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On imbedding theorem for non-separable metric spaces

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The chief purpose of this note is to generalize Menger-Nöbeling's theorem[1] on imbedding of separable metric spaces of dimension$\leqq n$ in E_{2n+1}. We shall prove that every n-dimensional metric space with a σ-star-countable (finite) basis[2] can be imbedded in the topological product $N(\Omega)\times I_{2n+1}$ of a generalized Baire's zero-dimensional space[3] $N(\Omega)$ and $2n+1$-dimensional Euclidean cube I_{2n+1}. Moreover, from the proof of this theorem we deduce a universal n-dimensional space of such metric spaces and an analogous theorem to Szpilrajn's theorem on connections between dimension and n-mesure[4].

DEFINITION. We call a covering $\mathfrak{U}$ *star-finite* (*star-countable*) if every set of $\mathfrak{U}$ intersects finitely (countably) many sets of $\mathfrak{U}$.

An open basis consisting of an enumerable number of star-finite (star-countable) open coverings is called a *σ-star-finite* (*σ-star-countable*) *open basis*[5].

REMARK. A regular space R has a σ-star-finite basis if and only if R has a σ-star-countable basis. Moreover K. Morita has proved the following theorem: A regular space having σ-star-finite (σ-star-countable) basis can be imbedded in the topological product $N(\Omega)\times I^{\omega}$ of a generalized Baire's 0-dimensional space $N(\Omega)$ and Hilbert cube I^{ω}, and the converse is also true[6].

REMARK. A metric space having a σ-star-finite basis is not needed to have the star-finite property or the star-countable property[7]. For example, $N(\Omega)\times\{x|0<x<1\}$ has obviously a σ-star-finite basis, but it has not the star-countable property if the cardinal number of Ω is greater than $\aleph_0$. For if we put $S(\alpha_1,\alpha_2,\cdots,\alpha_k)=\{p|p=(\alpha_1,\alpha_2,\cdots,\alpha_k,\cdots)\in N(\Omega)\}$, then it is easily seen that the open covering

1) Cf. W. Hurewicz and H. Wallman: Dimension Theorem (1941).

2) Cf. the following definition. In this note we use Lebesque's dimension.

3) This notion is due to K. Morita: Normal Families and Dimension Theory for Metric Spaces, Math. Ann. Bd. 128(1954). For any two sequences of elements from an abstract set Ω $\alpha=(\alpha_1,\alpha_2,\cdots)$, $\beta=(\beta_1,\beta_2,\cdots)$, α_i, $\beta_j\in\Omega$, we define the metric $\rho(\alpha,\beta)$ by $\rho(\alpha,\beta)=1/\min\{k|\alpha_k\neq\beta_k\}$, $\rho(\alpha,\alpha)=0$. Then the set $N(\Omega)$ of all such sequences turns out to be a zero-dimensional metric space and is called a generalized Baire's zero-dimensional space.

4) La dimension et la mesure, Fund- Math. 28 (1937).

5) From now forth we omit the word "open" for breviiy.

6) The proof of this theorem is unpublished. We express our thanks to Prof. Morita, who wrote us this unpublished theorem.

7) We call R has the star-finite (star-countable) property if and only if every open covering of R has a star-finite (star-countable) open refinement.

$\{N(\Omega)\times\{x|1/2<x<1\},\ \ S(\alpha_1)\times\{x|1/2^2<x<1/2+1/2^2\},\cdots,S(\alpha_1,\cdots,\alpha_k)\times\{x|1/2^{k+1}<x<1/2^k+1/2^{k+1}\},\cdots|\alpha_i\in\Omega(i=1,2,\cdots)\}$ of this space has no star-countable refinement and accordingly no star-finite refinement. To see this we assume that $\mathfrak{U}$ is a star-countable refinement of this covering. Then $\bigcup_{n=1}^{\infty}S^n(U,\mathfrak{U})$ for an arbitrary $U\in\mathfrak{U}$ consists of countably many sets of $\mathfrak{U}$. We can select $S(\alpha_1,\cdots,\alpha_k)\times I\subseteqq U$. It follows from the connectedness of $\{x|0<x<1\}$ that $\bigcup_{n=1}^{\infty}S^n(U,\mathfrak{U})\supseteqq S(\alpha_1,\cdots,\alpha_k)\times\{x|0<x<1\}$. Hence $\bigcup_{n=1}^{\infty}S^n(U,\mathfrak{U})$ contains every set of $\mathfrak{U}$ contained in $S(\alpha_1,\cdots,\alpha,\alpha_{k+1})\times\{x|1/2^{k+2}<x<1/2^{k+1}+1/2^{k+2}\}$ for some $\alpha_{k+1}\in\Omega$, which is a contradiction.

