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On the groups with the same table of characters as symmetric groups.

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The finite groups G and G' are said to have the same table of characters when there exist one to one correspondences $\chi_\mu \leftrightarrow \chi_\mu'$ between their (ordinary) irreducible characters and $C_\alpha \leftrightarrow C_\alpha'$ between their conjugate classes, and $\chi_\mu(C_\alpha) = \chi_\mu'(C_\alpha')$ for all μ and α. Even if G and G' have the same table of characters, they are not necessarily isomorphic to each other, for instance the two types of non-abelian group of order p^3 (p ; a prime number) have the same table of characters.

In this paper, we shall prove the following theorem.

THEOREM. *If a finite group G has the same table of characters as a symmetric group S_n, then G is isomorphic to S_n.*

To prove the theorem, we shall prove some propositions and lemmas.

PROPOSITION 1. *The symmetric group of degree n is isomorphic to a group generated by the generators $a_1, \dots, a_{n-1}$ with the defining relations*

(1) $$a_1^2 = a_2^2 = \cdots = a_{n-1}^2 = 1,$$

(2) $$(a_i\ a_j)^2 = 1 \qquad \textit{if } 1 < j-i,$$

(3) $$(a_i a_{i+1})^3 = 1.$$

For the proof, see Dickson; Linear Groups p. 287.

PROPOSITION 2. *If G and G' have the same table of characters, then the orders of them coincide with each other, and the degree of any irreducible character of G and the number of elements of any conjugate class of G are equal to the degree of the corresponding character of G' and the number of elements of the corresponding conjugate class of G' respectively.*

Proof. Let χ_i and C_i $(i = 1, 2, \cdots, k)$ be respectively the irreducible characters and the conjugate classes of G. We denote by χ_i' and C_i' the character and conjugate class of G' corresponding to χ_i and C_i respectively. Especially we shall denote by C_1 and C_j', the conjugate classes of the unit elements of G and G' respectively.

Since $\chi_\mu\ (C_1) = \chi_\mu'(C_1') = \deg\ \chi_\mu$, we have

$$\sum_{\mu=1}^{k} \chi_\mu'(C_1') \overline{\chi_\mu'(C_j')} = \sum_{\mu=1}^{k} \deg\ \chi_\mu \deg \chi_\mu' \gneq 0$$

Therefore, from the orthogonality relations of characters, it is seen that C_1' coincides with C_j', and hence $\deg\ \chi_\mu = \deg \chi_\mu'$.

Let g and g' be the orders of G and G' respectively.
Then

$$g=\sum_{\mu=1}^{k}(\deg \chi_{\mu})^2=\sum_{\mu=1}^{k}(\deg \chi_{\mu}')^2=g'.$$

Further, if we denote by h_i and h_i' the numbers of elements of C_i and C_i' respectively, then from the orthogonality relations of characters, we have

$$g/h_{\alpha}=\sum_{\mu=1}^{k}\chi_{\mu}(C_{\alpha})\overline{\chi_{\mu}(C_{\alpha})}=\sum_{\mu=1}^{k}\chi_{\mu}'(C_{\alpha}')\overline{\chi_{\mu}'(C_{\alpha}')}=g'/h_{\alpha}'.$$

Since, as proved above, $g=g'$, we have $h_{\alpha}=h_{\alpha}'$.

PROPOSITION 3. *If two finite groups G and G' have the same table of characters, then they have the same multiplication table of the conjugate classes.*

Proof. Assume that

$$C_{\alpha}C_{\beta}=\sum_{\gamma}c_{\alpha\beta\gamma}C_{\gamma}.$$

Then

$$\frac{h_{\alpha}\chi_{\mu}(C_{\alpha})}{f_{\mu}}\ \frac{h_{\beta}\chi_{\mu}(C_{\beta})}{f_{\mu}}=\sum_{\delta}c_{\alpha\beta\delta}\frac{h_{\delta}\chi_{\mu}(C_{\delta})}{f_{\mu}}$$

where f_{μ} is the degree of χ_{μ}. Hence

$$\sum_{\mu}\frac{\overline{\chi_{\mu}(C_{\gamma})}}{f_{\mu}}(h_{\alpha}\chi_{\mu}(C_{\alpha})\,h_{\beta}\chi_{\mu}(C_{\beta}))$$
$$=\sum_{\mu,\delta}c_{\alpha\beta\delta}\,h_{\delta}\chi_{\mu}(C_{\delta})\overline{\chi_{\mu}(C_{\gamma})}=c_{\alpha\beta\gamma}\,g$$

