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On complete metric space

By Jun-iti NAGATA

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We characterized the uniform topology of a complete uniform space by the lattice of uniform coverings satisfying some conditions in previous papers.[1] But then we assumed that the uniform space had no isolated point. While the purpose of this paper is to take away this restriction, it is an attempt to establish a unity between the theory of characterization by uniform coverings and that by real valued functions. Now we concern ourselves only with metric spaces.

In §1 we shall characterize a complete metric space by the lattice of "uniform nbds (in the extended meaning)". In §2 we shall give corollaries derived from the result in §1 and especially the theory of characterization by uniform coverings.

1. From now forth we denote by R a complete metric space.

DEFINITION. We mean by *a uniform nbd (in the extended meaning)* a real valued function $f(x)$ of R satisfying

I) $f(x)\geqq\varepsilon$ for some $\varepsilon>0$,

II) $f(x)\leqq\frac{1}{2n}$, $d(x, y)\leqq\frac{1}{2n}$ imply $f(y)\leqq\frac{1}{n}$ for every natural number n.

We consider a directed set $D(R)$ of uniform nbds satisfying

1) there exist $e_n\in D(R)(n=1, 2, \cdots)$ such that

a) $e_n\geqq e_{n+1}$,

b) $\{e_n|n=1, 2, \cdots\}$ is cofinal in $D(R)$,

c) $\lim_{n\to\infty} e_n(x)=0$ $(x\in R)$,

d) for every $\varepsilon>0$, $x\in R$ there exists $f\in D(R)$: $f(x)<\varepsilon$, $f(y)\geqq e_n(y)\left(d(x, y)\geqq\frac{1}{n}\right)$,

2) $f, g\in D(R)$ implies $f\vee g\in D(R)$[2].

DEFINITION. We call a sequence $\{F_n|n=1, 2, \cdots\}$ of subsets $F_n=\{f|f\vee f_n\ngeqq b_n,\ f\in D(R)\}$ of $D(R)$ *a cauchy sequence by* $\{f_n, b_n\}$ when it

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1) On uniform homeomorphism between two uniform spaces, this journal Vol. 3, No. 1-2, 1952. On relations between lattices of finite uniform coverings of a metric space and the uniform topology, this journal Vol. 4, No. 1, 1953.

2) The relation $f\geqq g$ for two elements f, g of $D(R)$ means $f(x)\geqq g(x)$ for every If for $x\in R$ every $f\in D(R)$ there exists e_n such that $e_n\leqq f$, then we call $\{e_n\}$ cofinal in $D(R)$. For example, $e_n=\frac{1}{n}$ $(n=1, 2, \cdots)$ satisfy the conditions a), b), c) of 1). $d(x, y)$ denotes the distance between x and y.

satisfies the following conditions,

i) $f_n, b_n \in D(R)(n=1, 2, \cdots)$,

ii) $\{b_n | n=1, 2, \cdots\}$ is cofinal in $D(R)$, $b_n \geqq b_{n+1}$,

iii) $F_n \neq \phi$,

iv) for every $h \in D(R)$ there exists n_0 such that $f \in F_m$, $g \in F_n$ and $m, n \geqq n_0$ imply $f \vee g \not\geqq h$.

LEMMA 1. *Let $e_n(n=1, 2, \cdots)$ be a sequence of elements of satisfying the condition* 1) *and let f_n be an element satisfying the condition of f for $\varepsilon = e_n(x_0)$ in d), then $F_n = \{f | f \vee f_n \not\geqq e_n\} (n=1, 2, \cdots)$ is a cauchy sequence by $\{f_n, e_n\}$.*

Proof. The conditions i)– iii) are obviously satisfied.