THEOREM 1. *Suppose R is a regular space having a σ-star-finite (σ-star-countable) basis and $\dim R\leqq n$. Then R is homeomorphic to a subset of $N(\Omega)\times I_{2n+1}$, wher I_{2n+1} is $2n+1$-dimensional Euclidean cube and $N(\Omega)$ is the generalized Baire's 0-dimensional space for a set Ω whose cardinal number is not less than the cardinal number of an open basis of R.*

Proof. 1. There exists, as is seen from the above Morita's theorem, a sequence $\mathfrak{N}_1>\mathfrak{N}_2>\mathfrak{N}_3>\cdots$ of star-finite open coverings $\mathfrak{N}_m$ of R such that $S(p,\mathfrak{N}_m)$ $(m=1,2\cdots)$ is a neighborhood basis of every point p of R. We define an open decomposition $\mathfrak{S}_m$ of R by $\mathfrak{S}_m=\{S^\infty(N,\mathfrak{N}_m)|N\in\mathfrak{N}_m\}$, where $S^\infty(N,\mathfrak{N}_m)=\bigcup_{n=1}^{\infty}S^n(N,\mathfrak{N}_m)$[8]. Let $\mathfrak{S}_m=\{S_\alpha|\alpha\in A_m\}$ and $S_\alpha\frown S_\beta=\phi$ $(\alpha\neq\beta)$, then for every $\alpha\in A_m$ S_α is a countable sum of sets of $\mathfrak{N}_m$, i. e. $S_\alpha=\cup\{N_{\alpha i}^{(m)}\,|\,i=1,2\cdots\}$, $N_{\alpha i}^{(m)}\in\mathfrak{N}_m(i=1,2\cdots)$. Since $\mathfrak{N}_m$ is locally finite, there exists an open covering $\mathfrak{P}_m$ of R such that $\mathfrak{P}_m=\{P_{\alpha i}^{(m)}|\alpha\in A_m, i=1,2\cdots\}$, $\bar{P}_{\alpha i}^{(m)}\subseteqq N_{\alpha i}^{(m)}$. Next, we difine a sequence of open coverings by $\mathfrak{U}_{i,m}=\{N_{\alpha i}^{(m)},S_\alpha-\bar{P}_{\alpha i}^{(m)}|\alpha\in A_m\}$, $\mathfrak{U}_1=\mathfrak{U}_{1,1}$, $\mathfrak{U}_2=\mathfrak{U}_1\wedge\mathfrak{U}_{2,1}\wedge\mathfrak{U}_{1,2},\cdots$, $\mathfrak{U}_m=\mathfrak{U}_{m-1}\wedge\mathfrak{U}_{m,1}\wedge\mathfrak{U}_{m-1,2}\wedge\cdots\wedge\mathfrak{U}_{1,m},\cdots$. Then $\mathfrak{U}_1>\mathfrak{U}_2>\mathfrak{U}_3>\cdots$, and $S(p,\mathfrak{U}_m)$ $(m=1,2\cdots)$ is a neighborhood basis of each point p of R, and $\mathfrak{U}_m$ is fiinite in every $S_\alpha(\alpha\in A_m)$. Let $\Omega=\bigcup_{m=1}^{\infty}A_m$, then it is clear that $|\Omega|\leqq$ the cardinal number of any open basis of R. We define a continuous mapping $c(x)$ of R into $N(\Omega)$ by $c(x)=(\alpha_1,\alpha_2,\cdots)$ $(x\in S_{\alpha m}, \alpha_m\in A_m\ (m=1,2\cdots))$ and denote by $M(R)$ the totality of continuous mappings φ of R into $N(\Omega)\times I_{2n+1}$ such that $\varphi(x)=(c(x),\phi(x))$ $(x\in R)$ for a continuous mapping $\phi(x)$ of R into I_{2n+1}.

