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On the cell structure of the octanion projective plane Π

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1. Introduction.

As the projective plane, we have four main examples, namely, the real, complex, quaternion and octanion projective planes, which are denoted by P, M, Q and Π respectively. Since the real, complex, and quaternion number fields are associative, the projective planes over these fields can be treated by the quite similar methods and have the similar properties. However, since the octanion number field (*i.e.* Cayley number field) is non-associative, we can not sometimes treat it as well as the formers. H. Freudenthal [2], [3] and the others [1] firstly have constructed the octanion projective plane by the matrix method. This construction is also applicable in the associative cases. Our first purpose of this note is to give the connection between the usual construction and the above construction of P, M and Q (see §6).

The topological properties of the spaces P, M and Q are well known as the space with the simplest structure, for example, they are CW-complex[1)] in the sense of J. H. C. Whitehead [6], [7]. The second purpose of this note is to show that Π *is also a CW-complex in which the 16-dimensional cell* e^{16} *is attached to the 8-dimensional sphere* S^8 *by the Hopf map* $\eta: S^{15}\to S^8$ (see §4).

2. Definition of the octanion projective plane Π.

In §§2–5, the latin letters $x, y, z, s, t, \cdots\cdots$ will denote the octanion numbers and the Greek letters $\xi, \eta, \zeta, \sigma, \tau, \cdots\cdots$ the real numbers.

Let $\mathfrak{J}$ be the set of all hermitian matrices of three order

$$X=\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix}^{2)}$$

with coefficients in the octanion number field. We define the multiplication in $\mathfrak{J}$ by

$$X\circ Y=\frac{1}{2}(XY+YX),$$

where XY is the usual matrix multiplication of X and Y. Then $\mathfrak{J}$ becomes a 27-dimensional distributive, commutative and non-associative algebra over the real

1) For the definition of CW-complex, see [8].
2) The symbols $\bar{x}$ and $|x|$ indicate the conjugate number and the absolute value of x respectively.

number field. And we define the inner product in $\mathfrak{J}$ by

$$(X,\ Y)=tr\ (X\circ Y)^{3)}$$

If X is an element of $\mathfrak{J}$, then each of the following five conditions is equivalent to each other[4]:

1) X is an irreducible idempotent, *i. e.* if $X=X^2\neq 0^{5)}$ and $X=X_1+X_2$, where $X\in\mathfrak{J}$, $X_i^2=X_i$ $(i=1, 2)$, and $X_1\circ X_2=0$, then $X_1=0$ or $X_2=0$.

2) $tr(X)=tr(X^2)=tr(X^3)=1^{5)}$.

3) $X=X^2$ and $tr(X)=1$.

4) $X=\alpha(E_1)$, where $E_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ and $\alpha\in F_4$:[6]

α is an automorphism of $\mathfrak{J}$, *i. e.* a non-singular linear transformation of $\mathfrak{J}$ which satisfies

$$\alpha(X\circ Y)=\alpha X\circ\alpha Y.$$

5) $\xi_1+\xi_2+\xi_3=1$,

$$\xi_2\xi_3=|x_1|^2,\qquad \xi_3\xi_1=|x_2|^2,\qquad \xi_1\xi_2=|x_3|^2,$$
$$\xi_3\bar{x}_3=x_1x_2,\qquad \xi_2\bar{x}_2=x_3x_1,\qquad \xi_1\bar{x}_1=x_2x_3.$$

Now, let Π be the set of all elements X satisfying one of the above conditions 1)–5). Π is called the octanion projective plane. Each element of Π defines a point and also a line of Π and we call that a point X and a line Y are incident if $(X,\ Y)=0$. Then, H. Freudenthal [2], [3] show that Π becomes a projective plane (in which the Desargues' axiom is not satisfied).

Using the condition 4) and Y. Matsushima's result [5] which states that the subgroup of F_4 consisting of all automorphisms α such that $\alpha E_1=E_1$ is isomorphic to the universal covering group spin (9) of the proper orthogonal group SO (9), then we have $\Pi=F_4/\mathrm{spin}(9)$. Thus Π is a compact homogeneous manifold.