Let h be an arbitrary element of $D(R)$, then $h(x_0) > \frac{1}{p}$ for some natural number p. From c), d) of 1) there exists n_0 such that $n \geqq n_0$ implies $e_n(x_0) < \frac{1}{4p}$, $f_n(y) \geqq e_n(y) (d(x_0, y) \geqq \frac{1}{4p})$. If $f \vee f_m \not\geqq e_m$, $g \vee f_n \not\geqq e_n$ for some $m, n \geqq n_0$, then it must be $f(y) < a_m(y)$, $g(z) < a_n(z)$ for some y, z such that $d(x_0, y) < \frac{1}{4p}$, $d(x_0, z) < \frac{1}{4p}$. Since $a_m(x_0) < \frac{1}{4p}$ and $a_n(x_0) < \frac{1}{4p}$, from II) we get $a_m(y) \leqq \frac{1}{2p}$, $a_n(z) \leqq \frac{1}{2p}$, and hence $f(y) < \frac{1}{2p}$, $g(z) < \frac{1}{2p}$. Therefore $f(x_0) \leqq \frac{1}{p}$, $g(x_0) \leqq \frac{1}{p}$ and $f(x_0) \vee g(x_0) \leqq \frac{1}{p} < h(x_0)$. Thus we get $f \vee g \not\geqq h$.

LEMMA 2. *If $\{F_n | n=1, 2, \cdots\}$ is a cauchy sequence by $\{f_n, b_n\}$, then $A_n = \{x | x \in R, f_m(x) < b_m(x)$ for some $m \geqq n\} (n=1, 2, \cdots)$ is a cauchy filter of R.*

Proof. Let p be an arbitrary natural number, then using iv) for $h = e_p$, we get n_0 such that $f \in F_m$, $g \in F_n$ and $m, n \geqq n_0$ imply $f \vee g \not\geqq e_p$. Now we shall show that $x, y \in A_{n_0}$ implies $d(x, y) < \frac{2}{p}$. To show this, we assume the contrary, *i.e.* $f_m(x) < b_m(x)$, $f_n(y) < b_n(y)$, $m, n \geqq n_0$ and $d(x, y) \geqq \frac{2}{p}$. Then there exist $f, g \in D(R)$ such that $f(x) < b_m(x)$, $f(z) \geqq e_p(z) \left(d(x, z) \geqq \frac{1}{p}\right)$; $g(y) < b_n(y)$, $g(z) \geqq e_p(z) \left(d(y, z) \geqq \frac{1}{p}\right)$ from d) of 1). Since $d(x, y) \geqq \frac{2}{p}$, there hold $f \vee g \geqq e_p$, $f \vee f_m \not\geqq b_m$ and $g \vee f_n \not\geqq b_m$ simultaneously, but this is impossible. Hence $d(x, y) < \frac{2}{p}$, and hence $\{A_n\}$ is a cauchy filter.

DEFINITION. We denote by $\{F_n\} \sim \{G_n\}$ the relation between two cauchy sequences $\{F_n\}$ and $\{G_n\}$ by $\{f_n, b_n,\}$ and $\{g_n, c_n\}$ respectively such that

for every $h \in D(R)$ there exists n_0 such that $n \geqq n_0$, $f \in F_n$ and $g \in G_n$ imply $f \vee g \not\geqq h$.

LEMMA 3. *In order that $\{F_n\} \sim \{G_n\}$ it is necessary and sufficient that cauchy filters $A_n = \{x | f_m(x) < b_m(x)$ for some $m \geqq n\}$ $(n=1, 2, \cdots)$ and $B_n = \{x | g_m(x) < c_m(x)$*

for some $m \geqq n\}(n=1, 2, \cdots)$ *converge to a point* x_0.