Moreover we difine the following notions which will be needed later on. $T_\alpha=c(S_\alpha)$, $\mathfrak{T}_\alpha=\{T_\alpha\times S_{1/m}(x)|x\in I_{2n+1}\}$ for $\alpha\in A_m$, where we denote by $S_{1/m}(x)$ the spherical neighborhood of radius $1/m$ around x in I_{2n+1}. We mean by a *star-decomposition* an open decomposition of R consisting of sets contained in $\bigcup_{m=1}^{\infty}\mathfrak{S}_m$ and intersecting no one another. Let C: $\{S_\gamma|\gamma\in C(\subseteqq\bigcup_{m=1}^{\infty}A_m)\}$ be a star-decomposition, then for every $\gamma\in C$ we denote by $m(\gamma)$ the number such that $\gamma\in A_{m(\gamma)}$.

We denote by $M(R,m)$ the totality of mappings of $M(R)$ satisfying $f^{-1}(\mathfrak{T}_\gamma)$

8) Cf. J. W. Tukey: Convergence and uniformity in toplogy (1940).

$=\{f^{-1}(T)\mid T\in\mathfrak{T}_\gamma\}<\mathfrak{U}_m\ (\gamma\in C)$ for some star-decomposition C: $\{S_\gamma|\gamma\in C\}$ of R. Last we define *C-neighborhood* $N_C(f)$ of $f\in M(R)$ by $N_C(f)=\{g\mid g\in M(R),\ \sup\{d(\pi f(x),\ \pi g(x))|x\in S_\gamma\}<1/m(\gamma)\}$ for a star-decomposition C: $\{S_\gamma|\gamma\in C\}$, where π and d denote the projection of $N(\Omega)\times I_{2n+1}$ onto I_{2n+1} and the metric of I_{2n+1} respectively.

2. First we prove $N_C(f)\frown M(R,\ m)\neq\phi$ for every $f\in M(R)$, every star-decomposition C and every positive integer m. Take $l(\gamma)=\max(6m(\gamma),\ m)$ for every $\gamma\in C$ and put $D_\gamma=\{\delta|\delta\in A_{l(\gamma)},\ T_\delta\subseteqq T_\gamma$ (or $S_\delta\subseteqq S_\gamma$ as the same)$\}$. Since we can cover I_{2n+1} by a finite subcovering of $\{S_{1/l(\gamma)}(x)|x\in I_{2n+1}\}$, we denote by $\{S_{1/l(\gamma)}(x_i)|i=1,\ 2,\cdots,a(\gamma)\}$ such a covering, then $\mathfrak{T}'_\delta=\{T_\delta\times S_{1/l(\gamma)}(x_i)|i=1,\cdots,a(\gamma)\}$ is a finite subcovering of $\mathfrak{T}_\delta(\delta\in A_{l(\gamma)})$. Since $f^{-1}(\mathfrak{T}'_\delta)=\{f^{-1}(T')|T'\in\mathfrak{T}'_\delta\}$ and $\mathfrak{U}_m$ are, from $l(\Upsilon)\geqq m$, finite open coverings of S_δ, we have an open finite covering $\mathfrak{V}_\delta$ of S_δ satisfying order $\mathfrak{V}_\delta\leqq n+1$, $\mathfrak{V}^\Delta_\delta<\mathfrak{U}_m\wedge f^{-1}(\mathfrak{T}'_\delta)$. $\mathfrak{V}=\cup\{\mathfrak{V}_\delta|\delta\in D_\gamma,\ \Upsilon\in C\}$ is an open covering of R of order $\leqq n+1$.