3. Hopf map $\eta: S^{15}\to S^8$.

We take a 15-dimensional sphere S^{15} as the space of all pairs of octanion numbers $(x,\ y)$ with $|x|^2+|y|^2=1$, and an 8-dimensional sphere S^8 as the space of all pairs $(s,\ \sigma)$ of octanion number s and real number σ with $|s|^2=\sigma(1-\sigma)$. We as-

3) $t_r(X)=\xi_1+\xi_2+\xi_3$. $(X,\ Y)=(Y,\ X)=\xi_1\eta_1+\xi_2\eta_2+\xi_3\eta_3+2((x_1, y_1)+(x_2, y_2)+(x_3, y_3))$ where $(x,\ y)$ indicates the inner product of x and y.

4) For a proof of this proposition, see [2] and [3].

5) $X^2=X\circ X=X\ X$. $X^3=X^2\circ X=X\circ X^2$ (if $X\in\mathfrak{J}$).

6) The group consisting of all automorphisms of $\mathfrak{J}$ is denoted by F_4. This group is a compact, connected, simply connected and the exceptional F_4-type simple Lie group in the Cartan's classification of the simple Lie groups, and is called the elliptic projective group of the octanion projective plane Π.

sociate the point (x, y) of S^{15} with the point $(x\bar{y}, |y|^2)$ of S^8. Thus we obtain a mapping $\eta: S^{15}\to S^8$ called Hopf map, and S^{15} becomes a fibre space with base space S^8, fibre S^7 and projection η.

4. Cellular decomposition of Π.

We take $\bar{\mathcal{E}}^{16}$ as the space of pairs of octanion numbers (x, y) with $|x|^2+|y|^2=1$ $\mathcal{E}^{16}$ with $|x|^2+|y|^2<1$ and $\dot{\mathcal{E}}^{16}=S^{15}$ with $|x|^2+|y|^2=1$. A representation of Π as a CW-complex composed of three cells e^0, e^8, e^{16} is obtained as follows: The vertex e^0 is defined by the point $E_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$. The line E_1 is the set of all points X such that $(X, E_1)=0$, namely,

$$X=\begin{pmatrix}0&0&0\\0&\xi_2&x_1\\0&\bar{x}_1&\xi_3\end{pmatrix}$$

with $|x_1|^2=\xi_2\xi_3$ and $\xi_2+\xi_3=1$, so that the line E_1 defines an 8-dimensional sphere S^8 and e^8 is defined as the cell S^8-e^0.

Now, we define a continuous map $f: \bar{\mathcal{E}}^{16}\to\Pi$ by

$$f(x, y)=\begin{pmatrix}1-|x|^2-|y|^2 & \bar{x}\sqrt{1-|x|^2-|y|^2} & \bar{y}\sqrt{1-|x|^2-|y|^2}\\ x\sqrt{1-|x|^2-|y|^2} & |x|^2 & x\bar{y}\\ y\sqrt{1-|x|^2-|y|^2} & \bar{y}x & |y|^2\end{pmatrix}.$$

Obviously $f'=f\,|\,S^{15}$ map S^{15} onto the line E_1, in fact,

$$f'(x, y)=\begin{pmatrix}0&0&0\\0&|x|^2&x\bar{y}\\0&\bar{y}x&|y|^2\end{pmatrix}.$$

This map may be regaded as the Hopf map $\eta: S^{15}\to S^8$.

Next, in order to define the 16-dimensional cell e^{16}, we shall show that $f\,|\,\mathcal{E}^{16}$ maps $\mathcal{E}^{16}$ homeomorphically onto $\Pi-S^8$. To this purpose, we take a 16-dimensional cell $'\mathcal{E}^{16}$ as the set of all pairs (s, t) of any octanion numbers. We define two mappings $h: \mathcal{E}^{16}\to{}'\mathcal{E}^{16}$ and $h': {}'\mathcal{E}^{16}\to\mathcal{E}^{16}$ by

$$h(x, y)=\left(\frac{x}{\sqrt{1-|x|^2-|y|^2}},\ \frac{y}{\sqrt{1-|x|^2-|y|^2}}\right)$$