Consequently

$$c_{\alpha\beta\gamma}=\frac{1}{g}\sum_{\mu}\frac{\overline{\chi_{\mu}(C_{\gamma})}}{f_{\mu}}(h_{\alpha}h_{\beta}\chi_{\mu}(C_{\alpha})\chi_{\mu}(C_{\beta})).$$

If we assume in G' that

$$C_{\alpha}'C_{\beta}'=\sum c'_{\alpha\beta\gamma}C_{\gamma}',$$

then, in the same way as above, we have

$$c'_{\alpha\beta\gamma}=\frac{1}{g'}\sum_{\mu}\frac{\overline{\chi_{\mu}'(C_{\gamma}')}}{f_{\mu}'}(h_{\alpha}'h_{\beta}'\chi_{\mu}'(C_{\alpha}')\chi_{\mu}'(C_{\beta}')).$$

Hence, from Proposition 2, we have

$$c_{\alpha\beta\gamma}=c'_{\alpha\beta\gamma}.$$

In the following lemmas, we shall always assume that G is a group with the same table of characters as the symmetric group S_n of degree n, where $n\geqq 3$ and $n\neq 4$, and $C(i_1^{\alpha_1}, i_2^{\alpha_2}, \cdots)$ denotes the conjugate class of G corresponding to the conjugate class of S_n whose element can be expressed as a product of α_1 cycles of length i_1, α_2 cycles of length i_2, $\cdots$ such as each of letters occurs in only one cycle of them, for instance, $C(2^2, 3)$ denotes the conjugate class of G corresponding to the conjugate class of S_n containing (1 2) (3 4) (5 6 7). From Proposition 3, G and

S_n has the same multiplication table of conjugate classes, therefore we can know completely the multiplication table of conjugate classes of G by investigating that of S_n.

LEMMA 1. *The order of an element a of $C(2)$ is 2.*

Proof. It is easily seen that if $n \geqq 4$, then

$$C(2)^2 = 3C(3) + 2C(2^2) + \frac{n(n-1)}{2}C(1),$$

and if $n=3$, then

$$C(2)^2 = 3C(3) + 3C(1).$$

Denote by $n(i_1^{\alpha_1}, i_2^{\alpha_2}, \cdots)$ the order of the normalizer of an element of $C(i_1^{\alpha_1}, i_2^{\alpha_2}, \cdots)$. If a^2 is contained in $C(3)$, $n(2)$ divides $n(3)$ because the normalizer of a^2 contains that of a. Since $n(2)=2(n-2)!$ and $n(3)=3(n-3)!$, $2(n-2)!$ divides $3(n-3)!$ and hence 3 is a multiple of $2(n-2)$. This is impossible. Hence a^2 is contained in $C(2^2)$ or $C(1)$. In the fommer case, $n(2)$ divides $n(2^2)$. Since $n(2^2) = 8(n-4)!$, $2(n-2)!$ divides $8(n-4)!$ and hence 4 is a multiple of $(n-2)$ $(n-3)$. But this is impossible unless $n=4$. Therefor a^2 must be contained in $C(1)$ i.e. $a^2=1$.

LEMMA 2. *The order of an element of $C(3)$ is 3.*

Proof. From the multiplication table of conjugate classes, we can see that any element of $C(3)$ can be expressed in exactly three ways as a product of two elements of $C(2)$. Let x be an element of $C(3)$ and $x=ab$ with two elements a and b of $C(2)$.

If $x^2=1$, then $ab=ba$. Since $a \neq b$, x can be expressed in different ways of an even number as a product of two elements of $C(2)$, which is a contradiction. Therefore $x^2 \neq 1$. Now

$$x = ab = b(bab) = (bab)(babab)$$
$$= (babab)\ (bababab)\ .$$

It is easily seen that a, b and bab are all distinct and $babab$ is distinct from b and bab. Hence $a = babab$, i.e. $x^3 = (ab)^3 = 1$.