Let us consider fixed $\Upsilon\in C$ and $\delta\in D_\gamma$, and assume that $V_1,\cdots,V_s$ are all the members of $\mathfrak{V}_\delta$. Then we select vertices $x(V_i)$ $(i=1\cdots s)$ in I_{2n+1} for which it is true that $d(\pi f(V_i),\ x(V_i))<1/3m(\Upsilon)$ $(i=1,\cdots,s)$, the $x(V_i)$ are in general position in E_{2n+1}, *i. e.* no $m+2$ of the vertices $x(V_i)$ $(m=0,\ 1,\cdots,2n)$ lie in an m-dimentional linear subspace of E_{2n+1}. We define a barycentric mapping ϕ_δ of S_δ into I_{2n+1} by

$$\phi_\delta(p)=\frac{\sum_{i=1}^{s}\rho(p,\ V_i^c)x(V_i)}{\sum_{i=1}^{s}\rho(p,\ V_i^c)}\quad(p\in S_\delta),$$

where we consider $x(V_i)$ as a vector and denote by $\rho(p,V_i^c)$ $\inf\{\rho(p,\ q)|q\in V_i^c\}$[9]. Thus we get a continous mapping $\phi(p)=\phi_\delta(p)\,(p\in S_\delta,\ \delta\in D_\gamma,\ \Upsilon\in C)$ of R into I_{2n+1}. We now prove the mapping $\varphi(p)=(c(p),\ \phi(p))\in M(R)$ is contained in the common part of $N_C(f)$ and $M(R,\ m)$.

To prove $\varphi\in N_C(f)$, we take an arbitrary point $p\in S_\gamma$ for $\Upsilon\in C$, then $p\in S_\delta$ for some $\delta\in D_\gamma$. Assume the V_i are so numbered that $\{V_1,\cdots,V_t\}$ is the set of all the $V_i\in\mathfrak{V}_\delta$ which contain p. Then $\rho(p,\ V_i^c)=0$ for $i>t$. From $\delta(\pi f(V_i))\leqq 2/l(\Upsilon)\leqq 1/3m(\Upsilon)$ and $d(\pi f(V_i),\ x(V_i))<1/3m(\Upsilon)$, we get $d(x(V_i),\ \pi f(p))<2/3m(\Upsilon)$ $(i=1,\ 2,\cdots,t)$. A foriorti, the center of gravity $\phi(p)$ of the $x(V_i)$ satisfies $d(\phi(p),\ \pi f(p))<2/3m(\Upsilon)<1/m(\Upsilon)$. Therefore $\varphi\in N_C(f)$.

Next to show $\varphi\in M(R,\ m)$, we fix $\Upsilon\in C$ and $\delta\in D_\gamma$ and suppose $V_{i_1},\cdots,V_{i_t}$ are all the members of $\mathfrak{V}_\delta$ containing a given point p of S_δ. Consider the linear $(t-1)$ space $L_\delta(x)$ in I_{2n+1} spanned by the vertices $x(V_{i_1}),\cdots,x(V_{i_t})$, then $t\leqq n+1$ and $\phi_\delta(p)\in L_\delta(p)$ are obvious.

Since there are only a finite number of the linear subspaces $L_\delta(p)$, there exists a positive number $h(\delta)>0$ such that any two of these linear subspaces

9) $\rho(p,\ q)$ denotes the metric of R. V^c denotes the complement set of V.

$L_\delta(p)$ and $L_\delta(p')$ either meet or else have a distance $\geqq 2/h(\delta)$ from each other.