and

$$h'(s, t)=\left(\frac{s}{\sqrt{1+|s|^2+|t|^2}},\ \frac{t}{\sqrt{1+|s|^2+|t|^2}}\right)$$

respectively, and define two mappings $g: {}'\mathcal{E}^{16}\to\Pi-S^8$ and $g': \Pi-S^8\to{}'\mathcal{E}^{16}$ by

$$g(s, t) = \frac{1}{1+|s|^2+|t|^2}\begin{pmatrix} 1 & \bar{s} & \bar{t} \\ s & |s|^2 & s\bar{t} \\ t & t\bar{s} & |t|^2 \end{pmatrix}^{7)}$$

and

$$g'\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \left(\frac{\bar{x}_3}{\xi_1}, \frac{x_2}{\xi_1}\right)$$

respectively. The fact that X is not lying on the line E_1 shows $\xi_1 \neq 0$, so g' is well defined. Now, we can easily verify that $h \circ h' = 1$, $h' \circ h = 1$, $g \circ g' = 1$ and $g' \circ g = 1$. The continuities of h, h', g and g' are obvious from their definitions. Therefore h and g are homeomorphisms. Since $f \mid \mathcal{E}^{16} = g \circ h$, $f \mid \mathcal{E}^{16}$ is also a homeomorphism. So, we may define e^{16} by the cell $\Pi - S^8$.

Thus we have the following proposition:

The octanion projective plane Π is a CW-complex in which the 16-dimensional cell e^{16} is attached to the 8-dimensional sphere S^8 by the Hopf map $\eta : S^{15} \rightarrow S^8$.

5. Homology, cohomology and some homotopy groups of Π.

With the above cell structure of Π, the homology and cohomology groups are easily computed. The homology groups with the integral coefficient are

$$H_0(\Pi) = H_8(\Pi) = H_{16}(\Pi) = Z,^{8)}$$
$$H_i(\Pi) = 0 \quad \text{otherwise.}$$

The cohomology groups with the integral coefficient are also

$$H^0(\Pi) = H^8(\Pi) = H^{16}(\Pi) = Z,$$
$$H^i(\Pi) = 0 \quad \text{otherwise.}$$

Since Π is a connected compact manifold, if we wish to compute the cup product $e^8 \vee e^8$,[9] we can use so called Poincaré duality theorem. In the projective plane, any two distincct projective lines intersect exactly just one point, so that we have $e^8 \vee e^8 = e^{16}$ [9] if e^{16} is suitably oriented.

Some homotopy groups of Π are

$$\pi_i(\Pi) = \pi_{i+1}(S^7) \qquad \text{for} \quad 1 \leq i \leq 15.$$

In fact, with the above cellular decomposition, it is obvious that the injection homomorphisms $i^* : \pi_i(S^8) \rightarrow \pi_i(\Pi)$ are onto-isomorphisms for $0 \leq i \leq 14$ and onto-

7) $\lambda(a_{ij}) = (\lambda a_{ij})$
8) Z is the infinite cylic group.
9) We assume that e^8 and e^{16} represent also the cohomological class containing the cell e^8 and e^{16} respectively.

homomorphism for $i=15$. By the suspension isomorphism $\pi_{i-1}(S^7)\to\pi_i(S^8)$, the results for $1\leq i\leq 14$ are obtained. For $i=15$, since $\pi_{15}(S^8)=\pi_{15}(S^{15})+\pi_{14}(S^7)$ and the first term is the kernel of the injection $i^*: \pi_{15}(S^8)\to\pi_{15}(\Pi)$, we have $\pi_{15}(\Pi)=\pi_{14}(S^7)$.

Since $\pi_i(\Pi)=0$ for $0\leq i\leq 7$, by making use of the homotopy exact sequence of $F_4/\mathrm{spin}(9)=\Pi$, some homotopy groups of F_4 are obtained as follows:

$$\pi_i(F_4)=0 \quad \text{for} \quad 0\leq i\leq 1,$$
$$\pi_i(F_4)=\pi_i(\mathrm{SO}(9)) \quad \text{for} \quad 2\leq i\leq 6,$$

$\pi_7(\mathrm{SO}(9))=\pi_7(\mathrm{spin}(9))\to\pi_7(F_4)$ is onto by the injection.