LEMMA 3. *The order of an element of $C(2^2)$ is 2.*

Proof. From the multiplication table of conjugate classes, we can see that any element of $C(2^2)$ can be expressed in exactly two ways as a product of two elements of $C(2)$. If an element x is the product of two elements a and b of $C(2)$, then

$$x = ab = b(bab) = (aba)a.$$

Clearly $a \neq b$, hence aba is equal to a or b. In the former case, a must be equal to b, which is a contradiction. Therefore $aba=b$, i.e. $x^2 = (ab)^2 = 1$.

LEMMA 4. *Let a and b be two distinct element of $C(2)$. If a and b are commutative whith each other then $ab \in C(2^2)$. If a and b are not commutative with each other then $ab \in C(3)$.*

Proof. From the multiplication table of conjugate classes, we can see that ab is contained in $C(2^2)$ or $C(3)$. The order of ab is 2 if and only if $ab=ba$, and hence $ab \in C(2^2)$ if and only if $ab=ba$.

LEMMA 5. *The order of an element of* $C(4)$ *is* 4.

Proof. From the multiplication table of conjugate classes, we can see that any element of $C(4)$ can be expressed as a product of an element of $C(2)$ and an element of $C(3)$. Let x be an element of $C(4)$ and $x=ay$ with an element a of $C(2)$ and an element y of $C(3)$. If the order of x is 2, $x^2=ay\,ay=1$ and hence $y\,a\,y=a$. Since $y^3=1$, $y\,a\,y^{-1}=ay=x$, which is a contradiction. Hence $x^2\neq 1$.

From the multiplication table of conjugate classes, it is seen that x can be expressed in exactly two ways as a product of an element b of $C(2)$ and an element z of $C(2^2)$. If $b=x\,b\,x^{-1}$, i.e. $b=b\,z\,b\,z\,b$, then $b\,z\,b\,z=x^2=1$, which is a contradition, hence $b\neq x\,b\,x^{-1}$. Since

$$x=b\,z=(x\,b\,x^{-1})\ (x\,z\,x^{-1})=(x^2\,b\,x^{-2})\ (x^2\,z\,x^{-2})$$

$x^2\,b\,x^{-2}$ is equal to b or $x\,b\,x^{-1}$. In the latter case, $b=x\,b\,x^{-1}$, which is a contradiction. Therefore $x^2\,b\,x^{-2}$ must be equal to b, i.e. $b=b\,z\,b\,z\,b\,z\,b\,z\,b$. Hence $x^4=(b\,z)^4=1$. Since $x^2\neq 1$ as proved above, the order of x is 4.

LEMMA 6. *Let* $k\leqq[\frac{n}{2}]$. *If* a_i $(i=1,\ 2,\cdots,k)$ *are elements of* $C(2)$ *and* $a_1a_2\cdots a_k$ *is contained in* $C(2^k)$, *then* $a_1,\ a_2,\cdots,a_k$ *are all distinct and commutative. Further any element of* $C(2^k)$ *can be uniquely expressed as such a product of* k *elements of* $C(2)$ *disregarding their arrangement.*

Conversely, if k *elements* $a_1,\cdots,a_k$ *of* $C(2)$ *are all distinct and commutative, then* $a_1\cdots a_k$ *is contained in* $C(2^k)$.

Proof. In the case $k=1$, our assertion is trivial. To prove the lemma in the case $k=2$, let a_1 and a_2 be two elements of $C(2)$ and assume that $a_1\,a_2$ is contained in $C(2^2)$. Clearly $a_1\neq a_2$, and, from Lemma 3, $(a_1\,a_2)^2=1$, i.e. $a_1\,a_2=a_2\,a_1$. Since any element of $C(2^2)$ can be expressed in exactly two ways as a product of two elements of $C(2)$, any element of $C(2^2)$ can be expressed uniquely as a product of two elements of $C(2)$ disregarding their arrangement. Conversely, if a_1 and a_2 are two distinct elements of $C(2)$ and they are commutative, then it is immediately seen from Lemma 4 that $a_1\,a_2\in C(2^2)$.