Proof. If $\{A_n\}$ and $\{B_n\}$ converge to a point $x_0 \in R$, then for an arbitrary element h of $D(R)$ we can take a natural number p such that $h(x_0) > \frac{1}{p}$. Since $\{b_n\}$, $\{c_n\}$ are cofinal in $D(R)$, there exists n_0 such that $y \in A_{n_0}$ and $z \in B_{n_0}$ imply $d(x_0, y) < \frac{1}{4p}$ and $d(x_0, z) < \frac{1}{4p}$ respectively, and $b_n(x_0) < \frac{1}{4p}$, $c_n(x_0) < \frac{1}{4p}$ $(n \geqq n_0)$. Hence $f \in F_n$, $g \in G_n$ and $n \geqq n_0$ imply $f(y) < b_n(y)$, $g(z) < c_n(z)$ for some y, z such that $d(x_0, y) < \frac{1}{4p}$, $d(x_0, z) < \frac{1}{4p}$, and hence $b_n(y) \leqq \frac{1}{2p}$, $c_n(z) \leqq \frac{1}{2p}$, *i.e.* $f(y) < \frac{1}{2p}$, $g(z) < \frac{1}{2p}$. Therefore we get $f(x_0) \leqq \frac{1}{p}$, $g(x_0) \leqq \frac{1}{p}$ and $f \vee g \ngeqq h$.

Conversely, if $\{A_n\}$ and $\{B_n\}$ converge to distinct points x and y respectively, then there exists some natural number p such that $d(x, y) > \frac{2}{p}$. For every n_0 there exists $n \geqq n_0$ such that $x' \in A_n$ and $y' \in B_n$ imply $d(x', y') \geqq \frac{2}{p}$. Hence there exist x', y' such that $f_n(x') < b_n(x')$, $g_n(y') < c_n(y')$; $d(x', y') \geqq \frac{2}{p}$. Now we get $f, g \in D(R)$ satisfying $f \in F_n$, $g \in G_n$ and $f \vee g \geqq e_p$ simultaneously as in the proof of Lemma 2. Namely, there holds the negation of $\{F_n\} \sim \{G_n\}$.

From Lemma 3 we can classify all the cauchy sequences of $D(R)$ by the relation $\sim$. We denote by $\mathfrak{D}(R)$ the set of all such classes. From this lemma and the completeness of R we get a one-to-one correspondence between R and $\mathfrak{D}(R)$; hence we denote by $\mathfrak{D}(A)$ the image of a subset A of R in $\mathfrak{D}(R)$ by this correspondence.

DEFINITION. We call $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$ *u-disjoint sets of* $\mathfrak{D}(R)$ when for some $h \in D(R)$ and every $\{F_n\} \in \mathfrak{D}(x) \in \mathfrak{D}(A)$, $\{G_n\} \in \mathfrak{D}(y) \in \mathfrak{D}(B)$ there exist $f \in F_n$, $g \in G_n$ satisfying $f \vee g \geqq h$ for an infinite number of n.

LEMMA 4. $\mathfrak{D}(A)$ *and* $\mathfrak{D}(B)$ *are u-disjoint if and only if* A *and* B *are u-disjoint sets of* R.[3)].

Proof. If A and B are u-disjoint, then $d(A, B) > \frac{2}{p}$ for some natural number p. For every $\{F_n\} \in \mathfrak{D}(x) \in \mathfrak{D}(A)$, $\{G_n\} \in \mathfrak{D}(y) \in \mathfrak{D}(B)$ and n_0 there exists $n \geqq n_0$ such that $x \in A_n$ and $y \in B_n$[4)] imply $d(x, y) \geqq \frac{2}{p}$, for $\{A_n\} \to x \in A$, $\{B_n\} \to y \in B$ and $d(x, y) > \frac{1}{p}$. Since we get $f, g \in D(R)$ satisfying $f \in F_n$, $g \in G_n$ and $f \vee g \geqq e_p$ as in the proof of Lemma 2, $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$ are u-disjoint according to the definition.

If A and B are not u-disjoint, then for an arbitrary element h of $D(R)$ we can take $x \in A$, $y \in B$: $d(x, y) < \frac{1}{4p}$ for a natural number p such that $\frac{1}{p} < \varepsilon \leqq h(x) (x \in R)$. Let $\{F_n\} \in \mathfrak{D}(x)$ and $\{G_n\} \in \mathfrak{D}(y)$, then $\{A_n\} \to x$ and $\{B_n\} \to y$; hence for some n_0,

3) We say A and B are u-disjoint when $d(A\ B) = \inf\{d(x, y) \mid x \in A, y \in B\} > 0$

4) In this proof we denote by $\{A_n\}$, $\{B_n\}$ the same caucy filters as in Lemma 3.