Putting $E_\delta=\{\varepsilon|\varepsilon\in A_{h(\delta)},\ \ T_\varepsilon\subseteqq T_\delta\}$ we consider a star-decomposition E: $\{S_\varepsilon|\varepsilon\in E_\delta,\ \delta\in D_\gamma,\ \Upsilon\in C\}$. If $\varphi(p),\ \varphi(p')\in T_\varepsilon\times S_{1/h(\delta)}(x)$ for $p,\ p'\in R$, then it follows that $c(p),\ c(p')\in T_\varepsilon$ and $d(\phi(p),\ x)<1/h(\delta),\ d(\phi(p'),\ x)<1/h(\delta)$; hence $p,\ p'\in S_\varepsilon\subseteqq S_\delta$. Therefore $d(\phi(p),\ \phi(p'))<2/h(\delta)$ implies $L_\delta(p)\frown L_\delta(p')\neq\phi$. If we suppose $L_\delta(p')$ is spanned by $x(V_{j_1}),\cdots,x(V_{j_u}),\ u\leqq n+1$, then since $x(V_{i_1}),\cdots,x(V_{i_t}),\ x(V_{j_1}),\cdots,x(V_{j_u})$ are in general position in E_{2n+1}, it follows that at least one of $x(V_{j_1}),\cdots,x(V_{j_u})$ is also one of $x(V_{i_1}),\cdots,x(V_{i_t})$. Hence p and p' are contained in a common member V_i of $\mathfrak{V}_\delta$, *i. e.* $p'\in S(p,\ \mathfrak{V}_\delta)$. Since $\mathfrak{V}_\delta^\Delta<\mathfrak{U}_m$, it holds $\varphi^{-1}(T_\varepsilon\times S_{1/h(\delta)}(x))\subseteqq U$ for some $U\in\mathfrak{U}_m$. Thus we get $\varphi^{-1}(\mathfrak{T}_\varepsilon)<\mathfrak{U}_m$ for every $\varepsilon\in E$, proving $\varphi\in M(R,\ m)$.

We prove now that for a given $\varphi\in N_C(f)\frown M(R,\ m)$ there exists a star-decomposition C' satisfying $N_{C'}(\varphi)\subseteqq N_C(f)\frown M(R,\ m)$. Since $\varphi\in N_C(f)$ implies $\sup\ \{d(\pi f(p),\ \pi\varphi(p))|p\in S_\gamma\}=a_\gamma<1/m(\Upsilon)\ \ (\Upsilon\in C)$, we take a positive integer $h(\Upsilon)$ for $\Upsilon\in C$: $1/h_\gamma<1/m(\Upsilon)-a_\gamma$ and define a star-decomposition D by
D: $\{S_\delta|\delta\in D_\gamma,\ \Upsilon\in C\}\ \ (D_\gamma=\{\delta|\delta\in A_{h(\gamma)},\ T_\delta\subseteqq T_\gamma\})$.

Let $\psi\in N_D(\varphi)$, then taking $p\in S_\delta,\ \delta\in D_\gamma$ for a given point p of S_γ, we get $d(\pi\varphi(p),\ \pi\psi(p))<1/h(\Upsilon)$. Therefore $\sup\ \{d(\pi f(p),\ \pi\psi(p))|p\in S_\gamma\}\leqq a_\gamma+1/h(\Upsilon)<1/m(\Upsilon)$, proving $\psi\in N_C(f)$, *i. e.* $N_D(\varphi)\subseteqq N_C(f)$.