Now, we shall consider the case $k\geqq 3$ and prove the lemma for the cases by an induction on k. Any element of $C(2^k)$ can be expressed in exactly $k!$ ways as a product of k elements of $C(2)$ and also can be expressed in exactly k ways as a product of an element of $C(2^{k-1})$ and an element of $C(2)$. If a product of an element x of $C(2)^{k-1}$ and an element a of $C(2)$ is contained in $C(2^k)$, then x must be contained in $C(2^{k-1})$, for otherwise $x\ a$ can be expressed as a product of k elements of $C(2)$ in more ways than $k!$. If $a_1,\cdots,a_k$ are elements of $C(2)$ and $a_1\cdots a_k$ is contained in $C(2^k)$ then $a_1\cdots a_{k-1}$ is contained in $C(2^{k-1})$, therefore, by the induction hypotheses, $a_1,\cdots,a_{k-1}$ are all distinct and commutative. Since $a_2\cdots a_k\ a_1=a_1(a_1\cdots a_k)a_1$ is also contained in $C(2^k)$, $a_2,\cdots,a_k$ are all distinct and commutative, and since $a_3 \cdots a_k a_1 a_2$ is an element of $C(2^k)$, a_k and a_3 are distinct and commutative to each other. Thus it is proved that $a_1, \cdots, a_k$ are all distinct and commutative. All arrangements of a_i in the product $a_1 \cdots a_k$ give $k!$ expressions of the element $a_1 \cdots a_k$ as a product of k elements of $C(2)$, and hence there is not any such expression other than these.

To prove the latter half, we shall consider $C(2^{k-1})C(2)$. From the multiplication table of conjugate classes, we can easily see that it is made up of elements of $C(2^k)$, $C(2^{k-2}, 3)$, $C(2^{k-2})$ and $C(2^{k-3}, 4)$. Any element of $C(2^{k-2}, 3)$ can be uniquely expressed as a product of an element x of $C(2^{k-2})$ and an element y of $C(3)$. Since $xy = (xy)\, x\, (xy)^{-1}(xy)\, y\, (xy)^{-1}$, $(xy)\, y\, (xy)^{-1} = xyx^{-1} = y$, i. e. x and y are commutative. From the first half of the lemma and Lemma 3, orders of x and y are 2 and 3 respectively, therefore the order of xy is 6. In a similar way, the order of an element of $C(2^{k-3}, 4)$ is 4.

Now assume that k elements $a_1, \cdots, a_k$ are all distinct and commutative. Then, by the induction hypothesis $a_1 \cdots a_{k-1}$ is contained in $C(2^{k-1})$, therefore $a_1 \cdots a_{k-1}$ $a^k \in C(2^{k-1})C(2)$. Since its order is 2, it is contained in $C(2^k)$ or $C(2^{k-2})$. In the latter case, $a_1 \cdots a_k = b_1 \cdots b_{k-2}$ with $b_i \in C(2)$ and then

$$a_1 \cdots a_{k-1} = b_1 \cdots b_{k-2}\, a_k.$$

From the first half of the lemma, a_k is equal to some a_i $(1 \leqq i \leqq k-1)$. This is a contradiction. Hence $a_1 \cdots a_k$ must be contained in $C(2^k)$.

LEMMA 7. *If a_1 and a_2 are two elements of $C(2)$ such that $a_1 a_2$ is contained in $C(2^2)$, then there exists an element b of $C(2)$ such that $a_1 a_2 b \in C(4)$, and then $a_1\, b$ and $a_2\, b$ must be contained in $C(3)$.*

Proof. Since $C(2^2)C(2)$ contains $C(4)$, there exists an element b of $C(2)$ such that $a_1 a_2 b \in C(4)$. Clearly b is distinct from a_1 and a_2, and hence $a_i\, b$ are contained in $C(2^2)$or $C(3)$. If $a_1 b$ and $a_2 b$ are both contained in $C(2^2)$, then b is commutative with a_1 and a_2, and hence, from Lemma 6, $a_1 a_2\; b \in C(2^3)$, which is a contradiction. If $a_1 b$ is contained in $C(2^2)$ and $a_2 b$ is contained in $C(3)$, then a_1 is commutative with a_2 and b, and hence the order of $a_1 a_2 b$ is 6, which is a contradition. Thus it is proved that $a_1 b$ and $a_2 b$ are both contained in $C(3)$.

