·12)'. Then $P_{1*}(\nu) = E((-1)^{p''q'+q+1}E^q\alpha) = P_{1*}(\mu_{1*}(\partial^{-1}((-1)^{p''q'+q+1}E^q\alpha)))$. Similarly $P_{2*}(\nu)=P_{2*}(\mu_{2*}(\partial^{-1}((-1)^{p''q'+q+1+p(q+n)}E^p\beta)))$. Therefore

$$(-1)^{p''q'+q+1}\nu=\mu_{1*}(\partial^{-1}(E^q\alpha))+(-1)^{p(q+n)}\mu_{2*}(\partial^{-1}(E^p\beta)).$$

From the naturality of the boundary operator ∂, we have that

$$\begin{aligned}(-1)^{p''q'+q+1}\partial\nu &= \partial(\mu_{1*}(\partial^{-1}E^q\alpha))+(-1)^{p(q+n)}\partial(\mu_{2*}(\partial^{-1}E^p\beta)) \\ &= \mu_{1*}(E^q\alpha)+(-1)^{p(q+n)}\mu_{2*}(E^p\beta) \\ &= (-\iota_m \divideontimes \beta)\circ E^q\alpha+(-1)^{p(q+n)}(\alpha\divideontimes\iota_n)\circ E^p\beta \\ &= (-1)^{m(q+n)+1}E^m\beta\circ E^q\alpha+(-1)^{p(q+n)}E^n\alpha\circ E^p\beta,\end{aligned}$$

by (4·2) and (1·7). Then by (4·4),

$$\begin{aligned}E^n\alpha\circ E^p\beta-(-1)^{(p+m)(q+n)}E^m\beta\circ E^q\alpha &= (-1)^{p''q'+q+1+p(q+n)}\partial\nu \\ &= (-1)^{p''q'+p(q+n)}[E(\alpha'\divideontimes\beta'), E(\alpha''\divideontimes\beta'')].\end{aligned}$$

By (1·14), (1·13), iii) of (1·5), (1·7) and by (2·12)',

$$\begin{aligned}[E(\alpha'\divideontimes\beta'), E(\alpha''\divideontimes\beta'')] &= [\iota_{m+n+1}, \iota_{m+n+1}]\circ((\alpha'\divideontimes\beta')*(\alpha''\divideontimes\beta'')) \\ &= (-1)^{p''+q''+2n}[\iota_{m+n+1}, \iota_{m+n+1}]\circ E(\alpha'\divideontimes\beta'\divideontimes\alpha''\divideontimes\beta'') \\ &= (-1)^{p''+q''+p''q'+mn}[\iota_{m+n+1}, \iota_{m+n+1}]\circ E(\alpha'\divideontimes\alpha''\divideontimes\beta'\divideontimes\beta'') \\ &= (-1)^{p''+q''+p''q'+mn+p(q+2n)}[\iota_{m+n+1}, \iota_{m+n+1}]E^{n+1}(\alpha'\divideontimes\alpha'')\circ E^{p+1}(\beta'\divideontimes\beta'') \\ &= (-1)^{p''q'+p(q+n)+(p+m)n+p''+q''}[\iota_{m+n+1}, \iota_{m+n+1}]\circ E^{2n}H\alpha\circ E^pH\beta.\end{aligned}$$

Consequently

$$\begin{aligned}E^n\alpha\circ E^p\beta-(-1)^{(m+n)(q+n)}E^m\beta\circ E^q\alpha \\ = (-1)^{(p+m)n+p''+q''}[\iota_{m+n+1}, \iota_{m+n+1}]\circ E^{2n}H\alpha\circ E^pH\beta.\end{aligned}$$

q. e. d.

THEOREM (4·6). *Suppose that* $p\leqq\text{Min.}(n, m-1)+2m-1$ *and* $q\leqq\text{Min.}(m, n-1)+2n-1$ *for* $\alpha\in\pi_{p+1}(S^{m+1})$ *and* $\beta\in\pi_{q+1}(S^{n+1})$, *then*

$$E^n\alpha\circ E^p\beta-(-1)^{(p+m)(q+n)}E^m\beta\circ E^q\alpha$$
$$=[\iota_{m+n+1},\ \iota_{m+n+1}]\circ E^{2n}H\alpha\circ E^pH\beta$$
$$=-[\iota_{m+n+1},\ \iota_{m+n+1}]\circ E^{2n}H\alpha\circ E^pH\beta\,.$$

Proof. Since $p\leqq 3m-2$ and $q\leqq 3n-2$, α and β are represented by Hopf constructions of some mappings by (2·13). Then the proposition (4·5) is applied in this case, and it is sufficient to prove that $2[\iota_{m+n+1},\ \iota_{m+n+1}]\circ E^{2n}H\alpha\circ E^pH\beta=0$. If m is even, then $2H\alpha=0$ by Theorem 5.42 of [9]. Also if n is even, $2H\beta=0$. If m and n are odd, then $m+n+1$ is odd and $2[\iota_{m+n+1},\ \iota_{m+n+1}]=0$ by the anti-commutativity of the Whitehead product operation. In all cases $2[\iota_{m+n+1},\ \iota_{m+n+1}]\circ E^{2n}H\alpha\circ E^{2n}H\beta=0$. q. e. d.

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