$f_n(z)\geqq b_n(z)\left(d(x, z)\geqq\frac{1}{4p}\right)$, $g_n(z)\geqq c_n(z)\left(d(y, z)\geqq\frac{1}{4p}\right)$; $b_n(x)<\frac{1}{4p}$, $c_n(y)<\frac{1}{4p}(n\geqq n_0)$. Therefore $f\in F_n$, $g\in G_n$ imply $f(z)<b_n(z)\leqq\frac{1}{2p}$, $g(z')<c_n(z')\leqq\frac{1}{2p}$ for some z, z' such that $d(x, z)\leqq\frac{1}{4p}$, $d(y, z')\leqq\frac{1}{4p}$. Since $d(x, z')<\frac{1}{2p}$, there holds $g(x)\leqq\frac{1}{p}$ from II), and this combining with $f(x)\leqq\frac{1}{p}$ leads to $f\vee g\ngeqq h$. Namely $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$ are not u-disjoint.

Since we showed previously that the uniform topology of a metric space is defined by u-disjointness[5], from this lemma we can define in $\mathfrak{D}(R)$ the uniform topology uniformly homeomorphic with that of R. Hence we get the following

THEOREM 1. *In order that two complete metric spaces R_1 and R_2 are uniformly homeomorphic it is necessary and sufficient that $D(R_1)$ and $D(R_2)$ are isomorphic, where $D(R_1)$ and $D(R_2)$ are directed sets of uniform nbds satisfying* 1), 2).

2. COROLLARY 1. *The uniform topology of a metric space R is characterized by the lattice $L_a(R)$ of all uniform nbds (in the extended meaning), i.e. of all real valued functions satisfying I), II).*

Proof. Let $e_n=\frac{1}{n}$ and define $f(x)=\frac{\varepsilon}{2}+d(x_0, x)$ for each $x_0\in R$ and $\varepsilon>0$, then conditions 1), 2) are clearly satisfied. Since $f(x)=\frac{\varepsilon}{2}+d(x_0, x)\leqq\frac{1}{2n}$ and $d(x, y)\leqq\frac{1}{2n}$ imply $f(y)=\frac{\varepsilon}{2}+d(x_0, y)\leqq\frac{\varepsilon}{2}+d(x_0, x)+d(x, y)\leqq\frac{1}{n}$, f satisfies II)

COROLLARY 2. *The uniform topology of a metric space R is characterized by the lattice $L_d(R)$ of all real valued functions satisfying I) and $|f(x)-f(y)|\leqq d(x, y)$ $(x, y\in R)$.*

Proof. It is obvious.

COROLLARY 3. *The uniform topology of a metric space R is characterized by the lattice $L'(R)$ of all mappings of R into $N=\left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$ satisfying I), II).*

Proof. Since $e_n=\frac{1}{n}\in L'(R)$, if we define $f(x)$ such that $f(x)=\frac{1}{n}\left(\frac{1}{n}\leqq\frac{\varepsilon}{2}+d(x_0, x)<\frac{1}{n-1}\right)$ for each $x_0\in R$ and $\varepsilon>0$, we can see easily that all conditions are satisfied. We show only that f satisfies II). If $f(x)\leqq\frac{1}{2n}$, $d(x, y)\leqq\frac{1}{2n}$, then $\frac{\varepsilon}{2}+d(x_0, x)<\frac{1}{2n-1}$, and hence $\frac{\varepsilon}{2}+d(x_0, y)<\frac{1}{2n-1}+\frac{1}{2n}<\frac{1}{n-1}$. Therefore $f(y)\leqq\frac{\varepsilon}{2}+d(x_0 y)<\frac{1}{n-1}$, and namely $f(y)\leqq n$.

Next, we investigate relations between uniform nbds (in the extended meaning) and uniform coverings. We consider a uniform covering $\mathfrak{U}$ consisting of spheres

5) See " On relations … ".