LEMMA 8. *Let a_1, a_2 and b be elements of $C(2)$. If $a_1 a_2$ is contained in $C(2^2)$ and if $a_1 b$ and $a_2 b$ are both contained in $C(3)$, then $x = a_1 a_2 b$ is an element of $C(4)$.*

Proof. Since $C(2)\; C(3)$ is made up of elements of $C(2, 3)$, $C(4)$ and $C(2)$, x is contained in one of $C(2,3)$, $C(4)$, and $C(2)$.

(1) Assume that x is contained in $C(2)$. Since $a_1 a_2 = xb$ is an element of $C(2^2)$, it is seen by Lemma 6 that b is equal to a_1 or a_2, which is a contradiction.

(2) Assume that x is contained in $C(2,3)$. Then x can be expressed uniquely as a product of an element of $C(2)$ and an element of $C(3)$. Since $x = a_1(a_2 b) = a_2$ $(a_1 b)$, a_1 is equal to a_2, which is a contradiction.

Thus it is proved that x must be contained in $C(4)$.

LEMMA 9. *Let a_1, a_2 and a_3 be elements of $C(2)$ such that $a_1a_2a_3$ is contained in $C(2^3)$. If an element b of $C(2)$ is not commutative with a_1 and a_2, then b is commutative with a_3.*

Proof. If b is not commutative with a_3, then, from Lemma 4, $a_i b (i = 1, 2, 3)$ are all contained in $C(3)$, and hence, from Lemma 8, $a_2 a_3 b$ is contained in $C(4)$. Therefore $x = a_1a_2a_3b$ is contained in $C(2^3)$ $C(2)$ and $C(2)$ $C(4)$. Since $C(2^3)$ $C(2)$ is made up of elements of $C(2^4)$, $C(2^2, 3)$, $C(2^2)$ and $C(2,4)$, and since $C(2)$ $C(4)$ is made up of elements of $C(2,4)$, $C(5)$, $C(3)$ and $C(2^2)$, x is contained in $C(2^2)$ or $C(2,4)$. If x is an element of $C(2^2)$ then $a_1a_2a_3b = a_1' a_2'$ with $a_1', a_2' \in C(2)$. Since $a_1a_2a_3 = a_1' a_2' b$ is an element of $C(2^3)$, from Lemma 6, b is equal to some a_i. This is a contradiction. Next assume that x is contained in $C(2,4)$. x can be uniquely expressed as a product of an element of $C(2)$ and an element of $C(4)$. Since $x = a_1(a_2a_3b) = a_2(a_1a_3b)$ and a_2a_3b and a_1a_3b are both contained in $C(4)$, we have $a_1 = a_2$, which is a contradiction.

LEMMA 10. *Let a_1, a_2 and a_3 be elements of $C(2)$. If a_1 is not commutative with a_2, and a_3 is commutative with a_1 and a_2, then there exists an element b of $C(2)$ which is commutative with a_1 and not commutative with a_2 and a_3.*

Proof. From Lemma 7, there exists an element b of $C(2)$ such that $a_2ba_3 = a_3^{-1}(a_3a_2b)a_3 \in C(4)$. Then b is not commutative with a_2 and a_3. If b is commutative with a_1, then b is an asking element. Now we assume that b is not commutative with a_1. Since $a_1 a_2 \in C(3)$, $x = a_1a_2b$ is contained in $C(3)C(2)$. $C(3)C(2)$ is made up of elements of $C(3,2)$, $C(4)$ and $C(2)$. Therefore x is contained in one of $C(3,2)$, $C(4)$ and $C(2)$.