Moreover, since $\varphi\in M(R,\ m)$, it holds $\varphi^{-1}(\mathfrak{T}_\beta)<\mathfrak{U}_m\ \ (\beta\in B)$ for some star-decomposition B, where $\mathfrak{T}_\beta=\{T_\beta\times S_{1/m(\beta)}(x)|x\in I_{2n+1}\}$ as above mentioned. Putting $D_\beta=\{\delta|\delta\in A_{2m(\beta)},\ T_\delta\subseteqq T_\beta\}$ for every $\beta\in B$, we have a star-decomposition E: $\{S_\varepsilon|\varepsilon\in D_\beta,\ \beta\in B\}$. Let $\psi\in N_E(\varphi)$, then we see easily that $\varepsilon\in D_\beta$ implies $\psi^{-1}(T_\varepsilon\times S_{1/2m(\beta)}(x))\subseteqq\varphi^{-1}(T_\beta\times S_{1/m(\beta)}(x))$ for every $x\in I_{2n+1}$. For if we assume the contrary, then there exists a point p of R such that $\psi(p)\in T_\varepsilon\times S_{1/2m(\beta)}(x),\ \varphi(p)\notin T_\beta\times S_{1/m(\beta)}(x)$. Hence $p\in S_\varepsilon,\ \pi\psi(p)\in S_{1/2m(\beta)}(x)$ and $\pi\varphi(p)\notin S_{1/m(\beta)}(x)$, and hence $d(\pi\psi(p),\ \pi\varphi(p))\geqq 1/2m(\beta)$, which contradicts $\psi\in N_E(\varphi)$. Thus it must be $\psi^{-1}(T_\varepsilon\times S_{1/2m(\beta)}(x))\subseteqq\varphi^{-1}(T_\beta\times S_{1/m(\beta)}(x))\subseteqq U$ for some $U\in\mathfrak{U}_m$. Therefore $\psi^{-1}(\mathfrak{T}_\varepsilon)<\mathfrak{U}_m(\varepsilon\in E)$, proving $\psi\in M(R,\ m)$, *i. e.* $N_E(\varphi)\subseteqq M(R,\ m)$. Now $C'=D\wedge E$: $\{S_\delta\frown S_\varepsilon|\delta\in D,\ \varepsilon\in E\}$ is obviously a star-decomposition satisfying $N_{C'}(\varphi)\subseteqq N_C(f)\frown M(R,\ m)$.

3. We can select by the consequences of 2 two sequences $C_1>C_2>C_3>\cdots$, $D_1>D_2>D_3\cdots$ of star-decompositions and a sequence $f_1,\ f_2,\ \cdots$ of elements of $M(R)$ such that

$\Upsilon\in C_m$ implies $m(\Upsilon)>m$,

$N_{C_1}(f_1)\subseteqq M(R,\ 1)$,

D_1: $\{T_\delta|\delta\in D_{1\gamma},\ \Upsilon\in C_1\}\quad(D_{1\gamma}=\{\delta|\delta\in A_{2m(\gamma)},\ T_\delta\subseteqq T_\gamma\}(\Upsilon\in C_1))$,

$N_{C_2}(f_2)\subseteqq N_{D_1}(f_1)\frown M(R,\ 2)$,

D_2: $\{T_\delta|\delta\in D_{2\gamma},\ \Upsilon\in C_2\}\quad(D_{2\gamma}=\{\delta|\delta\in A_{2m(\gamma)},\ T_\delta\subseteqq T_\gamma\}(\Upsilon\in C_2))$,

....

$$N_{C_h}(f_h) \subseteqq N_{D_{h-1}}(f_{h-1}) \cap M(R, h),$$

$$D_h: \{T_\delta | \delta \in D_{h\gamma}, \ \gamma \in C_h\} \quad (D_{h\gamma} = \{\delta | \delta \in A_{2m(\gamma)}, \ T_\delta \subseteqq T_\gamma\} (\gamma \in C_h)),$$

....

Then since $f_h \in N_{D_k}(f_k)$ $(h \geqq k)$, it holds $d(\pi f_k(p), \pi f_h(p)) < 1/2m(\gamma) < 1/k$ $(h \geqq k)$ for some $\gamma \in C_k$, and hence $\pi f_h(p)$ $(h=1, 2\cdots)$ uniformly converges to a continuous mapping $\phi(p)$ of R into I_{2n+1}.

Let us show $(c(p), \phi(p)) = \varphi(p) \in \bigcap_{m=1}^{\infty} M(R, m)$. Since $f_h \in N_{D_k}(f_k)$ $(h \geqq k)$, if we take $\delta \in D_{k\gamma}$: $p \in S_\delta$ for given $\gamma \in C_k$ and $p \in S_\gamma$, then $d(\pi f_h(p), \pi f_k(p)) < 1/2m(\gamma)$; hence $d(\phi(p), \pi(f_k(p)) \leqq 1/2m(\gamma) < 1/m(\gamma)$. Therefore $\varphi(p) \in N_{C_k}(f_k) \subseteqq M(R, k)$, *i. e.* $\varphi(p) \in \bigcap_{m=1}^{\infty} M(R, m)$. Thus we get a homeomorphic mapping φ of R into $N(\Omega) \times I_{2n+1}$.