6. Comparison between Π and the associative cases.

Since P, M and Q are treated by the quite similar methods, we shall treat only the complex projective plane M in the following.

The usual definition of M is the following: In the set of all triples of complex numbers $[x_1, x_2, x_3]$, not all zero, two such triples $[x_1, x_2, x_3]$ and $[y_1, y_2, y_3]$ are said to be equivalent if there exists a complex number a such that $y_i=x_i a$ $(i=1, 2, 3)$. The set of equivalent classes forms a complex projective plane M. Any element $[x_1, x_2, x_3]$ of M defines a point and also a line of M, and a point $[x_1, x_2, x_3]$ and a line $[y_1, y_2, y_3]$ are called incident if $x_1\bar{y}_1+x_2\bar{y}_2+x_3\bar{y}_3=0$. As is known, M becomes a projective plane by this incidence, and is a connected compact manifold.

In the notation of §§2–5, let the latin letters x, y, z, s, ... denote the complex numbers. Then $\mathfrak{J}(M)$ is a 9-dimensional commutative, distributive and associative algebra consisting of all complex hermitian matrices of three order. Let $F(M)$ be the group consisting of its automorphisms. The Hopf map mentioned in §3 becomes the Hopf map $\eta(M): S^3\to S^2$ defined by $\eta(M)\,(x, y)=(x\bar{y}, |y|^2)$, and this is equivalent with the usual definition of the Hopf map $\eta': S^3\to S^2$ such that $\eta'(x, y)=[x, y, 0]$. The definition and the consideration mentioned in §§2–4 is applicable if we replace $\mathfrak{J}$, F_4 and η by $\mathfrak{J}(M)$, $F(M)$ and $\eta(M)$ respectively. Thus we obtain an another construction of the complex projective plane M_1.

The connection between the classical construction and the new one is given as follows: We define a continuous mapping ϕ; $M\to M_1$ by

$$\phi([a, b, c])=\frac{1}{|a|^2+|b|^2+|c|^2}\begin{pmatrix}|a|^2 & a\bar{b} & a\bar{c}\\ b\bar{a} & |b|^2 & b\bar{c}\\ c\bar{a} & c\bar{b} & |c|^2\end{pmatrix}.$$

ϕ does not depend upon the choise of the representatives of $[a, b, c]$. Conversely, since any point X of M_1 contains at least one non zero column vector, we define a continuous mapping ψ: $M_1\to M$ assigning X to the point of M which contains the above non zero vector as its representative. ψ does not depend upon the choice of non zero column vectors of X. Since $\phi\circ\psi=1$ and $\psi\circ\phi=1$, ϕ and ψ are homeomorphisms. Furthermore, the incidencies between the corresponding points and the corresponding lines are equivalent to each other. Therefore M and M_1 are quite coincided as a projective plane by ϕ and ψ.

The automorphisms of $\mathfrak{J}(M)$ are never new operations, *i. e.* they are closely connected with the linear unitary transformations of the 3-dimensional complex linear space. These connection will be clear by the following lemma:

If $\alpha \in F(M)$, i. e. α is a non-singular linear transformation of $\mathfrak{J}(M)$ such that $\alpha(X \circ Y) = \alpha X \circ \alpha Y$ for any $X, Y \in \mathfrak{J}(M)$, then there exists an unitary matrix U of three order such that $\alpha X = UXU^*$[10] or $\alpha X = U\bar{X}U^*$ for all $X \in \mathfrak{J}(M)$, and conversely.

Proof. The relations $E_i \circ E_j = 0 (i \neq j)$, $E_i \circ E_i = E_i$ imply that $\alpha E_i \in M_1 (i = 1, 2, 3)$ and αE_i are mutually incident. Therefore $\psi(\alpha E_1) = [a, b, c]$, $\psi(\alpha E_2) = [p, q, r]$ and $\psi(\alpha E_3) = [u, v, w]$ are unitary orthogonal with respect to the usual unitary inner product defined in the 3-dimensional complex linear space. We may assume that they are normalized.[11] If we take a matrix U such that

$$U = \begin{pmatrix} a & p & u \\ b & q & v \\ c & r & w \end{pmatrix},$$

then U is an unitary matrix and $\alpha E_i = UE_iU^*$ $(i = 1, 2, 3)$. Therefore we may assume that α invariant E_i $(i = 1, 2, 3)$.