(1) Assume that $x \in C(3, 2)$. Any element of $C(3, 2)$ can be expressed uniquely as a product of an element of $C(3)$ and an element of $C(2)$. Since $x = (a_1 a_2)b = x^{-1}(a_1a_2)xx^{-1} bx$ and $a_1 a_2 \in C(3)$, we have $a_1 a_2 = x^{-1}(a_1 a_2)x = b^{-1}(a_1 a_2)b$ and hence b is commutative with $a_1 a_2$. Therefore $x = (a_1 a_2)b = (b a_1) a_2$. Since $b a_1$ is an element of $C(3)$, b is equal to a_2, which is a contradiction.

(2) Assume that $x = a_1a_2b$ is contained in $C(2)$. $a_1a_2 = xb$ is contained in $C(3)$ and any element of $C(3)$ can be expressed as a product of two elements of $C(2)$ in exactly three ways. Since $xb = a_1a_2 = a_2(a_2^{-1}a_1a_2) = (a_1a_2a_1^{-1})a_1$, b must be equal to one of a_2, $a_2^{-1}a_1a_2$ and a_1, and in either case b is commutative with a_3. This is a contradiction.

Thus x must be contained in $C(4)$. Since $C(4)$ is contained in $C(2^2)C(2)$, there are three elements c_1', c_2', c_3' of $C(2)$ such that $c_1' c_2' \in C(2^2)$ and $c_1' c_2' c_3' \in C(4)$. Then, from Lemma 7, $c_1' c_3'$ and $c_2' c_3'$ are both contained in $C(3)$. Since $c_2' c_3' c_1' = c_1'^{-1}(c_1' c_2' c_3')c_3'$ is also an element of $C(4)$, there are three elements c_1, c_2 and c_3 of $C(2)$ such that $x = c_1c_2c_3$ and $c_1c_3 \in C(2^2)$ and c_1c_2 and c_2c_3 are contained in $C(3)$. Since c_1 is commutative with c_3 and $(c_1c_2)^3 = 1$,

$$xc_1x^{-1}=c_1^x=c_1c_2c_3c_1c_3c_2c_1=c_1c_2c_1c_2c_1=c_2.$$

Since $(c_2c_3)^3=1$,

$$c_2^x=c_1c_2c_3c_2c_3c_2c_1=c_1c_3c_1=c_3.$$

Any element of $C(4)$ can be expressed as a product of an element of $C(2)$ and an element of $C(3)$ in exactly 4 ways, and

$$x=c_1(c_1{}^xc_1{}^{x^2})=c_1{}^x(c_1{}^{x^2}c_1{}^{x^3})=c_1{}^{x^2}(c_1{}^{x^3}c_1)=c^{x^3}(c_1c_1{}^x).$$

It is easily seen that $c_1, c_1{}^x, c_1{}^{x^2}$ and $c_1{}^{x^3}$ are all distinct. Therefore a_1 must be equal to some $c_1{}^{x^i}$, and there are two elements a_2', b' of $C(2)$ such that $x=a_1a_2'b'$ and $a_1b'\in C(2^2)$. Then $a_2b=a_2'b'=b(ba_2b)=(a_2ba_2)a_2$ and b' is not equal to either of b and a_2 because a_1 is commutative with neither of b and a_2. Any element of $C(3)$ can be expressed as a product of two elements of $C(2)$ in exactly three ways. Therefore b' must be equal to ba_2b. Since $(ba_2)^3=1, b'=ba_2b=a_2ba_2$ and hence b' is not commutative with either of a_2 and $a_3=a_2a_3a_2$. Thus b' is an asking element.

Proof of Theorem

(1) The case where n is even and $n\neq 4$.