We now consider a countable number of n-dimensional linear spaces L_1, L_2. $\cdots$ in I_{2n+1}. Replacing $M(R, m)$ in this proof with $N(R, m) = \{\varphi | \varphi \in M(R), \varphi^{-1}(\mathfrak{T}_\gamma) < \mathfrak{U}_m, \overline{\pi\varphi(S_\gamma)} \cap L_m = \phi (\gamma \in C)$ for some star-decomposition $C\}$, we can show that R is homeomorphic with a subset of $N(\Omega) \times (I_{2n+1} - \bigcup_{m=1}^{\infty} L_m)$. Its proof is a modification of the above proof, and hence it is left to the reader. If L_1, L_2, $\cdots$ are all the linear spaces in I_{2n+1} of the form $x_{i_1} = r_1, \cdots, x_{i_{n+1}} = r_{n+1}$, the r's being rational, then we get the following

THEOREM 2. *In order that a metric space R with a σ-star-finite (countable) basis has dimension $\leqq n$ and has an open basis whose cardinal number is not greater than $\mathfrak{m}$ it is necessary and sufficient that R is homeomorphic with a subset of $N(\Omega) \times M_{2n+1}^n$, where Ω is a set with $|\Omega| = \mathfrak{m}$, and M_{2n+1}^n is the set of points in I_{2n+1} at most n of whose cordinates are rational*[10].

DEFINITION. We say that the *p-dimensional density* of a subset S of a metric space is zero if and only if for every $\varepsilon > 0$ there exist a decomposition $S = \cup \{A_{i\gamma} | \gamma \in C, i = 1, 2\cdots\}$ such that $\delta(A_{i\gamma}) < \varepsilon$ $(\gamma \in C, i = 1, 2\cdots)$, $\sum_{i=1}^{\infty} [\delta(A_{i\gamma})]^p < \varepsilon$ $(\gamma \in C)$ and such that $\bigcup_{i=1}^{\infty} A_{i\gamma} = S_\gamma$ is open in S for every $\gamma \in C$ and $S_\gamma \cap S_{\gamma'} = \phi$ $(\gamma \neq \gamma')$[11].

THEOREM 3. *If a metric space R has dimention $\leqq n$ and has a σ-star-finite (countable) basis, then it is homeomorphic to a subset of $N(\Omega) \times I_{2n+1}$ of $(n+1)$-density zero.*

Proof. Replacing $M(R, m)$ in the proof of Theorem 1 with $O(R, m) = \{\varphi | \varphi \in M(R)$, for some star-decomposition C $\varphi^{-1}(\mathfrak{T}_\gamma) < \mathfrak{U}_m$, and there exist decompositions

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10) It is well known that dim $M_{2n+1}^n = n$, and hence dim $N(\Omega) \times M_{2n+1}^n = n$ from the general product theorem due to M. Katětov: On the dimension of non-separable spaces *I*, Czechoslovak Math. Journ. T. 2 (1952) and K. Morita, loc. cit.

11) $\delta(A)$ denotes the diameter of A. We shall agree to set $[\delta(A)]^0 = 0$ if A is empty and $[\delta(A)]^0 = 1$ otherwise.

$\overline{\varphi(R)}\frown S_\gamma=\bigcup_{i=1}^{\infty}A_{i\gamma}$ $(\gamma\in C)$ such that $\delta(A_{i\gamma})<1/m$, $\sum_{m=1}^{\infty}[\delta(A_{i\gamma})]^{n+1}<1/m$ $(\gamma\in C)\}$, we get $\bigcap_{m=1}^{\infty}O(R, m)\neq\phi$, proving Theorem 3. This proof is similar to that of Theorem 1, and it is left to the reader.