Let F_i^x be the point X of M_1 with $x_i = x$ and all numbers except x_i are zero. Then $E_i \circ F_i^x = 0$, $2E_j \circ F_i^x = F_i^x$ if $i \neq j$. Whence we have $E_i \circ \alpha F_i^x = 0$ and $2E_j \circ \alpha F_i^x = \alpha F_i^x$. It follows

$$\alpha F_i^x = F_i^{\alpha_i x} \qquad (i = 1, 2, 3),$$

where α_i are linear transformations of the complex number field.

Now, $F_i^x \circ F_i^y = (x, y)(E_{i+1} + E_{i+2})$, where $\{i, i+1, i+2\}$ is a permutation of $\{1, 2, 3\}$, implies

$$(\alpha_i x, \alpha_i y) = (x, y).$$

Further $F_1^{2x} \circ F_2^{2y} = F_3^{\overline{2(xy)}}$ implies

$$\alpha_1(x)\, \alpha_2(y) = \overline{\alpha_3(\overline{xy})}.$$

If we put $\alpha_1(1) = a$, $\alpha_2(1) = b$, $\alpha_3(1) = c$, $\alpha_1(\sqrt{-1}) = a_1$, $\alpha_2(\sqrt{-1}) = b_1$, and $\alpha_3(\sqrt{-1}) = c_1$, then we have $ab = \bar{c}$, $ab_1 = -\bar{c}_1$, $a_1b_1 = -\bar{c}$ and $a_1b = -\bar{c}_1$. These imply $a^2 + a_1^2 = 0$ with $|a| = |a_1| = 1$ and $b^2 + b_1^2 = 0$ with $|b| = |b_1| = 1$. Therefore, $a_1 = a\sqrt{-1}$

10) U^* is the transposed conjugate matrix of U.

11) Any point $x = [x_1, x_2, x_3]$ of M contains a representative $a = [a_1, a_2, a_3]$ such that $|a_1|^2 + |a_2|^2 + |a_3|^2 = 1$. This process $x \to a$ is called the normalization of x.

or $a_1=-a\sqrt{-1}$ and $b_1=b\sqrt{-1}$ or $b_1=-b\sqrt{-1}$.

At last, we have

$$\begin{cases} \alpha_1(x)=ax \\ \alpha_2(x)=bx \\ \alpha_3(x)=cx \end{cases} \quad \text{or} \quad \begin{cases} \alpha_1(x)=a\bar{x} \\ \alpha_2(x)=b\bar{x} \\ \alpha_3(x)=c\bar{x}\,. \end{cases}$$

with $ab=\bar{c}$, $bc=\bar{a}$ and $ca=\bar{b}$. Therefore if we take an unitary matrix V such that

$$V=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{c} & 0 \\ 0 & 0 & b \end{pmatrix},$$

Then we have $\alpha X=VXV^*$ or $\alpha X=V\bar{X}V^*$. q. e. d.

Therefore, as the group which operates transitively on M_1, we may take the group $PU(3)$ consisting of all automorphisms u of $\mathfrak{J}(M)$ such that $u: X\to UXU^*$ where $X\in\mathfrak{J}(M)$ and U is an unitary matrix of three order. This group $PU(3)$ is isomorphic to the $U(3)/ZU(3)$ (where $U(3)$ is the unitary group of three order and $ZU(3)$ is its central normal subgroup), and is called the elliptic projective group of M_1.

7. Conjecture.

In §5, we have

$$\pi_i(\Pi)=\pi_{i-1}(S^7) \qquad \text{for} \quad 1\leq i\leq 15.$$

However, we conjecture that

$$\pi_i(\Pi)=\pi_i(S^{23})+\pi_{i-1}(S^7)\,.$$

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