Since the theorem is trivial in the case $n=2$, we may assume that $n=2m>4$. Let $a_1,\cdots,a_m$ be such elements of $C(2)$ as $a_1a_2\cdots a_m\in C(2^m)$. From Lemma 7, there exists an element b_1 of $C(2)$ such that $a_1a_2b_1\in C(4)$. Then b_1 is different from a_i $(3\leqq i\leqq m)$ and, from Lemma 9, is commutative with them. Now we assume that $b_1, b_2,\cdots, b_k (k<m)$ are elements of $C(2)$ such that $a_i\,a_{i+1}\,b_i\in C(4)$ and b_i is commutative with all b_j and a_l except a_i and a_{i+1}. From Lemma 10, there is an element b_{k+1} of $C(2)$ such that $a_{k+1}\,a_{k+2}\,b_{k+1}$ is contained in $C(4)$, and b_{k+1} is commutative with b_k, Then, from Lemma 9, b_{k+1} is commutative with all b_j and a_l except a_{k+1} and a_{k+2}. Thus it is proved that there are $m-1$ elements $b_1,\cdots,b_{m-1}$ of $C(2)$ such that the orders of $a_i\,b_i$ and $b_i\,a_{i+1}$ are both 3 and b_i is commutative with all b_j and a_l except a_i and a_{i+1}. Set $c_{2k-1}=a_k$ and $c_{2k}=b_k$. Then $c_1,\cdots,c_{n-1}$ satisfy the relatians;

(1) $$c_1^2=\cdots c_{n-1}^2=1,$$

(2) $$(c_i\,c_j)^2=1 \quad \text{if} \quad 1<j-i$$

(3) $$(c_i\,c_{i+1})^3=1.$$

Hence, from Proposition 1, there is a homomorphism from S_n to the subgroup H of G generated by $c_1,\cdots,c_{n-1}$. Since $n>4$, H is isomorphic to S_n/A_n (A_n; the alternative group of degree n) or S_n. Clearly the order of H is greater than 2, therefore H is isomorphic to S_n. Compairing the orders we have $H=G$.
Thus, in this case, G is isomorphic to S_n.

(2) The case $n=4$.

Let a be an element of $C(2)$. As proved in Lemma 1, a^2 is contained in $C(1)$

or $C(2^2)$. If $a^2=1$, then it is proved, as in (1), that there are three elements c_1, c_2, c_3 of $C(2)$ which satisfy the relations

1) $c_1^2=c_2^2=c_3^2=1,$

2) $(c_1 c_3)^2=1,$

3) $(c_1 c_2)^3=(c_2 c_3)^3=1.$

Now, we assume that $x=a^2$ is contained in $C(2^2)$. The normalizer $N(x)$ of x is a subgroup of order 8. Since $N(x)$ contains an elements of $C(2)$ and the order of normalizer of an element of $C(2)$ is 4, $N(x)$ is not abelian.

The normal subgroup N of order 4 which consists of elements of $C(2^2)$ and 1 is contained in $N(x)$ and an abelian group of the type (2, 2). Let $N=(x)\times(y)$. Then aya^{-1} must be equal to xy, and hence $(ay)^2=1$. Then ay is contained in $C(4)$ and the order of an element of $C(4)$ is 2.

Now the multiplication table of conjugate classes of S_4 is left unchanged by exchanging $C(2)$ and $C(4)$. Using $C(4)$ instead of $C(2)$, we can prove in a similar way as above that there are three elements c_1,c_2,c_3 of $C(4)$ which satisfy the above relations 1), 2), 3)

Thus, in either case, there are three elements c_1, c_2, c_3, in G which satisfy the above relations 1), 2), 3).

Let H be the subgroup of G generated by c_1, c_2 and c_3. Then H is homomorphic to S_4, and hence H is isomorphic to one of S_4, S_4/A_4 and S_4/M, where A_4 is the alternative group of degree 4 and M is the normal subgroup of order 4. Since the order of H is greater than 4, H is not isomorphic to S_4/A_4. If H is isomorphic to S_4/M then $c_1 c_3$ must be equal to 1. But $c_1 c_3$ is contained in $C(2^2)$, and hence H must be isomorphic to S_4. Compairing the orders, we can see that G is isomorphic to S_4.

(3) The case where n is odd.

Since the theorem is trivial in the case $n=1$, we may assume that $n=2m+1>1$. Then, as proved above, there are $n-2$ elements $c_1, \cdots, c_{n-2}$ which satisfy the relations 1), 2), 3) in (1), and these generates a subgroup G_1 of G which is isomorphic to S_{n-1}. The index of G_1 is n. Since G contains only one proper normal subgroup and its index is 2, $\bigcap_{x\in G} G_1^x=1$. Therefore G is isomorphic to some subgroup of S_n. Since the order of G is $n!$, G is isomorphic to S_n.