THEOREM 4. *Every metric space R of $(n+1)$-density zero has dimension $\leqq n$.*

Proof. Let us show ind dim $R\leqq n$ [12]. We consider an arbitrary pair F, G of closed sets with $\rho(F, G)>0$. Select a positive integer with $1/m<\rho(F, G)$, and let $R=\cup\{A_{i\gamma}|\gamma\in C,\ i=1,2\cdots\}$ be a decomposition of R such that $\delta(A_{i\gamma})<1/m^2$, $\sum_{i=1}^{\infty}[\delta(A_{i\gamma})]^{n+1}<1/m^2$ and such that $S_\gamma=\cup\{A_{i\gamma}|i=1,2\cdots\}$ is open for every $\gamma\in C$. We put $u_{i\gamma}=\sup\{\rho(F, x)|x\in A_{i\gamma}\}$, $v_{i\gamma}=\inf\{\rho(F, x)|x\in A_{i\gamma}\}$. Then it is easily seen that $u_{i\gamma}-v_{i\gamma}\leqq\delta(A_{i\gamma})$. We define a non-negative valued function $d_\gamma(r)$ for every $\gamma\in C$ by

$$d_{i\gamma}(r)=\begin{cases}0 & (0\leqq r<v_{i\gamma} \text{ or } u_{i\gamma}<r)\\ [\delta(A_{i\gamma})]^n & (v_{i\gamma}\leqq r\leqq u_{i\gamma}),\end{cases}$$

$$d_\gamma(r)=\sum_{i=1}^{\infty}d_{i\gamma}(r).$$

Since $d_{i\gamma}(r)\geqq 0$, we may interchange integration and summation by Lebesgue's theorem, and hence from $\int_0^{\frac{1}{m}}d_{i\gamma}(r)dr\leqq[\delta(A_{i\gamma})]^{n+1}$ it follows that $\int_0^{\frac{1}{m}}d_\gamma(r)\,dr$ $=\int_0^{\frac{1}{m}}\sum_{i=1}^{\infty}d_{i\gamma}(r)dr=\sum_{i=1}^{\infty}\int_0^{\frac{1}{m}}d_{i\gamma}(r)dr\leqq\sum_{i=1}^{\infty}[\delta(A_{i\gamma})]^{n+1}<1/m^2(\gamma\in C)$. This implies $d_\gamma(r(\gamma))<1/m$ for some $r(\gamma)$: $0<r(\gamma)\leqq 1/m$. We denote by $S(F, r)$ the set of all the points satisfying $\rho(F, x)<r$ and by $S(r)$ the boundary of $S(F, r)$. Then $[\delta(A_{i\gamma}\frown S(r(\gamma)))]^n\leqq d_{i\gamma}(r(\gamma))$ combining with $d_\gamma(r(\gamma))<1/m$ implies $\sum_{i=1}^{\infty}[\delta(A_{i\gamma}\frown S(r(\gamma)))]^n<1/m$. $\cup\{S_\gamma\frown S(F, r(\gamma))|\gamma\in C\}=U$ is obviously an open set of R such that $F\subseteqq U\subseteqq G^C$. Since $\bar{U}-U=\cup\{A_{i\gamma}\frown S(r(\gamma))|\gamma\in C,\ i=1,2,\cdots\}$, the n-dimensional density of $\bar{U}-U$ is zero. The above arguement is also valid for $n=0$. Hence $\bar{U}-U=\phi$ for a space R of 1-dimensional density zero. Consequently ind dim $R\leqq 0$. Thus we can establish this theorem inductively by Morita's theorem[13].

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12) Ind dim $\phi=-1$, ind dim $R\leqq n$ if and only if for any pair of a closed set F and an open set G with $F\subseteqq G$ there exists an open set U such that $F\subseteqq U\subseteqq G$, ind dim $(\bar{U}-U)$ $\leqq n-1$. The equivalence of this dimension with Lebesgue's dimension was proved by M. Katětov, loc. cit. and by K. Morita, loc. cit. independently.

13) Loc. cit. Ind dim $R\leqq n$ if and only if R has a σ-locally-finite open basis $\mathfrak{B}$ such that the boundary of each set of $\mathfrak{B}$ has ind dimension $\leqq n-1$.