$S_{n(x)}(x)=\left\{y \mid d(x, y)<\frac{1}{n(x)} \in N\right\}$ $(x \in R)$. For $\mathfrak{U}$ we define a function $f(\mathfrak{U}, x)$ such that $f(\mathfrak{U}, x)=\operatorname{Max}\left\{\frac{1}{n} \mid S_n(x) \subseteq S \text{ for some } S \in \mathfrak{U}\right\}$. Then $f(\mathfrak{U}, x)$ satisfies clearly I).

LEMMA 1. *$f(\mathfrak{U}, x)$ satisfies II) for every $\mathfrak{U}$.*

Proof. If we assume $f(y)>\frac{1}{n}$, $d(x, y) \leqq \frac{1}{2n}$, then $S_{n-1}(y) \subseteq S$ for some $S \in \mathfrak{U}$. Since $d(x, z)<\frac{1}{2n-2}$ implies $d(y, z)<\frac{1}{2n}+\frac{1}{2n-2}<\frac{1}{n-1}$, $S_{2n-2}(x) \subseteqq S$. Namely, we get $f(\mathfrak{U}, x) \geqq \frac{1}{2n-2}>\frac{1}{2n}$ and condition II).

Hence $f(\mathfrak{U}, x) \in L'(R)$ for every $\mathfrak{U}$.

LEMMA 2. *$f(\mathfrak{U}, x) \geqq f(\mathfrak{V}, x)(x \in R)$, if and only if $\mathfrak{U}>\mathfrak{V}$.*[6]

Proof. If $\mathfrak{U}>\mathfrak{V}$, then $S_n(x) \subseteq S \in \mathfrak{V}$ implies $S_n(x) \subseteq S' \in \mathfrak{U}$, and hence $f(\mathfrak{V}, x) \leqq f(\mathfrak{U}, x)$. If $\mathfrak{U} \not> \mathfrak{V}$, then there exists $S_n(x) \in \mathfrak{V}$ such that $S_n(x) \not\subseteq S$ for every $S \in \mathfrak{U}$, and hence $f(\mathfrak{V}, x) \geqq \frac{1}{n}$, $f(\mathfrak{U}, x)<\frac{1}{n}$, *i.e.* $f(\mathfrak{V}, x) \not\leqq f(\mathfrak{U}, x)$.

LEMMA 3. *$f(\mathfrak{U} \vee \mathfrak{V}, x)=f(\mathfrak{U}, x) \vee f(\mathfrak{V}, x)$.*[7]

Proof. Let $f(\mathfrak{U} \vee \mathfrak{V}, x)=\frac{1}{n}$, then $S_n(x) \subseteq S \in \mathfrak{U} \vee \mathfrak{V}$. Since $S \in \mathfrak{U}$ and $S \in \mathfrak{V}$ imply $\frac{1}{n} \leqq f(\mathfrak{U}, x)$ and $\frac{1}{n} \leqq f(\mathfrak{V}, x)$ respectively, we obtain $\frac{1}{n} \leqq f(\mathfrak{V}, x) \vee f(\mathfrak{U}, x)$. On the other hand $f(\mathfrak{V}, x) \vee f(\mathfrak{U}, x) \leqq f(\mathfrak{U} \vee \mathfrak{V}, x)$ is an immediate consequence of Lemma 2, and hence this lemma is proved.

LEMMA 4. *$f(\mathfrak{U}_\wedge \mathfrak{V}, x)=f(\mathfrak{U}, x)_\wedge f(\mathfrak{V}, x)$, where $\mathfrak{U}_\wedge \mathfrak{V}=\{S_n(x) \mid S_n(x) \subseteq S, S'$ for some $S \in \mathfrak{U}$ and $S' \in \mathfrak{V}\}$.*

Proof. $f(\mathfrak{U}_\wedge \mathfrak{V}, x) \leqq f(\mathfrak{U}, x)_\wedge f(\mathfrak{V}, x)$ is an immediate consequence of Lemma 1. Conversely, let $\frac{1}{n}=\operatorname{Min}\{f(\mathfrak{U}, x), f(\mathfrak{V}, x)\}$, then $S_n(x) \subseteq S_\frown S'$ for some $S \in \mathfrak{U}$ and $S' \in \mathfrak{V}$ Hence according to the definition of $\mathfrak{U}_\wedge \mathfrak{V}$, we obtain $\frac{1}{n} \leqq f(\mathfrak{U}_\wedge \mathfrak{V}, x)$.

Combining Lemma 1–Lemma 4, we get

THEOREM 2. *The totality $L_u(R)$ of uniform coverings consisting of spheres is isomorphic to a sublattice of $L'(R)$.*

We denote by $L(R)$ a subset of $L_u(R)$ satisfying the following conditions,

1)′ $L(R)$ is cofinal in $L_u(R)$,

2)′ if $\mathfrak{U}, \mathfrak{V} \in L(R)$, then $\mathfrak{U} \vee \mathfrak{V} \in L(R)$,

3)′ for every $\mathfrak{U} \in L(R)$ and an open set S, there exist $\mathfrak{V} \in L(R)$ such that $S_n(x) \in \mathfrak{V}$ implies $S_n(x) \not\supseteqq S$, and $S_n(x) \in \mathfrak{U}$ and $S_n(x)_\frown S=\phi$ imply $S_n(x) \in \mathfrak{V}$.

Then we obtain

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6) We denote by $\mathfrak{V}<\mathfrak{U}$ the relation that for every $S \in \mathfrak{V}$ there exists some $S' \in \mathfrak{U}$: $S \subseteq S'$.

7) $\mathfrak{U} \vee \mathfrak{V}=\{S \mid S \in \mathfrak{U}$ or $S \in \mathfrak{V}\}$.

LEMMA 5. *$\{f(\mathfrak{U},x)\,|\,\mathfrak{U}\in L(R)\}$ satisfies 1), 2) for every metric space R without isolated point.*

Proof. 2) is immediately deduced from 2)′ and Lemma 3. If we take $\mathfrak{U}_m\in L(R)$ such that $\mathfrak{U}_n<\{S_{3n}(x)\,|\,x\in R\}$, $\mathfrak{U}_{n+1}<\mathfrak{U}_n$, then $e_n=f(\mathfrak{U}_n, x)$ $(n=1, 2, \cdots)$ satisfy clearly a), b) of 1). Next, since an arbitrary point x_0 of R is no isolated point, for every n there exist $x\in S_n(x_0)-x_0$ and m such that $x\notin S(x_0, \mathfrak{U}_m)$.[8] Since $S_n(x_0)\subsetneqq S$ for every $S\in\mathfrak{U}_m$, $e_m(x_0)=f(\mathfrak{U}_m, x_0)<\frac{1}{n}$. This implies $\lim_{n\to\infty} e_n(x_0)=0$.

Lastly, to see the validity of d), for e_n and $\varepsilon'>0$ we denote by $\mathfrak{V}$ an element of $L(R)$ satisfying the condition of $\mathfrak{V}$ in 3)′ for $\mathfrak{U}_n$ and $S_\varepsilon(x_0)=\left\{y\,|\,d(x_0, y)<\varepsilon=\mathrm{Min}\left(\varepsilon', \frac{1}{3n}\right)\right\}$. Then we can easily show that $f(\mathfrak{V}, x)$ satisfies the condition of f in d). $f(\mathfrak{V}, x_0)<\varepsilon$ is obvious from the property of $\mathfrak{V}$ and $S_\varepsilon(x_0)$. If $d(x_0, x)\geqq\frac{1}{n}$ and $f(\mathfrak{U}, x)=\frac{1}{m}$, then $S_m(x)\subseteq S_p(y)$ for certain $S_p(y)\in\mathfrak{U}$. To show $S_\varepsilon(x_0)\frown S_p(y)=\phi$, we assume the contrary. Since $\mathfrak{U}_n<\{S_{3n}(x)\}$, the assumption that $S_\varepsilon(x_0)\frown S_p(y)\neq\phi$ leads to the existence of $y\in R$ such that $d(x_0, y)<\varepsilon, d(y, x)<\frac{2}{3n}$ and to $d(x_0, x)<\frac{1}{n}$, but this is a contradiction. Hence it must be $S_\varepsilon(x_0)\frown S_p(y)=\phi$, and hence $S_m(x)\subseteq S_p(y)\in\mathfrak{V}$ from the property of $\mathfrak{V}$, which implies $\frac{1}{m}\leqq f(\mathfrak{V},x)$. Thus d) of 1) is valid for $L(R)$.

From Theorem 1, Theorem 2 and this lemma we get the following proposition previously obtained by the author,[9]

THEOREM 3. *In order that two complete metric spaces R_1 and R_2 without isolated point are uniformly homeomorphic it is necessary and sufficient that $L(R_1)$ and $L(R_2)$ are isomorphic, where $L(R_1)$ and $L(R_2)$ are lattices of uniform coverings satisfying* 1)′, 2)′, 3)′.

From now forth we denote by R a metric space and by R^* the completion of R. Let f be a uniform nbd of R, *i.e.* a real valued function satisfying I), II), then defining f^*: $f^*(x)=f(x)$ $(x\in R)$, $f^*(z)=\lim_{n\to\infty}\sup\{f(x)-d(x, z)\,|\,d(x, z)<\frac{1}{n}, x\in R\}$, we see easily that f^* satisfies I) and II)′ $f^*(x)\leqq\frac{1}{4n}$, $d(x, y)\leqq\frac{1}{4n}$ imply $f^*(y)\leqq\frac{1}{n}$ $(x, y\in R^*)$.

Furthermore we obtain easily the following lemmas.

LEMMA 6. *$f^*\geqq g^*$, if and only if $f\geqq g$.*

LEMMA 7. $f^*\vee g^*=(f\vee g)^*$.

LEMMA 8. *If $\{e_n(x)\}$ is cofinal in $D(R)$, then $\{e_n{}^*(x)\}$ is cofinal in $D^*(R)=\{f^*\,|\,f\in D(R)\}$.*

LEMMA 9. *If $\lim_{n\to\infty} e_n(x_0)=0$ $(x_0\in R)$, then $\lim_{n\to\infty} e_n{}^*(x_0)=0$ $(x_0\in R^*)$.*

8) $S(x_0, \mathfrak{U}_m)=\smile\{S\,|\,x_0\in S\in\mathfrak{U}_m\}$

9) See "On uniform homeomorphism…". In this paper we proved the theorem generally in a complete uniform space without isolated point.

LEMMA 10. *If* $\{e_n\}$ *satisfies* d) *of* 1), *then for every* $\varepsilon > 0$ *and* $x \in R^*$ *there exists* $f^* \in D^*(R)$ *such that* $f^*(x) < \varepsilon$, $f^*(y) \geqq e^*_n(y)$ $\left(d(x, y) \geqq \frac{1}{n}\right)$.

We omit the proofs of these lemmas.

Therefore, if $D(R)$ is a directed set of uniform nbds satisfying 1), 2), tnen $D^*(R) = \{f^* \mid f \in D(R)\}$ is a directed set satisfying 1), 2) for R^*, which elements satisfy I), II)'. Let R_1 and R_2 be metric spaces, then an isomorphism between $D(R_1)$ and $D(R_2)$ implies an isomorphism between $D^*(R_1)$ and $D^*(R_2)$ from Lemma 6, and hence we obtain the following

THEOREM 4. *If* R_1 *and* R_2 *are metric spaces and if* $D(R_1)$ *and* $D(R_2)$ *are isomorphic, then* R_1^* *and* R_2^* *are uniformly homeomorphic, where* $D(R_1)$ *and* $D(R_2)$ *are directed sets of uniform nbds of* R *satisfying* I), II).

COROLLARY 4. $L_a(R)$, $L_d(R)$ *and* $L'(R)$ *of a metric space* R *characterize the uniform topology of the completion* R^* *respectively.*